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Minimal solutions of variational problems on a torus

by

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1. INTRODUCTION

a) The variational problem.

In this paper we consider a special class of extremals, the so-called minimal solutions, of a variational problem on a torus. We view the torus T^{n+1} as the quotient of its universal covering manifold $\mathbb{R}^{n+1}$ by the group $\mathbb{Z}^{n+1}$. Denoting the points in $\mathbb{R}^{n+1}$ by $\bar{x} = (x_1, x_2, \ldots, x_{n+1})$ and setting $x = (x_1, x_2, \ldots, x_n)$ we consider n-dimensional hypersurfaces in $\mathbb{R}^{n+1}$ which can be represented as the graph of a function $u(x)$ over $\mathbb{R}^n$ by

$$x_{n+1} = u(x), \qquad x \in \mathbb{R}^n. \tag{1.1}$$

No periodicity is required for $u(x)$.

Such a function will be called an extremal for a variational problem

$$\int F(x, u, u_x)dx \tag{1.2}$$

if it is a solution of the Euler equation

$$\sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu} F_{u_{x_\nu}}(x, u, u_x) = F_u(x, u, u_x). \tag{1.3}$$

Here it is required that the integrand $F = F(x, x_{n+1}, p)$ has period 1 in the variables $x_1, x_2, \ldots, x_{n+1}$ so that the differential equation (1.3) is invariant under the translations $\bar{x} \to \bar{x} + \bar{j}$, $\bar{j} \in \mathbb{Z}^{n+1}$ and can be viewed as a differential equation on the torus $T^{n+1} = \mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$. Moreover, we will assume that

$$F \in C^{l,\varepsilon}(T^{n+1} \times \mathbb{R}^n), \qquad l \geq 2, \quad 0 < \varepsilon < 1,$$

i. e. F has derivatives up to order l which are Hölder continuous with Hölder exponent ε. Moreover, F satisfies the Legendre condition

$$\delta \, | \, \xi \, |^2 \leq F_{p_\nu p_\mu}(x, x_{n+1}, p)\xi_\nu \xi_\mu \leq \delta^{-1} \, | \, \xi \, |^2 \tag{1.4}$$

with some constant $\delta \in (0, 1)$, for all $(\bar{x}, p) \in \mathbb{R}^{2n+1}$. The function F should grow roughly like $| p |^2$ for large $| p |$; the precise conditions are given in Section 3, (3.1). The typical example is given by

$$F(x, x_{n+1}, p) = \Sigma a_{\nu\mu}(\bar{x})p_\nu p_\mu + 2\Sigma b_\nu(\bar{x})p_\nu + c(\bar{x}) \tag{1.5}$$

where $a_{\nu\mu}$, b_ν, c belong to $C^{l,\varepsilon}(T^{n+1})$ and $a_{\nu\mu}$ is positive definite.

b) Minimal solutions.

As a rule one considers a variational problem like (1.2) over a compact domain. Since we want to consider noncompact hypersurfaces (1.1) and the domain of definition of u is $\mathbb{R}^n$ the question arises in which sense the variational principle is to be understood. Here we follow Giaquinta and Giusti [*8*] in defining « minimal solutions » of the variational problem. We require $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$, the space of u for which the first derivatives belong to $L^2_{\mathrm{loc}}(\mathbb{R}^n)$.

DEFINITION 1.1. — An element $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ is called a minimal solution of the variational problem (1.2) if

$$\int_{\mathbb{R}^n} (F(x, u + \varphi, u_x + \varphi_x) - F(x, u, u_x))dx \geq 0$$

for every $\varphi \in W^{1,2}_{\mathrm{comp}}(\mathbb{R}^n)$.

In other words, fixing u at the boundary of any domain $\Omega \subset \mathbb{R}^n$ it gives the minimal value to the integral

$$I_\Omega(u) = \int_\Omega F(x, u, u_x)dx \, .$$

Therefore every minimal solution will be an extremal but not every extremal is a minimal solution. In general, extremals are minimal only with respect to sufficiently small domains and it is not at all clear whether minimal solutions exist.

Since the variational problem is invariant under the group of translations $\mathbb{Z}^{n+1}$, say $\bar{x} \to \bar{x} + \bar{j}$, any minimal $x_{n+1} = u(x)$ will be transformed into another minimal solution

$$x_{n+1} = u(x + j) - j_{n+1} = \tau_{\bar{j}} u, \quad \bar{j} \in \mathbb{Z}^{n+1}.$$

Of course, it gives rise to the same surface on T^{n+1}. We will require that this surface on T^{n+1} has no selfintersections which amounts to the following

DEFINITION 1.2. — The surface $x_{n+1} = u(x)$ has no selfintersections on T^{n+1} if for every $\bar{j} \in \mathbb{Z}^{n+1}$

$$\tau_{\bar{j}} u(x) - u(x)$$

has a fixed, i. e. is for all x either positive or negative or identically zero.

The concept of minimal solutions without selfintersection is natural in connection with foliations of extremals, as we shall see. Our first goal will be to study the set $\mathcal{M}$ of minimal solutions without selfintersections, prove *a priori* estimates for them and establish their existence.

c) Some properties of minimal solutions without selfintersection.

In the case of the Dirichlet integral, i. e. $F = |p|^2$, the Euler equation becomes the Laplace equation $\Delta u = 0$ and, in this case, one verifies that every harmonic function is a minimal solution. However, the only minimal solutions without selfintersections are the linear functions

$$u_0(x) = \alpha \cdot x + \beta, \qquad \alpha \in \mathbb{R}^n, \quad \beta \in \mathbb{R}.$$

The same assertion is true for any integrand $F = F(p)$ which is independent of x, u (see Section 2).

The first main result can be viewed as a comparison statement, comparing a minimal solution u without selfintersection for a general variational problem with those of a translation invariant one, say $F = |p|^2$.

THEOREM. — For any minimal solution u without selfintersection there exists a vector $\alpha \in \mathbb{R}^n$ such that the surfaces

$$x_{n+1} = u(x) \quad \text{and} \quad x_{n+1} = \alpha \cdot x + u(0)$$

have a distance in $\mathbb{R}^{n+1}$ less than a constant c depending on F only, but not the individual $u \in \mathcal{M}$ (see Section 2).

Thus to every $u \in \mathcal{M}$ one can associate a vector $\alpha \in \mathbb{R}^n$, $(\alpha, -1)$ being the normal to the hyperplane $x_{n+1} = \alpha \cdot x + u(0)$. We show that conversely to every $\alpha \in \mathbb{R}^n$ there exists a $u \in \mathcal{M}$ (see Section 5 and 6). We denote the set of $u \in \mathcal{M}$ belonging to $\alpha \in \mathbb{R}^n$ by $\mathcal{M}(\alpha)$. Moreover u can be chosen so that the set of periods of $u(x)$ and $u_0(x) = \alpha \cdot x + \beta$ agree, i. e. that

(1.6′) $\quad u(x + j) - j_{n+1} = u(x) \qquad \bar{j} \in \mathbb{Z}^{n+1}$ holds if and only if

(1.6″) $\quad u_0(x + j) - j_{n+1} = u_0(x) \qquad \bar{j} \in \mathbb{Z}^{n+1}$ i. e. if $\alpha \cdot j - j_{n+1} = 0$.

The relation between u and u_0 reaches farther. If u has the just mentioned property then the relation

$$u_0(x + j) - j_{n+1} \rightarrow u(x + j) - j_{n+1}$$

is monotone, i. e. the ordering of these translates is independent of the particular integrand (see Section 6).

d) Foliations of minimals.

The hypersurface $x_{n+1} = u(x)$, $u \in \mathcal{M}(\alpha)$ when viewed on T^{n+1} is, in general, not compact. A necessary condition for compactness is that α is rational, i. e. all components are rational. If, on the other hand, α is not rational then the translates of any hyperplane $x_{n+1} = \alpha \cdot x + \beta_0$ are dense on $\mathbb{R}^{n+1}$ and by considering their limits we obtain all parallel hyperplanes $x_{n+1} = \alpha \cdot x + \beta$ where α is fixed and β varies over $\mathbb{R}$. These hyperplanes form a foliation, given by $u_{x_\nu} = \alpha_\nu$, whose leaves are extremals for $F = |p|^2$. More generally, one can ask whether for an arbitrary variational problem the translates $u(x + j) - j_{n+1}$ of $u \in \mathcal{M}(\alpha)$ generate a foliation of minimals. A foliation of T^{n+1} of codimension 1 is given by a one-parameter family of surfaces

$$x_{n+1} = u(x, \beta), \qquad \beta \in \mathbb{R}$$

with $u(x, \beta) < u(x, \beta')$ if $\beta < \beta'$, which is invariant under the translations $\bar{x} \rightarrow \bar{x} + \bar{j}$; in particular we assume $u(x, \beta + 1) = u(x, \beta) + 1$. If $u(x, \beta)$ is an extremal for each β we call it an extremal foliation, if $u(x, \beta)$ is minimal for each β we call it a minimal foliation. It is clear that the leaves $x_{n+1} = u(x, \beta)$ have no selfintersections and this is one reason for considering nonselfintersecting solutions. It is also a standard result of the theory of calculus of variations that any extremal foliation is a minimal foliation; this can be proven with the help of Hilbert's invariant integral, noting that $x_{n+1} = u(x, \beta)$ represents a field of extremals.

For a $u \in \mathcal{M}(\alpha)$ satisfying (1.6) we consider the limit set $\mathcal{L} \subset T^{n+1}$ of the translates $(x, u(x + j) - j_{n+1})$ under the fundamental group $\mathbb{Z}^{n+1}$ in an appropriate topology; $\mathcal{L}$ is sometimes called the hull of u. Two cases arise:

A) If $x_{n+1} = u(x)$ is dense on the torus, i. e. if $\mathcal{L} = T^{n+1}$ then the translates $\tau_{\bar{j}} u$ generate a minimal foliation $u_{x_\nu} = \psi_\nu(x, u)$, ψ_ν Lipschitz continuous on T^{n+1}. Moreover, this foliation can be mapped by a homeomorphism

$$(x, \theta) \rightarrow (x, x_{n+1} = U(x, \theta))$$

with $U(x, \theta) - \theta \in C(T^{n+1})$ into the trivial foliation $\theta = \alpha \cdot x + \beta$. In other words, the leaves of the foliation are given by

$$x_{n+1} = U(x, \alpha \cdot x + \beta), \qquad \beta \in \mathbb{R}. \tag{1.7}$$

B) If α is not rational, but $x_{n+1} = u(x)$ is not dense on T^{n+1} then the limit set is a Cantor set, invariant under $\mathbb{Z}^{n+1}$, which is foliated by minimal solutions

$$x_{n+1} = u(x, \beta) = U(x, \alpha \cdot x + \beta)$$

where, however, the function $U(x, \theta)$, which is strictly increasing with respect to θ, is not continuous. Both cases can occur, and B) has to be considered the « general » situation.

e) **Quasiperiodic solutions.**

A continuous function $g(x)$, $x \in \mathbb{R}^n$ which can be written in the form

$$g(x) = G(x, \theta) \quad \text{with} \quad \theta = \alpha \cdot x = \sum_{\nu=1}^{n} \alpha_\nu x_\nu \tag{1.8}$$

where $G \in C(T^{n+1})$, is called quasi-periodic with frequencies $\alpha_1, \alpha_2, \ldots, \alpha_n$. Thus, by (1.7), in case A) the functions $\exp(2\pi i u)$ are quasi-periodic; for simplicity we will in this case call u itself quasi-periodic. Our results can be viewed as a construction of generalized quasi-periodic solutions of the Euler equations. While in case A) the solutions (1.7) are indeed quasi-periodic they are not in the case B) since $U(x, \theta)$ fails to be continuous. In particular, we find such solutions for every $\alpha \in \mathbb{R}^n$ for the nonlinear differential equation

$$\Delta u = V_u(x, u)\,; \qquad V \in C^{l,\varepsilon}(T^{n+1})$$

with periodic right hand side, e. g.

$$\Delta u = \lambda \sin(2\pi u) \prod_{\nu=1}^{n} \sin(2\pi x_\nu)\,.$$

f) **Alternate variational principle.**

One can try to construct the function $U(x, \theta)$ for a given α by a direct construction, avoiding the previous steps. In Section 7 we sketch such an approach which is based on the regularized variational principles

$$\begin{cases} \displaystyle\iint_{T^{n+1}} \frac{\varepsilon}{2}\left(\frac{\partial U}{\partial \theta}\right)^2 + F(x, U, DU)\,dx\,d\theta \\ \displaystyle D_\nu = \frac{\partial}{\partial x_\nu} + \alpha_\nu \frac{\partial}{\partial \theta}\,. \end{cases} \tag{1.9}$$

Minimizing this functional over all $U = U(x, \theta)$, $U - \theta \in W^{1,2}(T^{n+1})$, one obtains for $\varepsilon > 0$ a smooth function $U^{(\varepsilon)}(x, \theta)$ which is monotone in θ. The desired function can be obtained by taking the limit of a subsequence $U^{(\varepsilon_s)}$, $\varepsilon_s \to 0$. However, this section is fragmentary and we restrict ourselves to proving the strict monotonicity of $U^{(\varepsilon)}$ for $\varepsilon > 0$.

g) Connection with the theory by Aubry and Mather.

These results as well as their proofs are inspired by the work by Aubry [*2*] and Mather [*16*] and this paper can be viewed as a generalization of their ideas. Their work is concerned with area-preserving mappings φ of an annulus or cylinder which have the monotone twist property. One of their basic achievements is the construction of a closed invariant set for a prescribed rotation number α, the so-called Mather set, which is either a closed invariant Lipschitz curve or an invariant Cantor set lying on such a curve. Both these authors devised a different construction for these sets. Aubry based his construction on a variational problem for one-dimensional sequences u_i, $i \in \mathbb{Z}$ and his so-called minimal energy orbits. The definition of these minimal energy orbits and their construction is generalized by our minimal solutions. We dropped the term « energy » since the variational expression may represent the « action » in mechanics or some other physical quantity. There is a difference, however, which is basic. In Aubry's theory every minimal energy orbit has a monotonicity property which corresponds to the absence of selfintersections in our picture. This is due to the fact that Aubry's theory refers to one dimensional discrete orbits, corresponding to $n = 1$ in our situation, while for $n > 1$ the nonselfintersection property has to be imposed. We showed in an earlier note [*20*] that the variational problem underlying Aubry's theory [*2*] for discrete orbits can be replaced by a variational problem (1.2) for $n = 1$ where the monotone twist property translates into the Legendre condition.

There is a translation of the other concepts: An invariant curve of φ corresponds to a minimal foliation for (1.2). A Mather set which is not an invariant curve corresponds to a minimal foliation on a Cantor set $\mathscr{L}$. The rotation number α corresponds to the rotation vector $\alpha \in \mathbb{R}^n$.

Mather's construction of his invariant sets is based on a variational problem which had been studied earlier by Percival for numerical purposes. It translates into a degenerate variational problem which is described in a regularized form by (1.9).

Thus this work can be viewed as a generalization of [*2*] [*16*] to higher dimensions where it is important that the one-dimensional orbits are replaced by surfaces of *codimension 1*. This is crucial not only for the ordering of the orbits, but also for the maximum principle for scalar elliptic partial differential equations. We did not discuss examples showing that both cases A) and B) occur and refer to such examples for $n = 1$ [*21*].

h) Tools from calculus of variations.

For variational problems (1.2) an extensive theory has been developed and it is known that all minimals bounded in a ball B belong to $C^{l,\varepsilon}(B)$ and satisfy the Euler equations. This is the consequence of the regularity theory for such problems. This difficult theory is presented in the book [*15*] by Ladyzhenskaya and Ural'tseva in a form most appropriate for our purpose. It is built on the basic work by De Giorgi [*4*] who developed the first approach to obtaining *pointwise estimates* for weak solutions of elliptic partial differential equations. These estimates are used to prove the Hölder continuity of the solutions. This technique has been developed by many mathematicians, e. g. Morrey [*17*], Gilbarg-Trudinger [*12*], Giaquinta, Giusti, Di Benedetto-Trudinger who proved a Harnack inequality in great generality. We will not reprove the *qualitative* statements about the regularity of the minimals but use the pointwise estimates, in particular the Harnack inequality, to get *quantitative* information. In order to get bounds for minimals, as stated in Theorem 2.1, we can use to advantage the beautiful work of Giaquinta and Giusti [*7*] [*9*] who established that quasi-minima – a generalization of the concept of minimals – belong to the so-called De Giorgi class for which Di Benedetto and Trudinger [*5*] proved their Harnack inequality, using earlier ideas of Krylov and Safonov. With the help of these deep results the proofs are quite simple and natural. One may say that this work is a combination of a study of the action of the fundamental group on the set of minimals with the pointwise estimates and the strong maximum principle for elliptic partial differential equations.

i) Open problems.

It would be desirable to develop a theory of this type for minimal surfaces on T^{n+1} with respect to a Riemannian metric. The corresponding integrand F in (1.2) grows like $|p|$, however, and the theory breaks down. In that case one would like to consider minimal surfaces which are not the graph of a function. Such a theory would require genuine extensions; it would be interesting because it could lead to a theory of minimal surfaces of codimension 1 on other higher dimensional manifolds, also with non-commutative fundamental group. Actually for $n = 1$ such a theory has been carried out by G. Hedlund [*13*] who studied minimal geodesics (called geodesics of class A) on a torus. These results were generalized by H. Buseman to Finsler metrics and G-spaces. For a recent exposition of these ideas we refer to Bangert [3] where further literature is quoted. Also his theory has very similar aspects as that of Aubry, but his orbits need not be graphs of functions. In this respect his work is more general than Aubry's. For compact surfaces of higher genus such a theory was developed by M. Morse [18] already in 1924.

We mention that we derived only the first basic steps of such a theory of minimal solutions on a torus. We did not show, for example, that the Cantor set $\mathcal{L}$ is independent of the minimal solution $u \in \mathcal{M}(\alpha)$ generating it. For $n = 1$ this is true, however the proof of this fact does not carry over directly. Recently V. Bangert [1] succeeded in proving such a statement for $n \geq 2$; more precisely he showed that a minimal set $\mathcal{L}_0$ of the translates $u(x + j) - j_{n+1}, \bar{j} \in \mathbb{Z}^{n+1}$ of a recurrent minimal $u \in \mathcal{M}(\alpha)$ is independent of the choice of u. Thus the set $\mathcal{L}_0 = \mathcal{L}_0(\alpha)$ which is a Cantor set or the torus T^{n+1} is associated with the variational problem and the rotation vector α, and not any particular solution u.

There is another possible generalization of Aubry's and Mather's theory which asks for invariant Mather sets for Hamiltonian systems of more than two degrees of freedom. In this direction one has only a perturbation theory, the so-called KAM theory, but no theory in general. Our paper does not contribute to this question; we are concerned with a higher dimensional generalization where the solutions are hypersurfaces of codimension 1 and not one-dimensional curves. Incidentally, also in our situation a perturbation theory generalizing the perturbation of invariant tori can be developed; it will be formulated in Section 8.

2. MINIMAL SOLUTIONS ON A TORUS

On the torus $T^{n+1} = \mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$ we consider a variational problem

$$\int F(x, u, u_x)dx \tag{2.1}$$

where the integrand $F(x, u, p)$ is required to be continuous in u, p and measurable in x; moreover, it is required to satisfy the inequalities

$$\delta_0 |p|^2 - c_0 \leq F(x, u, p) \leq \delta_0^{-1} |p|^2 + c_0 \tag{2.2}$$

for all x, u, p where $\delta_0 \in (0, 1)$, $c_0 > 0$ are constants. Finally F is assumed

(1) « A uniqueness Theorem for $\mathbb{Z}^n$-periodic variational problems », preprint Bern 1986. Bangert uses a more restrictive concept of recurrence than we do (definition 6.4).

to be of period 1 in $x_1, x_2, \ldots, x_n$ and u, so that (2.1) can be viewed as a variational problem on the torus T^{n+1}.

Our aim is to derive properties for minimal solutions defined above with the additional property that $u = u(x)$ has no selfintersection on T^{n+1}. We say that u has no selfintersection on T^{n+1} if the translated solution

$$\tau_{\bar{j}} u = u(x + j) - j_{n+1} \quad \text{with} \quad \bar{j} = (j_1, \ldots, j_{n+1}) \in \mathbb{Z}^{n+1}$$

does not intersect u, i. e. if for every $\bar{j} \in \mathbb{Z}^{n+1}$

$$(2.3) \qquad \tau_{\bar{j}} u - u \quad \text{does not change signs}.$$

In other words, this difference is either positive or negative or identically zero.

For orientation we consider the example of the Dirichlet integral

$$(2.4') \qquad \int | u_x |^2 dx$$

for which every harmonic function u is a minimal solution. It is without selfintersection if and only if it is a linear function

$$(2.4'') \qquad u(x) = \alpha \cdot x + \beta \qquad \alpha \in \mathbb{R}^n, \quad \beta \in \mathbb{R}.$$

Indeed, the harmonic function $u(x + j) - u(x)$ would have to be a constant, since any in $\mathbb{R}^n$ positive harmonic function is—by the Harnack inequality—a constant; i. e.

$$u(x + j) - u(x) = c(j).$$

Here the constant $c(j)$ satisfies $c(j + h) = c(j) + c(h)$, and therefore has the form $c(j) = \alpha \cdot j$ with some $\alpha \in \mathbb{R}^n$. In other words, the function

$$\hat{u}(x) = u(x) - \alpha \cdot x$$

is harmonic in $\mathbb{R}^n$ and satisfies $\hat{u}(x + j) = \hat{u}(x)$ and therefore is a constant, say β, proving the claim.

Actually, for $n = 1$ the linear functions are the only harmonic functions which are automatically nonselfintersecting. This holds generally for variational problems (2.1) satisfying (2.2) if $n = 1$. However, for $n \geq 2$ it is easy to find minimal solutions with selfintersections; e. g. the harmonic function $u = x_1 x_2$. In other words, the condition that the minimal energy orbits have no selfintersections has to be imposed only for $n \geq 2$.

The main result of this section is contained in the following

THEOREM 2.1. — If $u = u(x)$, $x \in \mathbb{R}^n$ is a minimal solution of (2.1) without selfintersections and if (2.2) holds then there exists a unique vector $\alpha \in \mathbb{R}^n$ such that

$$| u(x) - \alpha \cdot x |$$

is bounded for all $x \in \mathbb{R}^n$. Moreover, there exists a constant c_1, depending on c_0, δ_0 only so that

$$| u(x + y) - u(x) - \alpha \cdot y | \leq c_1 \sqrt{1 + |\alpha|^2}$$

for all x, y.

In geometric terms this means that the surface $z = u(x)$ has a distance $\leq c_1$ from the hyperplane $z = \alpha . x + u(0)$. We will refer to α as the rotation vector of u.

The proof depends strongly on the basic work by De Giorgi, Ladyzhenskaya and Ural'tseva, Giaquinta and Giusti, Trudinger and others. First of all, according to Giaquinta and Giusti every minimal solution is locally bounded and even Hölder continuous, so that it makes sense to speak of its value at a point. This is proven by Giaquinta and Giusti by verifying that these minimal solutions u (and more generally quasi-minima) as well as $-u$ satisfy the inequalities (see [*7*], Section 4; we specialize to $m = 2$; $g = c_0$, $\sigma = \infty$).

$$(2.5) \qquad \int_{A_y(k,\rho)} u_x^2 dx \leq \gamma \left\{ (r - \rho)^{-2} \int_{A_y(k,r)} (u - k)^2 dx + | A_y(k, r) | \right\}$$

for all $0 < \rho < r$ and all real k where

$$(2.6) \qquad A_y(k, \rho) = \{ x \in \mathbb{R}^n, | x - y | < \rho, u(x) > k \}$$

and $| A_y(k, \rho) |$ denotes the Lebesgue measure of this set. The constant γ depends on c_0, δ_0 only. Functions $u \in W^{1,2}_{loc}(\mathbb{R}^n)$ satisfying such a set of inequalities (2.5) are called of De Giorgi class DG_2^+ because of De Giorgi's fundamental work on the regularity of elliptic differential equations [*4*]. As a matter of fact, if u and $-u$ belong to DG_2^+ it follows that u is locally bounded and even Hölder continuous (see [*15*], Chap. II, Sect. 6 or [*10*] [*11*]). Moreover, according to [*15*] Lemma 6.2 of Chap. II one has

$$(2.7') \qquad \operatorname*{osc}_{|x| \leq r} u \leq \theta\left(\operatorname*{osc}_{|x| \leq 2r} u + c_2 r \right)$$

where $\theta \in (0, 1)$, $c_2 > 0$ depend on γ only and not on the function u or on r.

In the above result one can replace the spheres $| x | \leq r$, $| x | \leq 2r$ by cubes, for example, and we take

$$Q = \left\{ x \in \mathbb{R}^n, \ | x_\nu | \leq \frac{1}{2} \right\}$$

$$3Q = \left\{ x \in \mathbb{R}^n, \ | x_\nu | \leq \frac{3}{2} \right\}.$$

Then one has

$$(2.7'') \qquad \operatorname*{osc}_{Q} u \leq \theta(\operatorname*{osc}_{3Q} u + c_2)$$

with some other constants $\theta \in (0, 1)$, $c_2 > 0$.

On the other hand we use that u has no selfintersections. For a fixed $x \in \mathbb{R}^n$ we consider the denumerable set

$$S_x = \{ u(x + j) - j_{n+1}, \bar{j} \in \mathbb{Z}^{n+1} \}$$

and consider the translations $\tau_\nu : S_x \to S_x$ defined by

$$\tau_\nu(u(x + j) - j_{n+1}) = u(x + e_\nu + j) - j_{n+1}$$

where e_ν is the ν^{th} unit vector. Then

$$\tau_\nu(s + 1) = \tau_\nu(s) + 1 \qquad \text{for} \quad s \in S_x$$

and $\tau_\nu(s) < \tau_\nu(s')$ for $s < s'$, $s, s' \in S_x$, since u has no selfintersections. If $\tau_\nu(s)$ would be defined for all real s it would define a mapping of the circle $\mathbb{R}/\mathbb{Z}$ into itself for which one can define the Poincaré rotation number. By the same standard arguments (see below) one shows that the rotation number

$$\alpha_\nu = \lim_{m \to \infty} \frac{\tau_\nu^m(s)}{m}, \quad s \in S_x \tag{2.8'}$$

exists and is independent of s. More generally, since $\tau_1, \tau_2, \ldots, \tau_n$ commute one has

$$\lim_{m \to \infty} \frac{\tau^{km}}{m} = \sum_{\nu=1}^{n} k_\nu \alpha_\nu = k \cdot \alpha \quad \text{where} \quad \tau^k = \tau_1^{k_1} \ldots \tau_n^{k_n}.$$

Moreover, for any such monotone mapping $\tau^k = \tau_1^{k_1} \ldots \tau_n^{k_n}$ of S_x onto itself one has

$$| \tau^k(s) - s - k \cdot \alpha | \leq 1 \tag{2.8''}$$

and hence

$$| u(x + k) - u(x) - k \cdot \alpha | \leq 1 \quad \text{for all} \quad k \in \mathbb{Z}^n. \tag{2.9}$$

Moreover, α is independent of x since the mappings τ_ν for different values of $x \in \mathbb{R}^n$ are conjugate.

To complete the proof we have to verify a similar estimate as (2.9) with $k \in \mathbb{Z}^n$ replaced by any $y \in \mathbb{R}^n$. For such a $y \in \mathbb{R}^n$ determine $k \in \mathbb{Z}^n$ so that

$$y - k \in Q$$

so that from (2.9)

$$| u(x + y) - u(x) - y \cdot \alpha | \leq \operatorname*{osc}_{x+k+Q} u + \sum_{\nu=1}^{n} | \alpha_\nu | + 1. \tag{2.10}$$

Since all our assumptions are invariant under translations $x \to x + a$ it suffices to find an estimate for $\operatorname*{osc}_{Q} u$ in terms of $| \alpha |$.

From (2.9) with $x \in Q$ and $k = \pm e_\nu$ one can deduce

$$\text{(2.11)} \qquad \underset{3Q}{\operatorname{osc}}\, u \leq \underset{Q}{\operatorname{osc}}\, u + 2 \sum_{\nu=1}^{n} (1 + |\alpha_\nu|).$$

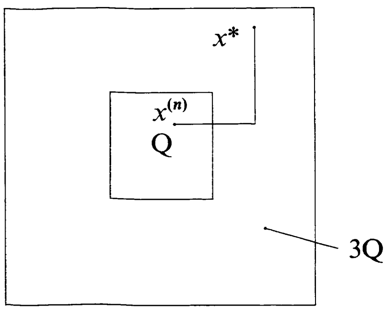
Indeed, if

$$\max_{3Q} u = u(x^*), \quad \min_{3Q} u = u(x_*)$$

one can find points $x^{(\nu)} \in 3Q$, $\nu = 0, 1, \ldots, n$ such that

$$x^{(0)} = x^*, \ x^{(\nu)} - x^{(\nu-1)} = \eta_\nu e_\nu; \quad \eta_\nu = \pm 1 \text{ or } 0$$

and such that $x^{(n)} \in Q$ (see figure).



Hence, by (2.9),

$$\max_{3Q} u = u(x^{(0)}) \leq u(x^{(n)}) + \sum_{\nu=1}^{n} |u(x^{(\nu)}) - u(x^{(\nu-1)})|$$

$$\leq \max_{Q} u + \sum_{\nu=1}^{n} (1 + |\alpha_\nu|).$$

With a similar lower estimate for $\min_{3Q} u$ we obtain for $\underset{3Q}{\operatorname{osc}}\, u = \max_{3Q} u - \min_{3Q} u$ the estimate (2.11).

Combining (2.11) with (2.7″) we find

$$\underset{Q}{\operatorname{osc}}\, u \leq \theta\left(\underset{Q}{\operatorname{osc}}\, u + 2 \sum_{\nu=1}^{n} (1 + |\alpha_\nu|) + c_2 \right)$$

hence

$$\underset{Q}{\operatorname{osc}}\, u \leq \frac{\theta}{1-\theta}\left(2 \sum_{\nu=1}^{n} (1 + |\alpha_\nu|) + c_2 \right) \leq c_3 \sqrt{1 + |\alpha|^2}.$$

Since this estimate holds for any translated cube $x + Q$ we obtain from (2.10) the desired estimate, proving theorem 2.1.

Repeated application of the inequality (2.7') gives Hölder continuity of u, more explicitly

$$\begin{aligned}\operatorname*{osc}_{|x| \leq r2^{-\nu}} u &\leq \theta^\nu \operatorname*{osc}_{|x| \leq r} u + c_2 r \frac{\theta}{2^\nu}(1 + 2\theta + \ldots + (2\theta)^{\nu-1}) \\ &\leq \theta^\nu \operatorname*{osc}_{|x| \leq r} u + \nu\theta^\nu c_2 r\end{aligned}$$

where we assumed without loss of generality $\theta \in \left(\frac{1}{2}, 1\right)$. Combining this with theorem 2.1 we can sharpen that theorem to

THEOREM 2.2. — *For any non-selfintersecting minimal solution $u(x)$ there exists a Hölder exponent $\varepsilon > 0$ and a constant $c_4 > 0$, depending on c_0, δ_0 only, such that*

$$| u(x + y) - u(x) - \alpha \cdot y | \leq c_4 \sqrt{1 + | \alpha |^2} \min(1, | y |^\varepsilon)$$

holds for all $x, y \in \mathbb{R}^n$.

We want to extend the characterization of minimal solutions without selfintersections to the case where $F = F(p)$ is independent of x, u.

THEOREM 2.3. — *If $F = F(p) \in C^2(\mathbb{R}^{n+1})$ and*

$$\delta | \xi |^2 \leq \sum_{\nu,\mu=1}^{n} F_{p_\nu p_\mu}(p)\xi_\nu\xi_\mu \leq \delta^{-1} | \xi |^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n$$

then any minimal without selfintersections is of the form $u(x) = \alpha \cdot x + \beta$.

Proof. — Let u be such a minimal without selfintersections. Then for any $(j, j_{n+1}) \in \mathbb{Z}^{n+1}$

$$v(x) = u(x + j) - j_{n+1}$$

has the same properties and $v(x) - u(x)$ is > 0, < 0 or $\equiv 0$ for all x. We claim that $w(x) = v(x) - u(x)$ is a constant. Indeed, since both u, v are weak solutions of the Euler equation

$$\sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu} F_{p_\nu}(u_x) = 0$$

one finds for w the elliptic differential equation

$$\sum_{\nu=1}^{n} (a_{\nu\mu}(x) w_{x_\mu})_{x_\nu} = 0$$

with

$$a_{\nu\mu}(x) = \int_0^1 F_{p_\nu p_\mu}((1-t)u_x + tv_x)dt.$$

From a generalized Harnack inequality (see [*19*], remark at the end of Section 6) it follows that a positive solution w of such an equation must be a constant. Hence

$$u(x+j) - u(x) = c(j).$$

As before we conclude that $c(j) = \alpha \cdot j$ for some $\alpha \in \mathbb{R}^n$. Now $u_0(x) = \alpha \cdot x$ is obviously a minimal without selfintersection of our problem and

$$u(x) - u_0(x) = u(x) - \alpha \cdot x$$

has period 1 in all variables $x_1, \ldots, x_n$. Since also $u - u_0$ satisfies an elliptic partial differential equation it must be a constant, i. e. $u(x) = \alpha \cdot x + \beta$.

Appendix to Section 2.

For completeness we supply the proof of (2.8′), (2.8″) which usually is given only in the case that the function $\tau_\nu(s)$ is defined for all real s and is continuous.

Dropping the subscript ν we write $\tau(s) = \tau_\nu(s)$ and assume that $\tau(s)$ is defined on a denumerable set S,

$$\tau(s) \in S \quad \text{for} \quad s \in S, \quad \tau(s+1) = \tau(s) + 1 \quad \text{and}$$
$$\tau(s) < \tau(s') \quad \text{for} \quad s < s', \quad s, s' \in S.$$

Then $\tau(s) - s$ has the period 1 and

a) $$|\tau(s) - s - \tau(s') + s'| < 1 \quad \text{for all} \quad s, s' \in S.$$

Indeed, otherwise, because of the periodicity, one could find

$$s_1, s_2 \in S \quad \text{with} \quad s_1 < s_2 < s_1 + 1 \quad \text{with} \quad \tau(s_2) - s_2 \geq \tau(s_1) - s_1 + 1.$$

The right hand side equals $\tau(s_1 + 1) - s_1 > \tau(s_2) - s_1$ which yields $-s_2 > -s_1$, a contradiction.

The same inequality holds for τ^m and we set

$$a_m = \inf_{s \in S}(\tau^m(s) - s); \quad b_m = \sup_{s \in S}(\tau^m(s) - s)$$

so that $0 \leq b_m - a_m \leq 1$ by *a*), and

b) $$a_m \leq \tau^m(s) - s \leq b_m.$$

Because of

$$\tau^{mp}(s) - s = \sum_{\nu=1}^{p}(\tau^{m\nu}(s) - \tau^{m(\nu-1)}(s))$$

for all natural numbers m, p one has

c) $$pa_m \leq a_{mp} \leq \tau^{mp}(s) - s \leq b_{mp} \leq pb_m.$$

Exchanging m, p we have also $b_{mp} \leq m b_p$ hence

$$\frac{a_m}{m} \leq \frac{b_p}{p}.$$

With $a_p \geq b_p - 1$ this implies

$$\frac{a_m}{m} - \frac{a_p}{p} \leq \frac{a_m}{m} - \frac{b_p - 1}{p} \leq \frac{1}{p}$$

hence

$$\left| \frac{a_m}{m} - \frac{a_p}{p} \right| \leq \max\left(\frac{1}{p}, \frac{1}{m}\right).$$

Thus $\frac{a_m}{m}$ converges to a number, say α, and because of $0 \leq b_m - a_m \leq 1$ also $\frac{b_m}{m} \to \alpha$. On account of *b*) also $\tau^m(s)/m \to \alpha$ for all $s \in S$, proving (2.8′).

Now using *c*) for $m = 1$ gives

$$a_1 \leq \frac{\tau^p(s) - s}{p} \leq b_1 .$$

For $p \to \infty$, we obtain $a_1 \leq \alpha \leq b_1$. This means, that both numbers $\tau(s) - s$ and α lie in the interval $[a_1, b_1]$ of length ≤ 1, hence

$$| \tau(s) - s - \alpha | \leq 1 \quad \text{for all} \quad s \in S$$

proving (2.8″).

We apply these inequalities to $\tau_1^{k_1} . \tau_2^{k_2} \ldots \tau_n^{k_n} = \tau^k$ and find

$$| \tau^k(s) - s - \alpha(k) | \leq 1 \quad \text{for all} \quad s \in S$$

where $\quad \alpha(k) = k_1\alpha_1 + \ldots + k_n\alpha_n, \quad \alpha_\nu = \alpha(e_\nu)\,.$

3. COMPACTNESS OF THE SET OF MINIMAL SOLUTIONS

The above estimates were valid under the very general assumption (2.2) which does not allow for the formulation of the Euler equations, much less imply their elliptic character. In the following we strengthen the assumptions and require that

$$(3.1) \quad \begin{cases} i) \ F \in C^{l,\varepsilon}(\mathbb{R}^{2n+1}), \quad l \geq 2, \quad \varepsilon > 0 \\ ii) \ F \text{ has period 1 in } x_1, x_2, \ldots, x_n, u\,. \\ iii) \ \delta\, |\xi|^2 \leq \displaystyle\sum_{\nu,\mu=1}^{n} F_{p_\nu p_\mu}(x, u, p)\xi_\nu\xi_\mu \leq \delta^{-1}\, |\xi|^2 \\ \quad\ \ |F_{pu}| + |F_{px}| \leq c(1 + |p|) \\ \quad\ \ |F_{uu}| + |F_{ux}| + |F_{xx}| \leq c(1 + |p|^2) \end{cases}$$

with some constants $\delta \in (0, 1)$, $c > 0$.

Obviously these assumptions imply (2.2) with some positive constants δ_0, c_0 so that our previous estimates hold for arbitrary minimal solutions $u(x)$. In particular,

$$\hat{u}(x) = u(x) - \alpha \cdot x - u(0)$$

are bounded uniformly by $c_1\sqrt{1 + |\alpha|^2}$. Under the assumptions (3.1) it is possible to show that u have Hölder continuous derivatives which can be uniformly estimated. For this purpose the periodicity condition (3.1) *ii*) is irrelevant once pointwise bounds of $\hat{u}$ have been found. The delicate estimate technique for pointwise bounds of $\hat{u}_x$ in terms of $\sup |\hat{u}|$ were developed by Ladyzhenskaya and Ural'tseva [*15*] and later extended and modified by Morrey [*17*], Trudinger, Giaquinta and Giusti and others. For a generalization involving obstacles and such conditions see Eisen [*6*], whose arguments are based on Ladyzhenskaya-Ural'tseva's approach. Another approach not using the divergence structure was developed by Amann and Crandall [*1*] based on an idea by Tomi.

THEOREM 3.1. — Let u be a nonselfintersecting minimal solution of (2.1) with rotation vector α, where $F = F(x, u, u_x)$ satisfies (3.1) *i*) and *iii*) but not necessarily *ii*). If u satisfies

$$|u(x + y) - u(x) - \alpha \cdot y| \leq c_1\sqrt{1 + |\alpha|^2},$$

then $u \in C^{1,\varepsilon}(\mathbb{R}^n)$ for some positive ε independent of u but depending on $|\alpha|$, c, δ, and

$$|u_x|_{C^\varepsilon} \leq \gamma_1$$

where γ_1 is a constant depending on c, δ and $|\alpha|$.

In the following we will denote by γ, γ_1 etc. such constants depending only on c, δ and $|\alpha|$, where it is understood that they are monotone increasing functions of $|\alpha|$. We note that generally γ_1 grows faster than linear; even for $n = 1$ it may grow exponentially with $|\alpha|$.

We reduce the proof to the results of Ladyzhenskaya and Ural'tseva. We note that

$$v(x) = u(x) - u(0) - \alpha \cdot x$$

is a minimal solution for the variational problem

$$\int \tilde{F}(x, v, v_x)dx$$

with

$$\tilde{F}(x, v, v_x) = F(x, u(0) + \alpha \cdot x + v, \alpha + v_x).$$

On account of (3.1) *iii)* one has

(3.2′) $$\delta |\xi|^2 \leq \Sigma \tilde{F}_{v_{x_\nu} v_{x_\mu}} \xi_\nu \xi_\mu \leq \delta^{-1} |\xi|^2$$

(3.2″) $$\begin{aligned} |\tilde{F}_{v_x v}| + |\tilde{F}_{v_x x}| &= |F_{pu}| + |F_{px} + \alpha F_{pu}| \\ &\leq c(1 + |\alpha|)(1 + |\alpha + v_x|) \\ &\leq \gamma(1 + |v_x|) \end{aligned}$$

and similarly

(3.2‴) $$|\tilde{F}_{vv}| + |\tilde{F}_{vx}| + |\tilde{F}_{xx}| \leq \gamma(1 + |v_x|^2),$$

with $\gamma = c(1 + |\alpha|)^4$.

Now we use the results of Chap. 4 of Ladyzhenskaya and Ural'tseva [*15*] which apply more generally to weak solutions of quasilinear differential equations

$$\sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu} a_\nu(x, v, v_x) + a(x, v, v_x) = 0 .$$

In our case we have

$$a_\nu(x, v, v_x) = \tilde{F}_{v_{x_\nu}}(x, v, v_x), \quad a(x, v, v_x) = -\tilde{F}_v(x, v, v_x)$$

which on account of (3.2′)-(3.2‴) satisfy the required estimates (3.1), (3.2) and (5.7) for $m = 2$ in the book [*15*]. Moreover, by our assumption we have $|v(x)| \leq c_1\sqrt{1 + \alpha^2} = \gamma_0$. Hence by theorem 5.2 in [*15*], Chap. 4 $v \in C^1(\mathbb{R}^n)$, and there exists a constant γ_1 depending on c, δ and $|\alpha|$ only such that

$$|v_x| \leq \gamma_1$$

for all $x \in \mathbb{R}^n$. Moreover, by theorem 6.1 of [*15*] one has $v \in C^{1,\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon > 0$, also depending on c, δ and $|\alpha|$ only, an estimate

$$|v|_{C^{1,\varepsilon}} \leq \gamma_1 .$$

This proves theorem 3.1.

In this argument the periodicity condition (3.1) *ii)* was irrelevant. But if we reinstate it we can apply theorem 2.1 and obtain the

COROLLARY 3.2. — Let u be a nonselfintersecting minimal solution of (2.1) with the rotation vector α, where F satisfies (3.1) *i.*), *ii*), *iii*). Then $u \in C^{1,\varepsilon}$ and

$$|u(x) - u(0) - \alpha \cdot x|_{C^{1,\varepsilon}(\mathbb{R}^n)} \leq \gamma_1' .$$

Moreover, all minimal solutions satisfy the weak Euler equations

$$\int_{\mathbb{R}^n} \left(\sum_{\nu=1}^{n} \varphi_{x_\nu} F_{p_\nu}(x, u, u_x) + \varphi F_u(x, u, u_x) \right) dx = 0$$

for all $\varphi \in C^1_{\text{comp}}(\mathbb{R}^n)$. These formulae are meaningful since u, u_x are continuous. As a matter of fact, from the general theory it is an easy consequence that $u \in C^2(\mathbb{R}^n)$ and u is a classical solution of the Euler equation

$$\sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu} F_{p_\nu}(x, u, u_x) - F_u(x, u, u_x) = 0 .$$

If $F \in C^{l,\varepsilon}$ it follows even that $u \in C^{l,\varepsilon}$. For us the quantitative estimates of theorem 3.1 will be more important than these qualitative statements.

We associate with any minimal solution u a rotation vector α and will now denote by $\mathscr{M}(\alpha)$ the set of all nonselfintersecting minimal solutions belonging to the rotation vector α. Moreover, for $A > 0$ we set

$$\mathscr{M}_A = \bigcup_{|\alpha| \leq A} \mathscr{M}(\alpha)$$

$$\mathscr{M} = \bigcup_{\alpha \in \mathbb{R}^n} \mathscr{M}(\alpha).$$

In $\mathscr{M}$ we use the C^1-topology on compact sets.

COROLLARY 3.3. — The set $\mathscr{M}_A/\mathbb{Z}$ is compact with respect to the C^1-topology on compact sets in $\mathbb{R}^n$. In other words, any sequence $u^{(s)} \in \mathscr{M}_A$ possesses a subsequence, say $u^{(s_\nu)}$, and an integer k_ν for which $u^{(s_\nu)} - k_\nu$ converges with first derivatives uniformly on any compact set to a function $u^* \in \mathscr{M}_A$.

Proof. — This is an immediate consequence of Corollary 3.2: Replacing $u^{(s)}$ by $u^{(s)}$ + integer we can assume $0 \leq u^{(s)}(0) < 1$. Since for $u^{(s)} \in \mathscr{M}_A$ the rotation vector $\alpha^{(s)}$ satisfies $|\alpha^{(s)}| \leq A$ one can take a subsequence for which $\alpha^{(s)}$ converges to a vector α^*, and by the theorem of Ascoli-Arzela for any closed ball $B_\mu = \{ x \in \mathbb{R}^n, |x| \leq \mu \}$ there is a sequence $s = s_\nu^{(\mu)} \to \infty$ so that $u^{(s)}$ converges in $C^1(B_\mu)$ for $\nu \to \infty$. Thus the diagonal sequence $u^{(s)}$ for $s = s_\nu^{(\nu)}$ converges to a function $u^* \in C^1(\mathbb{R}^n)$ in the given topology.

If u^* would not be minimal there would exist a $\varphi \in H^1_{\text{comp}}(\mathbb{R}^n)$ so that

$$I_B(u^* + \varphi) < I_B(u^*) \quad \text{where} \quad I_\Omega(u) = \int_\Omega F(x, u, u_x)dx$$

where B is a closed ball containing supp φ. If $u^{(s)}$ denotes the above subsequence converging to u^* then

$$I_B(u^{(s)} + \varphi) \geq I_B(u^{(s)})$$

since $u^{(s)} \in \mathscr{M}$. Since $u^{(s)} \to u^*$ in $C^1(B)$ one has

$$I_B(u^{(s)} + \varphi) \to I_B(u^* + \varphi)$$

by the dominated convergence theorem. Since also $I_B(u^{(s)}) \to I_B(u^*)$ we conclude $I_B(u^* + \varphi) \geq I_B(u^*)$, a contradiction. Hence u^* is a minimal solution. It also has no selfintersection. Indeed, for any $\bar{j} \in \mathbb{Z}^{n+1}$

$$\tau_{\bar{j}} u^{(s)} - u^{(s)} = u^{(s)}(x + j) - j_{n+1} - u^{(s)}(x)$$

is > 0, < 0 or $\equiv 0$ and therefore

$$\tau_{\bar{j}} u^* - u^* \geq 0 \text{ or } \leq 0 .$$

In the next section, Lemma 4.2 we will show that this implies

$$\tau_{\bar{j}} u^* - u^* > 0 \text{ or } < 0 \text{ or } \equiv 0 ,$$

i. e. $u^* \in \mathcal{M}$.

To show that $u^* \in \mathcal{M}_A$ we show that $\alpha^* = \lim_{s\to\infty} \alpha^{(s)}$ is the rotation vector for u^*. This follows from

LEMMA 3.4. — The function

$$\alpha : \mathcal{M}_A \to \mathbb{R}^n$$

assigning to $u \in \mathcal{M}$ its rotation vector $\alpha = \alpha(u)$ is continuous in the uniform topology on all compact sets of $\mathbb{R}^n$, hence *a fortiori* also in the above topology of C^1-convergence on compact sets.

Proof. — Let $u, v \in \mathcal{M}_A$ with rotation vectors α, β respectively. We set $w = v - u$ and $\gamma = \beta - \alpha$ so that we have by theorem 3.1

$$| w(x) - w(0) - \gamma \cdot x | \leq c^* . A$$

hence

$$| \beta - \alpha | = | \gamma | = \frac{1}{R} \max_{|x| \leq R} (\gamma \cdot x) \leq \frac{c^* A}{R} + \frac{1}{R} \max_{|x| \leq R} | w(x) - w(0) | .$$

The right hand side can be made smaller than 2ε by first choosing R so large that $c^* A R^{-1} < \varepsilon$ and then, for fixed R, making the second term small. Thus lemma 3.4 and corollary 3.3 are proven.

4. PAIRS OF MINIMAL SOLUTIONS

We consider two minimal solutions u, v and study the possibility of their intersections in $\mathbb{R}^n$, i. e. points x where $u(x) = v(x)$. Clearly, if $\alpha(u) \neq \alpha(v)$ then by theorem 2.1 $u - v$ changes sign and u, v do intersect.

THEOREM 4.1. — If u, $v \in \mathcal{M}$ then the open set

$$\{ x \in \mathbb{R}^n, u(x) < v(x) \}$$

has no bounded components.

Proof. — Let V be such a bounded component. Then we set

$$\bar{u} = \begin{cases} v & \text{in V} \\ u & \text{otherwise} \end{cases}$$

so that $\varphi = \bar{u} - u \in W^{1,2}$, supp $\varphi \subset \bar{V}$. Hence

$$I_V(v) = I_V(u + \varphi) \geq I_V(u)$$
$$I_V(u) = I_V(v - \varphi) \geq I_V(v),$$

i. e. $I_V(u) = I_V(v) = I_V(\bar{u})$.
Thus for any open set W containing $\bar{V}$ we have

$$I_W(\bar{u}) = I_W(u)$$

i. e. $\bar{u}$ is also minimal. From $u, \bar{u} \in \mathcal{M}$, $u \leq \bar{u}$ follows either $u = \bar{u}$ or $u < \bar{u}$ in $\mathbb{R}^n$ as the next lemma shows. Hence $u < \bar{u}$ in $\mathbb{R}^n$ and $V = \mathbb{R}^n$, in contradiction to the boundedness of V.

LEMMA 4.2. — If $u, v \in \mathcal{M}$, $u \leq v$ then either $u \equiv v$ or $u < v$.

Proof. — This follows from the strong maximum principle for elliptic partial differential equation. We set $w(x) = v(x) - u(x) \geq 0$ and assume that for some point x^* we have $w(x^*) = 0$. Then w has an absolute minimum. w is the solution of an elliptic partial differential equation

$$\sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu}(F_{p_\nu}(x, u + w, u_x + w_x) - F_{p_\nu}(x, u, u_x)) - (F_u(x, u + w, u_x + w_x) - F_u(x, u, u_x)) = \sum_{\nu,\mu=1}^{n} a_{\nu\mu} w_{x_\nu x_\mu} + \sum_{\nu=1}^{n} b_\nu w_{x_\nu} + cw = 0$$

with continuous coefficients. By the maximum principle (see appendix to section 4) it follows that $w \equiv 0$. Thus we have $w > 0$ or $w \equiv 0$.

There is a stronger result which shows that for two minimals which do not intersect in $\mathbb{R}^n$ the difference $|u(x) - v(x)|$ is of the same magnitude for all x if x stays in a bounded region, e. g. 3Q.

THEOREM 4.3. — If $u, v \in \mathcal{M}_A$ and $u < v$ in 4Q then there exists a positive constant γ depending on c, δ and A such that

$$v(x) - u(x) \leq \gamma(v(y) - u(y)) \quad \text{for all} \quad x, y \in 3Q.$$

This implies that $\sup_{3Q}(v(x) - u(x)) = \Delta_+ < \infty$, $\inf_{3Q}(v(x) - u(x)) = \Delta_- > 0$ and $\Delta_+ \leq \gamma\Delta_-$.

Proof. — This follows from the Harnack inequality in a situation in which it was first proven by N. Trudinger [26]. More recently E. Di Benedetto and N. Trudinger [5] proved such a Harnack inequality for functions in the De Giorgi class. Although this deeper result could be circumvented we find it convenient to use it here.

For this purpose we fix $u \in \mathcal{M}$ and consider v variable and set $w = v - u > 0$. Clearly w is a minimal solution for

$$\int F(x, u + w, u_x + w_x)dx. \tag{4.1}$$

We will replace this variational problem by another one:

$$\int G(x, w, w_x)dx \tag{4.2}$$

for which w is again a minimal solution. For this purpose we set

$$G(x, w, w_x) = F(x, u+w, u_x+w_x) - F(x, u, u_x) - \sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu}(F_{p_\nu}(x, u, u_x)w). \tag{4.3}$$

Then

$$J_V(w) = \int G(x, w, w_x) = I_V(u + w) - I_V(u) + R(w)$$

where $R(w)$ depends only on the boundary values of w, hence does not affect the property of w being a minimal. Thus if w is minimal for (4.1) then also for (4.2) and vice versa.

Since u satisfies the Euler equations we find from (4.3)

$$\begin{aligned} G(x, w, w_x) &= F(x, u+w, u_x+w_x) - F(x, u, u_x) - \sum_{\nu=1}^{n} F_{p_\nu}(x, u, u_x)w_{x_\nu} - F_u(x, u, u_x)w \\ &= \int_0^1 (1-t)\left(\frac{d}{dt}\right)^2 F(x, u + tw, u_x + tw_x)dt \\ &= \sum_{\nu,\mu=1}^{n} a_{\nu\mu}(x)w_{x_\nu}w_{x_\mu} + 2\sum_{\nu=1}^{n} b_\nu w_{x_\nu} w + cw^2 \end{aligned}$$

where

$$\begin{aligned} a_{\nu\mu}(x) &= \int_0^1 (1 - t)F_{p_\nu p_\mu}(x, u + tw, u_x + tw_x)dt \\ b_\nu(x) &= \int_0^1 (1 - t)F_{p_\nu u}(x, u + tw, u_x + tw_x)dt \\ c(x) &= \int_0^1 (1 - t)F_{uu}(x, u + tw, u_x + tw_x)dt. \end{aligned}$$

From the assumption (3.1) *iii*) and Theorem 3.1 we conclude that

$$\frac{1}{4}\delta\,|\,w_x\,|^2 - \gamma_1 w^2 \leq \mathrm{G}(x, w, w_x) \leq \delta^{-1}\,|\,w_x\,|^2 + \gamma_1 w^2 \tag{4.4}$$

with some positive constant γ_1 depending on c, δ, A only.

Following again the ideas of Giaquinta and Giusti one shows that any minimal w of (4.2), where G satisfies (4.4), as well as $-\,w$ fulfill the inequalities

$$\int_{\mathrm{A}_y(k,\rho)} w_x^2 \leq \gamma_2 \left\{ \frac{1}{(r-\rho)^2} \int_{\mathrm{A}_y(k,r)} (w-k)^2 dx + k^2\,|\,\mathrm{A}_y(k,r)\,| \right\} \tag{4.5}$$

for all real k and for all $0 < \rho < r$ where again

$$\mathrm{A}_j(k, r) = \{\, x \in \mathbb{R}^n,\, |\, x - y\,| < r,\, w(x) > k \,\}\,.$$

The constant γ_2 depends on c, δ, A only.

The assumption for the theorem of Di Benedetto and Trudinger is that the function $w \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega)$, $w \geq 0$ in Ω and $\pm\, w$ satisfy

$$\int_{\mathrm{A}_y(k,\rho)} w_x^2 dx \leq \gamma_3 \left\{ \frac{1}{(r-\rho)^2} \int_{\mathrm{A}_y(k,r)} (w-k)^2 dx + \left(\frac{k}{r}\right)^2 |\,\mathrm{A}_y(k,r)\,| \right\} \tag{4.6}$$

for all real k, $0 < \rho < r$ and all domains $\mathrm{A}_y(k, r)$ for which $\mathrm{B}_y(r)$ belong to Ω. The difference to (4.5) lies in the replacement of k^2 by $(k/r)^2$. Hence if we restrict ourselves to a bounded domain, say $\Omega = 4\mathrm{Q}$, then the radius r of balls $\mathrm{B}_y(r)$ belonging to Ω is bounded, e. g. here by 2 and (4.6) follows from (4.5) with $\gamma_3 = 4\gamma_2$.

The theorem of Di Benedetto and Trudinger asserts that any function $w \in \mathrm{W}^{1,2}_{\mathrm{loc}}(4\mathrm{Q})$, $w \geq 0$ in 4Q for which (4.6) holds satisfies in a compact subdomain, e. g. 3Q the inequality

$$w(x) \leq \gamma_4 w(y) \quad \text{for} \quad x, y \in 3\mathrm{Q}\,,$$

with a constant γ_4 depending on γ_3 only, i. e. depending on c, δ, A only. This proves theorem 4.3.

COROLLARY 4.4. — If $u \in \mathscr{M}_{\mathrm{A}}$ let $\tau_\nu : \mathrm{S}_x \to \mathrm{S}_x$ be the mappings

$$\tau_\nu : u(x + j) - j_{n+1} \to u(x + j + e_\nu) - j_{n+1}$$

defined in Section 2. Then τ_ν is Lipschitz continuous, and satisfies

$$|\,\tau_\nu(s_2) - \tau_\nu(s_1)\,| \leq \gamma\,|\,s_2 - s_1\,|, \qquad \nu = 1, 2, \ldots, n\,,$$

where γ is the constant of theorem 4.3.

This is an immediate consequence of theorem 4.3.

Thus the τ_ν can be extended uniquely to the closure of S_x as Lipschitz continuous mappings, and these extensions still commute pairwise.

We conclude this section with another estimate which expresses the

Lipschitz continuity of the foliation which will be constructed in Section 6. For this purpose we will have to assume that the third derivatives of F are Hölder continuous, i. e. that (3.1) holds with $l \geq 3$.

THEOREM 4.5. — If $u, v \in \mathcal{M}_A$ and $u < v$ in a ball B then there exists a constant γ depending on c, δ and A only such that

$$| v_x(x) - u_x(x) | \leq \gamma(v(x) - u(x)) \quad \text{for} \quad x \in \frac{1}{2} B .$$

Proof. — By theorem 3.1 one has in $\mathbb{R}^n$

$$| u |_{C^{1,\varepsilon}}, | v |_{C^{1,\varepsilon}} \leq \gamma_1$$

and by a general result $u \in C^{l,\varepsilon}$ since $F \in C^{l,\varepsilon}$ and

$$(4.7) \qquad | u |_{C^{2,\varepsilon}}, | v |_{C^{2,\varepsilon}} \leq \gamma_2$$

see, for example, [*15*], p. 336.

Since $u, v > u$ satisfy the Euler equations

$$\sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu} F_{p_\nu}(x, u, u_x) - F_u(x, u, u_x) = 0$$

$$\sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu} F_{p_\nu}(x, v, v_x) - F_u(x, v, v_x) = 0$$

we obtain by taking the difference a partial differential equation for $w = v - u > 0$:

$$(4.8) \qquad \sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu}\left(\sum_{\mu=1}^{n} a_{\nu\mu} w_{x_\mu} + b_\nu w\right) - \sum_{\nu=1}^{n} b_\nu w_{x_\nu} - cw = 0$$

where

$$a_{\nu\mu} = \int_0^1 F_{p_\nu p_\mu}(x, (1-t)u + tv, (1-t)u_x + tv_x)dt$$

$$b_\nu = \int_0^1 F_{p_\nu u}(\ldots)dt$$

$$c = \int_0^1 F_{uu}(\ldots)dt\,;$$

where the arguments are the same as in the first line.

Because of our assumptions these coefficients $a_{\nu\mu}$, b_ν, c are in $C^{1,\varepsilon}(B)$ and

$$| a_{\nu\mu} |_{C^{1,\varepsilon}(B)}, \qquad | b_\nu |_{C^{1,\varepsilon}(B)}, \qquad | c |_{C^{1,\varepsilon}(B)} \leq \gamma_3$$

with some constant γ_3 depending on c, δ and A. Therefore (4.8) can be written explicitly as

$$\sum_{\nu,\mu=1}^{n} a_{\nu\mu} w_{x_\nu x_\mu} + \sum_{\nu=1}^{n} \mathrm{B}_\nu w_{x_\nu} + \mathrm{C} w = 0$$

with Hölder continuous coefficients, whose ε-Hölder norm can be estimated by γ_3. Since the equation is uniformly elliptic and the quadratic form satisfies $\sum_{\nu,\mu=1}^{n} a_{\nu\mu}\xi_\nu\xi_\mu \geq \delta \mid \xi \mid^2$ one obtains by the Schauder estimate (Gilbarg-Trudinger [12], p. 85)

$$\mid w_x(x) \mid \leq \gamma_4 \max_{x\in \mathrm{B}} w \qquad \text{for} \qquad x \in \frac{1}{2}\mathrm{B}.$$

Using theorem 4.3 we find

$$\mid w_x(x) \mid \leq \gamma . \gamma_4 w(x) \qquad \text{for} \qquad x \in \frac{1}{2}\mathrm{B}$$

which proves the statement.

In the case of foliations of minimal surfaces this argument was used by B. Solomon [25]. Clearly it is unessential that one has a foliation and we will need this estimate for arbitrary pairs of minimals. Also one has to require u, v to be minimal solutions in B only.

Appendix to Section 4.

For completeness we prove a simple consequence of the maximum principle in a form as it was needed for Lemma 4.2:

LEMMA 4.6. — Let Ω be an open and connected subset of $\mathbb{R}^n$ and let $u \in \mathrm{C}^2(\Omega)$ satisfy an elliptic partial differential equation

$$\mathrm{L}u = \sum_{\nu,\mu=1}^{n} a_{\nu\mu}(x) u_{x_\nu x_\mu} + \sum_{\nu=1}^{n} b_\nu(x) u_{x_\nu} + c(x) u = 0$$

where the coefficients are continuous in Ω and $\Sigma a_{\nu\mu}\xi_\nu\xi_\mu$ positive definite. If $u \geq 0$ in Ω then $u > 0$ or $u \equiv 0$ in Ω.

Proof. — If $c \geq 0$ this follows immediately from E. Hopf's strong maximum principle, since then

$$\sum_{\nu,\mu=1}^{n} a_{\nu\mu}(x) u_{x_\nu x_\mu} + \sum_{\nu=1}^{n} b_\nu(x) u_{x_\nu} \leq 0$$

and $u \geq 0$ in Ω. Thus if u assumes the minimum value 0 it is identically zero (see [24], p. 61).

The general case can be reduced to this situation by a standard trick; We assume $0 \in \Omega$, $u(0) = 0$ and have to show $u(x) = 0$ for any $x \in \Omega$. It suffices to establish this for all x in an open domain $D \subset \Omega$, with compact closure in Ω. The function $v = e^{-\lambda x_1} u$ satisfies the elliptic differential equation

$$0 = e^{-\lambda x_1} L(e^{\lambda x_1} v) = \tilde{L}(v)$$

where $\tilde{L}$ is a differential operator of the same form as L with coefficients $\tilde{a}_{\nu\mu} = a_{\nu\mu}$ and

$$\tilde{c} = e^{-\lambda x_1} L(e^{\lambda x_1}) = a_{11}(x)\lambda^2 + b_1(x)\lambda + c(x).$$

For λ sufficiently large we have $\tilde{c} > 0$ in D. Therefore, if $0 \in D$ and D is connected, we conclude $v = 0$, hence $u = 0$ in D.

5. EXISTENCE OF MINIMAL SOLUTIONS; RATIONAL α

In order to construct minimal solutions for a given rotation vector we begin with rational α, i. e. a vector with rational components and construct first minimal subtori. For other $\alpha \in \mathbb{R}^n$ the minimal solutions are found by approximation with those with rational α.

For a given minimal solution u we associate the group $\bar{\Gamma}$ of all periods $\bar{\gamma} = (\gamma, \gamma_{n+1})$, where $\gamma = (\gamma_1, \ldots, \gamma_n)$, i. e. the set of $\bar{\gamma} \in \bar{\Gamma}$ for which

$$\tau_{\bar{\gamma}} u = u(x + \gamma) - \gamma_{n+1} = u(x)$$

holds. This is a subgroup of $\mathbb{Z}^{n+1}$ which contains no point on the x_{n+1}-axis but the origin. Therefore $\dim_{\mathbb{Z}} \bar{\Gamma} \leq n$. Let us denote by $\bar{E}$ the smallest linear subspace of $\mathbb{R}^{n+1}$ containing $\bar{\Gamma}$, so that $\dim_{\mathbb{R}} \bar{E} = \dim_{\mathbb{Z}} \bar{\Gamma} \leq n$.

Clearly $\bar{\Gamma}$ is a subgroup of $\mathbb{Z}^{n+1} \cap \bar{E}$ and we will call $\bar{\Gamma}$ « maximal » if $\Gamma = \mathbb{Z}^{n+1} \cap \bar{E}$. Obviously every $\bar{\Gamma}$ is contained in a maximal lattice, namely in $\mathbb{Z}^{n+1} \cap \bar{E}$.

If α denotes the rotation vector for u then $\bar{\alpha} = (\alpha, -1) \in \mathbb{R}^{n+1}$ is orthogonal to $\bar{E}$ since for $\bar{\gamma} \in \bar{\Gamma}$, $m \in \mathbb{Z}$,

$$u(x) = u(x + m\gamma) - m\gamma_{n+1} = m(\gamma \cdot \alpha - \gamma_{n+1}) + O(1)$$

i. e.

$$\gamma \cdot \alpha - \gamma_{n+1} = \bar{\gamma} \cdot \bar{\alpha} = 0 \qquad \text{for all} \qquad \bar{\gamma} \in \bar{\Gamma}.$$

More generally, if $\bar{\Gamma}$ is any subgroup of $\mathbb{Z}^{n+1}$ containing no point on the x_{n+1}-axis but the origin then $\dim \bar{\Gamma} \leq n$ and $\bar{\Gamma}$ possesses a normal of the form $\bar{\alpha} = (\alpha, -1)$, i. e. it is contained in a hyperplane $x_{n+1} = \alpha \cdot x$.

Denoting by π the projection $\pi(x, x_{n+1}) = x$ of $\mathbb{R}^{n+1}$ onto $\mathbb{R}^n$ we consider the group $\Gamma = \pi\overline{\Gamma}$. From Γ and α one obtains

$$\overline{\Gamma} = \{ \overline{\gamma} = (\gamma, \gamma_{n+1}) \in \mathbb{Z}^{n+1}, \quad \gamma \in \Gamma, \quad \gamma_{n+1} = \gamma \cdot \alpha \}.$$

Clearly $\overline{\Gamma}$ is maximal if and only if Γ is maximal.

In the special case of the Dirichlet integral the minimal solutions without self intersections are linear functions $u = \alpha \cdot x + \beta$. Thus given any $\alpha \in \mathbb{R}^n$ there exists a $u \in \mathcal{M}$. Also the corresponding period lattice

$$\overline{\Gamma} = \{ \overline{\gamma} \in \mathbb{Z}^{n+1}, \gamma_{n+1} = \gamma \cdot \alpha \}$$

is maximal. We will establish both facts for general variational problems on the torus.

Before proving these facts we remark:
The hyperplane

$$x_{n+1} = \alpha \cdot x + \beta \tag{5.1}$$

is dense on the torus $T^{n+1} = \mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$ if and only if α is not rational, or equivalently if and only if

$$\dim \overline{\Gamma} < n \qquad \text{or} \qquad \dim \Gamma < n. \tag{5.2}$$

On the other hand the hyperplane (5.1) represents a subtorus of T^{n+1} if and only if α is rational, i. e.

$$\dim \overline{\Gamma} = n \qquad \text{or} \qquad \dim \Gamma = n. \tag{5.3}$$

In this case $\mathbb{R}^n/\Gamma = T^n(\Gamma)$ is a torus and for any $\gamma \in \Gamma$

$$\alpha \cdot (x + \gamma) + \beta = \alpha \cdot x + \beta + \gamma_{n+1}$$

since $\gamma_{n+1} = \alpha \cdot \gamma \in \mathbb{Z}$.

To prove that the hyperplane (5.1) is dense on T^{n+1} if (5.2) holds we just have to establish that the hyperplanes

$$x_{n+1} = \alpha \cdot x + \beta + \alpha \cdot j - j_{n+1}$$

are dense on $\mathbb{R}^{n+1}$, i. e. the set $\alpha \cdot j - j_{n+1}$ is dense on $\mathbb{R}^1$. Since α is not rational there is at least one irrational α_ν, say α_1 and it is well-known that already the set $\{ \alpha_1 j_1 - j_{n+1} \}$ is dense which implies the statement.

Finally we record that $\alpha_1, \alpha_2, \ldots, \alpha_n, -1$ are rationally independent if and only if

$$\dim \overline{\Gamma} = \dim \Gamma = 0,$$

a case we will discuss in the next section.

We turn to the construction of minimal subtori for an arbitrary variational problem (2.1) satisfying the conditions (3.1). We prescribe

$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ with rational components and a subgroup Γ' of

$$(5.4) \qquad \begin{cases} \Gamma = \{ \gamma \in \mathbb{Z}^n, \alpha\cdot\gamma \in \mathbb{Z} \} \\ \dim \Gamma' = \dim \Gamma = n \,. \end{cases}$$

We are looking for subtori

$$x_{n+1} = u(x) = \alpha\cdot x + \hat{u}(x)$$

with the translation group

$$\overline{\Gamma}' = \{ \overline{\gamma} = (\gamma, \gamma_{n+1}) \mid \gamma \in \Gamma' \,;\, \gamma_{n+1} = \alpha\cdot\gamma \} \,.$$

In other words, we require

$$(5.5) \qquad \hat{u}(x + \gamma) = \hat{u}(x) \qquad \text{for all} \quad \gamma \in \Gamma' \,.$$

Let Ω' be a fundamental domain of $\mathbb{R}^n/\Gamma'$ and consider

$$(5.6) \qquad \hat{u} \in W^{1,2}(\mathbb{R}^n/\Gamma') \,,$$

i. e. $\hat{u}$ satisfies (5.5) and

$$\int_{\Omega'} (\hat{u}^2 + \hat{u}_x^2)dx < \infty \,.$$

THEOREM 5.1. — Given a rational vector $\alpha \in \mathbb{R}^n$ and a subgroup Γ' of $\Gamma = \{ \gamma \in \mathbb{R}^n, \alpha\cdot\gamma = \mathbb{Z} \}$ satisfying (5.4) there exists an element $\hat{u}^* \in W^{1,2}(\mathbb{R}^n/\Gamma')$ such that $u^* = \alpha\cdot x + \hat{u}^*$ minimizes

$$I_{\Omega'}(u) = \int_{\Omega'} F(x, u, u_x)dx$$

in the class of u with $u - \alpha\cdot x \in W^{1,2}(\mathbb{R}^n/\Gamma)$, provided F satisfies (3.1). Moreover, $u^* \in C^2(\mathbb{R}^n)$ satisfies the Euler equation.

Proof. — This is a standard result proven by the direct methods of the calculus of variations where the compactness of Ω' plays an important role. For the existence one needs only $F(x, u, p)$ to satisfy (2.2) and *a* convexity condition with respect to p. We refer to Chap. 5 of Ladyzhenskaya and Ural'tseva where the boundary value problem is treated while we require the periodicity conditions (5.5) instead. We make the necessary additional remarks.

Because of (2.2) we have

$$(5.7) \qquad I_{\Omega'}(u) \geq \delta_0 \int_{\Omega'} u_x^2 dx - c_0 \mid \Omega' \mid$$

so that $I_{\Omega'}(u)$ is bounded from below. The class of admissible function is not empty since $u = \alpha\cdot x$ belongs to it. Now one picks a minimizing sequence

$$u_m = \alpha\cdot x + \hat{u}_m$$

with

$$I_{\Omega'}(u_m) \to \inf I_{\Omega'}(u).$$

Because of (5.7) we conclude that

$$\int_{\Omega'} |\hat{u}_{mx}|^2 dx \leq c \tag{5.8}$$

is bounded. We can replace $\hat{u}$ by $\hat{u} + k$, $k \in \mathbb{Z}$ and achieve that the mean-value

$$0 \leq |\Omega'|^{-1} \int_{\Omega'} \hat{u}_m dx = \mu_m < 1.$$

By Poincaré's inequality one has

$$\int_{\Omega'} (u_m - \mu_m)^2 dx \leq b^2 \int_{\Omega'} |u_{mx}|^2 dx$$

where b denotes the diameter of Ω'. Hence

$$\begin{aligned}\int_{\Omega'} \hat{u}_m^2 dx &\leq 2\left(|\Omega'| \mu_m^2 + b^2 \int_{\Omega'} |u_{mx}|^2 dx \right)\\ &\leq 2(|\Omega'| + b^2 c) = c'.\end{aligned}$$

Thus

$$\| \hat{u}_m \|_{W^{1,2}(\Omega')} \leq c'$$

is bounded.

By choosing a subsequence which converges weakly in $W^{1,2}(\Omega')$ and using the lower semicontinuity one obtains the desired $\hat{u}^* \in W^{1,2}(\Omega')$ as in [15], Chap. 5.

For the regularity we can apply the results of Chap. 6 in [*15*] assuming (3.1). As a matter of fact, since $\mathbb{R}^n/\Gamma'$ is a torus one can forget about the more difficult part of regularity at the boundary and gets away with the « interior regularity ».

Temporarily let us denote the minimal periodic solutions belonging to α, Γ' by $\mathcal{M}(\alpha, \Gamma')$ minimizing $I_{\Omega'}$. Then we have

THEOREM 5.2. — The set $\mathcal{M}(\alpha, \Gamma')$ of minimal periodic solutions $u(x) = \alpha . x + \hat{u}(x)$ is totally ordered, i. e. if $u, v \in \mathcal{M}(\alpha, \Gamma')$ then one has for all x either $u(x) < v(x)$ or $u(x) > v(x)$ or $u \equiv v$.

Proof. — For $u, v \in \mathcal{M}(\alpha, \Gamma')$ we have

$$I_{\Omega'}(u) = I_{\Omega'}(v) = d$$

where d denotes the minimum of $I_{\Omega'}(u)$ in this class of functions. We set

$$w^+(x) = \max(u(x), v(x)) = \begin{cases} u(x) & \text{in A} \\ v(x) & \text{in B} \end{cases}$$

$$w^-(x) = \min(u(x), v(x)) = \begin{cases} v(x) & \text{in A} \\ u(x) & \text{in B} \end{cases}$$

where

$$A = \{ x \in \Omega', u(x) > v(x) \}$$
$$B = \{ x \in \Omega', u(x) \le v(x) \}.$$

Hence

$$I_{\Omega'}(w^+) = I_A(u) + I_B(v)$$
$$I_{\Omega'}(w^-) = I_A(v) + I_B(u)$$

and adding

$$I_{\Omega'}(w^+) + I_{\Omega'}(w^-) = I_{\Omega'}(u) + I_{\Omega'}(v) = 2d.$$

Since by the minimality of d also $I_{\Omega'}(w^\pm) \ge d$ we conclude

$$I_{\Omega'}(w^+) = I_{\Omega'}(w^-) = d.$$

Since $w^\pm$ belong also to the class of admissible functions we conclude that

$$w^+, w^- \in \mathcal{M}(\alpha, \Gamma').$$

Using the argument of lemma 4.2 we conclude from $u \le w^+$ that $u < w^+$ or $u \equiv w^+$. In the second case we have $v \le u$, hence $v < u$ or $v \equiv u$ as claimed. In the first case we have $u < v$ which proves the statement.

COROLLARY 5.3. — If $u \in \mathcal{M}(\alpha, \Gamma')$ then u has no selfintersections. Indeed $\tau_{\bar{j}} u = u(x + j) - j_{n+1}$ belongs to $\mathcal{M}(\alpha, \Gamma')$ also, hence $\tau_{\bar{j}} u > u$ or $< u$ or $\equiv u$.

THEOREM 5.4. — $\mathcal{M}(\alpha, \Gamma') = \mathcal{M}(\alpha, \Gamma)$ where $\Gamma \supset \Gamma'$ is maximal.
Consequently the class of minimal periodic orbits is characterized by α alone; therefore we will from now on denote it by $\mathcal{M}_{\text{per}}(\alpha)$.

Proof. — We observe that for $u \in \mathcal{M}(\alpha, \Gamma')$ and $\gamma \in \Gamma$ also

$$v(x) = u(x + \gamma) - \alpha \cdot \gamma$$

belongs to $\mathcal{M}(\alpha, \Gamma')$. Indeed $\alpha \cdot \gamma$ is an integer and $\gamma \in \mathbb{Z}^n$. Writing

$$v(x) = \alpha \cdot x + \hat{v}(x)$$

we have $\hat{v}(x) = \hat{u}(x + \gamma)$ and therefore

$$\int_{\Omega'} \hat{v} dx = \int_{\Omega'} \hat{u} dx.$$

Therefore $\hat{v} - \hat{u} = v - u$ must have zeroes, and by theorem 5.2, $v = u$ or

$$u(x + \gamma) = u(x) + \alpha.\gamma \qquad \text{for all} \quad \gamma \in \Gamma$$

proving $u \in \mathcal{M}(\alpha, \Gamma)$. Moreover, $\min \{ I_{\Omega'}, \mathcal{M}(\alpha, \Gamma') \} = \frac{|\Omega'|}{|\Omega|} \min \{ I_\Omega, \mathcal{M}(\alpha, \Gamma) \}$ simplying redily Theorem 5.4.

COROLLARY 5.5. — $\mathcal{M}_{per}(\alpha) \subset \mathcal{M}(\alpha)$.

So far the elements u in $\mathcal{M}_{per}(\alpha)$ were characterized by minimality with respect to the class of u for which $u - \alpha \cdot x$ have a fixed period lattice and it is not clear that they are minimal solutions as defined in Section 1. That is the content of the above corollary. We point out, however, that the above containment is proper, in general, and $\mathcal{M}(\alpha)$ may contain u for which $u - \alpha \cdot x$ is not periodic.

To prove the corollary we observe that by corollary 5.3 $u \in \mathcal{M}_{per}(\alpha)$ has no selfintersection and since $u - \alpha \cdot x$ is periodic, α is the rotation vector for u. It remains to show that

$$I_B(u + \varphi) \geq I_B(u) \tag{5.9}$$

for arbitrary $\varphi \in W^{1,2}_{comp}$, where B is a ball, $|x| < R$, containing the supp φ.

For this purpose we set

$$\Gamma' = N\Gamma = \{ \gamma, \gamma N^{-1} \in \Gamma \}$$

for a large integer N. Then according to theorem 5.4 the function $u \in \mathcal{M}_{per}(\alpha)$ can be viewed as element of $\mathcal{M}(\alpha, \Gamma')$, i. e.

$$I_{\Omega'}(u + \psi) \geq I_{\Omega'}(u) \tag{5.10}$$

for all $\psi \in W^{1,2}(\mathbb{R}^n/\Gamma')$ and Ω' denoting a fundamental domain of $\mathbb{R}^n/\Gamma'$. We choose N so large that supp $\varphi \subset B \subset \Omega'$, then (5.9) follows from (5.10).

After we have shown that $\mathcal{M}(\alpha)$ is not empty for α rational, since it contains $\mathcal{M}_{per}(\alpha)$, it is not hard to construct minimal solutions without selfintersections for arbitrary α.

THEOREM 5.6. — Under the conditions (3.1) the variational problem has minimal solutions without selfintersections for every prescribed $\alpha \in \mathbb{R}^n$, i. e.

$$\mathcal{M}(\alpha) \neq \emptyset \quad \text{for all} \quad \alpha \in \mathbb{R}^n.$$

Proof. — Given $\alpha \in \mathbb{R}^n$ we set $A = |\alpha| + 1$ and pick a sequence of rational $\alpha^{(s)} \in \mathbb{R}^n$ with $\alpha^{(s)} \to \alpha$ for $s \to \infty$. Pick $u^{(s)} \in \mathcal{M}_{per}(\alpha^{(s)})$, so that $u^{(s)} \in \mathcal{M}_A$ for large s. By corollary 3.3 $\mathcal{M}_A/\mathbb{Z}$ is compact and there exists a subsequence $u^{(s_\nu)} - m_\nu$ converging to an element $u \in \mathcal{M}_A$ in the C^1-topology on compact sets. According to lemma 3.4 one has $u \in \mathcal{M}(\alpha)$.

6. THE ACTION OF THE FUNDAMENTAL GROUP

We consider a minimal energy solution $u \in \mathcal{M}(\alpha)$ first in the case that $\alpha_1, \alpha_2, \ldots, \alpha_n, -1$ are rationally independent so that u does not admit any period. In this case the translated solutions

$$\tau_{\bar{j}} u, \qquad \bar{j} \in \mathbb{Z}^{n+1}$$

are distinct since u has no selfintersections on T^{n+1}. Thus they give rise to an ordering of the fundamental group, defined by $\tau_{\bar{j}} u < \tau_{\bar{k}} u$. It is remarkable that this ordering is independent of the solution and even independent of the variational problem. For example, for $F = |u_x|^2$ we saw that $u^{(0)} = \alpha \cdot x$ belongs to $\mathcal{M}(\alpha)$, and our claim amounts to the statement that

$$\tau_{\bar{j}} u < \tau_{\bar{k}} u \quad \text{if and only if} \quad \tau_{\bar{j}} u^{(0)} < \tau_{\bar{k}} u^{(0)} .$$

In other words, we assert

LEMMA 6.1. — If $u \in \mathcal{M}(\alpha)$ and $\alpha_1, \alpha_2, \ldots, \alpha_n, -1$ rationally independent then

$$u(x + j) - j_{n+1} < u(x + k) - k_{n+1} \quad \text{iff} \quad j \cdot \alpha - j_{n+1} < k \cdot \alpha - k_{n+1} .$$

Proof. — It suffices to prove the statement for $x = 0$ and for $\bar{j} = 0$, or

$$(6.1) \qquad u(k) - u(0) > k_{n+1} \quad \text{iff} \quad k \cdot \alpha > k_{n+1} .$$

With the previous notation the mapping

$$\tau^k : u(j) \to u(j + k) \quad \text{for} \quad j \in \mathbb{Z}^n$$

has the rotation number $k \cdot \alpha$ which is not an integer. Moreover, by corollary 4.4 this mapping τ^k is Lipschitz continuous on the set $S = \{u(j), j \in \mathbb{Z}^n\}$ and can be extended, by continuity, to $\bar{S}$ and by defining it linear in the intervals of $\mathbb{R} \backslash \bar{S}$ to a mapping on $\mathbb{R}$. This extension is monotone, continuous and satisfies $\tau^k(s + 1) = \tau^k(s) + 1$ and also has the rotation number $k \cdot \alpha$. It is well-known that for an integer g

$$g + 1 > \tau^k(s) - s > g \quad \text{iff} \quad g + 1 > k \cdot \alpha > g$$

which is equivalent to the statement (6.1).

The family of linear functions

$$\theta = \alpha \cdot x + \alpha \cdot j - j_{n+1}$$

is mapped to the translates

$$x_{n+1} = \tau_{\bar{j}} u = u(x + j) - j_{n+1}$$

of $u \in \mathcal{M}(\alpha)$ by the mapping

$$(x, \theta) \to (x, \mathrm{U}(x, \theta)) \tag{6.2}$$

where $\mathrm{U}(x, \theta)$ is defined by

$$\mathrm{U}(x, \alpha \cdot x + j \cdot \alpha - j_{n+1}) = u(x + j) - j_{n+1} \tag{6.3}$$

for the dense values of $\theta = \alpha \cdot x + j \cdot \alpha - j_{n+1}$. This definition is unambiguous since $\alpha_1, \ldots, \alpha_n, -1$ are rationally independent. By lemma 6.1 this function $\mathrm{U}(x, \theta)$ is monotone in θ and therefore can be extended to monotone functions

$$\mathrm{U}^+(x, \theta) = \lim_{\theta' \searrow \theta} \mathrm{U}(x, \theta'), \quad \mathrm{U}^-(x, \theta) = \lim_{\theta'' \nearrow \theta} \mathrm{U}(x, \theta'') \tag{6.4}$$

where θ', θ'' are decreasing resp. increasing sequences taken from the dense set on which U is defined. Clearly, for fixed x one has

$$\mathrm{U}^+(x, \theta) = \mathrm{U}^-(x, \theta)$$

except for a denumerable set and the discontinuities lie on hyperplanes $\theta = \alpha \cdot x + \beta$. In general, $\mathrm{U}^+(x, \theta) \geq \mathrm{U}^-(x, \theta)$, and U^+, U^- are continuous if and only if they are equal.

LEMMA 6.2. — The above defined functions $\mathrm{U}^+, \mathrm{U}^-$ are strictly monotone in θ and satisfy

$$\mathrm{U}^{\pm}(x + e_\nu, \theta) = \mathrm{U}^{\pm}(x, \theta)$$
$$\mathrm{U}^{\pm}(x, \theta + 1) = \mathrm{U}^{\pm}(x, \theta) + 1 .$$

Therefore the mappings

$$(x, \theta) \to (x, \mathrm{U}^{\pm}(x, \theta))$$

can be viewed as mappings of T^{n+1} into T^{n+1}.

Proof. — The above periodicity conditions are immediately verified for U from the definition 6.3 and therefore follow for the extensions $\mathrm{U}^{\pm}$. Also $\mathrm{U}^{\pm}$ are obviously monotone; they are strictly monotone since $\mathrm{U}(x, \theta)$ is strictly monotone on the dense set $\{ \alpha \cdot j - j_{n+1} \}$.

We want to free ourselves from the condition that the $\alpha_1, \alpha_2, \ldots, \alpha_n, -1$ are rationally independent. If these quantities are rationally dependent then the mapping (6.3) is generally not well defined. To avoid this difficulty we construct a minimal solution u with the additional property that

$$u(x + j) - j_{n+1} = u(x) \quad \text{if} \quad \alpha \cdot j - j_{n+1} = 0 . \tag{6.5}$$

If this condition is satisfied then the definition (6.3) is again unambiguous for all $\theta = \alpha \cdot x + j \cdot \alpha - j_{n+1}$. These are dense if not all $\alpha_1, \alpha_2, \ldots, \alpha_n$ are rational which we will assume.

To construct a $u \in \mathcal{M}(\alpha)$ satisfying (6.5) we introduce the maximal lattice

$$\Gamma = \{ \gamma \in \mathbb{Z}^n, \alpha \cdot \gamma \in \mathbb{Z} \}$$

and

$$\overline{\Gamma} = \{ \overline{\gamma} \in \mathbb{Z}^{n+1}, \alpha \cdot \gamma - \gamma_{n+1} = 0 \} .$$

Let $r = \dim_{\mathbb{Z}} \Gamma < n$ and $\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(r)}$ a basis of Γ and $\overline{\gamma}^{(1)}, \overline{\gamma}^{(2)}, \ldots, \overline{\gamma}^{(r)}$ the corresponding basis in $\overline{\Gamma}$. We may assume that $\det (\gamma_\nu^{(\rho)})_{\nu,\rho \leq r} \neq 0$.

Now we approximate α by rational $\alpha^{(s)}$ which satisfy

$$\alpha^{(s)} \cdot \gamma - \gamma_{n+1} = 0 \quad \text{for all } \overline{\gamma} \in \overline{\Gamma} . \tag{6.6}$$

In other words, the corresponding lattices $\overline{\Gamma}^{(s)}$ should contain $\overline{\Gamma}$. For this purpose we choose $\alpha_{r+1}^{(s)}, \ldots, \alpha_n^{(s)}$ as rational numbers, tending to $\alpha_{r+1}, \ldots, \alpha_n$ as $s \to \infty$. Then the equations

$$\sum_{\nu=1}^{n} \alpha_\nu^{(s)} \gamma_\nu^{(\rho)} - \gamma_{n+1}^{(\rho)} = 0 \quad \text{for} \quad \rho = 1, 2, \ldots, r$$

determine $\alpha_1^{(s)}, \alpha_2^{(s)}, \ldots, \alpha_r^{(s)}$ uniquely as rational numbers since $(\gamma_\nu^{(\rho)})$, ν, $\rho = 1, 2, \ldots, r$ is non singular. These $\alpha^{(s)}$ clearly satisfy (6.6); moreover, $\alpha^{(s)} \to \alpha$ for $s \to \infty$.

For these $\alpha^{(s)}$ we construct minimal solutions $u^{(s)} \in \mathcal{M}_{\text{per}}(\alpha^{(s)})$ whose period lattice $\overline{\Gamma}^{(s)}$ contains $\overline{\Gamma}$ by theorem 5.4 i. e. we have

$$u^{(s)}(x + j) - j_{n+1} = u^{(s)}(x) \quad \text{for all} \quad \overline{j} \in \overline{\Gamma} .$$

We can assume $0 \leq u^{(s)}(0) < 1$ and conclude that a subsequence of $u^{(s)}$ converges to an element $u \in \mathcal{M}(\alpha)$ satisfying (6.5) since these relations hold for all approximations.

With this remark we can define $U(x, \theta)$ via (6.3) for α whose components are not all rational. In this case $U^{\pm}(x, \theta)$ are similarly defined and satisfy the properties of lemma 6.2.

If α is a rational vector then we define $U^{\pm}(x, \theta)$ in the analogue way. We just have to note that the set $\{ \alpha \cdot j - j_{n+1} \}$ is not dense. Thus if $u \in \mathcal{M}_{\text{per}}(\alpha)$ then (6.3) allows the definition of $U(x, \theta)$ for $\theta = \alpha \cdot x + \alpha \cdot j - j_{n+1}$. Now we define $U^{+}(x, \theta)$ as the largest monotone function in $\theta \in \mathbb{R}$ which extends $U(x, \theta)$. Similarly, $U^{-}(x, \theta)$ denotes the smallest monotone function extending $U(x, \theta)$. Thus $U^{\pm}(x, \theta)$ is defined for all $\alpha \in \mathbb{R}^n$.

THEOREM 6.3. — For every $\alpha \in \mathbb{R}^n$ there exists a function $U(x, \theta)$, strictly monotone in θ, and satisfying

$$U(x + e_\nu, \theta) = U(x, \theta)$$
$$U(x, \theta + 1) = U(x, \theta) + 1$$

such that for every $\beta \in \mathbb{R}$

$$U(x, \alpha \cdot x + \beta)$$

belongs to $\mathcal{M}(\alpha)$.

This is an obvious consequence of (6.3) with $U = U^+$ or U^-. Indeed, if α is not rational and $\beta' = \alpha \cdot j - j_{n+1}$ then

$$U(x, \alpha \cdot x + \beta') = u(x + j) - j_{n+1} \in \mathcal{M}(\alpha)$$

and if β' is an increasing sequence tending to β then the corresponding sequence $U(x, \alpha \cdot x + \beta')$ increases to $U^+(x, \alpha \cdot x + \beta)$. Incidentally, by theorem 4.3, the convergence is uniform on compact sets. By compactness of $\mathcal{M}_A/\mathbb{Z}$ it follows that also $U^+(x, \alpha \cdot x + \beta)$ belongs to $\mathcal{M}_A$ and by lemma 3.4 it belongs to $\mathcal{M}(\alpha)$. The same argument applies to $U^-(x, \alpha \cdot x + \beta)$.

In case α is rational $U^+(x, \theta)$ is a step function and $U^+(x, \alpha \cdot x + \beta)$ agrees with

$$U^+(x, \alpha \cdot x + \alpha \cdot j - j_{n+1}) = u(x + j) - j_{n+1}$$

for some $\bar{j} \in \mathbb{Z}^{n+1}$. Therefore in this case the statement is trivial.

The solutions $U^\pm(x, \alpha \cdot x + \beta)$ of theorem 6.3 need not contain the function $u \in \mathcal{M}(\alpha)$ generating them, if α is not rational. They have an additional property. We take over the terminology of dynamical systems $(n = 1)$:

DEFINITION 6.4. — A minimal $u \in \mathcal{M}(\alpha)$ is called recurrent, if u is the limit of a sequence of translates

$$\tau_{\bar{j}^{(s)}} u \quad \text{with} \quad |\bar{j}^{(s)}| \to \infty, \quad \bar{j}^{(s)} \in \mathbb{Z}^{n+1}.$$

THEOREM 6.5. — The solutions $U^\pm(x, \alpha \cdot x + \beta)$ are recurrent.

Proof. — This is an obvious consequence of the fact that $U^+(x, \alpha \cdot x + \beta)$, for example, is the limit of

$$U(x, \alpha \cdot x + \alpha \cdot j - j_{n+1}) = u(x + j) - j_{n+1}$$

for an increasing sequence $\beta' = \alpha \cdot j - j_{n+1}$ with $|j| + |j_{n+1}| \to \infty$. Moreover, the limit set of any limit set, such as S, agrees with itself.

We have to distinguish two different cases:

A) The minimal solution $x_{n+1} = u(x)$ is dense on T^{n+1}.

B) $x_{n+1} = u(x)$ is not dense on T^{n+1}.

Both cases occur and even for $n = 1$ one can give examples illustrating both situations. The first case occurs, for example, for an integrand $F = F(p)$ independent of x and u and for an α not rational (*).

The case A) is characterized by the following

THEOREM 6.6. — For a given $u \in \mathcal{M}(\alpha)$ the following assertions are equivalent:

(*) Also, if F is independent of u or independent of some $x_\nu \neq 0$ and $\alpha_\nu \neq 0$ then only case A occurs.

i) $(x, x_{n+1}) = (x, u(x))$ is dense in T^{n+1}.
ii) The set $\{ u(j) - j_{n+1}, (j, j_{n+1}) \in \mathbb{Z}^{n+1} \}$ is dense on $\mathbb{R}$.
iii) $U^+(x, \theta) = U^-(x, \theta)$, i. e. both functions are continuous on $\mathbb{R}^{n+1}$.
iv) $x_{n+1} = U^{\pm}(x, \alpha \cdot x + \beta), \beta \in \mathbb{R}$, defines a foliation of minimal solutions which, via the homeomorphism

$$(x, \theta) \to (x, U(x, \theta)),$$

is taken into the foliation $\theta = \alpha \cdot x + \beta$.

Proof. — The assertion *i)* amounts to the density of $u(x + j) - j_{n+1}$ for fixed x and varying j, j_{n+1} which by theorem 4.3 is equivalent to *ii)*.

If $U^+(x, \theta) > U^-(x, \theta)$ then the interval $(U^-(x, \theta), U^+(x, \theta))$ does not contain any cluster point of $u(x + j) - j_{n+1}$. Indeed, if $j \cdot \alpha - j_{n+1}$ runs through an increasing sequence tending to $\theta - \alpha \cdot x$ then $u(x + j) - j_{n+1}$ tends to $U^-(x, \theta)$ and for a decreasing sequence one obtains $U^+(x, \theta)$. This is in conflict to *i)*, that is *i)* implies *iii)*. The converse is obvious.

Finally, *iv)* is just a rephrasing of theorem 6.3. One just has to notice that

$$(x, \theta) \to (x, U^+(x, \theta))$$

is a homeomorphism. This follows from the strict monotonicity of $U^+(x, \theta)$, which follows from the strict monotonicity of $U(x, \theta)$ on a dense set. We remark that the situation described in theorem 6.6 occurs only if α is not rational.

In case α is not rational and $u \in \mathcal{M}(\alpha)$ is dense in T^{n+1} then $U^+ = U^- = U$ and

$$x_{n+1} = U(x, \alpha \cdot x + \beta) \tag{6.7}$$

defines a foliation of T^{n+1}: Since $U(x, \theta)$ is continuous and strictly monotone in θ one can find for a given $\bar{x} = (x, x_{n+1})$ a unique β such that (6.7) holds. We define $u(x, \beta) = U(x, \alpha \cdot x + \beta)$; by theorem 6.3 this represents a minimal solution of $\mathcal{M}(\alpha)$ for each β. We define $\psi_\nu = \psi_\nu(x, x_{n+1})$ by the relation

$$\psi_\nu(x, x_{n+1}) = u_{x_\nu}(x, \beta) \quad \text{if} \quad x_{n+1} = u(x, \beta).$$

Then the differential equation

$$u_{x_\nu} = \psi_\nu(x, u) \tag{6.8}$$

defines the foliation, whose leaves are $u = u(x, \beta), \beta \in \mathbb{R}$. From the definition it follows that ψ_ν has period 1 in $x_1, x_2, \ldots, x_{n+1}$ and $\psi_\nu \in C(T^{n+1})$. Thus (6.8) defines a foliation on T^{n+1}. By theorem 4.5 the functions ψ_ν are Lipschitz continuous in u. Since $u(x, \beta)$ is for each fixed β twice continuously differentiable in x we conclude:

THEOREM 6.7. — If $u \in \mathcal{M}(\alpha)$ is dense in T^{n+1} and α not rational then the functions $\psi_\nu(x, u)$ defining the foliation (6.7) are Lipschitz continuous on T^{n+1}.

Proof. — If $\bar{x} = (x, x_{n+1})$, $\bar{x}' = (x', x'_{n+1}) \in \mathbb{R}^{n+1}$ we determine β, β' so that

$$x_{n+1} = u(x, \beta)\,; \quad x'_{n+1} = u(x', \beta')$$

where $u(x, \beta) = U(x, \alpha . x + \beta)$. With the intermediate point $\bar{y} = (x', u(x', \beta))$ we have by theorem 4.5

$$\begin{aligned} |\psi_\nu(\bar{x}') - \psi_\nu(\bar{x})| &\leq |\psi_\nu(\bar{x}') - \psi_\nu(\bar{y})| + |\psi_\nu(\bar{y}) - \psi_\nu(\bar{x})| \\ &\leq \gamma |u(x', \beta') - u(x', \beta)| + |u_{x_\nu}(x', \beta) - u_{x_\nu}(x, \beta)| \\ &\leq \gamma(|x'_{n+1} - x_{n+1}| + |u(x, \beta) - u(x', \beta)|) + \sup |u_{xx}|\, |x' - x| \\ &\leq \gamma |x'_{n+1} - x_{n+1}| + (\sup |u_x| \gamma + \sup |u_{xx}|)\, |x' - x|\,. \end{aligned}$$

Because of periodicity we can restrict ourselves to $|x_\nu| \leq \frac{1}{2}$, $\nu = 1, 2, \ldots, n+1$. Using the estimates (4.7) the right hand side can be estimated by $\gamma' |\bar{x} - \bar{x}'|$ which proves the theorem.

We remark that in this case the solutions $u(x, \beta)$ are quasi-periodic in the sense that there exists a function $\hat{U}(x, \theta) = U - \theta \in C(T^{n+1})$ such that

$$u(x, \beta) = \alpha \cdot x + \beta + \hat{U}(x, \alpha \cdot x + \beta)\,.$$

It would be more appropriate to say that $\exp(2\pi i u(x, \beta))$ is quasi-periodic since u is not even bounded for $\alpha \neq 0$, but we will consider this interpretation as understood.

We turn to the more interesting case B) in which the translates $\tau_{\bar{j}} u$ of a $u \in \mathcal{M}(\alpha)$ satisfying (6.5) are not dense, while α is not rational. By theorem 6.6 this amounts to the assumption that the set

$$S = \{ u(j) - j_{n+1}, \quad (j, j_{n+1}) \in \mathbb{Z}^{n+1} \}$$

is not dense in $\mathbb{R}$. This set can be viewed as the orbit through $u(0)$ under the commuting translations $\tau_1, \tau_2, \ldots, \tau_n$ and $\tau_{n+1} : x_{n+1} \to x_{n+1} - 1$. We consider the limit set L(S) of points which are cluster points of points in S for $|j| \to \infty$. Moreover, we define $L^+(S)$, $L^-(S)$ as the sets of $s \in \mathbb{R}$ for which there exist decreasing resp. increasing sequences $s^{(m)} \in S$ which converge to s as $m \to \infty$.

Obviously, one has

$$L(S) = L^+(S) \cup L^-(S)\,.$$

It is well-known that L(S) is a Cantor set, that is a perfect nowhere dense subset of $\mathbb{R}$ if, as we assume now, $L(S) \neq \mathbb{R}$. By our definition (6.4) of $U^\pm(x, \theta)$ one has

$$\begin{aligned} L^+(S) &= \{ U^+(0, \theta),\ \theta \in \mathbb{R} \} \\ L^-(S) &= \{ U^-(0, \theta),\ \theta \in \mathbb{R} \}\,. \end{aligned}$$

Indeed, if $s^{(m)} = u(j^{(m)}) - j^{(m)}_{n+1}$ is an increasing sequence tending to s so is $\alpha \cdot j^{(m)} - j^{(m)}_{n+1} = \theta^{(m)}$ increasing and if $\theta^{(m)} \to \theta$ one has

$$s = \lim_{m\to\infty} s^{(m)} = \lim_{m\to\infty} \mathrm{U}(0, \theta^{(m)}) = \mathrm{U}^-(0, \theta).$$

Conversely, any $\mathrm{U}^-(0, \theta)$ belongs to $\mathrm{L}^-(\mathrm{S})$. The same arguments apply to $\mathrm{L}^+(\mathrm{S})$.

More generally, we define the limit set $\mathscr{L} \subset \mathbb{R}^{n+1}$ as the set of (x, x_{n+1}) for which x_{n+1} is the limit of $u(x + j^{(m)}) - j^{(m)}_{n+1}$, $|\bar{j}^{(m)}| \to \infty$. With $\mathscr{L}^+$, $\mathscr{L}^-$ we denote the limit sets where these sequences are decreasing, increasing, respectively. If $\mathrm{E}_{n+1} = \{ \bar{x} \in \mathbb{R}^{n+1}, x = 0 \}$ denotes the x_{n+1}-axis one has $\mathscr{L} = \mathscr{L}^+ \cup \mathscr{L}^-$ and

$$\mathrm{L}^{\pm}(\mathrm{S}) = \mathscr{L}^{\pm} \cap \mathrm{E}_{n+1}.$$

Moreover, one has

$$\mathscr{L}^{\pm} = \{ (x, \mathrm{U}^{\pm}(x, \theta)), \ \theta \in \mathbb{R} \}$$

and these sets are invariant under the translations $\bar{x} \to \bar{x} + e_\nu$, $\nu = 1, 2, \ldots, n+1$ and can be viewed as sets on the torus T^{n+1}; $\mathscr{L}$ is a Cantor set on T^{n+1} if it is not equal to T^{n+1}.

The « gaps » of this Cantor set $\mathscr{L}$ are given by the discontinuities of $\mathrm{U}^{\pm}(x, \theta)$. These discontinuities occur along hyperplanes $\theta = \alpha \cdot x + \beta$. Let $\theta = \alpha \cdot x + \beta^*$ be such a discontinuity and

$$u^{\pm} = \mathrm{U}^{\pm}(x, \alpha \cdot x + \beta^*), \quad u^-(x) < u^+(x).$$

Then the gap width $\delta(x) = u^+(x) - u^-(x) > 0$ satisfies by theorem 4.5

$$\max_{x \in \mathrm{Q}} \delta(x) \leq \gamma \min_{x \in \mathrm{Q}} \delta(x)$$

with a constant γ depending on c, δ and $|\alpha|$ only. This means that the ratio of the gap width is uniformly bounded over Q, independently of u and the particular gap.

We mention that in case $\alpha_1, \alpha_2, \ldots, \alpha_n, -1$ are rationally independent

$$\int_{\mathbb{R}^n} (u^+(x) - u^-(x))dx \leq 1 \ (*).$$

Indeed this integral agrees with

$$\sum_j \int_{\mathrm{Q}} [(u^+(x + j) - j_{n+1}) - (u^-(x + j) - j_{n+1})]dx$$

and the sets $u^-(x) - j_{n+1} < x_{n+1} < u^+(x + j) - j_{n+1}$ are disjoint. By an appropriate choice of j_{n+1} one can bring these sets into $\mathrm{Q} \times [0, 1]$ so that the above integral is at most 1.

On the set $\mathscr{L}$ we can define the foliation $\psi_\nu(\bar{x})$ as before so that

(6.9) $$u_{x_\nu} = \psi_\nu(x, u)$$

(*) Together with Theorem 4.3 this implies $u^+(x) - u^-(x) \to 0$ for $|x| \to \infty$.

is satisfied by $u(x, \beta) = U^{\pm}(x, \alpha \cdot x + \beta)$ for all $\beta \in \mathbb{R}$. As before $\psi_\nu(\bar{x})$ is invariant under the translation $\bar{x} \to \bar{x} + e_\nu$ $(\nu = 1, 2, \ldots, n+1)$ and therefore is defined on $\mathscr{L}/\mathbb{Z}^{n+1}$. By the same argument as before we see that $\psi_\nu(\bar{x})$ is Lipschitz continuous:

$$|\psi_\nu(\bar{x}') - \psi_\nu(\bar{x})| \leq \gamma |\bar{x}' - \bar{x}| \quad \text{for all} \quad \bar{x}, \bar{x}' \in \mathscr{L}.$$

We summarize: In case B) we have a foliation (6.9) defined on a Cantor set $\mathscr{L} \subset T^{n+1}$ which is given by Lipschitz continuous functions.

We point out that for $n = 1$ this statement corresponds to the fact that Mather sets of monotone twist mappings are subsets of Lipschitz continuous curves (see [*14*]).

7. AN ALTERNATE VARIATIONAL PRINCIPLE

We describe another approach to construct the function $U(x, \theta)$ of the previous section. The difficulty is that this function is, in general, not even continuous. We will construct U as the limit of a smooth minimal $U^{(\varepsilon)}$ of a different variational principle, which is obtained by regularization. We will not prove here that the limit function agrees with the function $U^{\pm}$ of the previous section (at the points of continuity) but only discuss this variational principle in its own right. The main point is that the minimals $U^{(\varepsilon)}$ of this variational problem are monotone in θ, the crucial property of the function $U^{\pm}$.

We consider the class of functions U for which

$$U(x, \theta) - \theta \in W^{1,2}(T^{n+1}),$$

i.e. $U - \theta$ has period 1 in $x_1, \ldots, x_n, \theta$ and consider on $W^{1,2}(T^{n+1})$ the functional

$$\text{(7.1)} \qquad J(U) = \iint_Q \frac{\varepsilon}{2}\left(\frac{\partial U}{\partial \theta}\right)^2 + F(x, U, DU)\,dx\,d\theta$$

where

$$\bar{Q} = \left\{ (x, \theta) \in \mathbb{R}^{n+1},\ |x_\nu| \leq \frac{1}{2},\ |\theta| \leq \frac{1}{2} \right\}$$

$$D_\nu = \frac{\partial}{\partial x_\nu} + \alpha_\nu \frac{\partial}{\partial \theta}.$$

Note that the functional J depends on ε and a vector $\alpha \in \mathbb{R}^n$, which is not indicated in our notation.

In contrast to our previous variational principle this integral is taken over a compact domain, namely the torus $T^{n+1} = \mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$.

For $\varepsilon > 0$ this is a regular variational principle and under the previous assumptions on F (see (3.1)) the standard theory guarantees the existence of a minimal $U = U(x, \theta, \varepsilon)$ minimizing $J(U)$. Moreover, by the regularity theory, one has $U - \theta \in C^{2,\varepsilon}(T^{n+1})$.

THEOREM 7.1. — If $U = U(x, \theta, \varepsilon)$ is a minimal of the functional (7.1) with $\varepsilon > 0$ then

$$\frac{\partial U}{\partial \theta} > 0 .$$

Before proving this theorem we remark:

LEMMA 7.2. — If U, V are minimals of (7.1) then also

$$U^* = \max(U, V), \quad U_* = \min(U, V)$$

are minimals.

Proof. — The argument is identical with that of theorem 5.2. We show again that

$$J(U^*) + J(U_*) = J(U) + J(V)$$

and since $J(U) = J(V)$ is the minimum of the functional J we conclude

$$J(U^*) = J(U_*) = J(U)$$

i. e. U^*, U_* are minimals, since they also belong to the class of admissible functions.

LEMMA 7.3. — If U, V are minimals of (7.1) and $U \leq V$ then we have either $U < V$ or $U \equiv V$.

Proof. — This is a consequence of the maximum principle for elliptic partial differential equations. We show that

$$\varepsilon\left(\frac{\partial}{\partial \theta}\right)^2 U + \sum_{\nu=1}^{n} D_\nu F_{p_\nu}(x, U, DU) - F_u(x, U, DU) = 0$$

$$\varepsilon\left(\frac{\partial}{\partial \theta}\right)^2 V + \Sigma D_\nu F_{p_\nu}(x, V, DV) - F_u(x, V, DV) = 0$$

and therefore $W = V - U > 0$ satisfies an elliptic partial differential equation

$$\varepsilon\left(\frac{\partial}{\partial \theta}\right)^2 W + \Sigma D_\nu(a_{\nu\mu}(x, \theta) D_\mu W) + \Sigma b_\nu D_\nu W + cW = 0 .$$

The assertion now follows from Lemma 4.6.

Proof of Theorem 7.1. — We first show that $U(x, \theta)$ is strictly monotone in θ. Note that with $U(x, \theta)$ also $V(x, \theta) = U(x, \theta + c)$ is a minimal for every constant c, since the variational principle does not explicitly depend on θ. Moreover,

$$\hat{U}(x, \theta) = U(x, \theta) - \theta, \quad \hat{V}(x, \theta) = V(x, \theta) - \theta = c + \hat{U}(x, \theta + c)$$

have period 1 in all variables. Hence

$$\iint_{\bar{Q}} (V - U) dx d\theta = \iint_{\bar{Q}} (\hat{V} - \hat{U}) dx d\theta = c .$$

Thus if $c > 0$ we conclude that

$$\max_{\bar{Q}} (V - U) > 0 .$$

We claim that $V - U > 0$ everywhere. If not $V - U$ would take on the value 0, hence also $U^* - U$ would have a zero if U^* is defined by

$$U^* = \max (U, V) \geq U .$$

By Lemmas 7.2 and 7.3 we conclude $U^* \equiv U$, a contradiction. Thus $U(\theta + c) - U(\theta) > 0$ for $c > 0$, and therefore

$$\frac{\partial U}{\partial \theta} \geq 0 .$$

Since $U_\theta = \dfrac{\partial U}{\partial \theta}$ is also the solution of an elliptic partial differential equation one concludes again that $U_\theta \equiv 0$ or $U_\theta > 0$. The first case can not occur since $U(x, \theta + 1) = U(x, \theta) + 1$. This completes the proof of the theorem.

Now one could study the limit of the minimals $U^{(\varepsilon)} = U^{(\varepsilon)}(x, \theta)$ for $\varepsilon \to 0, \varepsilon > 0$, and show that the limit function $U^{(0)}$ minimizes the variational principle

$$J_0(U) = \iint_{\bar{Q}} F(x, U, DU) dx d\theta$$

over all functions U with $U - \theta \in W^{1,2}(T^{n+1})$ for which $U(x, \theta') \geq U(x, \theta)$ for $\theta' \geq \theta$. This is the generalization of a variational principle suggested by Percival [*22*] [*23*] and which was the basis of Mather's paper. This approach has the advantage that it is applicable for all $\alpha \in \mathbb{R}^n$, whether the $\alpha_1, \alpha_2, \ldots, \alpha_n$ are rationally independent or not. We will not pursue this approach in this paper.

8. A STABILITY THEOREM FOR MINIMAL FOLIATIONS

In this section we present without proof a perturbation theorem about minimal foliations. We begin with an unperturbed Lagrange function

$F^0 = F^0(x, u, p)$ for which $u = \alpha \cdot x + \beta$ for a fixed $\alpha \in \mathbb{R}^n$ and all $\beta \in \mathbb{R}$ is an extremal, i. e.

$$(8.1) \qquad \begin{cases} \sum_{\nu=1}^{\nu} (\partial_{x_\nu} + \alpha_\nu \partial_\theta) F^0_{p_\nu}(x, \theta, \alpha) = F^0_u(x, \theta, \alpha) \\ \sum_{\nu,\mu=1}^{n} F^0_{p_\nu p_\mu}(x, \theta, \alpha) \xi_\nu \xi_\mu > 0 \quad \text{for all} \quad \xi \in \mathbb{R}^n \backslash (0) . \end{cases}$$

For example, if $F^0 = F^0(p)$ is independent of u, p this is true for all $\alpha \in \mathbb{R}^n$. We ask whether for small perturbations

$$F = F^0 + \varepsilon G$$

and fixed α there is still a smooth foliation belonging to the same α. This is a « small divisor » problem and the result below can be viewed as a generalization of the existence theorem of invariant tori for near integrable Hamiltonian systems of two degrees of freedom. In particular, $\alpha = (\alpha_1, \ldots, \alpha_n)$ has to be restricted by Diophantine inequalities. We will assume that there exist positive constants c_0, τ such that

$$(8.2) \qquad \sum_{\nu=1} (j_{n+1}\alpha_\nu - j_\nu)^2 \geq c_0^{-1}(1 + j_{n+1}^2)^{-\tau}$$

for all $\bar{j} \in \mathbb{Z}^{n+1} \backslash (0)$.

Let $B = B_r(\alpha)$ be an open ball in $\mathbb{R}^n$ with the center α and assume that

$$(8.3) \qquad F^0, G \in C^\infty(T^{n+1} \times B) .$$

THEOREM 8.1 (2). — If $F^0 = F^0(x, u, p)$, $G = G(x, u, p)$ satisfy (8.1), (8.3) and $\alpha \in \mathbb{R}^n$ satisfies (8.2) then there exists an $\varepsilon_0 > 0$ such that for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ there exists a smooth function $U = U(x, \theta, \varepsilon)$ with $U - \theta \in C^\infty(T^{n+1})$ and

$$| U - \theta |_{C^1(T^{n+1})} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

such that for each $\beta \in \mathbb{R}$

$$u(x, \beta) = U(x, \alpha \cdot x + \beta, \varepsilon)$$

(2) This theorem was presented in September 1984 at the conference « Contemporary Problems in Algebra and Analysis held at the Steklov Institute in Moscow.

is a solution of the Euler equation

$$\sum_{\nu=1}^{n} \partial_{x_\nu} F_{p_\nu}(x, u, u_{x_\nu}) = F_u(x, u, u_x)$$

$$F = F^0 + \varepsilon G .$$

In other words $u = u(x, \beta)$ defines a smooth minimal foliation for the perturbed problem; note that

$$\frac{\partial u}{\partial \beta} = U_\theta(x, \alpha \cdot x + \beta, \varepsilon) \to 1 \quad \text{as} \quad \varepsilon \to 0$$

hence $\dfrac{\partial u}{\partial \beta} > 0$ for small ε. Hence $\mathscr{L}(\alpha) = T^{n+1}$ in this case, if $|\varepsilon|$ is sufficiently small.

One can view this result as a stability theorem for the foliation under perturbation of the Lagrangian. Indeed, under the coordinate transformation

$$(x, x_{n+1}) \to (x, U(x, x_{n+1}))$$

the unperturbed foliation $x_{n+1} = \alpha . x + \beta$ goes over into the perturbed foliation $x_{n+1} = u(x, \beta)$. Thus we can say, that under the assumptions of the above theorem the foliation survives under small perturbations and, moreover, remains in the same conjugacy class.

The proof of the above theorem—which will be published elsewhere—uses the rapid iteration technique in conjunction with careful L^2-estimates of the approximations to the solution $U = U(x, \theta)$ of the degenerate partial differential equation

$$\sum_{\nu=1}^{n} D_\nu F_{p_\nu}(x, U, DU) = F_u(x, U, DU)$$

$$D_\nu = \frac{\partial}{\partial x_\nu} + \alpha_\nu \frac{\partial}{\partial \theta} .$$

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