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On a free boundary problem for the stationary Navier-Stokes equations

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ABSTRACT. — We investigate stationary flows in a fluid body together with its free boundary. The fluid is assumed to be viscous and incompressible; the free boundary is governed by continuity of the normal stress. Hence the configuration to be considered here can be regarded as a generalization of a classical equilibrium figure. The main tool in proving existence of a regular solution consists in a hard implicit function theorem.

Key words : 76 D 05, 76 U 05, 35 Q 10.

RÉSUMÉ. — Nous étudions les flots stationnaires dans un corps fluide à frontière libre. Le liquide est supposé visqueux et incompressible; la frontière libre est régie par la continuité de la contrainte normale. Les configurations que nous examinons ici peuvent être regardées comme une généralisation d'une figure d'équilibre classique. L'outil principal pour démontré l'existence d'une solution régulière est un théorème de fonction implicite à la Nash-Moser.

1. INTRODUCTION

Consider a drop of a viscous, incompressible fluid under the influence of some exterior force density f. A stationary flow inside the drop can be described by the Navier-Stokes system

$$\left.\begin{aligned} -\nu\Delta v + \mathrm{D}p + v\,.\,\mathrm{D}v &= f \\ \operatorname{div} v &= 0 \end{aligned}\right\} \text{ in } \Omega \tag{1}$$

together with the boundary conditions

$$v\,.\,n=0, \qquad t_k\,.\,\mathrm{T}\,.\,n=0 \text{ on } \Sigma, \quad k=1,2, \tag{2}$$

$$n\,.\,\mathrm{T}\,.\,n=p_0 \text{ on } \Sigma. \tag{3}$$

As usual, $v=(v^1, v^2, v^3)=v(x)$, $x=(x^1, x^2, x^3)\in\mathbb{R}^3$, denotes the velocity, $p=p(x)$ the pressure, and $\nu>0$ is the kinematic viscosity. The unknown domain occupied by the fluid is called Ω, its boundary Σ; n is the outer normal to Σ, and t_1, t_2 span the tangent plane.

With T being the stress tensor,

$$\mathrm{T}_{ij} = -p\,\delta_{ij} + \nu(\mathrm{D}_i v^j + \mathrm{D}_j v^i), \tag{4}$$

$\mathrm{D}_i=\partial/\partial x^i$, $\mathrm{D}=(\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3)$, the dynamical boundary conditions in (2) state that the fluid cannot resist tangential stresses. Equation (3) governs the free boundary: Σ adjusts itself such that the fluid's normal stress equals the given pressure p_0 which is assumed to be constant throughout $\mathbb{R}^3\setminus\bar{\Omega}$. The volume $|\Omega|$ of the drop is prescribed, too; after a suitable transformation we can always obtain

$$|\Omega| = \frac{4}{3}\pi. \tag{5}$$

In this paper we prove the existence of classical solutions to the free boundary problem (1)-(3) for various configurations. A typical result is contained in

THEOREM 1. — *Let*

$$f_0(x) = \mathrm{D}\mathrm{U}(x), \qquad \mathrm{U}(x) = \int_\Omega g\,|x-y|^{-1}\,dy, \tag{6}$$

be the force of self-attraction. For $f=f_0+h$, $h\in C^{\lambda+\mu}$ *with* $\lambda>6$ *and* (in cylindrical coordinates r, θ, x^3)

$$h^\theta=h^\theta(r,x^3)=-h^\theta(r,-x^3), \qquad r^2=(x^1)^2+(x^2)^2,$$

and

$$h^3\equiv h^r\equiv 0 \tag{7}$$

$\|h\|_{C^{\lambda+\mu}}$ *small enough, there exists a unique solution* $v\in C^{5+\mu}(\bar\Omega)$, $p\in C^{4+\mu}(\bar\Omega)$, *and* $\Sigma\in C^{6+\mu}$ *to the free boundary problem* (1)-(3). v *and* p *are small in the sense that*

$$\|v\|_{C^{5+\mu}}+\|p-U\|_{C^{4+\mu}}\leqq C\|h\|_{C^{\lambda+\mu}}, \tag{8}$$

and Σ *lies in a* $C^{6+\mu}$*-neighborhood of the unit sphere* S; *the* $C^{6+\mu}$*-norm of the distance of* Σ *from* S *can again be estimated by* $C\|h\|_{C^{\lambda+\mu}}$.

Remark. — The solution established in this theorem can be interpreted as perturbation of a classical equilibrium figure Σ_0 of a rotating liquid. Although Σ_0 is the trivial solution, namely the unit sphere, which provides an equilibrium figure for zero angular velocity, the method of proof is based to a large extend on results for more general equilibrium figures. Therefore we can cover also other geometrical configurations by the same approach; the details are given in paragraph 6.

To motivate our method of proof we compare the problem (1)-(3) with related free boundary problems for the Navier-Stokes equations. The time-dependent analogue to (1)-(3) was solved by Solonnikov [14]. He used the transformation

$$(x,t)=\left(X+\int_0^t V(X,\tau)\,d\tau,\ t\right) \tag{9}$$

with $X\in\Omega(0)$, where $\Omega(t)$ is the (unknown) domain occupied by the fluid at time t; V denotes the velocity as a function of Lagrangian variables, i. e. the velocity of a particle at time t that was initially at $X\in\Omega(0)$. By (9) the domain $\bigcup_{0<t<T}\Omega(t)$ can be mapped onto the space-time cylinder $\Omega(0)\times(0,T)$ which is a known domain since the initial datum $\Omega(0)$ is given. As the unknown V is contained in (9) the transformation leads to a highly nonlinear system; however, the kinematical boundary condition for $v.n$ is automatically satisfied, and hence the Navier-Stokes equations with four conditions on the free boundary are reduced to a Neumann

problem on a given boundary. This elegant device [1] cannot be used for the stationary problem (1)-(3) because a flow is called stationary if v and p, regarded as functions of the position x rather than of the Lagrange variable X do not depend on time; as function of X a stationary velocity field does change in time except for trivial cases. Very often the method to handle a parabolic problem is modeled after the one used in the elliptic case; here we meet quite a different situation.

As a reduction to a boundary value problem on a fixed domain seems not to be available we will apply successive approximations. This method turned out to be useful in the related free boundary problem for stationary viscous flows where surface tension governs the free boundary, *cf.* [1]; then (3) has to be replaced by

$$n.\mathrm{T}.n=2\kappa\mathrm{H}\quad \text{on}\,\Sigma \tag{10}$$

where H denotes the mean curvature of Σ, and $\kappa=\text{Const.}$ is the coefficient of surface tension. To solve (1), (2), (10) we construct a sequence $\{v_n, p_n, \Sigma_n\}_{n=1}^{\infty}$, where (v_n, p_n) is the solution to the equations of motion (1), (2) in the domain Ω_{n-1} that is bounded by Σ_{n-1}, and Σ_n can be determined from (10) when T is evaluated at (v_n, p_n). To $v\in \mathrm{C}^{2+\mu}$, $p\in \mathrm{C}^{1+\mu}$ the mean curvature equation (10) yields a $\mathrm{C}^{3+\mu}$-surface, and on the other hand, for a solution v and p to (1), (2) the norm $\|v\|_{\mathrm{C}^{2+\mu}}+\|p\|_{\mathrm{C}^{1+\mu}}$ can be estimated by the $\mathrm{C}^{3+\mu}$-norm of the boundary Σ of the underlying domain. In this way the approximation procedure can be carried out in $\mathrm{C}^{2+\mu}\times \mathrm{C}^{1+\mu}\times \mathrm{C}^{3+\mu}$: all approximations $\{v_n, p_n, \Sigma_n\}$, if defined as above, lie in this space.

Successive approximations can yield only local existence theorems; therefore we can restrict ourselves to free boundaries Σ that lie in a neighborhood of a given boundary Σ_0. In the simplest case, when Σ_0 will be the unit sphere S, Σ can be represented as a graph over S:

$$\Sigma=\{(\xi,\rho):\xi\in \mathrm{S}, \rho=1+\zeta(\xi), \zeta:\mathrm{S}\to\mathbb{R}\}. \tag{11}$$

If we describe Σ in this way (3) becomes an equation for the scalar unknown ζ.

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[1] Lagrangian variables are commonly used when the kinematics and dynamics of fluid motions are studied in order to derive the equations of motions. Once these are established one usually works with Eulerian variables. This seems due to the fact that almost all mathematical contributions about the Navier-Stokes equations concern problems where the domain occupied by the fluid is fixed, and not free boundary problems.

If we insert ζ into (10) it becomes a second-order, elliptic equation with data of class $C^{1+\mu}$; hence the solution will be $C^{3+\mu}$ as is needed for the approximation scheme to converge. Formally, (3) follows from (10) by letting κ tend to zero. As κ is a coefficient in the principal part of (10) this indicates that (3) is no differential equation anymore for which a general existence theory has been developed. To get a solution of (3), the particular physical background must be taken into account, and it is obvious that for $T \in C^{1+\mu}$ the solution ζ will be at most of class $C^{1+\mu}$, too. Compared with the regularity of (10) above this property causes a loss of two derivatives in each approximation step. For such problems hard implicit function theorems provide a very powerful tool. This method was initiated by J. Nash [11] and J. Moser [10]; here a version of Moser's theorem due to Zehnder [18] will be applied, it is particularly suited to our problem as will be explained in paragraph 2.

We will close these introductory remarks with a brief discussion of the force of self-attraction f_0 and of the local nature of our existence results. The free boundary ζ is determined by a first order differential equation (3) which, as just remarked, seems not to be solvable in the general case. Therefore we introduce the force f_0 of self-attraction; as it is a gradient its potential U can be absorbed into the pressure p in (2) ([2]), and hence (3) becomes

$$\int_\Omega g\,|x-y|^{-1}\,dy = -p + \nu(D_i v^j + D_j v^i)\,n^i n^j. \tag{12}$$

Now the unknown ζ appears in the domain of integration Ω as well as in n. If we regard the left hand side as the principal part of equation (12), then, according to Lichtenstein [7], the integral $\int_\Omega g\,|x-y|^{-1}\,dy$ can be written in the form

$$\psi_0(\xi)\,\zeta(\xi) + \oint_S \frac{\zeta(\eta)}{d(\xi,\eta)}\,d\omega(\eta) + N(\zeta)(\xi) \equiv M\,\zeta(\xi) + N(\zeta)(\xi) \tag{13}$$

where $\psi_0(\xi)$ is the normal derivative of the Newtonian potential of S; $d(\xi, \eta)$ denotes the Euclidean distance between two points ξ, $\eta \in S$. The

([2]) A potential force in f does not affect v in (1), (2) since the pressure does not enter into the boundary condition (2).

nonlinear operator $N(\zeta)$ will be discussed in paragraphs 2, 4, where we will also show that the solvability of (12) can be reduced to the existence of the inverse of the integral operator $M\zeta$. In this way the introduction of f_0 as dominating force leads to an equation for the free boundary which can be handled. But also for physical reasons f_0 must be regarded as necessary. Self-attraction tends to hold the drop together and therefore balances other forces h that may act in the opposite way.

There exists a counterexample due to McCready [9] which, although found in quite a different context, supports our interpretation of f_0 as being necessary. McCready shows that it is impossible to give an *a priori* bound on Dirichlet's integral of a solution v to the Navier-Stokes equations if Neumann rather than Dirichlet conditions are given. Hence the basic *a priori* estimate which was first proven by Leray to establish the existence of a global solution (i. e. a solution for arbitrary large data f) fails in this case. In the context of free boundary problems this example indicates that only local solutions exist because there are no longer rigid walls which hold the fluid together regardless of the possibly very large forces that generate the fluid's motion.

2. THE LINEARIZED EQUATIONS

The free boundary problem (1)-(3) can be regarded as an abstract equation

$$F(f, z)=0, \tag{14}$$

where we define F by assigning to $z=(v, p, \Sigma)$ the right-hand sides of (1)-(3). As we want to solve (14) by means of a hard implicit function theorem we have to investigate the linearization of F

$$D_2 F(f, z)\tilde{z} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\{F(f, z+\varepsilon \tilde{z})-F(f, z)\}, \tag{15}$$

I am indebted to Professor S. Hildebrandt for aquainting me with Lichtenstein's paper [7] and to Professor E. Zehnder who introduced me into the method of hard implicit function theorems. This version of the paper was written at the Mittag-Leffler Institute in Djursholm. It is a great pleasure to thank the insiitute and its director, Professor L. Hörmander, for their kind hospitality.

but this can only be defined if the domain of F admits an affine structure: if (v, p) is given on $\bar{\Omega}$ where $\partial\Omega=\Sigma$, and if $\hat{v}$ and $\hat{p}$ are defined on the closure of another domain $\hat{\Omega}$ with $\partial\hat{\Omega}=\hat{\Sigma}$ we have to define their sum $z+\hat{z}\equiv(v, p, \Sigma)+(\hat{v}, \hat{p}, \hat{\Sigma})$, at least in a neighborhood of some given configuration $z_0=(v_0, p_0, \Sigma_0)$.

As we indicated already in (11) we restrict ourselves to surfaces Σ that are graphs over some known closed surface Σ_0:

$$\Sigma=\{x\in\mathbb{R}^3 : x=x_0+\zeta(x_0)\,n_0(x_0),\ x_0\in\Sigma_0, \zeta:\Sigma_0\to\mathbb{R}\}, \tag{16}$$

where $n_0(x_0)$ denotes the normal to Σ_0 at a point $x_0\in\Sigma_0$.

In certain cases we may use coordinates (ξ^1, ξ^2, ρ) on $\mathbb{R}^3$, where $\xi=(\xi^1, \xi^2)$ are local coordinates on Σ_0; this leads to the simpler expression

$$\Sigma=\{x\in\mathbb{R}^3 : x=(\xi^1, \xi^2, \rho) : \rho=1+\zeta(\xi),$$
$$\Sigma_0=\{(\xi, \rho) : \rho=1\}, \zeta:\Sigma_0\to\mathbb{R}\}. \tag{17}$$

Here we may think of spherical or elliptic coordinates. Due to the local structure of our existence theorems we can always assume $|\zeta|<1$; the domain included by Σ is then of the form

$$\Omega=\{x\in\mathbb{R}^3 : x=(\xi^1, \xi^2, \rho), 0\leqq\rho<1+\zeta(\xi), \xi\in\Sigma_0, \zeta:\Sigma_0\to\mathbb{R}\}, \tag{18}$$

and similarly in the general case. We now can define the «sum» of two boundaries Σ and $\hat{\Sigma}$ by

$$\Sigma+\hat{\Sigma}=\{(\xi, \rho) : \rho=1+\zeta(\xi)+\hat{\zeta}(\xi), \xi\in\Sigma_0\} \tag{19}$$

where ζ and $\hat{\zeta}$ are the scalar functions describing Σ and $\hat{\Sigma}$ resp. in (17). The representation of Σ by ζ leads immediately to a one-to-one transformation of Ω onto a standard domain Ω_0 (which is the one bounded by Σ_0); for this reason consider for $(\xi, \rho)\in\bar{\Omega}$ the mapping

$$(\xi, \rho)\to\sigma(\xi, \rho):=\left(\xi, \frac{\rho}{1+\zeta(\xi)}\right). \tag{20}$$

From our assumption $|\zeta|<1$ it is obvious that σ is invertible; furthermore σ as well as σ^{-1}, when considered as a function of ξ and ρ are as regular as $\zeta(\xi)$; derivatives $D^\alpha\sigma$, $D^\alpha\sigma^{-1}$ with $|\alpha|\leqq k$ can be estimated by derivatives of ζ of the same order and vice versa. If we assign to points $(\xi, \rho)\in\bar{\Omega}$ and $\sigma(\xi, \rho)\in\bar{\Omega}_0$ its cartesian coordinates x and y, the transformation (2) defines unambiguously an invertible relation between the cartesian coordinates $x \in \bar{\Omega}$ and $y \in \bar{\Omega}_0$ which we denote again by σ:

$$y = \sigma(x), \qquad x = \sigma^{-1}(y). \tag{21}$$

Remark. – We note that it is also possible to choose a domain Ω_τ instead of Ω_0 which belongs to the solution z_0 of (14), such that Ω, and all quantities defined on it, are represented by functions on Ω_τ. Here Ω_τ is defined by a 1-to-1 mapping $\tau : \Omega_0 \to \Omega_\tau$. If we call the transformation from Ω to Ω_τ again σ, the structure of the equations (25)-(35) below is precisely the same. As we will see in paragraph 3 an appropriate choice of Ω_τ sometimes yields a considerable advantage.

Using the transformations from (21) we can now write the dependent variables v and p as functions of $y \in \Omega_0$; in this way it becomes possible to define the sum of two velocity fields v and $\hat{v}$ that are initially given on different domains Ω and $\hat{\Omega}$. We set

$$u^i(y) := \left(\det \frac{\partial \sigma^i}{\partial x^j}\right)^{-1} \frac{\partial \sigma^i}{\partial x^j}(x)\, v^j(x) \tag{22}$$

$$q(y) := p(x)$$

again with $x \in \bar{\Omega}$, $y \in \bar{\Omega}_0$ and $y = \sigma(x)$.

Remark. – The advantage of the transformation (22) over the simpler one $u^i(y) := v^i(\sigma^{-1}(y))$ which one probably expects consists in the fact that (22) maps fields $v(x)$ with $\sum_{i=1}^{3} \frac{\partial}{\partial x^i} v^i(x) = 0$ into solenoidal fields u again, now divergence-free with respect to the new coordinates: $\sum_{i=1}^{3} \frac{\partial}{\partial y^i} u^i(y) = 0$. If on the other hand the regularity of v and u is considered, (22) seems to lead to a certain disadvantage since for $v \in C^2(\bar{\Omega})$ the new function u is of class C^2 only if σ is three-times differentiable. As we will see later in Lemma 4 we need three derivatives of Σ (that is of σ) anyway to bound second derivatives of u if u solves the equation of motions. Hence, as far as solutions to (1), (2) and their linearizations are concerned, (22) only simplifies the problem.

Now let (v, p) and $(\hat{v}, \hat{p})$ be given functions that are defined on Ω and $\hat{\Omega}$ resp., where the domains are characterized by scalar functions ζ and $\hat{\zeta}$ as in (18). Let σ and $\hat{\sigma}$ be the transformations (20) that are induced by ζ and $\hat{\zeta}$. If we insert v, p, σ and $\hat{v}$, $\hat{p}$, $\hat{\sigma}$ into (22), (23) we obtain as transformed velocity fields and pressures (u, q) and $(\hat{u}, \hat{q})$. These functions now have the same domain of definition Ω_0 such that $u + \hat{u}$ and $q + \hat{q}$ are well-defined quantities. As we will see next our free-boundary problem admits a formulation

$$\mathrm{F}(f, z)=0, \qquad z=(u, q, \zeta) \tag{24}$$

in terms of the functions $u: \bar{\Omega}_0 \to \mathbb{R}^3$, $q: \bar{\Omega}_0 \to \mathbb{R}$, $\zeta: \Sigma_0 \to \mathbb{R}$; it is then possible to define the linerarization $\mathrm{D}_2 \mathrm{F}(f, z)$.

If we apply the transformations (21)-(23) to the Navier-Stokes equations (1) we obtain after a formal but tedious calculation

$$\left.\begin{aligned} \mathrm{L} u^i + \bar{a}_{ij} \mathrm{D}_j q + \mathrm{N}_i(u, \mathrm{D} u) &= \tilde{a}_{ij} f^j \\ \mathrm{D}_j u^j &= 0 \end{aligned}\right\} \text{in } \Omega_0 \tag{25}$$

with

$$\mathrm{L} u^i = -\nu \mathrm{D}_k (a_{kl} \mathrm{D}_l u^i) + b_{ikl} \mathrm{D}_k u^l + c_{ij} u^j \tag{26}$$

$$\mathrm{N}_i(u, \mathrm{D}u) = a^{-1} u^j \mathrm{D}_j u^i + \tilde{b}_{ikl} u^k u^l. \tag{27}$$

Here D_j means partial differentiation with respect to the new variables y^j. The various coefficients depend on the transformation σ and its derivatives, namely

$$\left.\begin{aligned} a_{ij} &= \frac{\partial \sigma^i}{\partial x^k} \frac{\partial \sigma^j}{\partial x^k}, \qquad a = \left(\det \frac{\partial \sigma^i}{\partial x^j}\right)^{-1}, \\ \bar{a}_{ij} &= a_{ij} a \\ b_{ikl} &= \delta_{li} \frac{\partial}{\partial y^n} a_{nk} - \delta_{li} \Delta_x \sigma^k - 2 \frac{\partial \sigma^i}{\partial x^r} \frac{\partial \sigma^k}{\partial x^s} a \frac{\partial}{\partial x^s}\left(a^{-1} \frac{\partial (\sigma^{-1})^r}{\partial y^l}\right) \\ c_{ij} &= -\frac{\partial \sigma^i}{\partial x^n} a \Delta_x \left(\frac{\partial (\sigma^{-1})^n}{\partial y^j} a^{-1}\right), \\ \tilde{a}_{ij} &= \frac{\partial \sigma^i}{\partial x^j} a, \\ \tilde{b}_{ikl} &= \frac{\partial \sigma^i}{\partial x^n} \frac{\partial (\sigma^{-1})^m}{\partial y^k} \frac{\partial}{\partial x^m}\left(\frac{\partial (\sigma^{-1})^n}{\partial y^1} a^{-1}\right) \end{aligned}\right\} \tag{28}$$

To indicate that L and the coefficients depend on σ (and therefore on ζ) we sometimes write $\mathrm{L}(\zeta)$, $a_{ij}(\zeta)$ etc. It is understood that coefficients a_{ij}, a, and α_{ij} etc. (*see* below) depend on σ and its first derivatives, b_{ijk}, β_{ij} etc. on σ, $\mathrm{D}\sigma$, $\mathrm{D}^2\sigma$, and finally c_{ij} on σ and its derivatives up to order three.

In the same way the boundary conditions (2), (3) are transformed; we obtain

$$\alpha_i u^i = 0, \qquad \alpha_{ijk} \mathrm{D}_i u^j + \beta_{jk} u^j = 0 \text{ on } \Sigma_0, \quad k = 1, 2 \tag{29}$$

$$\alpha_{ij} \mathrm{D}_i u^j + \beta_j u^j + q = 0 \text{ on } \Sigma_0. \tag{30}$$

As we outlined in paragraph 1 a special form of the equation (3) for the free boundary arises when the force of self-attraction f_0 is introduced. Instead of (30) we have

$$\psi_0(\xi)\,\zeta(\xi) + \oint_{\Sigma_0} \frac{\zeta(\xi')}{d(\xi,\xi')} d\omega(\xi') + \mathrm{N}(\zeta)(\xi) - \mathrm{G}(u, q)(\xi) = 0 \tag{31}$$

where

$$\psi_0(\xi) = \frac{\partial}{\partial n_0} \mathrm{U}_0(\xi) \tag{32}$$

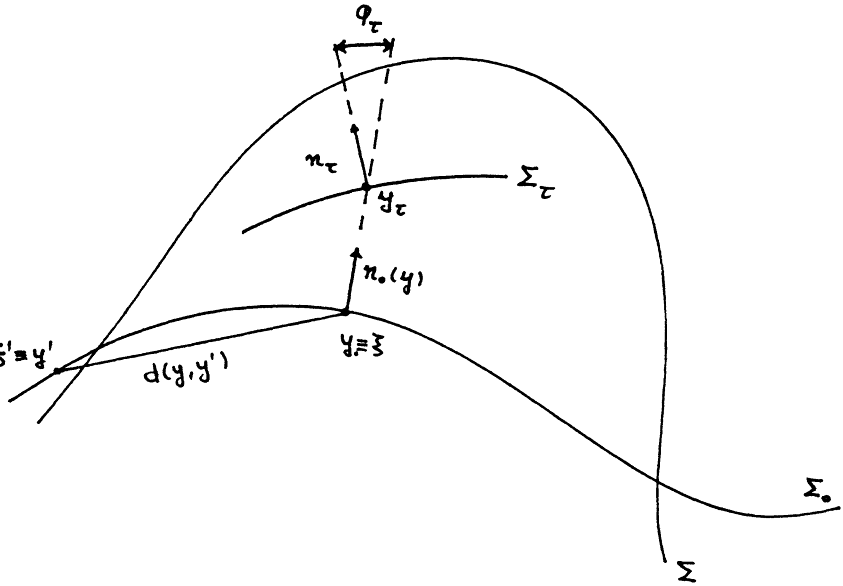
$$\mathrm{G}(u, q) = \alpha_{ij} \mathrm{D}_i u^j + \beta_j u^j + q \tag{33}$$

$$\mathrm{N}(\zeta) = \mathrm{R}_1(\zeta) + \mathrm{R}_2(\zeta) + \mathrm{R}_3(\zeta) \tag{34}$$

$$\left.\begin{aligned} \mathrm{R}_1(\zeta)(\xi) &= \int_0^{\zeta(\xi)} \left\{ \frac{\partial}{\partial n_0} \mathrm{U}(y_0 + \tau n_0) - \frac{\partial}{\partial n_0} \mathrm{U}(y_0) \right\} d\tau \\ \mathrm{R}_2(\zeta)(\xi) &= \zeta(\xi) \int_{\mathscr{S}} |y_0 - y'|^{-1} dy' \\ \mathrm{R}_3(\zeta)(\xi) &= \oint_{\Sigma_0} \int_0^{\zeta(\xi)} \left\{ \frac{1}{|y_0 - y'|} \frac{d\bar{\omega}_\tau}{d\omega} - \frac{1}{|y_0 - y'|} \right\} d\tau\, d\omega(\xi'). \end{aligned}\right\} \tag{35}$$

Here we use the same symbols as in Lichtenstein's paper [7], where the integral equation (31) is derived; *cf.* also Figure below.

$\mathrm{U}_0(y)$ and $\mathrm{U}(y)$ denote the Newtonian potentials $\int |y - y'|^{-1} dy'$ over Ω_0 and Ω resp.; for $y \in \Sigma_0$ we write $\mathrm{U}_0(\xi)$ instead of $\mathrm{U}_0(y(\xi))$. The domain $\mathscr{S}$ is defined as $(\Omega \setminus \Omega_0) \cup (\Omega_0 \setminus \Omega)$; $d\bar{\omega}_\tau$ is the surface element $d\omega_\tau$ of $\Sigma_\tau \equiv \{ y_\tau = y + \tau\zeta(y) n_0(y),\ y \in \Sigma_0 \}$ for $0 \leqq \tau \leqq 1$, multiplied by cos φ_τ, where φ_τ is the angle between the normals n_0 and $n_\tau(y_\tau)$. Finally $d(\xi, \xi')$ is the Euclidean distance between points ξ and ξ' of Σ_0.



In this way we obtain the following formulation for (14): $F^i(f, z)=0$, $i=1, \ldots, 4$ is of the form

$$\left.\begin{array}{c} L(\zeta)\,u^i+\bar{a}_{ij}(\zeta)\,D_j q+N_i(u, D u, \zeta)-\tilde{a}_{ij}(\zeta) f^j=0 \quad \text{in}\,\Omega_0, \\ i=1,2,3 \\ D_j u^j=0 \end{array}\right\} \tag{36}$$

together with the boundary conditions

$$\left.\begin{array}{c} \alpha_i(\zeta)\,u^i=0, \\ \alpha_{ijk}(\zeta)\,D_i u^j+\beta_{jk}(\zeta)\,u^j=0 \quad \text{on}\,\Sigma_0, \qquad k=1,2; \end{array}\right\} \tag{37}$$

$F^5(f, z)=0$ becomes

$$M\zeta+N(\zeta)-G(\zeta, u, q)=0 \quad \text{on}\,\Sigma_0, \tag{38}$$

with $M\zeta$ being the linear integral operator as in (13).

F is then a mapping from $\mathscr{Y}_0 \times \mathscr{Z}_0$ into $\mathscr{X}_0$, where

$$\left.\begin{array}{c} \mathscr{X}_0=C^{0+\mu}(\bar{\Omega}_0, \mathbb{R}^3) \times C^{0+\mu}(\bar{\Omega}_0, \mathbb{R}) \times C^{0+\mu}(\Sigma_0, \mathbb{R}) \\ \mathscr{Y}_0=C^{0+\mu}(\mathbb{R}^3, \mathbb{R}^3) \\ \mathscr{Z}_0=C^{2+\mu}(\bar{\Omega}_0, \mathbb{R}^3) \times C^{1+\mu}(\bar{\Omega}_0, \mathbb{R}) \times C^{3+\mu}(\Sigma_0, \mathbb{R}). \end{array}\right\} \tag{39}$$

Next we compute the linearized operator $D_2 F(f, z)\tilde{z}$; it is of the following form with $\tilde{z}=(\tilde{u}, \tilde{q}, \tilde{\zeta})$:

$$\left.\begin{aligned} D_2 F^i(f, z)\tilde{z} &= L(\zeta)\tilde{u}^i + \bar{a}_{ij}(\zeta) D_j \tilde{q} + l_{ij}(u, \zeta)\tilde{u}^j + l_j(u, \zeta) D_j \tilde{u}^i \\ &\quad + \sum_{|\gamma| \leqq 3} l_\gamma(u, q, \zeta) D^\gamma \tilde{\sigma} + \sum_{|\gamma| \leqq 1} m_\gamma(f, \zeta) D^\gamma \tilde{\sigma}, \qquad i=1, 2, 3 \\ D_2 F^4(f, z)\tilde{z} &= D_j \tilde{u}^j \end{aligned}\right\} \tag{40}$$

$$\begin{aligned} D_2 F^5(f, z)\tilde{z} &= M\tilde{\zeta} + m_0(\zeta)\tilde{\zeta} \\ &\quad + \sum_{|\gamma| \leqq 2} r_\gamma(u, q, \zeta) D^\gamma \tilde{\zeta} + m_{ij}(\zeta) D_i \tilde{u}^j + m(\zeta)\tilde{q} \end{aligned} \tag{41}$$

together with the boundary conditions for the operators in (40)

$$\left.\begin{aligned} \alpha_i(\zeta)\tilde{u}^i + \sum_{|\gamma| \leqq 1} \mu_\gamma(u, \zeta) D^\gamma \tilde{\zeta} &= 0 \\ \alpha_{ijk}(\zeta) D_i \tilde{u}^j + \beta_{kj}(\zeta)\tilde{u}^j + \sum_{|\gamma| \leqq 2} \kappa_\gamma(u, p, \zeta) D^\gamma \tilde{\zeta} &= 0. \end{aligned}\right\} \tag{42}$$

The coefficients l, m, n are obtained by a straightforward but lengthly calculation. For the purpose of this paper, however, it suffices to concentrate on their regularity in terms of the regularity of u, q and ζ and their smallness if u, q and ζ become small. These properties are listed below:

(i) $l_{ij}(u, \zeta)\tilde{u}^j + l_j(u, \zeta) D_j \tilde{u}^i$ is obtained by differentiating $N_i(u, Du, \zeta)$ from (36) with respect to u. This gives, *cf.* (27), (28):

$$l_{ij} = (b_{ikj} + b_{ijk}) u^k + a^{-1} D_j u^i$$
$$l_j = a^{-1} D_j u^i.$$

As a and b_{ijk} are analytic in ζ their $C^{0+\mu}$-norm is finite for $\zeta \in C^{3+\mu}$, and for $\| u \|_{C^{1+\mu}}, \| \zeta \|_{C^{2+\mu}}$ small, the coefficients l_{ij} and l_j are small in the $C^{0+\mu}$-norm, too.

(ii) $l_\gamma(u, p, \zeta)$ denotes the coefficients of $D^\gamma \sigma$ where γ is a multi-index with $|\gamma| = 0, \ldots, 3$, when we differentiate $L(\zeta) u^i + \bar{a}_{ij}(\zeta) D_j q + N_i(u, Du, \zeta)$ with respect to σ. As σ and its derivatives up to third order occur in these expressions we get the sum $\sum_{|\gamma| \leqq 3} l_\gamma(u, q, \zeta) D^\gamma \tilde{\sigma}$, where we collect in l_γ the derivatives of all coefficients with respect to $D^\gamma \sigma$. Regularity and smallness of the l_γ are clear as the expressions in (28) are analytic in σ and its derivatives. This holds for all coefficients below in the same way.

(iii) $m_\gamma(f, \zeta)$ is derived from differentiating $\tilde{a}_{ij}(\zeta) f^j$ with respect to σ.

(iv) $m_0(\zeta)$ consists of derivatives of the integral in $N(\zeta)$ from (34), (35) with respect to ζ. We obtain e. g.

$$\left(\frac{d}{d\zeta} R_1(\zeta)\right) = \left\{\frac{\partial}{\partial n_0} U(y_0 + \zeta n_0) - \frac{\partial}{\partial n_0} U(y_0)\right\} + \int_0^\zeta \frac{\partial}{\partial n_0} \frac{d}{d\zeta} (U(y_0 + \tau n_0) - U(y_0))\, d\tau.$$

Note that U depends on ζ, too. The smallness of $m_0(\zeta)$ has been proved by Lichtenstein [7] (25).

(v) $r_\gamma(u, q, \zeta)$ are the coefficients that occur when G in (31), (33) is differentiated with respect to ζ. As ζ and its derivatives up to second order occur we collect again the derivatives of the coefficients with respect to $D^\gamma \zeta$ and call them r_γ.

(vi) $m_{ij}(\zeta)$, $m(\zeta)$ is given by differentiating G with respect to u and q respectively.

In the boundary conditions (42) we finally have

(vii) $\mu_\gamma(u, \zeta)$ from differentiating α_i in (29) with respect to ζ, and

(viii) $\kappa_\gamma(u, p, \zeta)$ from α_{ijk} and β_{jk}.

To apply the implicit function theorem we must invert the operator $D_2 F(f, z)$ in a neighborhood of z_0. But only at z_0 itself the system (40)-(42) splits into a linearized Navier-Stokes system for $(\tilde{u}, \tilde{q})$ and an equation for $\tilde{\zeta}$. For $z \neq z_0$ all components of $\tilde{z}$ appear in each equation (except the fourth one).

This establishes the essential difference to the simpler approximation scheme from [1] which was mentioned already in paragraph 1. There one works only with the "diagonal" part of the gradient, namely $D_{z^i} F^j(f, z)$, $i, j = 1, \ldots, 4$ and $D_{z^5} F^5(f, z)$; this procedure however cannot be better than of first order and is therefore not appropriate for the hard implicit function theorem.

The expressions $l_\gamma(u, q, \zeta) D^\gamma \tilde{\sigma}$ cannot be regarded as small perturbations of the Navier-Stokes equations because they contain third derivatives in the unknown $\tilde{\sigma}$ (that is $\tilde{\zeta}$). Therefore the linearization $D_2 F(f, z) \tilde{z}$ admits only an approximate inverse; before we can define it we have to recall Zehnder's implicit function theorem.

3. ZEHNDER'S HARD IMPLICIT FUNCTION THEOREM AND THE APPROXIMATE INVERSE

The implicit function theorem on which this paper is based is formulated within the following framework, *cf.* Zehnder [18], pp. 118-121. Let $\{\mathscr{Z}_t\}_{t\geqq 0}$ be a one-parameter family of Banach spaces with norms $|\,.\,|_t$ such that for all t, t' with $0 \leqq t' \leqq t < \infty$ there holds

$$\mathscr{Z}_0 \supseteq \mathscr{Z}_{t'} \supseteq \mathscr{Z}_t \supseteq \mathscr{Z}_\infty \equiv \bigcap_{t>0} \mathscr{Z}_t \tag{43}$$

and

$$|z|_{t'} \leqq |z|_t \quad \text{for all } z \in \mathscr{Z}_t, \qquad t' \leqq t. \tag{44}$$

We assume the same properties to hold for $\{\mathscr{Y}_t\}_{t\geqq 0}$ and $\{\mathscr{X}_t\}_{t\geqq 0}$ also. As outlined already in (14), (39) we are given a mapping F from $\mathscr{D} \subset \mathscr{Y}_0 \times \mathscr{Z}_0$ with range $\mathscr{R} \subset \mathscr{X}_0$ such that

$$\mathrm{F}(f_0, z_0) = 0 \tag{45}$$

with f_0 and $z_0 = (0, \mathrm{U}_0(x), 0)$ as in (6).

HYPOTHESIS H.1. — *Assume* $\mathrm{F}: \mathscr{B}_0 \to \mathscr{X}_0$, *where*

$$\mathscr{B}_t = \{ (f, z) \in \mathscr{Y}_t \times \mathscr{Z}_t : |f - f_0|_t < 1, |z - z_0|_t < 1 \}, \tag{46}$$

to be continuous in (f, z) *and two times differentiable in* z; *furthermore there exists a constant* M_0 *such that*

$$\left.\begin{array}{c} |\mathrm{D}_2 \mathrm{F}(f, z)|_0, |\mathrm{D}_2^2 \mathrm{F}(f, z)|_0 \leqq \mathrm{M}_0, \\ \forall (f, z) \in \mathscr{B}_0. \end{array}\right\} \tag{47}$$

HYPOTHESIS H.2. — *For all* (f, z), $(f', z) \in \mathscr{B}_0$ *there holds*

$$|F(f, z) - \mathrm{F}(f', z)|_0 \leqq \mathrm{M}_0 |f - f'|_0. \tag{48}$$

HYPOTHESIS H.3. — (F, f_0, z_0) *is of order* s *with* $s > \gamma \geqq 1$, γ *as in* H.4 *below. This means*

$$(f_0, z_0) \in \mathscr{Y}_s \times \mathscr{Z}_s \tag{49}$$

$$\mathrm{F}(\mathscr{B}_0 \cap (\mathscr{Y}_t \times \mathscr{Z}_t)) \subset \mathscr{X}_t, \qquad \forall t \in [1, s]. \tag{50}$$

For every $t \in [1, s]$ *there exists a constant* M_t *such that*

$$|F(f, z)|_t \leqq M_t K \tag{51}$$

for all $(f, z) \in (\mathscr{Y}_t \times \mathscr{Z}_t) \cap \mathscr{B}_1$ *with*

$$|f - f_0|_t, |z - z_0|_t < K. \tag{52}$$

HYPOTHESIS H.4. — *For every* $(f, z) \in \mathscr{B}_\gamma$ *there exists a linear map* $H(f, z): \mathscr{X}_\gamma \to \mathscr{Z}_0$ *such that*

$$|H(f, z)(\varphi)|_0 \leqq M_0 |\varphi|_\gamma, \qquad \forall \varphi \in \mathscr{X}_\gamma. \tag{53}$$

$H(f, z)$ *is furthermore a continuous mapping from* $\mathscr{X}_t$ *into* $\mathscr{Z}_{t-\gamma}$ *provided* $(f, z) \in \mathscr{B}_\gamma \cap (\mathscr{Y}_t \times \mathscr{Z}_t)$. *Moreover there holds*

$$|H(f, z)(F(f, z))|_{t-\gamma} \leqq M_t K \tag{54}$$

for all (f, z) *that satisfy* (52).

$H(f, z)$ *is an approximate inverse to* $D_2 F(f, z)$ *in the sense that*

$$|[D_2 F(f, z) \circ H(f, z) - 1](\varphi)|_0 \leqq M_0 |F(f, z)|_\gamma |\varphi|_\gamma \tag{55}$$

holds for all $\varphi \in \mathscr{X}_\gamma$.

Under these hypotheses we have the following hard implicit function theorem:

THEOREM 2 (E. Zehnder [18], p. 121). — *Let* (F, f_0, z_0) *be of order s and satisfy* (H.1)-(H.4). *Then there exist an open neighborhood* $\mathscr{D}_\lambda = \{f \in \mathscr{Y}_\lambda : |f - f_0| < C\}$ *and a mapping* $\Psi: \mathscr{D}_\lambda \to \mathscr{Z}_\rho$ *such that for all* $f \in \mathscr{D}_\lambda$

$$F(f, z) = 0 \qquad \text{with} \quad z = \Psi(f) \tag{56}$$

and

$$|z - z_0|_\rho \leqq C^{-1} |f - f_0|_\lambda \tag{57}$$

$\Psi: \mathscr{D}_\lambda \to \mathscr{Z}_\rho$ *is continuous whenever* H *is.*

The numbers λ and ρ are chosen to be $\rho = 3$, $\lambda > 6$, *cf.* also paragraph 6 for a more detailed discussion of the minimal regularity assumptions.

To prove the hypothesis above we chose the quantities $\mathscr{X}_t$, $\mathscr{Y}_t$, $\mathscr{Z}_t$ in the following way:

$$\left.\begin{aligned}\mathscr{X}_t&=C^{t+\mu}(\tilde{\Omega}_0,\mathbb{R}^3)\times C^{t+\mu}(\bar{\Omega}_0,\mathbb{R}^3)\times C^{t+\mu}(\Sigma_0,\mathbb{R})\\ \mathscr{Y}_t&=C^{t+\mu}(\mathbb{R}^3,\mathbb{R}^3)\\ \mathscr{Z}_t&=C^{t+2+\mu}(\bar{\Omega}_0,\mathbb{R}^3)\times C^{t+1+\mu}(\bar{\Omega}_0,\mathbb{R})\times C^{t+3+\mu}(\Sigma_0,\mathbb{R}).\end{aligned}\right\}\quad(58)$$

If $F(f, z)$ is given via a transformation of Ω onto Ω_0 [or Ω_τ, *cf.* the remark about the mapping σ in paragraph 2 (21)] the coefficients are analytic functions in f, z, ζ, σ etc. Hence continuity in (f, z) and differentiability in z are obvious; (47) follows from the choice of the function spaces in (58). For $(f, z)\in\mathscr{B}_0$ implies that the components of z, i. e. u, q, and ζ, have as many derivatives more as the order of the differential operators in F. Hence H. 1 is satisfied.

Lipschitz continuity of F in the first argument follows easily from (36):

$$F(f,z)-F(f',z)=\tilde{a}_{ij}(\zeta)f^j-\tilde{a}_{ij}(\zeta)f'^j,$$

which proves H. 2.

H. 3 again is implied by the analyticity of f and the choice (58). For (F, f_0, z_0) is of infinite order, i. e. (49)-(51) is fulfilled for all $s>3$. $f_0=DU_0$, and $z_0=(0, U_0, 0)$, *cf.* [5] are clearly C^∞-functions, hence we have (49). The inclusion (50) states that for more regular functions f and z inserted into F the image is more regular, too. This is obvious from (36)-(38). The coefficients depend analytically on f and z, hence one can differentiate these expressions with respect to y^i as long as f and z allow it. The same reasoning applies to (51).

To define the approximate inverse $H(f, z)$ to $D_2 F(f, z)$ consider first the operator $D_2^* F(f, z)$ that includes all of $D_2(f, z)$ in (40)-(42) except the expressions

$$\left.\begin{matrix} l_\gamma D^\gamma\tilde{\sigma}, & m_\gamma D^\gamma\tilde{\sigma}, & r_\gamma D^\gamma\tilde{\zeta}, & m_{ij}D_0\tilde{u}^j,\\ m\tilde{q}, & \mu_\gamma D^\gamma\tilde{\zeta}, & \kappa_\gamma D^\gamma\tilde{\zeta}, & \end{matrix}\right\}\quad(59)$$

that is $D_2^* F(f, z)$ consists of a linearized Navier-Stokes system in its first four components solely for the unknown $\tilde{u}$ and $\tilde{q}$, and of an integral operator for $\tilde{\zeta}$ in its fifth component. We assume now, and this will be shown in paragraphs 4 and 5, that $D_2^* F(f, z)$ is invertible, and its inverse we denote by $H(f, z)$. Then (53) holds, for let $\varphi\in\mathscr{X}_\gamma$ be given with $\gamma=3$. Then φ^i, $i=1, \ldots, 4$, the right-hand side in the linearized Navier-Stokes system is of class $C^{3+\mu}$, hence $(\tilde{u}, \tilde{q})\in C^{5+\mu}\times C^{4+\mu}$, *cf.* Lemma 4, and (53) holds ([3]). The higher regularity of solutions $\tilde{z} = \mathrm{H}(f, z)(\varphi)$ of $\mathrm{D}_2^* \mathrm{F}(f, z)(\tilde{z}) = \varphi$ follows again from the theory of Navier-Stokes equations and from the integral equation in ζ, *cf.* Lemmata 4 and 12. To estimate $(\mathrm{D}_2 \mathrm{F}(f, z) \circ \mathrm{H}(f, z) - \mathbf{1})(\varphi)$ we write this expression formally as

$$\left[\begin{Bmatrix} \mathrm{A} & a \\ \bar{a} & \mathrm{B}+b \end{Bmatrix}\begin{Bmatrix} \mathrm{A}^{-1} & 0 \\ 0 & \mathrm{B}^{-1} \end{Bmatrix} - \mathbf{1}\right]\begin{Bmatrix} \varphi' \\ \varphi'' \end{Bmatrix},$$

where $\varphi' = (\varphi^1, \ldots, \varphi^4)$, $\varphi'' = \varphi^5$; $\mathrm{D}_2^* \mathrm{F}(f, z)$ is then $\begin{Bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{Bmatrix}$ and a, $\bar{a}$, and b denote the expressions that have been deleted from $\mathrm{D}_2\mathrm{F}$. Now consider $a\mathrm{B}^{-1}\varphi''$ and $\bar{a}\mathrm{A}^{-1}\varphi' + b\mathrm{B}^{-1}\varphi''$. For $\varphi \in \mathscr{X}_\gamma$ we have $\tilde{z} = (\mathrm{A}^{-1}\varphi', \mathrm{B}^{-1}\varphi'') \in \mathscr{Z}_\gamma$, therefore $\mathrm{D}_2\mathrm{F}(f, z)\tilde{z} \in \mathscr{X}_0$ because of the differentiations in $l_\gamma \mathrm{D}^\gamma$ etc. Now the expressions in (59) are small in the following sense;

$$\| l_\gamma \|_{\mathrm{C}^{0+\mu}} \leqq \mathrm{C} \| \zeta \|_{\mathrm{C}^{3+\mu}} \tag{60}$$

because ζ occurs at most with its third derivative, *cf.* (40)-(42). This is not sufficient to prove (55) because if ζ gets close to the solution of the nonlinear problem $\| l_\gamma \|$ does not tend to zero as required. At this point we exploit the possibility of choosing as reference domain another than Ω_0. All estimates will be applied to a sequence $\{z_k\}$ whose convergence towards a solution of (14) we show in paragraph 6. So when z_{k+1} is constructed we can choose Ω_{k-1} as reference domain (or any other close enough to it). Then the boundaries Σ_{k+1} and Σ_k are represented by functions which we call again ζ_{k+1}, ζ_k via the relation

$$\Sigma_{k+1} = \{ x \in \mathbb{R}^3 : x = y + \zeta_{k+1}(\xi)\, n_{(k-1)}(y),$$
$$y \in \Sigma_{k-1}, \zeta : \Sigma_{k-1} \to \mathbb{R},$$
$$(\xi^1, \xi^2) \text{ local coordinates on } \Sigma_{k-1} \}. \tag{61}$$

Then $\| l_\gamma \|_{\mathrm{C}^{0+\mu}}$, and all other expressions in (59) are still estimated by (60),

([3]) That we do not loose any derivatives here, as is allowed by H.4 does not lead to any improvement. The essential inequality is (54), where the loss of derivatives occurs.

but with the special choice in (61) we obtain

$$|a\mathrm{B}^{-1}\varphi''|_0,$$
$$|\bar{a}\mathrm{A}^{-1}\varphi'+b\mathrm{B}^{-1}\varphi''|_0 \leqq \mathrm{M}\,|\mathrm{F}(f, z_{k-1})|_3\,|\varphi|_3$$

as required by H.4.

Remark. — This procedure should be compared with the ones of Rabinowitz [13] and Kato [4], where a singular perturbation problem

$$-\sum_{i,j=1}^{n}(a_{ij}(x)\,u_{x^i})_{x^j}+u=\varepsilon f(x, u, \mathrm{D}u, \mathrm{D}^2 u, \mathrm{D}^3 u)$$

is studied for periodic functions a_{ij}, f, and u. This problem is similar to ours as the perturbation f contains higher derivatives than the elliptic operator $\mathrm{L}\,u=-(a_{ij}\,u_{x^i})_{x^j}+u$ that can be inverted. To obtain approximate solutions Rabinowitz introduces the elliptic regularization

$$(-1)^m\gamma\Delta^m v-\mathrm{L}\,v-\varepsilon\sum_{|\gamma|\leqq 3}\mathrm{A}_\gamma\mathrm{D}^\gamma v=g$$

where A_σ is the linearisation of f. This device yields even very regular approximate solutions depending on the m one chooses. A completely different approach is due to Kato [4], where a new abstract existence theorem is given. The singular perturbation problems above are locally well posed if considered in Sobolev spaces, and therefore Kato proves existence without any loss of derivatives. It is not known whether this method can be carried over to the free boundary problems we are interested in.

4. ESTIMATES FOR THE LINEARIZED EQUATIONS OF MOTIONS

The following results for equations of Stokes type are well known, *cf.* Solonnikov-Ščadilov [15], Bemelmans [1].

LEMMA 3. — (i) *Let* $u\in\mathrm{C}^{2+\mu}(\bar{\Omega}_\tau)$, $\zeta\in\mathrm{C}^{3+\mu}(\partial\Omega_\tau)$ *be given where* Ω_τ *is a domain with boundary of class* $\mathrm{C}^{3+\mu}$. *Then the boundary value problem*

$$\left.\begin{array}{c} L(\zeta)\tilde{u}^i+\bar{a}_{ij}(\zeta)D_j\tilde{q} \\ +l_{ij}(u,\zeta)\tilde{u}^j+l_j(u,\zeta)D_j\tilde{u}^i=\varphi^i, \\ i=1,2,3 \\ D_j\tilde{u}^j=0 \end{array}\right\} \quad in\ \Omega_\tau \tag{62}$$

$$\left.\begin{array}{c} \alpha_i(\zeta)\tilde{u}^i=0, \\ \alpha_{ijk}(\zeta)D_i\tilde{u}^j+\beta_{kj}(\delta\zeta)u^j=0 \quad on\ \partial\Omega_\tau, \\ k=1,2 \end{array}\right\} \tag{63}$$

with operators as in (40), (42) *admits to* $\varphi\in C^{0+\mu}$ *a classical solution* $\tilde{u}\in C^{2+\mu}(\bar{\Omega}_\tau)$, $\tilde{q}\in C^{1+\mu}(\bar{\Omega}_\tau)$ *as long as* $\|u\|_{C^{2+\mu}}$ *and* $\|\zeta\|_{C^{3+\mu}}$ *are small enough.*

(ii) *If* Ω_τ *is rotationally symmetric with respect to an axis* β, *the solution* $\tilde{u}$ *is uniquely determined modulo the rotation* $t\beta\wedge y$, $t\in\mathbb{R}$. *For domains without this symmetry the solution is always unique.*

(iii) *The results of* (i) *and* (ii) *remain true for non-homogeneous boundary data* ψ *as long as they satisfy* $\oint_{\partial\Omega_\tau}\psi.n\,d\sigma=0$.

LEMMA 4. — *The solution* $(\tilde{u},\tilde{q})$ *of* Lemma 3 *can be estimated by*

$$\|\tilde{u}\|_{C^{k+2+\mu}(\bar{\Omega}_\tau)}+\|\tilde{q}\|_{C^{k+1+\mu}(\bar{\Omega}_\tau)} \leqq C(\nu,k,\|\partial\Omega_\tau\|_{C^{k+3+\mu}},\|\zeta\|_{C^{k+3+\mu}},\|u\|_{C^{k+2+\mu}})\|\varphi\|_{C^{k+\mu}(\bar{\Omega}_\tau)}. \tag{64}$$

To extend Theorem 1 to the case where Σ_0 is not ncessarily the unit sphere we have to solve the equations of motion in a reference frame that rotates with constant angular velocity ω relatively to a fixed coordinate system; ω is not given but must be determined by prescribing the total torque exerted on the fluid body. If v denotes the velocity in the fixed frame, u the one in the accelerated system then u and v are related by $v(x)=u(x)+\omega\wedge x$. We then get the following equations of motion, *cf.* Weinberger [17]

$$\left.\begin{array}{c} \left.\begin{array}{c} -\nu\Delta v+Dp+[(v-\omega\wedge x).D]v+\omega\wedge u=f \\ \operatorname{div} v \end{array}\right\}\text{in }\Omega \\ v\,.\,n=(\omega\wedge x).n, \\ t_k.T(v,p).n=0 \quad \text{on } \Sigma, \qquad k=1,2 \end{array}\right\} \tag{65}$$

together with the side condition

$$\oint_{\Sigma} [T(v,p).n] \wedge x\, d\omega = 0 \tag{66}$$

In this formulation the coordinate system is chosen such that the torque on Ω vanishes.

If we denote by $\mathscr{J}'$ the space of all C^∞-functions on Ω that are divergence-free and satisfy $\varphi . n = (\omega \wedge x) . n$ on Σ, we obtain after multiplying the equations of motion by φ and integrating over Ω

$$\int_{\Omega} [-D_j T_{ij}(v,p)]\, \varphi^i\, dx + \varphi^i [(v - \omega \wedge x) . D]\, v^i + \varphi^i (\omega \wedge v)^i\, dx = \int_{\Omega} f^i \varphi^i\, dx. \tag{67}$$

Integration by parts gives

$$\int_{\Omega} -(D_j T_{ij})\, \varphi^i\, dx = \int_{\Omega} T_{ij} D_j \varphi^i\, dx - \oint_{\Sigma} T_{ij} n^j \varphi^i\, d\sigma$$

where the surface integral vanishes due to (66); furthermore

$$\int_{\Omega} \varphi^i [(v - \omega \wedge x) . D]\, v^i\, dx = -\int_{\Omega} v^i (D_j \varphi^i)(v^j - \varepsilon_{jkl}\, \omega^k x^l)\, dx$$

as the integrals

$$-\int_{\Omega} \varphi^i (D_j . v^j)\, u^i\, dx, \qquad \int_{\Omega} \varphi^i \varepsilon_{jkl}\, \omega^k \delta_{jl}\, dx$$

and

$$\oint_{\Sigma} \varphi^i (v^j - \varepsilon_{jkl}\, \omega^k x^l)\, n^i v^i\, d\omega$$

all vanish. Because of div $= 0$ we have

$$\int_{\Omega} T_{ij}(v,p)\, D_j \varphi^i\, dx = \frac{\nu}{2} \int_{\Omega} (D_i v^j + D_j v^i)(D_i \varphi^j + D_j \varphi^i)\, dx =: \mathscr{K}(v, \varphi).$$

If $\mathscr{J}$ is the closure of $\mathscr{J}'$ with respect to $\|\varphi\| = (\mathscr{K}(\varphi, \varphi))^{1/2}$ we call $v \in \mathscr{J}$ a weak solution of (65), (66) if there holds

$$\mathscr{K}(v, \varphi) - \int_\Omega v.[(v-\omega \wedge x).\mathrm{D}].\varphi\, dx - \int_\Omega \varphi.(\omega \wedge v)\, dx = \int f.\varphi\, dx \quad (68)$$

for all $\varphi \in \mathscr{J}'$. For the linearized equation a weak solution is defined similarly.

The equations (65), (66) can be treated as in [1]. If Σ is rotationally symmetric with respect to an axis ω, then $\omega \wedge x$ is always a tangential vector and we have $v.n=0$ as boundary conditions. Because of

$$\|v\|_{\mathrm{L}_2}^2 \leqq c\left\{\mathscr{K}(v, v) + \left(\oint_\Sigma v.n\, d\omega\right)^2\right\}, \quad \forall v \in \mathscr{L}(\Omega)/(\omega \wedge x) \quad (69)$$

where $\mathscr{L}(\Omega)$ is the closure of $\mathrm{C}_\sigma^\infty(\Omega) \cap \{v.n=0\}$ under $\mathscr{K}$-norm again, we can determine ω directly. If Σ does not possess rotational symmetry we use $\mathscr{J} = \varphi_0 \oplus \mathscr{L}(\Omega)$ where φ_0 are the pure rotations $\omega \wedge x$, and the coerciveness of $\mathscr{K}(v, v)$ follows agi an. Once a weak solution is established its regularity can be shown as for Lemma 4 above. We finally remark that if we transform the equations as in paragraph 2, the same existence and regularity results hold. We formulate these properties in

LEMMA 5. — *Let f be in $\mathrm{L}_2(\Omega)$. Then the linearization of problem* (65), (66) *admits a unique solution in the class $\mathscr{J}$. If the data are regular it can be estimated as in* (64).

5. ESTIMATES FOR LICHTENSTEIN'S INTEGRAL EQUATION

The integral equation

$$\mathrm{M}(\zeta)\tilde{\zeta} + m_0(\zeta)\tilde{\zeta} = \varphi \quad (70)$$

with the side condition that the volume included by $\tilde{\zeta}$ is prescribed can be solved for small data $\|\varphi\|_{\mathrm{C}^0(\Sigma_\tau)}$ by successive approximation, *cf.* e.g. Lichtenstein [5], pp. 14 ff. Again we take some Ω_τ as reference domain with boundary Σ_τ of class $\mathrm{C}^{3+\mu}$. We now prove some estimates for solutions of (70) we need later to apply the implicit function theorem.

LEMMA 6. — *The gradient of the gravitational potential points inward, i. e.*

$$\psi_\tau(\xi) < 0, \qquad \forall \xi \in \Sigma_\tau. \tag{71}$$

Proof. — This property is well-known for classifical equilibrium figures, *cf.* [6]. For Σ_τ close to an equilibrium figure Σ' we have

$$\left| \frac{\partial}{\partial n'} U'(y') - \frac{\partial}{\partial n_\tau} U_\tau(y_\tau) \right| < \varepsilon$$

according to Lichtenstein [7] (11). Here we denote by U′ the Newtonian potential of the domain bounded by Σ', n′ is the normal to Σ' etc.

(71) enables us to normalize the operator M; it can be written in the form

$$\tilde{Z}(\xi) + \oint_{\Sigma_\tau} \frac{\tilde{Z}(\eta)}{D(\xi, \eta)} d\omega(\eta) \tag{72}$$

with

$$\tilde{Z}(\xi) = \sqrt{-\psi_\tau(\xi)}\, \tilde{\zeta}(\xi),$$

$$D(\xi, \eta) = \sqrt{\psi_\tau(\xi)\, \psi_\tau(\eta)}\, d(\xi, \eta).$$

We will use $\tilde{Z}$ or rather $\tilde{\zeta}$ whenever it is more convenient.

LEMMA 7. — *The homogeneous integral equation*

$$\psi_\tau \tilde{\zeta} + \oint_{\Sigma_\tau} \tilde{\zeta}/d\, d\omega = 0 \tag{73}$$

admits trivial eigensolutions. These are the infinitesimal translations $\tilde{\zeta}_1$, $\tilde{\zeta}_2$, $\tilde{\zeta}_3$ in the direction of the coordinate axes and the rotations $\tilde{\zeta}_4$, $\tilde{\zeta}_5$, $\tilde{\zeta}_6$ about these axes. If Σ_τ is rotationally symmetric about the y^i-axis, then $\tilde{\zeta}_{3+i}$ is no eigensolution, $i = 1, 2, 3$.

Proof. — Let

$$\tilde{\Sigma}^\varepsilon = \{ x \in \mathbb{R}^3 : x = y_\tau + (\tilde{\zeta} + \varepsilon \tilde{\zeta}_1)\, n_\tau(y_\tau), y_\tau \in \Sigma_\tau \}$$

be the surface $\tilde{\Sigma}$ (which is given by $\tilde{\zeta}$) shifted into x^1-direction by the amount ε, and let $\tilde{\Omega}^\varepsilon$ be the domain bounded by $\tilde{\Sigma}^\varepsilon$. Then the Newtonian potentials of $\tilde{\Omega}$ and of $\tilde{\Omega}^\varepsilon$ are the same if evaluated at points that differ by the same shift εe_1:

$$U^\varepsilon(\xi, \tilde{\zeta}(\xi) + \varepsilon \tilde{\zeta}_1(\xi)) = U(\xi, \tilde{\zeta}(\xi)).$$

Therefore

$$0=\frac{1}{\varepsilon}\{\mathrm{U}^{\varepsilon}(\xi,\tilde{\zeta}(\xi)+\varepsilon\tilde{\zeta}_1(\xi))-\mathrm{U}(\xi,\tilde{\zeta}(\xi))\}$$

$$=\frac{1}{\varepsilon}\mathrm{N}(\varepsilon\tilde{\zeta}_1)(\xi)+\frac{1}{\varepsilon}\psi_\tau\varepsilon\tilde{\zeta}_1(\xi)+\frac{1}{\varepsilon}\oint_{\Sigma_\tau}\frac{\varepsilon\tilde{\zeta}_1(\eta)}{d(\xi,\eta)}d\omega(\eta)$$

according to [7] (24). As $\mathrm{N}(\varepsilon\tilde{\zeta}_1)=o(\varepsilon)$ for $\varepsilon\to 0$, *cf.* [4] (25), we obtain

$$\psi_\tau\tilde{\zeta}_1(\xi)+\oint_{\Sigma_\tau}\frac{\zeta_1(\eta)}{d(\xi,\eta)}d\omega(\eta)=0.$$

For the rotations the proof follows along the same lines. If Σ_τ is rotationally symmetric with respect to the y^3-axis then (y^1_τ, y^2_τ) and $(n^1_\tau(y^1_\tau, y^2_\tau), n^2_\tau(y^1_\tau, y^2_\tau))$ are parallel vectors on Σ_τ. Hence $\tilde{\zeta}_6$ vanishes and is therefore no eigensolution.

LEMMA 8. — *Consider the set $\mathscr{M}$ consisting of that part of the branch of MacLaurin ellipsoids that connects the sphere with the branch of Jacobi ellipsoids. For $\Sigma_\tau\in\mathscr{M}$ the equation* (73) *admits no other eigensolutions than the ones obtained in* Lemma 7.

Proof. — All MacLaurin and Jacobi ellipsoids for which (73) has in addition to $\tilde{\zeta}_1, \ldots, \tilde{\zeta}_6$ from above other eigensolutions have been characterized by Ljapounoff [8], paragraphs 34 and 78. On $\mathscr{M}$ the first one with nontrivial eigensolutions is the ellipsoid corresponding to the critical value of the angular velocity at which the Jacobi family branches off.

LEMMA 9. — *Let Σ_τ be in $\mathscr{M}$. Then*

$$\tilde{\mathrm{Z}}-\oint_{\Sigma_\tau}\tilde{\mathrm{Z}}/\mathrm{D}\,d\sigma\equiv[1-\mathscr{I}(\Sigma_\tau)]\tilde{\mathrm{Z}}=0 \tag{74}$$

and

$$[1-\mathscr{I}^2(\Sigma_\tau)]\tilde{\mathrm{Z}}=0 \tag{75}$$

admit the same solutions. Here $\mathscr{I}^2$ (Σ_τ) denotes the integral operator with the iterated kernel

$$\mathscr{I}^2(\Sigma_\tau)\tilde{\mathrm{Z}}=\oint_{\Sigma_\tau}\mathrm{Z}(\eta)\oint_{\Sigma_\tau}\mathrm{D}^{-1}(\xi,\eta')\mathrm{D}^{-1}(\eta',\eta)\,d\omega(\eta')\,d\omega(\eta). \tag{76}$$

Proof. — $[1+\mathscr{I}(\Sigma_0)]\tilde{\mathrm{Z}}=0$ has $\tilde{\mathrm{Z}}\equiv 0$ as only solution. Otherwise we had

$$0\geqq-\oint_{\Sigma_\tau}\tilde{\mathrm{Z}}^2\,d\sigma=\oint_{\Sigma_\tau}\oint_{\Sigma_\tau}\mathrm{D}^{-1}(\xi,\eta)\tilde{\mathrm{Z}}(\xi)\,\mathrm{Z}(\eta)\,d\omega(\xi)\,d\omega(\eta). \tag{77}$$

But according to Plemelj's theorem, *cf.* [12], p. 28, the single-layer potential $\mathscr{I}(\Sigma_\tau)$ is a positive operator.

Now let $Z \neq \tilde{Z}_i$, $\tilde{Z}_i$ a trivial eigensolution of (74), be a solution to (75). This implies

$$\begin{aligned} 0 &= [\mathbf{1} - \mathscr{I}^2(\Sigma_\tau)]\, Z \\ &= [\mathbf{1} + \mathscr{I}(\Sigma_\tau)]\, [1 - \mathscr{I}(\Sigma_\tau)]\, Z \\ &= [\mathbf{1} + \mathscr{I}(\Sigma_\tau)]\, Z' \end{aligned}$$

with $Z' \not\equiv 0$, as Z is no solution to (74) by assumption. This gives a contradiction to (77), and the lemma is proved.

Remark. — The iteration of kernels is a well-known tool in the study of integral equations. Usually, however, one does not know how many times one has to iterate such that $\mathbf{1} - \mathscr{I}$ and $\mathbf{1} - \mathscr{I}^k$ possess the same eigenfunctions. Because in (74) $\mathscr{I}$ has a real and symmetric kernel, we can determine k to be 2. This is not without interest because in the estimates $\| [\mathbf{1} - \mathscr{I}(\Sigma_0)]^{-1} \Phi \| \leqq C \| \Phi \|$ below the constant C depends on k, and hence can be calculated (in principle, at least) whereas in the general case it is obtained by some indirect reasoning.

LEMMA 10. — *Let* $\tilde{\Sigma}$ *be in a* $C^{2+\mu}$*-neighborhood of* $\Sigma_\tau \in \mathscr{M}$. *Then* $[\mathbf{1} - \mathscr{I}(\Sigma)]\, \tilde{Z} = 0$ *admits only the trivial solutions from* Lemma 7.

Proof. — According to Lemma 9 it is equivalent to study $\mathscr{I}(\Sigma_\tau)$ or $\mathscr{I}^2(\Sigma_\tau)$. Now $\mathscr{I}^2(\Sigma_\tau)$ is a Hilbert-Schmidt operator whose kernel grows like log $|y - y'|$. The eigenvalues of $\mathscr{I}^2(\Sigma_\tau)$ therefor cannot have -1 as accumulation point. As $\tilde{\Sigma}$ is close to Σ_τ we get

$$[\mathbf{1} - \mathscr{I}(\tilde{\Sigma})]\, \tilde{Z} = [\mathbf{1} - \mathscr{I}(\Sigma_\tau)]\, \tilde{Z} + K_\varepsilon \tilde{Z}$$

with

$$K_\varepsilon \tilde{Z} = (\tilde{\psi} - \psi_\tau)\, \tilde{Z} + \oint_{\tilde{\Sigma}} \tilde{Z}/D\, d\omega - \oint_{\Sigma_\tau} \tilde{Z}/D\, d\omega.$$

As K_ε is small the result follows immediately.

LEMMA 11. — *Let* $\tilde{Z}$ *be a solution to*

$$[\mathbf{1} - \mathscr{I}^2(\Sigma_\tau)]\, Z = \Phi \tag{78}$$

where Φ *is perpendicular to* $\tilde{Z}_i$:

$$\langle \Phi, \tilde{Z}_i \rangle = 0, \qquad i = 1, \ldots, 6; \tag{79}$$

here $\langle \Phi, \tilde{Z}_i \rangle$ *denotes the* L_2*-scalar product on* Σ_τ. *Then we have*

$$\| \tilde{Z} \|_{L_\infty (\Sigma_\tau)} \leqq C \| \Phi \|_{L_\infty (\Sigma_\tau)}. \tag{80}$$

Proof. — We start from

$$\begin{aligned} \| \tilde{Z} \|_{L_\infty} &\leqq \| \mathscr{I}^2 (\Sigma_\tau) \tilde{Z} \|_{L_\infty} + \| \Phi \|_{L_\infty} \\ &= \left\| \oint_{\Sigma_\tau} | I^2 (\,.\,, \eta) |^2 \, d\omega (\eta) \right\|_{L_\infty} \| \tilde{Z} \|_{L_2} + \| \Phi \|_{L_\infty} \\ &\leqq C \| \tilde{Z} \|_{L_2} + \| \Phi \|_{L_\infty} \end{aligned}$$

where $I_2 (\xi, \eta) = \oint_{\Sigma_\tau} D^{-1} (\xi, \eta') D^{-1} (\eta', \eta) \, d\omega (\eta')$.

The Hilbert-Schmidt operator $\mathscr{I}^2 (\Sigma_\tau)$ possesses infinitely many eigenvalues λ_n and corresponding eigenfunctions Φ_n with

$$\oint_{\Sigma_\tau} \Phi_n (\xi) \Phi_m (\xi) \, d\sigma (\xi) = \delta_{nm}.$$

The resolvent of $1 - \lambda \mathscr{I}^2$ is of the form

$$\Gamma (\xi, \eta; \lambda) = \sum_{n=1}^{\infty} \frac{\Phi_n (\xi) \Phi_n (\eta)}{\lambda_n - \lambda}, \tag{81}$$

hence for $\lambda = 1$

$$Z (\xi) = \Phi (\xi) + \oint_{\Sigma_\tau} \Phi (\eta) \Gamma (\xi, \eta; 1) \, d\omega (\eta).$$

Because of (79) we obtain

$$\begin{aligned} \langle \tilde{Z}, \tilde{Z} \rangle &= \langle \tilde{Z}, \Phi \rangle + \sum_{n=1}^{\infty} \langle \tilde{Z}, \Phi_n \rangle \left\langle \Phi_n \frac{1}{\lambda_n - 1}, \Phi \right\rangle \\ &\leqq \| \tilde{Z} \|_{L_2} \| \Phi \|_{L_2} + \sum_{n=1}^{\infty} \| \tilde{Z} \|_{L_2} \| \Phi_n \|_{L_2} \| \Phi_n \|_{L_2} \| \Phi \|_{L_2} \frac{1}{| 1 - \lambda_n |^2} \\ &\leqq (1 + C) \| \tilde{Z} \|_{L_2} \| \Phi \|_{L_2}; \end{aligned}$$

here we used Bessel's inequality

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = \left| \oint_{\Sigma_\tau} I^2 \, d\omega \right|^2 \leqq \text{Const.}$$

Higher regularity for solutions to $\psi_\tau \tilde{\zeta} + \oint_{\Sigma_\tau} \tilde{\zeta}/d\, d\omega = \varphi$ can now be obtained by standard methods. Differentiating the equation with respect to ξ^i we obtain an integral equation with kernel

$$\frac{\partial}{\partial \xi^i} \frac{1}{|\xi - \eta|} = - \frac{\xi^i - \eta^i}{|\xi - \eta|^3}.$$

An operator with this kernel is a bounded mapping from $C^{0+\mu}(\Sigma_\tau)$ onto itself provided Σ_τ is of class $C^{1+\mu}$. We get therefore

$$\begin{aligned} \|\tilde{Z}\|_{C^{1+\mu}} &\leqq \|\Phi\|_{C^{1+\mu}} + \|D \mathscr{I}(\Sigma_\tau) Z\|_{C^{0+\mu}} \\ &\leqq \|\Phi\|_{C^{1+\mu}} + C\|Z\|_{C^{0+\mu}}. \end{aligned}$$

From the Calderon-Zygmund inequality we obtain

$$\|\mathscr{I}(\Sigma_\tau)\tilde{Z}\|_{H^1_p} \leqq C\|\tilde{Z}\|_{L_p},$$

hence with $p > 2$

$$\begin{aligned} \|\tilde{Z}\|_{C^{0+\mu}} &\leqq \|\Phi\|_{C^{0+\mu}} + \|\mathscr{I}(\Sigma_\tau)\tilde{Z}\|_{C^{0+\mu}} \\ &\leqq \|\Phi\|_{C^{0+\mu}} + C\|\mathscr{I}(\Sigma_\tau)\tilde{Z}\|_{H^1_p} \\ &\leqq \|\Phi\|_{C^{0+\mu}} + C\|\tilde{Z}\|_{L_p} \\ &\leqq C\|\Phi\|_{C^{0+\mu}}, \end{aligned}$$

as $\|\tilde{Z}\|_{L_p}$ was already estimated in Lemma 11. We can state the regularity properties of solution to the integral equations as follows:

LEMMA 12. — *Let $\Phi \in C^{k+\mu}(\Sigma_\tau)$ with $\Sigma_\tau \in C^{k+1+\mu}$ satisfy the integrability condition* (79). *Then the solution $\tilde{Z}$ of $[1 - \mathscr{I}(\Sigma_\tau)]\tilde{Z} = \Phi$ can be estimated by Φ:*

$$\|Z\|_{C^{k+\mu}} \leqq C\|\Phi\|_{C^{k+\mu}}. \tag{82}$$

6. EXISTENCE RESULTS FOR THE FREE BOUNDARY PROBLEM

The first configuration we want to study is a perturbation of the sphere $\Sigma_0 = \mathrm{S}$. Clearly

$$z_0 = (v_0, p_0, \Sigma_0) \qquad \text{with} \qquad v_0(x) \equiv 0, \quad p_0(x) = \mathrm{U}_0(x) = \int_{\mathrm{B}} |x-y|^{-1}\,dy$$

is a solution of $\mathrm{F}(f_0, z_0) = 0$ as $f_0 = \mathrm{DU}_0$ is the only force acting here. After the existence of H has been established in the preceeding paragraphs it remains to investigate the sequence $\{z_k\}$ of successive approximations, especially the behavior of the smoothing operators. These are defined in Hörmander [3] where also the properties we need are proved for the case of Hölder spaces $\mathrm{C}^{m+\mu}$, $0 \leqq \mu \leqq 1$: let $w : \mathrm{K} \to \mathbb{R}$ be a function defined on a compact set $\mathrm{K} \subset \mathbb{R}$ which is of class $\mathscr{E}'(\mathrm{K}) \cap \mathrm{C}^a(\mathrm{K})$, $a = [a] + \mu > 0$, $\mu \in (0, 1)$. Then there is a mapping S: $\mathscr{E}'(\mathrm{K}) \cap \mathrm{C}^a(\mathrm{K}) \to \mathrm{C}_c^\infty(\mathbb{R}^n)$ with the following properties, *cf.* [4], Theorem A.10:

$$\text{(i)} \qquad \| \mathrm{S}_\theta w \|_b \leqq \mathrm{C}_1 \| w \|_a, \qquad \forall\, b \leqq a$$

$$\text{(ii)} \qquad \| \mathrm{S}_\theta w \|_b \leqq \mathrm{C}_2\, \theta^{b-a} \| w \|_a, \qquad \forall\, a \leqq b$$

$$\text{(iii)} \qquad \| u - \mathrm{S}_\theta w \|_b \leqq \mathrm{C}_3\, \theta^{b-a} \| w \|_a, \qquad \forall\, b \leqq a$$

$$\text{(iv)} \qquad \left\| \frac{d}{d\theta} \mathrm{S}_\theta w \right\|_b \leqq \mathrm{C}_4\, \theta^{b-a-1} \| w \|_a$$

The introduction of S_θ given in [3] is directly applicable to functions defined on a compact C^∞-manifold, such that the smoothing of the non-parametric representations ζ of the surface Σ can be done as in the work quoted above. The functions v and p, however, are defined on bounded domains $\bar{\Omega}$, but will not have compact support in Ω. According to Stein [16], pp. 176-180, we can extend such functions $v \in \mathrm{C}^{m+\mu}(\bar{\Omega})$ across the boundary to $\mathrm{V} \in \mathrm{C}^{m+\mu}(\mathrm{G})$ with $\Omega \subset\subset \mathrm{G}$; V then has the same Hölder norm as v. Next we consider $\chi . \mathrm{V}$ where χ is a C^∞-function on G that has compact support in G and satisfies

$$\chi(x) \equiv 1, \qquad \forall\, x \in \mathrm{G}' \quad \text{where} \quad \Omega \subset\subset \mathrm{G}' \subset\subset \mathrm{G}.$$

The Hölder norm $\| \chi \mathrm{V} \|_{\mathrm{C}^{m+\mu}(\mathrm{G})}$ can be estimated by $\mathrm{C}(m, \mu) \| \mathrm{V} \|_{\mathrm{C}^{m+\mu}(\mathrm{G})}$ where C does not depend on the function V; we then may apply the smoothing operators in the usual way.

The restriction on the exterior force h which we posed in (7) serves the following purpose. As the occurence of eigensolutions to the integral equation $M\zeta=0$, *cf.* (13), already indicates, there is an integrability condition on the datum φ when one wants to solve $M\zeta=\varphi$ (4). Physically this condition allows only such exterior forces that the center of mass of the fluid body is fixed. A condition that involves the datum h and the solution would be

$$\int_\Omega h(x)\,dx=0. \tag{83}$$

For integration of the Navier-Stokes equations gives

$$\int_\Omega -\nu\Delta v+\mathrm{D}p\,dx+\int_\Omega (v\,.\,\mathrm{D})\,v\,dx=\int_\Omega h\,dx=0.$$

Integrating by parts we get

$$\left.\begin{aligned}\int_\Omega (v\,.\,\mathrm{D})\,v\,dx&=-\int_\Omega v\operatorname{div} v\,dx+\oint_\Sigma v\,(v\,.\,n)\,d\omega=0\\ \int_\Omega -\nu\Delta v+\mathrm{D}p\,dx&=\oint_\Sigma \mathrm{T}(v,p)\,.\,n\,d\omega=0.\end{aligned}\right\} \tag{84}$$

As the tangential stresses vanish according to (2), we infer from (84)

$$\oint_\Sigma n\,.\,\mathrm{T}(v,p)\,n\,d\omega=0 \tag{85}$$

which states that the resultant of the forces that act on the fluid body vanishes. This implies in particular that h generates only interior motions in the fluid body but does not move it as a whole which would not be compatible with our assumption that the motion is stationary with respect to a fixed reference frame (5).

Now (83) involves already the solution Ω and therefore cannot be verified. One example for an admissible h is given in (7). Under this condition the flow and the domain occupied by the fluid has the same

(4) The nonlinearity $\mathrm{N}(\zeta)$ satisfies this condition always, as the discussion of the bifurcation equations in Lichtenstein [6] shows. The situation here is similar to the one studied in [6], pp. 76-78; the only difference lies in the fact that the exterior field then is explicitely given whereas in our case $n\,.\,\mathrm{T}\,.\,n$ involves the solution itself.

(5) For a further discussion *see* also [1], [2].

symmetry properties as was shown for the equations of motion in [2] and [17]; the corresponding property for the integral equation is known, *cf.* Lichtenstein [6]. Therefore theorem 1 is proved.

Remark. — The regularity properties for h and the solution (v, p, Σ) follow from Zehnder's discussion in [18], p. 126. As remarked there the relatively mild additional regularity that is required of h, namely $h \in C^{\lambda+\mu}$, $\lambda > 6$, instead of $C^{3+\mu}$ which would be optimal according to the linear theory, has the consequence that in $\| h \|_{C^{\lambda+\mu}} \leqq C$ the constant C becomes smaller as λ gets closer to 6. Zehnder's approach by analytic mappings in [18], paragraphs 1, 2 avoids this restriction on C, but it is not obvious whether this method can be applied here.

Our procedure can also be applied to other configurations than the one discussed in Theorem 1. So let Σ_1 be a MacLaurin ellipsoid corresponding to an angular velocity $\omega_1 \neq 0$. We further assume that Σ_1 is in $\mathcal{M}$ as stated in Lemma 8. If f_1 is again the force of self-attraction, $z_1 = (v_1, p_1, \Sigma_1)$ solves $F(f_1, z_1) = 0$ with $v_1(x) \equiv 0$, $p_1(x) = \int_\Omega \frac{g}{|x-y|} dy$, provided v_1 describes the relative motions in a reference frame that rotates with constant angular velocity ω_1, too. With Σ_1, ω_2, . . . we denote the same situation as above for a Jacobi ellipsoid; we assume that ω_2 is not a critical value, which means that the integral equation for the free boundary is uniquely solvable (6). If we now apply some exterior force density h, it is conceivable that the configuration we get will be stationary only in a reference frame rotating with some other angular velocity that is not known *a priori.* Take for instance $h = h_1 + h_2$ in the non-symmetric case where we choose h_1 to be the fictitious body force that corresponds to writing (v_2, p_2, Σ_2) in a frame that rotates with angular velocity $\omega_2' \neq \omega_2$, and let $h_2 \neq 0$ be such that the total torque exerted on the fluid body vanishes. This clearly produces a flow which is stationary in the reference system that rotates with angular velocity ω_2' but non-stationary in the original frame. In such a problem the unknown angular velocity ω for the appropriate coordinate system in which the solution is stationary can be determined by the equilibrium condition

$$\oint_\Sigma [T(v, p).n] \wedge x\, d\omega = 0. \tag{86}$$

(6) The critical values for which bifurcation may occur are isolated, *cf.* Ljapounov [8]. We therefore can assume that there is no bifurcation in a possibly small neighborhood of ω_2.

Navier-Stokes equations with this type of side condition were studied by Weinberger [17]; his results were used in the context of free boundary problem in [2].

If we apply rotationally symmetric forces to Σ_1, v_1, etc. there is no need to ask for a reference frame as above, because as long as the data share that symmetry the solution is stationary in any frame that rotates with constant angular velocity about the axis of symmetry. But also in this case it is more natural to work with an unknown frame and (86) as side conditions. When we estimate, as in (8), the difference between the velocity v of the flow and some rest solution $v_0 \equiv 0$ then v should measure deformations only. If we have $v = v_e + v_d$ with $v_e = \omega \wedge x$, $\omega = \text{Const}$, def $v_d \not\equiv 0$, the pure rotation should be compensated by choosing an appropriate reference system.

THEOREM 13. — *Let Σ_0 be a MacLaurin or Jacobi ellipsoid that corresponds to an angular velocity ω_0 which is not a critical value for the integral equation* (13), *and let f_0 be the force of self-attraction as in* (6). *For $f = f_0 + h$, where h satisfies the same hypotheses as in* Theorem 1, *there exists a constant ω such that in a coordinate system rotating with angular velocity ω there is a non-stationary solution (v, p, Σ) to the free boundary problem* (1)-(3). *It has the same regularity properties than the solution obtained in* Theorem 1.

Proof. — The only change with respect to Theorem 1 consists in formulating (and solving) the linearized Navier-Stokes equations in a coordinate system that rotates with an unknown angular velocity. Now our free boundary problem consists of (64), (65), (3). Using Lemma 5 instead of Lemma 3 we proceed in the same way as before.

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