ANNALES DE L'I. H. P., SECTION C

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Annales de l'I. H. P., section C, tome 4, n° 5 (1987), p. 405-421

<http://www.numdam.org/item?id=AIHPC_1987__4_5_405_0>

Ann. Inst. Henri Poincaré,
Vol. 4, n° 5, 1987, p. 405-421. *Analyse non linéaire*

A priori interior gradient bounds for solutions to elliptic Weingarten equations

by

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ABSTRACT. — In this paper a maximum principle approach is used to derive *a priori* interior gradient bounds for smooth solutions to the Weingarten equations

$$f(\lambda)=\sum_{i_1<i_2<\ldots<i_k}\lambda_{i_1}\ldots\lambda_{i_k}=\psi(x,\,u,\,\nu).$$

Here $\lambda=(\lambda_1,\ldots,\lambda_n)$ is the vector of principal curvatures of the graph of u at a point $(x,\,u(x))$ on the graph, with downward normal ν. One requires a one-sided height bound $(u<0)$, natural structure conditions on the prescribed function ψ, and the restriction that all λ lie in a certain cone of eigenvalues for which f is elliptic. The result generalizes what is known to be true for the prescribed mean curvature equation $(k=1)$.

Key words : Weingarten, nonlinear, elliptic, maximum principle, gradient.

RÉSUMÉ. — Dans cet article une méthode de principe du maximum est employée pour dériver une majoration *a priori* des gradients intérieurs pour les solutions C^3 d'équations elliptiques de Weingarten :

$$f(\lambda)=\sum_{i_1<i_2<\ldots<i_k}\lambda_{i_1}\ldots\lambda_{i_k}=\psi(x,\,u,\,\nu).$$

ici $\lambda=(\lambda_1, \ldots, \lambda_n)$ est le vecteur des courbatures du graphe de u au point $(x, u(x))$ de ce graphe avec un normal (vers le bas) ν. Il faut avoir une majoration $(u<0)$, des conditions naturelles sur la fonction ψ, et la contrainte que tous les λ se situent dans un certain cône de valeurs propres pour lesquelles f est elliptique. Ce résultat généralise ce qui est connu dans le cas de l'équation de la courbature moyenne $(k=1)$.

In this paper we extend a method (described in an earlier note [11]) that was used for the prescribed mean curvature equation to derive *a priori* interior gradient bounds for bounded solutions to the prescribed Weingarten equation

$$f(\lambda)=\sum_{i_1<\ldots<i_k} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k}=\psi(x, u, \nu). \tag{1}$$

In equation (1), $\lambda=(\lambda_1, \ldots, \lambda_n)$ is the vector of principal curvatures of the graph $S_u=\{z=u(x)\}\subset\mathbb{R}^{n+1}$, having downward normal $\nu=(\nu^1, \ldots, \nu^{n+1})$. We often write $\lambda=\lambda(S_u, P)$, $\nu=\nu(S_u, P)$ for $P\in S_u$. The integer k satisfies $1\leqq k<n$. The prescribed function $\psi(x, u, \nu)$ is assumed to be C^1, satisfying for some positive constants M, ψ_1,

$$\psi_u\geqq 0, \qquad |\psi|\leqq\psi_1, \qquad |\nabla\psi|\leqq M \tag{2}$$

and if $k\neq 1$, the additional inequality for constant $\psi_0>0$,

$$\psi\geqq\psi_0. \tag{3}$$

[In (2) one acutally only needs the one-sided bounds $\psi_u\geqq 0$, $\psi_{\nu^{n+1}}\leqq 0$ for those two partials.] There are also natural restrictions on the admissible values of λ, related to the ellipticity of (1). They are the requirements that

$$\frac{\partial f}{\partial\lambda_{j_1}}=f_{\lambda_{j_1}}\geqq 0, \qquad f_{\lambda_{j_1}\lambda_{j_2}}\geqq 0, \ldots, f_{\lambda_{j_1}\lambda_{j_2}\ldots\lambda_{j_{k-1}}}\geqq 0 \tag{4}$$

$\forall j_1, \ldots, j_{k-1}$ (distinct), $\forall\lambda$ that are principal curvature vectors of S_u. [This requirement that all derivatives of $f(\lambda)$ be nonnegative is discussed below.] The result of this paper is

MAIN THEOREM. — *Let* $u \in C^3(\overline{B_1(0)})$, *where* $\overline{B_1(0)} = \{x \in \mathbb{R}^n, \quad |x| \leqq 1\}$. *Let u solve* (1)-(4) *with* $u < 0$ *on* $\overline{B_1(0)}$ *and* $u(0) = -u_0$.

Then $\exists\, C = C(n, k, \psi_1, M, \psi_0, u_0)$ *so that*

$$|Du(0)| \leqq C.$$

Because a dilatation of $\mathbb{R}^{n+1}$ preserves the structure of statements (1)-(4), the main theorem yields *a priori* interior gradient bounds for balls of arbitrary radius. From the proof it will also be clear that one can derive local estimates near the boundary of a domain if one has local estimates on the boundary.

Weingarten and related nonlinear elliptic equations have generated much interest recently because they are a natural generalization of the prescribed mean cruvature and prescribed Gauss curvature equations. As of this writing, the question of existence and regularity for solutions to the Weingarten-Dirichlet problem does not seem to be completely solved, but the solution appears near. Some of the works in this progression are listed in the references [1], [2], [4] to [7], [10], [12], [13], [14] and [16]. In particular, L. Caffarelli, L. Nirenberg and J. Spruck (C.N.S.) have solved the problem for surfaces parameterized as graphs above a sphere (no Dirichlet data) [7], and for the Dirichlet problem in $\mathbb{R}^n$, when the vector λ of principal curvatures in (1) is replaced by the vector of eigenvalues of the Hessian [6].

The value of an *a priori* interior gradient estimate, aside from its natural geometric significance, is that in the presence of a complete theory, it yields interior compactness results for sequences of smooth solutions. One can then often extend existence and partial regularity theorems to domains for which they cannot at first be shown. Many papers have been written about the *a priori* interior gradient bounds for the prescribed mean curvature equation ([3], [8], [11], [15], [17]).

In the main theorem, the case $k = n$ is not included. This is because for the Gauss curvature equation ellipticity forces one to consider only convex functions, for which the interior gradient bounds of the form considered are trivial.

Before proving the main theorem, we explain the requirement (4). We show that λ satisfies (4) with strict inequalities and $f(\lambda) > 0$, if and only if $\lambda \in \Gamma$, the admissible cone of eigenvalues considered by C.N.S. in [6] :

$$\Gamma = \{\lambda \in \mathbb{R}^n \text{ s. t. } \lambda \text{ is in the component of } \{f > 0\} \text{ containing all positive } \lambda\, (\lambda_i > 0,\ \forall\, i)\}.$$

C.N.S. are led to this cone naturally by the two requirements that some function of f be a concave function of λ and that bounds $|\lambda| \leqq M$, $f \geqq \psi_0 > 0$ imply uniform ellipticity: $f_{\lambda_i} \geqq \delta > 0$, $\forall i$. These two requirements allow the method of continuity to work in the existence and regularity theorems that are proven in [6] and [7].

In Section 1 of [6], it is shown that $f^{1/k}$ is concave on the convex set Γ and that as a consequence $f_{\lambda_i} > 0$, $\forall \lambda \in \Gamma$. (Results from [9] are used.) This is the first inequality (strict) of (4). If $k \geqq 2$, then f_{λ_i} is itself a Weingarten curvature equation

$$f_{\lambda_i} = \sum_{\substack{i_l \neq i \\ i_1 < \ldots < i_{k-1}}} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_{k-1}},$$

defined for vectors $\tilde{\lambda} \in \mathbb{R}^{n-1}$, $\tilde{\lambda} = (\lambda_1, \ldots, \hat{\lambda_i}, \ldots, \lambda_n)$. The component Γ_{λ_i} of $\{f_{\lambda_i} > 0\} \subset \mathbb{R}^{n-1}$ containing all positive $\tilde{\lambda}$ contains $\Gamma \cap \mathbb{R}^{n-1}$ because $f_{\lambda_1} > 0$ on Γ and $\Gamma \cap \mathbb{R}_{n-1}$ is a convex set (hence component) of $\mathbb{R}^{n-1}$ containing positive $\tilde{\lambda}$. Thus since now $(f_{\lambda_i})_{\lambda_j} > 0$ on Γ_{λ_i}, it follows that $f_{\lambda_i \lambda_j} > 0$ on $\Gamma \subset \Gamma_{\lambda_i} \times \mathbb{R} \subset \mathbb{R}^n$. This is the second inequality (strict) of (4). If $k \geqq 3$, the remaining inequalities follow inductively.

Conversely, let Γ' be the set of vectors λ for which $f(\lambda) > 0$ and for which (4) holds strictly. We show $\Gamma' \subset \Gamma$. Let $\lambda \in \Gamma'$. If $\lambda > 0$, we are done. Otherwise, assume $\lambda_1 < 0$. Consider the path $\lambda(t) = (\lambda_1 + t(1 - \lambda_1), \lambda_2, \ldots, \lambda_n)$ in $\mathbb{R}^n$. We show that all $f_{\lambda_{i_1} \ldots \lambda_{i_l}}$ increase as t goes from 0 to 1 ($i_1 < i_2 < \ldots < i_l$, $0 \leqq l \leqq k-1$). Indeed

$$\frac{d}{dt} f_{\lambda_{i_1} \ldots \lambda_{i_l}}(\lambda(t)) = \begin{cases} 0 & \text{if some } i_r = 1 \\ (1 - \lambda_1) f_{\lambda_1 \lambda_{i_1} \ldots \lambda_{i_l}}(\lambda(t)) & \text{if no } i_r = 1. \end{cases}$$

Since any kth partial of f with respect to k distinct λ_i's is 1, the derivative formula implies that all $(k-1)$st partials are nondecreasing, hence positive. Inductively all $f_{\lambda_{i_1} \lambda_{i_l}}$ are nondecreasing ($l = k$, $k-1, \ldots, 0$). Thus $\lambda = (\lambda_1, \ldots, \lambda_n)$ is connected in Γ' to $(1, \lambda_2, \ldots, \lambda_n)$. Repeating this construction several times connects λ to a positive vector. Since Γ' contains all positive vectors and is connected, $\Gamma' \subset \Gamma$. Thus $\Gamma = \Gamma'$.

To prove the main theorem, we use the same test function and technique as in [11]. For simplicity of calculation, the computations are described via normal perturbations.

The idea is to construct an "interior barrier" surface $S_{\bar{u}}$ from S_u by perturbing it a small amount along its downward normal and then lifting this perturbed surface high enough. Let $\eta(x, z)$ be a continuous nonnegative function, smooth (with uniform C^2 bound) where it is positive. Let $\eta(x, u(x))$ have compact support on S_u. Perturb S_u by displacing the point $P=(x, u(x))$ along the downward normal $\nu=\nu(S_u, P)$ an amount $\varepsilon\eta(P)$. The resulting point $\bar{P}=(\bar{x}, \bar{u})$ is

$$\left.\begin{aligned} \bar{x}&=x+\varepsilon\eta(P)\,T, \qquad T=\frac{Du(x)}{\sqrt{1+|Du(x)|^2}} \\ \bar{u}&=u-\varepsilon\eta\,S, \qquad S=\frac{1}{\sqrt{1+|Du(x)|^2}} \end{aligned}\right\} \tag{5}$$

For small ε, the inverse function theorem implies that x is a smooth function of $\bar{x}$ when $\eta>0$. Thus for $\eta>0$ (and ε small), the points $(\bar{x}, \bar{u})$ describe the graph $S_{\bar{u}}$ of a smooth function $\bar{u}$. ($\bar{u}$ depends on ε but we suppress the dependence.) *Subsequent calculations are only assumed to make sense for* ε *sufficiently small.*

The two properties of $S_{\bar{u}}$ that enable it to be used as a barrier are that $f(\lambda(S_{\bar{u}}, \bar{P}))$, $\nu(S_{\bar{u}}, \bar{P})$ can be estimated from $f(\lambda(S_u, P))$, $\nu(S_u, P)$ and that the height difference between S_u and $S_{\bar{u}}$ at x (or $\bar{x}$) is $\varepsilon\eta(P)\sqrt{1+|Du(x)|^2}+O(\varepsilon^2)$. The second property follows because the difference in height (above $\bar{x}$) between $\bar{P}$ and the tangent plane $\pi(S_u, P)$ is exactly $\varepsilon\eta(P)\sqrt{1+|Du(x)|^2}$. [Here and later, $O(\varepsilon^2)$ terms are allowed to depend on C^3 norms of $\eta|_{\{\eta>0\}}$.] The first property is a consequence of Lemmas 1 and 2 below.

Let the letter w represent a function $w(x)$ whose graph in a fixed (x, z) coordinate system is S_w. If $A=(x_0, w(x_0))\in S_w$, then use the capital letter W to represent the function that (locally) parameterizes S_w above its tangent plane $\pi(S_w, A)$. That is, pick orthonormal coordinate vectors $f_1, \ldots, f_n$ for π and let f_{n+1} be the upward normal to π. Let A be the origin. Then for $y=(y^1, \ldots, y^n)$, $|y|$ small, require $y^i f_i+W(Y^i f_i) f_{n+1}$ to parameterize S_w near A. (In this paper, repeated indices other than n are summed from 1 to n.) We write $W(y)$ for $W(y^i f_i)$.

LEMMA 1. — *Let* $e_1, \ldots, e_n, e_{n+1}$ *be orthonormal coordinates of* $\mathbb{R}^{n+1}$, *with* e_{n+1} *pointing in the positive z direction and* $e_1, \ldots, e_n$ *chosen so that* $e_1, \ldots, e_{n-1}$ *are perpendicular to* $Dw(x_0)$, *which is a nonnegative multiple of* e_n. *Let* $f_1, \ldots, f_n$ *be corresponding coordinates in* $\pi(S_w, A)$,

$$f_i=e_i, \qquad 1\leq i\leq n-1 \tag{6}$$

$$f_n = S\, e_n + |T|\, e_{n+1},$$

$$S = \frac{1}{\sqrt{1 + |Dw(x_0)|^2}}, \qquad T = S\, Dw(x_0).$$

Then, letting subscripts refer to differentiation with respect to corresponding coordinates,

$$[W_{ij}] = \left[\begin{array}{c:c} S\, w_{ij} & S^2\, w_{in} \\ \hdashline S^2\, w_{nj} & S^3\, w_{nn} \end{array}\right].$$

We call the matrix $[W_{ij}] = \mathscr{D}^2 W$ *the tangential Hessian of* S_w *of* A. *It is one way of expressing the second fundamental form of* S_w *at* A. (See *Lemma* 1.1 *of* [7].)

The proof of Lemma 1 is straightforward. Points $\bar{A}$ near A on S_w can be expressed in both coordinate systems:

$$\tilde{A} = x^i e_i + w(x)\, e_{n+1} = y^i f_i + W(y) f_{n+1}. \tag{7}$$

Using (6), (7) yields

$$\left.\begin{aligned} & x^i = y^i, \qquad 1 \leqq i \leqq n-1 \\ & \begin{pmatrix} x^n \\ w(x) \end{pmatrix} = \begin{pmatrix} S & -|T| \\ |T| & S \end{pmatrix} \times \begin{pmatrix} y^n \\ W(y) \end{pmatrix}. \end{aligned}\right\} \tag{8}$$

Thus

$$\left.\begin{aligned} W(y) &= -|T|\, x^n + S\, w(x) \\ x^n &= S\, y^n - |T|\, W(y) \end{aligned}\right\} \tag{9}$$

so that

$$\begin{aligned} W_i &= \frac{\partial W}{\partial y^i} = -|T| \frac{\partial x^n}{\partial y^i} + S \frac{\partial w}{\partial x^k} \frac{\partial x^k}{\partial y^i} \\ W_{ij} &= -|T| \frac{\partial^2 x^n}{\partial y^i \partial y^j} + S\left(w_{kl} \frac{\partial x^l}{\partial y^j} \frac{\partial x^k}{\partial y^i} + w_k \frac{\partial^2 x^k}{\partial y^i \partial y^j} \right). \end{aligned} \tag{10}$$

Using (8), (9) one calculates

$$\frac{\partial^2 x^k}{\partial y^i \partial y^j} = \begin{cases} 0, & k < n \\ -|T|\, W_{ij}, & k = n \end{cases}$$

$$\frac{\partial x^l}{\partial y^j}=\begin{cases}\delta^{lj}, & j<n\\ \mathrm{S}, & l=j=n.\end{cases}$$

Using these formulas in (10) and the identity $\mathrm{S}\,w_n=|\mathrm{T}|$ yields Lemma 1:

$$\mathrm{W}_{ij}=\begin{cases}\mathrm{S}\,w_{ij}, & i,j<n\\ \mathrm{S}^2\,w_{ij}, & \text{one of } i,j=n\\ \mathrm{S}^3\,w_{nn}, & i=j=n.\end{cases}$$

LEMMA 2. — *Let* S_u *and* $\mathrm{S}_{\bar{u}}$ *be the solution surface and perturbed surface described earlier. Let,* $\mathrm{P}, \nu(\mathrm{S}_u, \mathrm{P})$, U *correspond to* $\bar{\mathrm{P}}$, $\nu(\mathrm{S}_{\bar{u}}, \bar{\mathrm{P}})$, $\bar{\mathrm{U}}$ *under the perturbation by* $\varepsilon\eta\nu$. *Then*

$$\nu(\bar{\mathrm{P}})=\nu(\mathrm{P})-\varepsilon\,\nabla_{\mathrm{T}}\,\eta+O(\varepsilon^2) \tag{11}$$

and given coordinates in $\pi(\mathrm{S}_u, \mathrm{P})$, *there are correpsonding coordinates in* $\pi(\mathrm{S}_{\bar{u}}, \bar{\mathrm{P}})$ *so that*

$$\bar{\mathrm{U}}_{ij}=\mathrm{U}_{ij}-\varepsilon(\eta\,\mathrm{U}_{ik}\,\mathrm{U}_{kj}+\eta_{ij}-\eta_\nu\,\mathrm{U}_{ij})+O(\varepsilon^2). \tag{12}$$

[*Again,* $O(\varepsilon^2)$ *terms depend at most on* C^3 *bounds for* u *and* C^2 *bounds for* $\eta|_{\{\eta>0\}}$. *The term* $\nabla_{\mathrm{T}}\,\eta$ *in* (11) *is the tangential gradient,* $\nabla_{\mathrm{T}}\,\eta=\nabla\eta-(\nabla\eta\,.\,\nu(\mathrm{S}_u, \mathrm{P}))\,\nu(\mathrm{S}_u, \mathrm{P})$. *The term* $\eta_\nu=\nabla\eta\,.\,\nu(\mathrm{S}_u, \mathrm{P})$.]

Proof. — Using the chain rule and (5), one can directly calculate first and second derivatives of $\bar{u}$ with respect to $\bar{x}$. This is done in [11]. From (26) there we have, in the case $\mathrm{D}u(x)=0$, (so $u_{ij}=\mathrm{U}_{ij}$) that at $\bar{\mathrm{P}}$ and P

$$\begin{aligned}\bar{u}_i&=-\varepsilon\eta_i+O(\varepsilon^2)\\ \bar{u}_{ij}&=\mathrm{U}_{ij}-\varepsilon(\eta\,\mathrm{U}_{ik}\,\mathrm{U}_{kj}+\eta_{ij}+\eta_{ij}+\eta_z\,\mathrm{U}_{ij})+O(\varepsilon^2).\end{aligned} \tag{13}$$

In these coordinates

$$\nu(\mathrm{S}_{\bar{u}}, \bar{\mathrm{P}})=(-\varepsilon\eta_1, \ldots, -\varepsilon\eta_n, -1)+O(\varepsilon^2)=\nu(S_u, \mathrm{P})-\varepsilon\,\nabla_{\mathrm{T}}\,\eta+O(\varepsilon^2),$$

so (11) holds.

Apply Lemma 1 to the function $w=\bar{u}$. Because $|\mathrm{D}\bar{u}|=O(\varepsilon)$, $\mathrm{S}=1+O(\varepsilon^2)$. If the coordinates for $\pi(\mathrm{S}_u, \mathrm{P})$ and $\pi(\mathrm{S}_{\bar{u}}, \bar{\mathrm{P}})$ are chosen as in the lemma, formula (12) follows from (13) (and $\eta_z=-\eta_\nu$). If any other coordinate system is used in $\pi(\mathrm{S}_u, \mathrm{P})$, it differs from the first by an orhtogonal transformation. Pick a corresponding system in $\pi(\mathrm{S}_{\bar{u}}, \bar{\mathrm{P}})$ differing from the one in Lemma 1 by an orthogonal transformation with the same matrix. In computing the matrices of $\mathscr{D}^2\,\bar{\mathrm{U}}$ and $\mathscr{D}^2\,\mathrm{U}$ with respect to these new coordinates, the original tangential Hessians will be conjugated by the same orthogonal matrix. So will be the three $O(\varepsilon)$ terms in (12). Thus (12) is true in the new coordinates also, and Lemma 2 is proven.

We wish to study $f(\lambda)$ where the height difference between $S_{\bar{u}}$ and S_u is maximized, using (1)-(4), (11), (12). In calculating, we will not be able to assume that the Hessians under consideration are diagonal. Thus it is important to write the function $f(\lambda)$ in terms of the tangential Hessian:

$$f(\lambda)=\sum_{i_1<\ldots<i_k} \lambda_{i_1}\ldots\lambda_{i_k}=\mathrm{F}(\mathscr{D}^2\mathrm{U})$$

$$=\sum_{\substack{\sigma\in S_k\\ i_1<\ldots<i_k}} (-1)^\sigma \mathrm{U}_{i_1 i_{\sigma(1)}}\mathrm{U}_{i_2 i_{\sigma(2)}}\ldots\mathrm{U}_{i_k i_{\sigma(k)}} \tag{14}$$

This formula is true because $\mathrm{F}(\mathscr{D}^2\mathrm{U})$ is a coefficient of the characteristic equation of the matrix $[\mathrm{U}_{ij}]$ and is invariant with respect to conjugation. Choosing coordinates in which $[\mathrm{U}_{ij}]$ is diagonal with entries $\lambda_1, \ldots, \lambda_n$, $\mathrm{F}(\mathscr{D}^2\mathrm{U})$ equals $f(\lambda)$.

From (14) the following important fact follows:

$$f_{\lambda_i}\geqq 0,\ \forall_i \quad\Leftrightarrow\quad [\mathrm{F}_{\mathrm{U}_{ij}}]\geqq 0. \tag{15}$$

This is because under changes of coordinates $\mathrm{F}_{\mathrm{U}_{ij}}$ is conjugated by orthogonal matrices so its positiveness is invariant. If coordinates are chosen so that $\mathscr{D}^2\mathrm{U}$ is diagonal, then from (14) one can see that $\mathrm{F}_{\mathrm{U}_{ij}}$ is a diagonal matrix with diagonal entries f_{λ_i}. In fact, the same reasoning shows that the other inequalities in (4) can also be stated in terms of derivatives of F, and we will need them.

LEMMA 3. — *We have the following equivalence:*

$$\left.\begin{array}{c} f_{\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_l}}\geqq 0,\\ \forall \textit{ collections } (i_1,\ldots,i_l) \textit{ of (distinct) indices}\\ \Leftrightarrow\\ \mathrm{F}_{\mathrm{U}_{i_1 j_1}\mathrm{U}_{i_2 j_2}\ldots\mathrm{U}_{i_l j_l}}\xi_{i_1 i_2\ldots i_l}\xi_{j_1 j_2\ldots j_l}\geqq 0\\ \forall \textit{ vectors } \xi=\{\xi_{i_1\ldots i_l}\} \textit{ with } l\cdot n \textit{ components.}\end{array}\right\} \tag{16}$$

To prove Lemma 3, we first show that the second condition is invariant under rotation of coordinates. Let θ be an orthogonal matrix and

$$\mathrm{U}_{km}=\mathrm{V}_{ij}\theta^{ki}\theta^{mj}.$$

Then

$$\frac{\partial}{\partial V_{ij}} = \theta^{ki}\,\theta^{mj}\frac{\partial}{\partial U_{km}}$$

so that

$$F_{V_{i_1 j_1} V_{i_2 j_2} \dots V_{i_l j_l}} \zeta_{i_1 i_2 \dots i_l} \zeta_{j_1 j_2 \dots j_l}$$

$$= F_{U_{k_1 m_1} U_{k_2 m_2} U_{k_l m_l}} \theta^{k_1 i_1}\,\theta^{m_1 j_1} \dots \theta^{k_l i_l}\,\theta^{m_l j_l} \zeta_{i_1 i_2 \dots i_l} \zeta_{j_1 j_2 \dots j_l}$$

$$= F_{U_{k_1 m_1} U_{k_2 m_2} \dots U_{k_l m_l}} \xi_{k_1 k_2 \dots k_l} \xi_{m_1 m_2 \dots m_l}$$

for

$$\xi_{k_1 \dots k_l} = \theta^{k_1 i_1}\,\theta^{k_2 i_2} \dots \theta^{k_l i_l} \zeta_{i_1 i_2 \dots i_l}.$$

Therefore it suffices to check (16) in the case $\mathscr{D}^2 U$ is diagonal, $U_{ij} = \lambda_i \delta^{ij}$. In these coordinates, $f_{\lambda_{i_1} \dots \lambda_{i_l}} = F_{U_{i_1 i_1} \dots U_{i_l i_l}}$. The implication $\Leftarrow$ is then immediate if for fixed $(i_1, \dots, i_l)$ we pick ξ by

$$\xi_{m_1 \dots m_l} = \begin{cases} 1 & \text{if } (m_1, \dots, m_l) = (i_1, \dots, i_l) \\ 0 & \text{otherwise.} \end{cases}$$

We show $\Rightarrow$ as follows. Realizing that $F_{U_{i_1 j_1} \dots U_{i_l j_l}}$ is the sum of all terms in (14) that contain the product $U_{i_1 j_1} \dots U_{i_l j_l}$, divided by this product, yields for $U_{ij} = \lambda_i \delta^{ij}$

$$F_{U_{i_1 j_1} \dots U_{i_l j_l}} = \begin{cases} 0 & \text{if } i_1, \dots, i_l \text{ not distinct;} \\ 0 & \text{if } i_1, \dots, i_l \text{ distinct,} \\ \text{but} & \\ \nexists\, \sigma \in S_l, & i_{\sigma(k)} = j_k, \quad k = 1, \dots, l; \\ (-1)^\sigma f_{\lambda_{i_1} \dots \lambda_{i_l}} & \text{if } i_1, \dots, i_l \text{ distinct,} \\ \text{and} & \\ \exists\, \sigma \in S_l, & i_{\sigma(k)} = j_k, \quad k = 1, \dots, l. \end{cases}$$

Therefore

$$F_{U_{i_1 j_1} \dots U_{i_l j_l} \xi_{i_1 \dots i_l} \xi_{j_1 \dots j_l}}$$
$$= \sum_{i_1 < \dots < i_l} f_{\lambda_{i_1} \dots \lambda_{i_l}} \Big(\sum_{\alpha, \beta \in S_l} (-1)^\alpha (-1)^\beta \xi_{i_{\alpha(1)} \dots i_{\alpha(l)}} \xi_{i_{\beta(1)} \dots i_{\beta(l)}} \Big)$$
$$= \sum_{i_1 < \dots < i_l} f_{\lambda_{i_1} \dots \lambda_{i_l}} \Big(\sum_{\alpha \in S_l} (-1)^\alpha \xi_{i_{\alpha(1)} \dots i_{\alpha(l)}} \Big)^2.$$

Thus $\Rightarrow$ holds and Lemma 3 is shown.

Since the perturbation function $\eta(x, u(x))$ has compact support on S_u, $\exists y$ where $u - \bar{u} = \varepsilon \eta \sqrt{1 + |Du|^2} + O(\varepsilon^2)$ is maximized. Then $y = \bar{x}$ for some x. Continue to write $P = (x, u(x))$, $\bar{P} = (\bar{x}, \bar{u}(\bar{x}))$ and call $(\bar{x}, u(\bar{x})) = \bar{\bar{P}}$. We almost have a maximum principle for f above $\bar{x}$:

LEMMA 4. — *Let* P, $\bar{P}$, $\bar{\bar{P}}$ *be as above. Then because* $u - \bar{u}$ *attains its maximum at* $\bar{x}$,

$$f(\lambda(S_{\bar{u}}, \bar{P})) \geqq f(\lambda(S_u, \bar{\bar{P}})) + O(\varepsilon^2). \tag{17}$$

Proof. — Since $u - \bar{u}$ is maximized at $\bar{x}$, calculus implies

$$\left. \begin{array}{c} \nu(S_{\bar{u}}, \bar{P}) = \nu(S_u, \bar{\bar{P}}) \\ [\bar{u}_{ij}(\bar{x})] \geqq [u_{ij}(\bar{x})] \end{array} \right\} \tag{18}$$

From Lemma 1 this implies that in corresponding coordinates

$$[\bar{U}_{ij}(\bar{P})] \geqq [U_{ij}(\bar{\bar{P}})]. \tag{19}$$

We also have

$$U_{ij}(\bar{\bar{P}}) - U_{ij}(P) = O(\varepsilon). \tag{20}$$

This is because $u_{ij}(\bar{x}) - u_{ij}(x)$ and $u_{ij}(x) - \bar{u}_{ij}(\bar{x})$ are both $O(\varepsilon)$, so their sum is, and by Lemma 1 so is the left-hand side of (20).

To prove the lemma we use (4), (15), (19), (20) and compute

$$f(\lambda(S_{\bar{u}}, \bar{P}) - f(\lambda(S_u, \bar{\bar{P}}))$$
$$= \int_0^1 \frac{d}{dt} F([U_{ij}(\bar{\bar{P}}) + t(\bar{U}_{ij}(\bar{P}) - U_{ij}(\bar{\bar{P}})]) \, dt$$
$$= \int_0^1 F_{U_{ij}}(U_{ij}(\bar{\bar{P}}) + O(\varepsilon))(\bar{U}_{ij}(\bar{P}) - U_{ij}(\bar{\bar{P}})) \, dt$$
$$= F_{U_{ij}}(U_{ij}(\bar{\bar{P}}))(\bar{U}_{ij}(\bar{P}) - U_{ij}(\bar{\bar{P}})) + O(\varepsilon^2) \geqq O(\varepsilon^2).$$

Both sides of (17) can be estimated in terms of $f(\lambda(\mathrm{S}_u, \mathrm{P}))$ and derivatives of η at P. For appropriate η, we will show that (17) cannot hold if $|\mathrm{D}u(x)|$ is too large. This will lead to the desired *a priori* bound. (In other words, if $\mathrm{S}_{\bar{u}}$ is lifted a large enough multiple of ε in the z-direction, this argument will show that the lifting lies above S_u, motivating the earlier use of the words "interior barrier".)

From calculus and (1),

$$f(\lambda(\mathrm{S}_u, \bar{\bar{\mathrm{P}}}) = \psi(\bar{\bar{\mathrm{P}}}, \nu(\mathrm{S}_u, \bar{\bar{\mathrm{P}}}))$$
$$= \psi(\mathrm{P}, \nu) + \nabla\psi(\mathrm{P}, \nu).(\bar{\bar{\mathrm{P}}} - \mathrm{P}, \nu(\mathrm{S}_u, \bar{\bar{\mathrm{P}}}) - \nu) + O(\varepsilon^2), \quad (21)$$

where $\nu = \nu(\mathrm{S}_u, \mathrm{P})$. From (12), (14),

$$f(\lambda(\mathrm{S}_{\bar{u}}, \bar{\mathrm{P}})) = \mathrm{F}(\mathscr{D}^2 \bar{\mathrm{U}}(\bar{\mathrm{P}}))$$
$$= \psi(\mathrm{P}, \nu) - \varepsilon\, \mathrm{F}_{\mathrm{U}_{ij}}(\eta\, \mathrm{U}_{ik}\mathrm{U}_{kj} + \eta_{ij} - \eta_\nu \mathrm{U}_{ij}) + O(\varepsilon^2). \quad (22)$$

[We write $\mathrm{F}_{\mathrm{U}_{ij}}$ for $\mathrm{F}_{\mathrm{U}_{ij}}(\mathscr{D}^2\mathrm{U}(\mathrm{P}))$.] Combining (17), (21), (22) along with (2) yields the estimate

$$\mathrm{F}_{\mathrm{U}_{ij}}(\eta\, \mathrm{U}_{ik}\mathrm{U}_{kj} + \eta_{ij} - \eta_\nu \mathrm{U}_{ij}) \leqq 1 + \mathrm{M}(\eta + |\nabla\eta|) \quad (23)$$

for ε sufficiently small. Because $\mathrm{F}_{\mathrm{U}_{ij}}\mathrm{U}_{ij} = k\,\mathrm{F}$ and because $[\mathrm{F}_{\mathrm{U}_{ij}}] \geqq 0$, (23) implies

$$\mathrm{F}_{\mathrm{U}_{ij}}\eta_{ij} \leqq 1 + \mathrm{M}\eta + (\mathrm{M} + k\psi_1)|\nabla\eta|. \quad (24)$$

As in [11], pick $\eta = h \circ \varphi$ where

$$\varphi(x, z) = \left[\frac{1}{2u_0} z + (1 - |x|^2)\right]^+ \qquad (+\text{means positive part}) \quad (25)$$

and

$$h(\varphi) = e^{\mathrm{K}\varphi} - 1, \qquad \mathrm{K}\ \text{large}. \quad (26)$$

Because $u \in \mathrm{C}^3(\bar{\mathrm{B}}_1)$, $u < 0$, $\eta(x, u(x))$ is continuous and has compact support on S_u, inside $\mathrm{B}_1 \times (-\infty, 0)$. From (25),

$$0 \leqq \varphi \leqq 1, \qquad \varphi_z = \frac{1}{2u_0}, \qquad \varphi_{zz} = \varphi_{iz} = 0,$$
$$\varphi_i\varphi_i \leqq 4, \qquad \varphi_{ij} = -2\,\delta^{ij}. \quad (27)$$

From (24), (25), (27)

$$\left.\begin{aligned} F_{U_{ij}}(h''\varphi_i\varphi_j+h'\varphi_{ij}) &\leqq 1+M\eta+(M+k\psi_1)|\nabla\eta| \\ h''F_{U_{ij}}\varphi_i\varphi_j &\leqq 1+Mh+(M_2+M_3\|F_{U_{ij}}\|)h' \end{aligned}\right\} \tag{28}$$

We show

LEMMA 5. — *If* P, $\bar{P}$, $\bar{\bar{P}}$, φ *are as above* $\exists M_4$ *s. t.* $|Du(x)|>M_4$ *implies* $\exists\delta>0$ *s. t.*

$$F_{U_{ij}}\varphi_i\varphi_j \geqq \delta(1+\|F_{U_{ij}}\|). \tag{29}$$

Here, as elsewhere, all constants depend only on n, k, ψ_1, M, ψ_0, u_0. *We are finished after proving the lemma, since we then pick* K *in* (26) *large enough so that* (28) *is violated. Therefore* $|Du(x)|\leqq M_4$ *and for* $\tilde{x}\in B_1$,

$$\begin{aligned}\eta(\tilde{x}, u(\tilde{x}))\sqrt{1+|Du(\tilde{x})|^2} &\leqq \eta(P)\sqrt{1+|Du(x)|^2}+O(\varepsilon) \\ &\leqq \eta(P)\sqrt{1+M_4^2}+O(\varepsilon),\end{aligned}$$

from which the desired bound follows:

$$\sqrt{1+|Du(0)|^2} \leqq \frac{\sup\limits_{B_1\times(-\infty,0]}\eta}{\eta(0,-u_0)}\sqrt{1+M_4^2}=C(n, k, \psi_1, M, \psi_0, u_0). \tag{30}$$

For the case of mean curvature, $k=1$, Lemma 5 is contained in the calculations of [11], so we restrict to $2\leqq k\leqq n-1$. We isolate the direction of steepest ascent in $\pi(S_u, P)$-the nth direction. Pick the first $n-1$ directions (along which π is horizontal) so that the submatrix $[U_{ij}]$, $1\leqq i, j\leqq n$, is diagonal with decreasing eigenvalues $\mu_1\geqq\mu_2\geqq\ldots\geqq\mu_{n-1}$. That is, with respect to these coordinates, $\mathcal{D}^2U(P)$ has the form

$$[U_{ij}]=\begin{bmatrix}\mu_1 & 0 & U_{1n} \\ 0 & \mu_{n-1} & \vdots \\ U_{n1} & \ldots & \mu_n\end{bmatrix}, \qquad \mu_1\geqq\mu_2\geqq\ldots\geqq\mu_{n-1}. \tag{31}$$

Since $U_{ii}=\mu_i$, we will often write F_{μ_i} for $F_{U_{ii}}$.

The first order contact (18) yields information about U_{ni}, $1\leqq i\leqq n$: Calculating in the coordinates of π, using (13) and calculus, yields

$$-\varepsilon\eta_i=\varepsilon\eta|Du(x)|U_{ni}+O(\varepsilon^2).$$

This is because to within $O(\varepsilon^2)$, one must travel an amount $\varepsilon\eta\,|Du(x)|$ along π in the nth-direction to get to the projection of $\bar{\bar{P}}$ onto π. From $\eta = h\circ\varphi$,

$$U_{ni} = -\frac{h'\,\varphi_i}{h\,|Du|} + O(\varepsilon). \tag{32}$$

In particular, since $\varphi_z = \dfrac{1}{2u_0} > 0$, it follows (*see* [11]) that for

$$|Du(x)| \geqq \max(12u_0, 3) = M_4, \ \varphi_n \geqq \frac{1}{\sqrt{10u_0}}.$$

Thus (for ε sufficiently small)

$$\mu_n = U_{nn} < 0 \qquad \text{if} \quad |Du(x)| \geqq M_4. \tag{33}$$

We need the following inequalities to prove Lemma 5.

$$\sum_{j_l \neq n,\, i_1 \ldots i_r} \mu_{j_1}\mu_{j_2}\ldots\mu_{j_{k-r}} = F_{\mu_n \mu_{i_1}\ldots\mu_{i_r}} \geqq 0 \tag{34}$$

$$\sum_{\substack{j_l \neq i_1 \ldots i_r \\ i_s \neq n \\ < \ <}} \mu_{j_1}\ldots\mu_{j_{k-r}} = F_{\mu_{i_1}\ldots\mu_{i_r}} + \sum_{m \neq n,\, i_1,\, \ldots,\, i_r} U^2_{mn} F_{\mu_n \mu_m \mu_{i_1}\ldots\mu_{i_r}} \geqq 0 \tag{35}$$

$$F([U_{ij}]) \leqq \binom{n-1}{k} \mu_1\mu_2\ldots\mu_k \qquad \text{if} \quad \mu_n < 0. \tag{36}$$

The symbols $<\ \ <$ in (34), (35) mean that $i_1 < \ldots < i_r$ and $j_1 < \ldots < j_r$.
To prove (34) then (35), note (4), (16) and that

$$F_{\mu_{i_1}\ldots\mu_{i_r}} = F_{U_{l_1 k_1} U_{l_2 k_2}\ldots U_{l_r k_r}} \xi_{l_1 l_2 \ldots l_r} \xi_{k_1 k_2 \ldots k_r}$$

for

$$\xi_{l_1 l_2 \ldots l_r} = \begin{cases} 1 & \text{if} \quad l_s = i_s \\ 0 & \text{otherwise.} \end{cases}$$

Thus $F_{\mu_{i_1}\ldots\mu_{i_r}} \geqq 0$. Using (31), compute this derivative in the case that some $i_s = n$ (34), and then if no $i_s = n$ (35).

Remark. — There is an alternate proof of (34), (35). One can show that if $\lambda=(\lambda_1, \ldots, \lambda_n)\in\Gamma$ and if $\mu=(\mu_1, \ldots, \mu_n)$ is defined by

$$\mu_k=\alpha_{ki}\lambda_i, \qquad \alpha_{ki}\geqq 0, \qquad \sum_k \alpha_{ki}=\sum_i \alpha_{ki}=1,$$

then also $\mu\in\Gamma$. (If $\mathcal{D}^2 U$ is the conjugate of $[\lambda_i\delta^{ij}]$ by θ, then $\mu_k=\theta_{ki}^2\lambda_i$.) The characterization (4) of Γ then implies $f_{\mu_{i_1}\ldots\mu_{i_r}}\geqq 0$, (34), (35).

Use (34), (35) to prove (36) as follows. From (14), (31),

$$F([U_{ij}])=\sum_{<}\mu_{j_1}\ldots\mu_{j_k}-\sum_{i\neq n}U_{ni}^2\sum_{j_l\neq i,n}\mu_{j_1}\ldots\mu_{j_{k-2}}$$

$$\leqq\sum_{<}\mu_{j_1}\ldots\mu_{j_k}=\frac{1}{k}\sum_i\Sigma\mu_i(\sum_{\substack{j_l\neq i\\ <}}\mu_{j_1}\ldots\mu_{j_{k-1}})$$

$$\leqq\frac{1}{k}\sum_{\mu_i>0}\mu_i(\sum_{\substack{j_l\neq i\\ <}}\mu_{j_1}\ldots\mu_{j_{k-1}})$$

$$=\frac{1}{k}\sum_{\mu_{i_1}>0}\mu_{i_1}\left(\frac{1}{k-1}\sum_{i_2\neq i_1}\mu_{i_2}(\sum_{\substack{j_j\neq i_1,i_2\\ <}}\mu_{j_1}\ldots\mu_{j_{k-2}})\right)$$

$$\leqq\frac{1}{k(k-1)}\sum_{\substack{\mu_{i_s}>0\\ i_s\text{ distinct}}}\mu_{i_1}\mu_{i_2}(\sum_{\substack{j_l\neq i_1,i_2\\ <}}\mu_{j_1}\ldots\mu_{j_{k-2}})\times\ldots$$

$$\times F([U_{ij}])\leqq\frac{1}{k!}\sum_{\substack{\mu_{i_s}>0\\ i_s\text{ distinct}}}\mu_{i_1}\mu_{i_2}\ldots\mu_{i_k}. \tag{37}$$

Since $F>0$, there must be some terms in the sum of (37), and the largest is $\mu_1\mu_2\ldots\mu_k$ (since $\mu_n<0$). There are at most $(n-1)(n-2)\ldots(n-k)$ terms possible. Thus (37) implies (36).

Returning to the lemma, note that

$$F_{U_{ij}}\varphi_i\varphi_j=F_{U_{nn}}\varphi_n\varphi_n+2\sum_{i\neq n}F_{U_{in}}\varphi_i\varphi_n+\sum_{i,j\neq n}F_{U_{ij}}\varphi_i\varphi_j. \tag{38}$$

The third term in the sum is nonnegative since $[F_{U_{ij}}]\geqq 0$. The condition (32) kindly makes the second term nonnegative to within $O(\varepsilon)$, for

$|Du(x)| \geqq M_4$: Since

$$F_{U_{in}} = -U_{ni} \sum_{\substack{j_l \neq i, n \\ <}} \mu_{j_1} \ldots \mu_{j_{k-2}},$$

we have

$$\sum_{i \neq n} F_{U_{in}} \varphi_i \varphi_n = \frac{h|Du|}{h'} \varphi_n \sum_{i \neq n} U_{in}^2 (F_{\mu_n \mu_i}) + O(\varepsilon).$$

Thus for $|Du(x)| \geqq M_4$, it suffices to show (29) by finding $\delta > 0$ so that

$$F_{U_{nn}} \varphi_n \varphi_n \geqq \delta (1 + \| F_{U_{ij}} \|).$$

Since $\varphi_n \geqq \dfrac{1}{\sqrt{10 u_0}}$, it suffices to find $\delta_1 > 0$ with

$$F_{U_{nn}} \geqq \delta_1 (1 + \| F_{U_{ij}} \|). \tag{39}$$

Using (34), (35) and $\mu_n \leqq 0$ yields

$$\begin{aligned}
F_{U_{nn}} &= \sum_{\substack{j_l \neq n \\ <}} \mu_{j_1} \ldots \mu_{j_{k-1}} \\
&= \sum_{\substack{j_l \neq 1 \\ <}} \mu_{j_1} \ldots \mu_{j_{k-1}} + (\mu_1 - \mu_n) \sum_{\substack{j_l \neq 1, n \\ <}} \mu_{j_1} \ldots \mu_{j_{k-2}} \\
&\geqq (\mu_1 - \mu_n) \sum_{\substack{j_l \neq 1, n \\ <}} \mu_{j_1} \ldots \mu_{j_{k-2}} \\
&= (\mu_1 - \mu_n) \Big(\sum_{\substack{j_l \neq 1, 2 \\ <}} \mu_{j_1} \ldots \mu_{j_{k-2}} + (\mu_2 - \mu_n) \sum_{\substack{j_l \neq 1, 2, n \\ <}} \mu_{j_1} \ldots \mu_{j_{k-3}} \Big) \\
&\geqq (\mu_1 - \mu_n)(\mu_2 - \mu_n) \sum_{\substack{j_l \neq 1, 2, n \\ <}} \mu_{j_1} \ldots \mu_{j_{k-3}},
\end{aligned}$$

$$F_{U_{nn}} \geqq (\mu_1 - \mu_n)(\mu_2 - \mu_n) \ldots (\mu_{k-1} - \mu_n) \geqq \mu_1 \mu_2 \ldots \mu_{k-1}. \tag{40}$$

Now use the apparently crucial hypothesis that ψ is strictly positive. From (36), (3), (40),

$$0 < \psi_0 \leqq F \leqq \binom{n-1}{k} \mu_1 \ldots \mu_k \leqq \binom{n-1}{k} (\mu_1 \ldots \mu_{k-1})^{1 + [1/(k-1)]}$$

so that

$$F_{U_{nn}} \geqq \left[\binom{n-1}{k}^{-1} \psi_0\right]^{(k-1)/k}. \tag{41}$$

Because $[F_{U_{ij}}]$ is positive semidefinite, $\sup_l |F_{U_{ll}}|$ dominates $\|F_{U_{ij}}\|$. But for any $l \neq n$, (35) implies

$$F_{U_{ll}} \leqq \sum_{\substack{j_m \neq l \\ <}} \mu_{j_1} \ldots \mu_{j_{k-1}}.$$

As in the proof of (36), equations (34), (35) imply after a chain of inequalities that

$$F_{U_{ll}} \leqq \binom{n-2}{k-1} \mu_1 \ldots \mu_{k-1}$$

so that by (40)

$$F_{U_{nn}} \geqq F_{U_{ll}} \binom{n-2}{k-1}^{-1}. \tag{42}$$

Combining (42), (41), one can pick δ_1 to make (39) true. Thus Lemma 5 and the main theorem are proven.

Isolating the dependence of the gradient bound (30) on u_0 and chasing constants, one can see that $K \sim \frac{1}{\delta} \sim u_0^2$ so that our gradient bound has the form

$$|Du(0)| \leqq C_1 e^{C_2 u_0^3}. \tag{43}$$

Since the best estimate for the prescribed mean curvature equation grows only like $C_1 e^{C_2 u_0}$, it is possible that the bound (43) is not optimal.

ACKNOWLEDGEMENTS

This work was supported by the National Science Foundation under grants No. MCS-8301906 and No. DMS 85-11478.

REFERENCES

[1] I. Bakelman and B. Kantor, Estimates for Solutions of Quasilinear Elliptic Equations Connected with Problems of Geometry in the Large, *Mat. Sbornik*, Vol. **91**, (133), 1973, pp. 336-349 = *Math. U.S.S.R.-Sbornik*, Vol. **20**, 1973, pp. 348-363.

[2] I. Bakelman and I. Kantor, Existence of Spherically Homeomorphic Hypersurfaces in Euclidean Space with Prescribed Mean Curvature, *Geometry and Topology*, Vol. **1**, 1974, pp. 3-10, Leningrad.

[3] E. Bombieri, E. Di Giorgi and M. Miranda, Una maggiorazione a priori relative alle ipersupertici minimali nonparametriche, *Arch. Rat. Mech. Anal*, Vol. **32**, 1969, pp. 255-269.

[4] L. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet Problem for Nonlinear Second Order Elliptic Equations. I. Monge-Ampère Equations, *Comm. Pure Appl. Math.*, Vol. **37**, 1984, pp. 369-402.

[5] L. Caffarelli, J. J. Kohn, L. Nirenberg and J. Spruck, The Dirichlet Problem for Nonlinear Second Order Equations. II. Complex Monge-Ampère, and uniformly Elliptic, Equations, *Comm. Pure Appl. Math.*, Vol. **38**, 1985, pp. 209-252.

[6] L. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet Problem for Nonlinear Second Order Elliptic Equations. III. Functions of eigenvalues of the Hessian, *Acta. Math.* (to appear).

[7] L. Caffarelli, L. Nirenberg and J. Spruck, *Nonlinear Second Order Elliptic Equations. IV. Starshaped Compact Weingarten Hypersurfaces* (to appear).

[8] R. Finn, New Estimates for Equations of Minimal Surface Type, *Arch. Rat. Mech. Anal.*, Vol. **14**, 1963, pp. 337-375.

[9] L. Gårding, An Inequality for Hyperbolic Polynomials, *J. Math. and Mechanics*, Vol. **8**, 1959, pp. 957-965.

[10] N. M. Ivočkina, The Integral Method of Barrier Functions and the Dirichlet Problem for Equations with Operators of Monge-Ampère Type, *Math. U.S.S.R.-Sbornik*, Vol. **40**, 1981, pp. 179-192.

[11] N. J. Korevaar, An Easy Proof of the Interior Gradient Bound for Solutions to the Prescribed Mean Curvature Problem, *Trans. A.M.S.* (to appear).

[12] N.V. Krylov, Boundedly Nonhomogeneous Elliptic and Parabolic Equations in a Domain, *Izvestia Math. Ser.*, Vol. **47**, 1983, pp. 75-108.

[13] V. I. Oliker, Hypersurfaces in $\mathbb{R}^{n+1}$ with Prescribed Gaussian Curvature and Related Equations of Monge-Ampère Type, *Comm. Part. Diff. Eqtns.*, Vol. **9**, 1984, pp. 807-838.

[14] A. V. Pogorelov, *The Minkowski Multi-Dimensional Problem*, Wiley, New York, 1978.

[15] L. Simon, Interior Gradient Bounds for Non-Uniformly Elliptic Equations, *Indiana U. Math. J.*, Vol. **25**, (9), 1976, pp. 821-855.

[16] A. E. Treibergs and S. W. Wei, Embedded Hypersurfaces with Prescribed Mean Curvature, *J. Diff. Geom.*, Vol. **18**, 1983, pp. 513-521.

[17] N. Trudinger, A New Proof of the Interior Gradient Bound for the Minimal Surface Equation in n Dimensions, *Proc. Nat. Acad. Sci. U.S.A.*, Vol. **69**, 1972, pp. 166-175.

(*Manuscrit reçu le* 29 *novembre* 1985.)