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Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems

by

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ABSTRACT. — Let X be a real Banach space and I a function on X such that $I = \Phi + \psi$ with $\Phi \in C^1(X, \mathbb{R})$ and $\psi : X \to (-\infty, +\infty]$ convex, proper and lower semicontinuous. A point $u \in X$ is said to be critical if $\psi(u) \neq +\infty$ and $\langle \Phi'(u), v - u \rangle + \psi(v) - \psi(u) \geqq 0 \ \forall v \in X$. The paper contains a number of existence theorems for critical points of functions of the above mentioned type. Critical levels of saddle type are characterized by minimax principles. The results are applied to variational inequalities and variational equations with single- and multivalued operators, which arise from studying certain elliptic boundary value problems.

RÉSUMÉ. — Soit X un espace de Banach et I une fonction sur X de la forme $I = \Phi + \psi$, où Φ est C^1 et ψ est convexe s. c. i., pouvant prendre la valeur $+\infty$. On définit une notion naturelle de point critique, et l'on démontre des théorèmes d'existence par des méthodes de minimax, du type Liusternik-Schnivelman. On applique ces résultats à des équations et inéquations variationnelles de type elliptique.

Mots-clés : Minimax principles. Liusternik-Schnivelman theory, convex functions, critical points. Ekeland's variational principle.

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INTRODUCTION

The purpose of this paper is to generalize some minimax methods in critical point theory to a class of functions which are not necessarily continuous. Let X be a real Banach space. Recall that for a continuously differentiable function $\Phi : X \to \mathbb{R}$, a point $u \in X$ is said to be *critical* if $\Phi'(u) = 0$. The corresponding number $\Phi(u)$ is called *a critical value.* It is well known that local maxima and minima are critical points. If Φ satisfies some appropriate compactness conditions (usually of Palais-Smale type), one may also find other critical points by minimaxing Φ over certain families of subsets of X. More precisely, if Γ is such a family, one can give sufficient conditions in order that the value

$$c = \inf_{A \in \Gamma} \sup_{u \in A} \Phi(u)$$

be critical. For an account of recent results in critical point theory for C^1 functions by minimax methods the reader is referred to [*20*] [*21*] [*24*].

Very recently critical point theory has been generalized by Chang [*8*] to locally Lipschitz continuous functions and by Struwe [*26*] [*27*] to functions which are of class C^1 with respect to certain families of subspaces of X. In this paper we present another generalization.

Let X be a real Banach space and $\psi : X \to (-\infty, +\infty]$ a convex lower semicontinuous function. The set $D(\psi) = \{ u \in X : \psi(u) < +\infty \}$ is called *the effective domain* of ψ. Denote by X^* the dual of X and by $\langle\ ,\ \rangle$ the duality pairing between X^* and X. For $u \in D(\psi)$ the set

$$\partial\psi(u) = \{ u^* \in X^* : \psi(v) - \psi(u) \geqq \langle u^*, v - u \rangle \qquad \forall v \in X \}$$

is called the *subdifferential* of ψ at u [*3*, § II.2]. We shall consider functions $I = \Phi + \psi$ with $\Phi \in C^1(X, \mathbb{R})$ and ψ as above. A point $u \in D(\psi)$ is said to be *critical* if $-\Phi'(u) \in \partial\psi(u)$, or equivalently, if u satisfies the inequality

$$\langle \Phi'(u), v - u \rangle + \psi(v) - \psi(u) \geqq 0 \qquad \forall v \in X.$$

Inequalities of this type arise in a number of problems of physics [*12*].

In [*10*] [*11*] Dias and Hernández invoked results from critical point theory for C^1 functions in order to study eigenvalue problems $\lambda u \in \partial\psi(u)$ for X a Hilbert space and ψ as above. They used the fact that the operator $\mathrm{id} + \partial\psi$ has a single-valued inverse which is of gradient type. Unfortunately, this approach does not seem to give results we want to obtain.

It is easy to see (cf. Proposition *1.1*) that local minima are critical points of I. In order to be able to obtain other critical points we need a compactness condition (which is introduced in Section 1) and a deformation result. For C^1 functions the required deformation is effected by moving along integral lines of a pseudogradient vector field [*21*, Theorem 1.9; *24*, Theorem 1.1]. In the case of functions which are only lower semicontinuous such a construction does not seem to be readily available, mainly because a noncritical value c may be « semicritical » in the sense that there may exist a critical point $\bar{u}$ with $I(\bar{u}) < c$ and a sequence $u_n \to \bar{u}$ with $I(u_n) \to c$. In Proposition 2.3 we obtain a result which in a sense is a weak version of the usual deformation theorem. Our deformation (denoted by α_s) has the inconvenient property that $I(\alpha_s(u))$ may increase for some u. The proofs of existence of nonminimum critical points (which become rather technical because of that) are effected by combining Proposition 2.3 with Ekeland's variational principle. The idea of using Ekeland's principle to obtain critical points other than local minima (actually, to prove the Mountain Pass Theorem), may be found in [*2*] [*6*].

The paper is organized as follows: Section 1 contains preliminary material. In particular, we introduce a compactness condition and recall Ekeland's variational principle. In Section 2 we prove a deformation result and in Section 3 we show that the Mountain Pass Theorem of Ambrosetti and Rabinowitz [*1*] [*21*] [*24*] and some related results [*22*] [*23*] [*24*] remain valid for functions satisfying our assumptions. Section 4 is devoted to generalizations of results of Clark [*9*] [*21*] [*24*] and Ambrosetti and Rabinowitz [*1*] [*21*] [*24*] concerning the existence of multiple critical points for even functions. In Section 5 we apply abstract results of Sections 3 and 4 to elliptic boundary value problems. Our examples include variational inequalities and variational equations with single- and multivalued operators.

After completing this paper I have been informed by I. Ekeland that for lower semicontinuous functions $I : X \to (-\infty, +\infty]$ having the property that $I(u) + c\,\|u\|^2$ is convex for some $c \geqq 0$, there is a regularization procedure due to J. M. Lasry [*28*, Lemma 7], which associates with I a family $(I_\varepsilon)_{0<\varepsilon<1/c}$ of functions such that $I_\varepsilon(u) \to I(u)\ \forall u \in X$ as $\varepsilon \to 0$ and $I_\varepsilon \in C^1(X, \mathbb{R})$. Furthermore, $I_\varepsilon(u) \leqq I(u)\ \forall u \in X$, I_ε and I have the same critical points and I_ε satisfies the Palais-Smale condition whenever I satisfies a condition of similar type (cf. (PS)′ below). Note that for such I our Theorem 4.3 is an easy consequence of the above-mentioned result of Clark: $I_\varepsilon(u) \leqq I(u)$, I satisfies the hypotheses of Clark's theorem and has therefore at least k pairs of nontrivial critical points; hence so does I. Note also that in some of the applications to boundary value problems in Section 5 (Theorem 5.1 and Theorem 5.8 with subsequent corollaries) $I(u) + c\,\|u\|^2$ will not be convex for any $c \geqq 0$.

1. PRELIMINARIES

Let X be a real Banach space and I a function on X satisfying the following hypothesis:

(H) $\mathrm{I} = \Phi + \psi$, *where* $\Phi \in \mathrm{C}^1(\mathrm{X}, \mathbb{R})$ *and* $\psi : \mathrm{X} \to (-\infty, +\infty]$ *is convex, proper* (*i. e.*, $\psi \not\equiv +\infty$) *and lower semicontinuous* (*l. s. c. in short*).

A point $u \in \mathrm{X}$ is said to be *a critical point* of I if $u \in \mathrm{D}(\psi)$ and if it satisfies the inequality

$$(1) \qquad \langle \Phi'(u), v - u \rangle + \psi(v) - \psi(u) \geqq 0 \qquad \forall v \in \mathrm{X}.$$

Note that X can be replaced by $\mathrm{D}(\psi)$ in (1). A number $c \in \mathbb{R}$ such that $\mathrm{I}^{-1}(c)$ contains a critical point will be called *a critical value*. We shall use the following notation:

$$\mathrm{K} = \{ u \in \mathrm{X} : u \text{ is a critical point} \},$$
$$\mathrm{I}_c = \{ u \in \mathrm{X} : \mathrm{I}(u) \leqq c \}, \qquad \mathrm{K}_c = \{ u \in \mathrm{K} : \mathrm{I}(u) = c \}.$$

1.1. PROPOSITION. — If I satisfies (H), each local minimum is necessarily a critical point of I.

Proof. — Let u be a local minimum of I. Using convexity of ψ, it follows that for all small $t > 0$,

$$0 \leqq \mathrm{I}((1-t)u + tv) - \mathrm{I}(u) = \Phi(u + t(v-u)) - \Phi(u) + \psi((1-t)u + tv) - \psi(u)$$
$$\leqq \Phi(u + t(v-u)) - \Phi(u) + t(\psi(v) - \psi(u)).$$

Dividing by t and letting $t \to 0$ we obtain (1). □

We shall assume that I satisfies the following compactness condition of Palais-Smale type:

(PS) *If* (u_n) *is a sequence such that* $\mathrm{I}(u_n) \to c \in \mathbb{R}$ *and*

$$(2) \qquad \langle \Phi'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geqq -\varepsilon_n \| v - u_n \| \qquad \forall v \in \mathrm{X},$$

where $\varepsilon_n \to 0$, *then* (u_n) *possesses a convergent subsequence.*

Condition (PS) can also be formulated as follows:

(PS′) *If* (u_n) *is a sequence such that* $\mathrm{I}(u_n) \to c \in \mathbb{R}$ *and*

$$(3) \qquad \langle \Phi'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geqq \langle z_n, v - u_n \rangle \qquad \forall v \in \mathrm{X},$$

where $z_n \to 0$, *then* (u_n) *possesses a convergent subsequence.*

1.2. PROPOSITION. — Conditions (PS) and (PS′) are equivalent.

Observe that (3) expresses the fact that $z_n - \Phi'(u_n) \in \partial\psi(u_n)$, or equivalently, that $z_n \in \Phi'(u_n) + \partial\psi(u_n)$. In this notation the similarity to (a version of) the usual Palais-Smale condition becomes more transparent. Observe also that if $\psi \equiv 0$, then $I \in C^1$ and our definition of critical point as well as our condition (PS) coincide with the usual ones.

In order to prove Proposition 1.2 we need an additional result. In what follows we shall use the same symbol $\| \ \|$ to denote the norms in X and X*.

1.3. LEMMA. — Let X be a real Banach space and $\chi : X \to (-\infty, +\infty]$ a l. s. c. convex function with $\chi(0) = 0$. If

$$\chi(x) \geqq -\|x\| \qquad \forall x \in X,$$

there exists a $z \in X^*$ such that $\|z\| \leqq 1$ and

$$\chi(x) \geqq \langle z, x \rangle \qquad \forall x \in X.$$

It is well known that χ is bounded below by an affine function, i. e., $\chi(x) \geqq \langle z, x \rangle - \beta$ for some $z \in X^*$ and $\beta \in \mathbb{R}$ [*3*, Proposition II.2.1]. The lemma asserts that under our assumptions we can choose z with norm $\leqq 1$ and $\beta = 0$.

Proof of lemma. — The proof was suggested to us by P. O. Lindberg. In the space $X \times \mathbb{R}$ define

$$A = \{ (x, t) : \|x\| < -t \} \qquad \text{and} \qquad B = \{ (x, t) : \chi(x) \leqq t \}.$$

It is easy to verify that A and B are convex (in fact, B is the epigraph of χ) and A is open. Moreover, $A \cap B = \phi$ because $\chi(x) \geqq -\|x\|$. Consequently, there exists a hyperplane separating A and B, i. e., we can find $\alpha, \beta \in \mathbb{R}$ and $w \in X^*$ such that

$$\langle w, x \rangle - \alpha t - \beta \geqq 0 \qquad \forall (x, t) \in \overline{A},$$
$$\langle w, x \rangle - \alpha t - \beta \leqq 0 \qquad \forall (x, t) \in B.$$

Since $(0, 0) \in \overline{A} \cap B$, $\beta = 0$. Taking $t = -\|x\|$ in the first of these inequalities gives

$$\langle w, x \rangle \geqq -\alpha \|x\| \qquad \forall x \in X.$$

It follows that $\alpha \geqq 0$ and $\|w\| \leqq \alpha$. If $\alpha = 0$, then $w = 0$ and there is no hyperplane. So $\alpha > 0$. Set $z = w/\alpha$ and $t = \chi(x)$ in the second of the above inequalities. Then $\|z\| \leqq 1$ and $\langle z, x \rangle \leqq t = \chi(x)$. □

Proof of Proposition 1.2. — It suffices to prove that (2) and (3) are equivalent and it is clear that (3) implies (2). So suppose that (2) is satisfied. If $\varepsilon_n \leqq 0$, we may take $z_n = 0$. If $\varepsilon_n > 0$, let $x = v - u_n$ and $\chi(x) = (\langle \Phi'(u_n), x \rangle + \psi(x + u_n) - \psi(u_n))/\varepsilon_n$. Then (2) reads

$$\chi(x) \geqq -\|x\| \qquad \forall x \in X.$$

According to Lemma 1.3 there is a $\zeta_n \in X^*$ with $\| \zeta_n \| \leqq 1$ and $\chi(x) \geqq \langle \zeta_n, x \rangle$. Setting $z_n = \varepsilon_n \zeta_n$ gives

$$(\langle \Phi'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n))/\varepsilon_n \geqq \langle z_n/\varepsilon_n, v - u_n \rangle .$$

Hence (3) is satisfied and $z_n \to 0$ because $\varepsilon_n \to 0$. □

1.4. PROPOSITION. — Suppose that I satisfies (H) and (PS) and let (u_n) be a sequence verifying the hypotheses of (PS). If u is an accumulation point of (u_n), then $u \in K_c$. In particular, K_c is a compact set.

Proof. — We may assume that $u_n \to u$. Passing to the limit in (2) and using the fact that $\lim \psi(u_n) \geqq \psi(u)$, we obtain (1). Hence $u \in K$. Moreover, since inequality (1) cannot be strict for $v = u$, $\lim \psi(u_n) = \psi(u)$. Consequently, $I(u_n) \to I(u) = c$ and $u \in K_c$.

If $(u_n) \subset K_c$, then $I(u_n) = c$ and (2) is satisfied with $\varepsilon_n = 0$. It follows that a subsequence of (u_n) converges to some $u \in X$. By the first part of the proposition, $u \in K_c$. Hence K_c is compact. □

1.5. REMARK. — Suppose that ψ is the indicator function of a nonempty closed convex sex $\mathbb{K}$, i.e., $\psi(u) = 0$ if $u \in \mathbb{K}$ and $\psi(u) = +\infty$ otherwise. Then u is a critical point of I if and only if $u \in \mathbb{K}$ and

$$\langle \Phi'(u), v - u \rangle \geqq 0 \qquad \forall v \in \mathbb{K} .$$

Furthermore, I satisfies (PS) if and only if each sequence $(u_n) \subset \mathbb{K}$ such that $\Phi(u_n) \to c \in \mathbb{R}$ and

$$\langle \Phi'(u_n), v - u_n \rangle \geqq \langle z_n, v - u_n \rangle \qquad \forall v \in \mathbb{K} ,$$

where $z_n \to 0$, has a convergent subsequence.

We shall make repeated use of the following variational principle of Ekeland.

1.6. PROPOSITION. [*13*, p. 444 and 456]. — Let (Z, d) be a complete metric space and $\Pi : Z \to (-\infty, +\infty]$ a proper l. s. c. function bounded below. Given $\delta, \lambda > 0$ and $x \in Z$ with

$$\Pi(x) \leqq \inf_{z \in Z} \Pi(z) + \delta ,$$

there exists a point $y \in Z$ such that

$$\Pi(y) \leqq \Pi(x), \quad d(x, y) \leqq 1/\lambda \qquad \text{and}$$
$$\Pi(z) - \Pi(y) \geqq -\delta\lambda d(y, z) \qquad \forall z \in Z .$$

An easy consequence of this result is the following

1.7. THEOREM. — If I is bounded below and satisfies (H) and (PS), then

$$c = \inf_{u \in X} I(u)$$

is a critical value.

Proof. — Let (w_n) be a sequence satisfying $I(w_n) \leqq c + 1/n$. By Proposition 1.6 with $\delta = 1/n$ and $\lambda = 1$, we find another sequence, (u_n), such that $I(u_n) \leqq c + 1/n$ and

$$I(w) - I(u_n) \geqq (-1/n) \| w - u_n \| \qquad \forall w \in X .$$

Set $w = (1 - t)u_n + tv$, $t \in (0, 1)$. Since ψ is convex,

$$\begin{aligned}\Phi(u_n + t(v - u_n)) - \Phi(u_n) + t(\psi(v) - \psi(u_n)) &\geqq I(w) - I(u_n) \\ &\geqq (-1/n) \| w - u_n \| = (-1/n)t \| v - u_n \| .\end{aligned}$$

Dividing by t and letting $t \to 0$ we obtain

$$\langle \Phi'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geqq (-1/n) \| v - u_n \| .$$

So by (PS) and Proposition 1.4, (u_n) possesses a subsequence converging to $u \in K_c$. $\square$

2. EXISTENCE OF DEFORMATIONS

2.1. LEMMA. — Suppose that I satisfies (H) and (PS) and let N be a neighbourhood of K_c. Then for each $\bar{\varepsilon} > 0$ there exists an $\varepsilon \in (0, \bar{\varepsilon})$ such that if $u_0 \notin N$ and $c - \varepsilon \leqq I(u_0) \leqq c + \varepsilon$, then

$$\langle \Phi'(u_0), v_0 - u_0 \rangle + \psi(v_0) - \psi(u_0) < -3\varepsilon \| v_0 - u_0 \| \tag{4}$$

for some $v_0 \in X$.

Proof. — If the conclusion is false, there exists a sequence $(u_n) \subset X - N$ such that $I(u_n) \to c$ and

$$\langle \Phi'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geqq (-1/n) \| v - u_n \| \qquad \forall v \in X .$$

So by (PS) and Proposition 1.4, a subsequence of (u_n) converges to $u \in K_c$. This, however, is impossible because $u_n \notin N$ for any n and N is a neighbourhood of K_c. $\square$

2.2. LEMMA. — Suppose that I satisfies (H) and (PS). Let N be a neighbourhood of K_c and ε a positive number such that (4) holds. Then for each $u_0 \in I_{c+\varepsilon} - N$ there exists a $v_0 \in X$ and an open neighbourhood U_0 of u_0 with the following properties:

$$(5) \qquad \langle \Phi'(u), v_0 - u \rangle + \psi(v_0) - \psi(u) \leqq \| v_0 - u \| \qquad \forall u \in U_0,$$

$$(6) \quad \langle \Phi'(u), v_0 - u \rangle + \psi(v_0) - \psi(u) \leqq -3\varepsilon \| v_0 - u \| \quad \forall u \in U_0 \text{ such that } I(u) \geqq c - \varepsilon.$$

Furthermore, if $u_0 \in K$, $v_0 = u_0$, otherwise v_0, U_0 and a number $\delta_0 > 0$ can be chosen so that $v_0 \notin \overline{U}_0$ and

$$(5') \quad \langle \Phi'(u), v_0 - u \rangle + \psi(v_0) - \psi(u) \leqq -\delta_0 \| v_0 - u \| \qquad \forall u \in U_0.$$

Proof. — Assume first $u_0 \in K$. Then u_0 satisfies (1), i. e.,

$$\langle \Phi'(u_0), u - u_0 \rangle + \psi(u) - \psi(u_0) \geqq 0 \qquad \forall u \in X.$$

It follows that if U_0 is a sufficiently small neighbourhood of u_0,

$$\begin{aligned} \langle \Phi'(u), u_0 - u \rangle + \psi(u_0) - \psi(u) &\leqq \langle \Phi'(u) - \Phi'(u_0), u_0 - u \rangle \\ &\leqq \| \Phi'(u) - \Phi'(u_0) \| \, \| u_0 - u \| \leqq \| u_0 - u \| \qquad \forall u \in U_0. \end{aligned}$$

So (5) with $v_0 = u_0$ is satisfied. By Lemma 2.1, $u \in N$ whenever $u \in K$ and $c - \varepsilon \leqq I(u) \leqq c + \varepsilon$. Hence $I(u_0) < c - \varepsilon$. If $I(u) < c - \varepsilon$ in a neighbourhood of u_0, U_0 may be chosen in this neighbourhood and condition (6) is empty. If each neighbourhood of u_0 contains points with $I(u) \geqq c - \varepsilon$, it follows from the continuity of Φ that $\psi(u) - \psi(u_0) \geqq d$ for some constant $d > 0$ and all u sufficiently close to u_0 and such that $I(u) \geqq c - \varepsilon$. So if U_0 is small enough,

$$\begin{aligned} \langle \Phi'(u), u_0 - u \rangle + \psi(u_0) - \psi(u) &\leqq \| \Phi'(u) \| \, \| u_0 - u \| - d \\ &\leqq -3\varepsilon \| u_0 - u \| \qquad \forall u \in U_0, \quad I(u) \geqq c - \varepsilon. \end{aligned}$$

Suppose now that $u_0 \notin K$ and $I(u_0) < c - \varepsilon$. Since u_0 is not a solution of (1), there exists a $v_0 \in X$ such that

$$(7) \qquad \langle \Phi'(u_0), v_0 - u_0 \rangle + \psi(v_0) - \psi(u_0) < 0.$$

Let $w_0 = tv_0 + (1 - t)u_0$, $0 < t < 1$. By convexity of ψ,

$$\langle \Phi'(u_0), w_0 - u_0 \rangle + \psi(w_0) - \psi(u_0) \leqq t(\langle \Phi'(u_0), v_0 - u_0 \rangle + \psi(v_0) - \psi(u_0)) < 0,$$

so we may assume that v_0 is arbitrarily close to u_0. As in the preceding part of the proof, $\psi(u) - \psi(u_0) \geqq d > 0$ for all u close enough to u_0 and such that $I(u) \geqq c - \varepsilon$. Using (7), this implies that if U_0 and $\| v_0 - u_0 \|$ are sufficiently small, then $(\psi(v_0) - \psi(u_0)) + (\psi(u_0) - \psi(u)) \leqq \frac{1}{2}d - d = -\frac{1}{2}d$ and

$$\langle \Phi'(u), v_0 - u \rangle + \psi(v_0) - \psi(u) \leqq \| \Phi'(u) \| \, \| v_0 - u \| - \frac{1}{2}d \leqq -3\varepsilon \| v_0 - u \| \qquad \forall u \in U_0, \quad I(u) \geqq c - \varepsilon.$$

Thus (6) is satisfied. Since $v_0 \neq u_0$, we may assume that $v_0 \notin \overline{U}_0$. In order to verify (5') note that since the left-hand side of (7) is negative,

$$\langle \Phi'(u_0), v_0 - u_0 \rangle + \psi(v_0) - \psi(u_0) < - \delta_0 \| v_0 - u_0 \|$$

for some $\delta_0 > 0$. Using continuity of Φ' and l. s. c. of ψ it follows that, after shrinking U_0 if necessary,

$$\langle \Phi'(u), v_0 - u \rangle + \psi(v_0) - \psi(u) < - \delta_0 \| v_0 - u \| \qquad \forall u \in U_0 .$$

The remaining case of $u_0 \notin K$, $I(u_0) \geqq c - \varepsilon$ is easy: by Lemma 2.1 we find v_0 such that (4) is satisfied. By continuity of Φ' and l. s. c. of ψ there is a neighbourhood U_0 of u_0, $v_0 \notin \overline{U}_0$, with the property that

$$\langle \Phi'(u), v_0 - u \rangle + \psi(v_0) - \psi(u) < - 3\varepsilon \| v_0 - u \| \qquad \forall u \in U_0 .$$

Hence (5') and (6). □

A family of mappings $\alpha(.,s) \equiv \alpha_s(.) : W \to X$, $0 \leqq s \leqq \bar{s}$, $\bar{s} > 0$, is said to be *a deformation* if $\alpha \in C(W \times [0, \bar{s}], X)$ and $\alpha_0 = \mathrm{id}_W$ (the identity on W).

2.3. Proposition. — Suppose that I satisfies (H) and (PS). Let N be a neighbourhood of K_c and $\bar{\varepsilon}$ a positive number. Then there exists an $\varepsilon \in (0, \bar{\varepsilon})$ such that for each compact subset A of $X - N$ with

$$c \leqq \sup_{u \in A} I(u) \leqq c + \varepsilon$$

we can find a closed set W containing A in its interior and a deformation $\alpha_s : W \to X$, $0 \leqq s \leqq \bar{s}$, having the following properties:

(8) $\qquad \| u - \alpha_s(u) \| \leqq s \qquad \forall u \in W,$

(9) $\qquad I(\alpha_s(u)) - I(u) \leqq 2s \qquad \forall u \in W,$

(10) $\qquad I(\alpha_s(u)) - I(u) \leqq - 2\varepsilon s \qquad \forall u \in W$ with $I(u) \geqq c - \varepsilon$

and

$$\sup_{u \in A} I(\alpha_s(u)) - \sup_{u \in A} I(u) \leqq - 2\varepsilon s . \tag{11}$$

Furthermore, if W_0 is a closed set and $W_0 \cap K = \phi$, W and α_s can be constructed so that

$$I(\alpha_s(u)) - I(u) \leqq 0 \qquad \forall u \in W \cap W_0 . \tag{12}$$

Proof. — Choose $\varepsilon \in (0, \bar{\varepsilon})$ so that Lemma 2.2 can be applied. For each $u_0 \in A$, let U_0 be the set constructed in that lemma. If $u_0 \in K$, we may assume U_0 is so small that $U_0 \cap W_0 = \phi$. The sets U_0 cover A. Let $(U_i)_{i \in J}$ be a finite subcovering. Denote by u_i and v_i the points corresponding to U_i in the same way as u_0 and v_0 correspond to U_0. We may assume that the subcovering has the property that if $i_0 \in J$ and $u_{i_0} \in K$, then the distance from u_{i_0} to each U_i with $i \neq i_0$ is positive. Indeed, if this is not the case for some u_{i_0}, we choose a closed neighbourhood D of u_{i_0} such that $D \subset U_{i_0}$, $u_i \notin D$ for $i \neq i_0$, and obtain a new covering by deleting D from all U_i with $i \neq i_0$ (strictly speaking, we obtain a refinement of the subcovering $(U_i)_{i \in J}$).

Let ρ_i be a continuous function such that $\rho_i(u) > 0$ $\forall u \in U_i$ and $\rho_i(u) = 0$ otherwise. Let $\sigma_i(u) = \rho_i(u) / \sum_{j \in J} \rho_j(u)$ $\forall u \in V = \bigcup_{i \in J} U_i$. Define the mappings α_s as follows. If $u_{i_0} \in A \cap K$ and $u \in U_{i_0} - \bigcup_{i \neq i_0} U_i$,

$$\text{(13)} \quad \alpha_s(u) = \begin{cases} u + s(u_{i_0} - u)/\| u_{i_0} - u \| & \text{for } 0 \leqq s < \| u_{i_0} - u \| \\ u_{i_0} & \text{for } s \geqq \| u_{i_0} - u \|. \end{cases}$$

For all other $u \in V$,

$$\text{(14)} \qquad \alpha_s(u) = u + s \sum_{i \in J} \sigma_i(u)(v_i - u)/\| v_i - u \| .$$

It is immediately seen that $\alpha_0 = \mathrm{id}$ and (8) is satisfied. Note that if $u \in U_{i_0} - \bigcup_{i \neq i_0} U_i$ and $u \neq u_{i_0}$, then for all $s \leqq \| u - u_{i_0} \|$, $\alpha_s(u)$ will be the same regardless which one of the formulas (13) and (14) we use. Consequently, α_s is well defined and continuous for sufficiently small positive s.

Suppose that $\alpha_s(u)$ is given by (14) and set $\alpha_s(u) = u + sw$. By Taylor's formula,

$$\text{(15)} \quad I(\alpha_s(u)) = I(u + sw) = \Phi(u) + s \langle \Phi'(u), w \rangle + r(s) + \psi(u + sw),$$

where

$$| r(s) | \leqq s \sup_{0 \leqq t \leqq s} \| \Phi'(u + tw) - \Phi'(u) \| .$$

Let δ be a given number such that

$$0 < 3\delta \leqq \min \{ 1, \varepsilon, \delta_i \}$$

(δ_i correspond to U_i in (5')). Since A is compact, there exists a closed set W containing A in its interior and an $\tilde{s} > 0$ such that

$$| r(s) | \leqq \delta s \qquad \forall s \leqq \tilde{s},\ u \in W \text{ and } w \in X \text{ with } \| w \| \leqq 1 .$$

By (14),

$$\alpha_s(u) = u + sw = \left(1 - s \sum_{i \in J} \sigma_i(u) \| v_i - u \|^{-1} \right) u + s \sum_{i \in J} \sigma_i(u) \| v_i - u \|^{-1} v_i .$$

If s is sufficiently small, $0 \leqq s \sum_{i \in J} \sigma_i(u) \| v_i - u \|^{-1} \leqq 1$ for all u such that

$\alpha_s(u)$ is given by (14). Using convexity of ψ and (15) it follows that

$$\mathrm{I}(\alpha_s(u)) \leqq \Phi(u) + s \sum_{i \in \mathrm{J}} \sigma_i(u) \| v_i - u \|^{-1} \langle \Phi'(u), v_i - u \rangle + \delta s$$

$$+ \left(1 - s \sum_{i \in \mathrm{J}} \sigma_i(u) \| v_i - u \|^{-1}\right) \psi(u) + s \sum_{i \in \mathrm{J}} \sigma_i(u) \| v_i - u \|^{-1} \psi(v_i)$$

$$= \mathrm{I}(u) + s \sum_{i \in \mathrm{J}} \sigma_i(u) \| v_i - u \|^{-1} (\langle \Phi'(u), v_i - u \rangle + \psi(v_i) - \psi(u)) + \delta s \,.$$

By (5), each term under the last summation sign is less than or equal to $\sigma_i(u)$. Hence

(16) $$\mathrm{I}(\alpha_s(u)) \leqq \mathrm{I}(u) + s + \delta s \,.$$

So (9) is satisfied for all small s. In the same way we see from (6) and (5′) that

(17) $$\mathrm{I}(\alpha_s(u)) \leqq \mathrm{I}(u) - 3\varepsilon s + \delta s \qquad \forall u \in \mathrm{W} \text{ with } \mathrm{I}(u) \geqq c - \varepsilon$$

and

(18) $$\mathrm{I}(\alpha_s(u)) \leqq \mathrm{I}(u) - 3\delta s + \delta s \qquad \forall u \in \mathrm{W} \cap \mathrm{W}_0$$

(recall that $\mathrm{U}_i \cap \mathrm{W}_0 = \phi$ whenever $u_i \in \mathrm{K}$). This gives (10) and (12).

Suppose now that $\alpha_s(u)$ is given by (13). Then

$$\alpha_s(u) = u + sw = (1 - s \| u_{i_0} - u \|^{-1})u + s \| u_{i_0} - u \|^{-1} u_{i_0}$$

whenever $s < \| u_{i_0} - u \| \equiv s_0$. So for such s (9) and (10) follow as in the preceding case. If $s \geqq s_0$, $\mathrm{I}(\alpha_s(u)) = \mathrm{I}(\alpha_{s_0}(u)) \leqq \mathrm{I}(u) + s_0 + \delta s_0 \leqq \mathrm{I}(u) + 2s$ and $\mathrm{I}(\alpha_s(u)) = \mathrm{I}(u_{i_0}) < c - \varepsilon$. So (9) and (10) are satisfied for small s. (Note that not only (9) and (10) but also (16) and (17) hold; this will be useful in the proof of Corollary 2.4.) To verify (12) recall that if $\alpha_s(u)$ is given by (13), then $u \in \mathrm{U}_{i_0}$ with $u_{i_0} \in \mathrm{K}$. Hence $\mathrm{U}_{i_0} \cap \mathrm{W}_0 = \phi$ and $u \notin \mathrm{W} \cap \mathrm{W}_0$.

It remains to prove (11). If $\sup_{u \in \mathrm{A}} \mathrm{I}(\alpha_s(u)) \leqq c - \frac{1}{2}\varepsilon$, (11) is satisfied for all $s \leqq 1/4$ because $\sup_{u \in \mathrm{A}} \mathrm{I}(u) \geqq c$. Suppose that $\sup_{u \in \mathrm{A}} \mathrm{I}(\alpha_s(u)) > c - \frac{1}{2}\varepsilon$. Set

$$\mathrm{B} = \{ u \in \mathrm{A} : \mathrm{I}(u) > c - \varepsilon \} \,.$$

By (9), $\sup_{u \in \mathrm{A}} \mathrm{I}(\alpha_s(u)) = \sup_{u \in \mathrm{B}} \mathrm{I}(\alpha_s(u))$ whenever s is small ($s \leqq \varepsilon/4$). Using this and (10) it follows that

(19) $$\sup_{u \in \mathrm{A}} \mathrm{I}(\alpha_s(u)) - \sup_{u \in \mathrm{A}} \mathrm{I}(u) = \sup_{u \in \mathrm{B}} \mathrm{I}(\alpha_s(u)) - \sup_{u \in \mathrm{B}} \mathrm{I}(u)$$
$$\leqq \sup_{u \in \mathrm{B}} (\mathrm{I}(\alpha_s(u)) - \mathrm{I}(u)) \leqq -2\varepsilon s \,. \qquad \square$$

2.4. COROLLARY. — Assume that Φ and ψ are even. If A is symmetric (i. e., $-A = A$), α_s may be chosen to be odd.

Proof. — We may assume that W is symmetric. Let

$$\beta_s(u) = \frac{1}{2}(\alpha_s(u) - \alpha_s(-u)).$$

Then β_s is odd and satisfies (8). Write $\alpha_s(u) = u + h_s(u)$. By Taylor's formula,

$$\begin{aligned} I(\beta_s(u)) = \Phi(u) + \frac{1}{2}\langle \Phi'(u),\ \ h_s(u) - h_s(-u) \rangle + r_1(s) \\ + \psi\left(\frac{1}{2}(u + h_s(u)) + \frac{1}{2}(u - h_s(-u))\right). \end{aligned}$$

Since Φ is even and ψ even and convex,

$$\begin{aligned} I(\beta_s(u)) \leqq \frac{1}{2}(\Phi(u) + \langle \Phi'(u), h_s(u) \rangle + \psi(u + h_s(u))) \\ + \frac{1}{2}(\Phi(-u) + \langle \Phi'(-u), h_s(-u) \rangle + \psi(-u + h_s(-u))) + \delta s. \end{aligned}$$

Hence by Taylor's formula again,

$$I(\beta_s(u)) \leqq \frac{1}{2} I(\alpha_s(u)) + \frac{1}{2} I(\alpha_s(-u)) + 2\delta s.$$

Using this and (16), $I(\beta_s(u)) \leqq I(u) + s + 3\delta s \leqq I(u) + 2s$ for s small. So β_s satisfies (9). In the same way one sees that (17) and (18) imply (10) and (12). Finally, (11) follows upon observing that (19) remains true whenever (9) and (10) hold. □

3. MOUNTAIN PASS THEOREM

Let Z be a topological space and X a real Banach space. A mapping $f : Z \to X$ is said to be *bounded* if the set $f(Z)$ is bounded in X. Denote by C(Z, X) the set of all continuous bounded mappings from Z to X, metrized by

$$d(f, g) = \sup_{z \in Z} \| f(z) - g(z) \|.$$

It is well known that C(Z, X) is a Banach space. Let $I : X \to (-\infty, +\infty]$ be a given function and define a new function $\Pi : C(Z, X) \to (-\infty, +\infty]$ by setting

$$\Pi(f) = \sup_{z \in Z} I(f(z)).$$

3.1. Lemma. — Suppose that $I : X \to (-\infty, +\infty]$ is l.s.c. Then also the function Π is l.s.c.

Proof. — Suppose that $f_n \to f$. Since I is l. s. c., $I(f(z)) \leqq \liminf I(f_n(z))$ $\forall z \in Z$. Hence

$$\Pi(f) = \sup_{z \in Z} I(f(z)) \leqq \liminf \sup_{z \in Z} I(f_n(z)) = \liminf \Pi(f_n). \qquad \square$$

Denote by $B_\rho(u)$ the open ball of radius ρ centered at u, by $\partial B_\rho(u)$ the boundary of $B_\rho(u)$ and let $B_\rho = B_\rho(0)$, $\partial B_\rho = \partial B_\rho(0)$.

3.2. Theorem (Mountain Pass Theorem). — Suppose that $I : X \to (-\infty, +\infty]$ is a function satisfying (H), (PS) and

i) $I(0) = 0$ and there exist $\alpha, \rho > 0$ such that $I|_{\partial B_\rho} \geqq \alpha$,
ii) $I(e) \leqq 0$ for some $e \notin \overline{B}_\rho$.

Then I has a critical value $c \geqq \alpha$ which may be characterized by

$$c = \inf_{f \in \Gamma} \sup_{t \in [0,1]} I(f(t)),$$

where $\Gamma = \{ f \in C([0, 1], X) : f(0) = 0, f(1) = e \}$.

Proof. — Since $f([0, 1]) \cap \partial B_\rho \neq \phi$ $\forall f \in \Gamma$, $c \geqq \alpha$. Suppose that c is not a critical value of I. Then $N = \phi$ is a neighbourhood of K_c. We may therefore use Proposition 2.3, with $N = \phi$ and $\bar{\varepsilon} = c$, to obtain a number $\varepsilon \in (0, \bar{\varepsilon})$. By the definition of c, $I_{c-\varepsilon/4}$ is not path connected and 0 and e lie in different path components, W_0 and W_e.

We shall need an auxiliary family of mappings from $[0, 1]$ to X (Γ is not suitable for our purposes because $\alpha_s \circ f$ may not be in Γ when f is). Let

$$\Gamma_1 = \{ f \in C([0, 1], X) : f(0) \in W_0 \cap I_{c-\varepsilon/2}, f(1) \in W_e \cap I_{c-\varepsilon/2} \}$$

and

$$c_1 = \inf_{f \in \Gamma_1} \sup_{t \in [0,1]} I(f(t)).$$

Since $\Gamma \subset \Gamma_1$, $c_1 \leqq c$. If $c_1 < c$, there exists an $f \in \Gamma_1$ such that $\sup_{t \in [0,1]} I(f(t)) < c$. Since $f(0)$ can be joined to 0 and $f(1)$ to e by paths lying in $I_{c-\varepsilon/4}$, there is a $g \in \Gamma$ such that $\sup_t I(g(t)) < c$, contradicting the definition of c. Hence $c = c_1$.

We claim that Γ_1 is a closed subset of $C([0, 1], X)$ (and in particular, Γ_1 is a complete metric space). To prove this, let (f_n) be a sequence in Γ_1 such that $f_n \to f$. Denote $f(0) = u$, $f_n(0) = u_n$. By l. s. c. of I, $I(u) \leqq \liminf I(u_n) \leqq c - \varepsilon/2$. Since ψ is convex and Φ continuous, there exist positive numbers $\delta_n \to 0$ such that $\forall t \in [0, 1]$,

$$\psi(tu_n + (1-t)u) \leqq t\psi(u_n) + (1-t)\psi(u)$$

and

$$\Phi(tu_n + (1-t)u) \leqq \Phi(u) + \delta_n \leqq t\Phi(u_n) + (1-t)\Phi(u) + 2\delta_n.$$

Therefore,

$$(20) \qquad I(tu_n+(1-t)u) \leqq tI(u_n)+(1-t)I(u)+2\delta_n \leqq c-\varepsilon/2+2\delta_n \leqq c-\varepsilon/4$$

for all large n. So for such n the segment joining u_n to u lies in W_0. In particular, $u \in W_0$. Since also $I(u) \leqq c - \varepsilon/2$, $u \in W_0 \cap I_{c-\varepsilon/2}$. Likewise, $f(1) \in W_e \cap I_{c-\varepsilon/2}$. Hence $f \in \Gamma_1$.

Since Γ_1 is a complete metric space and $\Pi(f) = \sup_t I(f(t))$ is l. s. c. (according to Lemma 3.1), we may use Proposition 1.6 with $Z = \Gamma_1$, $\delta = \varepsilon$ and $\lambda = 1$ in order to obtain an $f \in \Gamma_1$ such that $\Pi(f) \leqq c + \varepsilon$ and

$$(21) \qquad \Pi(g) - \Pi(f) \geqq -\varepsilon d(f,g) \qquad \forall g \in \Gamma_1 .$$

Let $A = f([0,1])$ and let α_s be the deformation given by Proposition 2.3 (note that $\Pi(f) \geqq c_1 = c$). Set $g = \alpha_s \circ f$. For sufficiently small s, $\alpha_s \circ f \in \Gamma_1$. Indeed, if $I(f(0)) \in (c-\varepsilon, c-\varepsilon/2]$, then $I(\alpha_s \circ f(0)) \leqq I(f(0)) \leqq c - \varepsilon/2$ by (10) and if $I(f(0)) \leqq c - \varepsilon$, $I(\alpha_s \circ f(0)) \leqq I(f(0)) + 2s \leqq c - \varepsilon/2$ according to (9). Hence $\alpha_s \circ f(0) \in W_0 \cap I_{c-\varepsilon/2}$. Likewise, $\alpha_s \circ f(1) \in W_e \cap I_{c-\varepsilon/2}$. So $\alpha_s \circ f \in \Gamma_1$. Since $d(f,g) \leqq s$ according to (8), it follows from (11) and (21) that

$$-2\varepsilon s \geqq \Pi(g) - \Pi(f) \geqq -\varepsilon d(f,g) \geqq -\varepsilon s .$$

This contradiction shows that $K_c \neq \phi$. $\square$

3.3. COROLLARY. — Suppose that $I : X \to (-\infty, +\infty]$ satisfies (H) and (PS). If 0 is a local minimum of I and if $I(e) \leqq I(0)$ for some $e \neq 0$, then I has a critical point different from 0 and e. In particular, if I has two local minima, then it has at least three critical points.

Proof. — We may assume without loss of generality that $I(0) = 0$. If one can find $\alpha, \rho > 0$ such that $\rho < \| e \|$ and $I|_{\partial B_\rho} \geqq \alpha$, the existence of a critical point different from 0 and e follows from Theorem 3.2. Suppose that such α, ρ do not exist. Let $r < \| e \|$ be a positive number such that $I|_{\overline{B}_r} \geqq 0$ and let $0 < \rho < r$. We shall use Proposition 1.6 with $Z = \overline{B}_r$, $\Pi = I|_{\overline{B}_r}$, $\delta = 1/n^2$ and $\lambda = n$. Since $\inf_{u \in \partial B_\rho} I(u) = 0$, there exist $w_n \in \partial B_\rho$, $u_n \in \overline{B}_r$ such that

$$0 \leqq I(u_n) \leqq I(w_n) \leqq 1/n^2, \qquad \| u_n - w_n \| \leqq 1/n$$

and

$$(22) \qquad I(z) - I(u_n) \geqq (-1/n) \| z - u_n \| \qquad \forall z \in \overline{B}_r .$$

Choosing n large enough we may assume that $u_n \in B_r$. Let $v \in X$ and $z = (1-t)u_n + tv$. If t is small positive, $z \in \overline{B}_r$. By (22) and the fact that ψ is convex,

$$\begin{aligned}(-1/n)t \| v - u_n \| &\leqq I((1-t)u_n + tv) - I(u_n) \\ &\leqq \Phi(u_n + t(v-u_n)) - \Phi(u_n) + t(\psi(v) - \psi(u_n)) .\end{aligned}$$

Dividing by t and letting $t \to 0$ we obtain

$$\langle \Phi'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geqq (-1/n) \| v - u_n \| .$$

Since v is arbitrary, it follows from (PS) that after passing to a subsequence, $u_n \to u \in \partial B_\rho$. So u is a critical point and $0 \neq u \neq e$. A similar argument using Proposition 1.6 can be found in [*14*, Proposition 5 and 19, Theorem 4].

If I has two local minima, u_0 and u_1, we may assume without loss of generality that $u_0 = 0$ and $I(u_1) \leqq I(u_0) = 0$. By the first part of the corollary, there exists a critical point u different from u_0 and u_1. Since also u_0 and u_1 are critical (by Proposition 1.1), the proof is complete. □

3.4. THEOREM (Generalized Mountain Pass Theorem). — Suppose that $I : X \to (-\infty, +\infty]$ satisfies (H) and (PS). Let $X = X_1 \oplus X_2$, where $\dim X_1 < \infty$, and suppose that

i) there are constants $\alpha, \rho > 0$ such that $I|_{\partial B_\rho \cap X_2} \geqq \alpha$,

ii) there is a constant $R > \rho$ and an $e \in X_2$, $\| e \| = 1$, such that the restriction of I to the boundary ∂Q of $Q = (\bar{B}_R \cap X_1) \oplus \{ re : 0 \leqq r \leqq R \}$ is nonpositive. Then I has a critical value $c \geqq \alpha$ which may be characterized by

$$c = \inf_{f \in \Gamma} \sup_{x \in Q} I(f(x)),$$

where $\Gamma = \{ f \in C(Q, X) : f|_{\partial Q} = \mathrm{id}_{\partial Q} \}$.

Proof. — Assume for the moment that $c \geqq \alpha$. Suppose c is not a critical value and apply Proposition 2.3 with $N = \phi$ and $\bar{\varepsilon} = c$ to obtain an $\varepsilon \in (0, \bar{\varepsilon})$. Denote by $\approx$ the homotopy relation and let

$$\Gamma_1 = \{ f \in C(Q, X) : f|_{\partial Q} \approx \mathrm{id}_{\partial Q} \text{ in } I_{c-\varepsilon/4}, \quad I \circ f|_{\partial Q} \leqq c - \varepsilon/2 \}$$

and

$$c_1 = \inf_{f \in \Gamma_1} \sup_{x \in Q} I(f(x)).$$

Since $\Gamma \subset \Gamma_1$, $c_1 \leqq c$. If $c_1 < c$, we find an $f \in \Gamma_1$ such that $\sup_{x \in Q} I(f(x)) < c$. Since $f|_{\partial Q} \approx \mathrm{id}_{\partial Q}$ in $I_{c-\varepsilon/4}$ and Q is homeomorphic to a closed finite dimensional Euclidean ball, $\mathrm{id}_{\partial Q}$ can be extended to a mapping $g : Q \to X$ such that $\sup_{x \in Q} I(g(x)) < c$ (the existence of such an extension follows from general results in homotopy theory [*16*, Proposition I.9.2]; one can also construct g explicitly—see the proof of Lemma 4.5). This contradicts the definition of c because $g \in \Gamma$. So $c_1 = c$.

We claim that Γ_1 is a closed subset of C(Q, X). Let (f_n) be a sequence in Γ_1 such that $f_n \to f$. Since I is l. s. c., $I \circ f|_{\partial Q} \leqq c - \varepsilon/2$. Using the argument of the proof of Theorem 3.2 (cf. (20)) and the fact that $f(\partial Q)$ is compact, one shows that for n large,

$$I(tf_n(x) + (1-t)f(x)) \leqq c - \varepsilon/4 \qquad \forall x \in \partial Q, \quad t \in [0, 1].$$

Hence $f|_{\partial Q} \approx f_n|_{\partial Q} \approx \mathrm{id}_{\partial Q}$ in $I_{c-\varepsilon/4}$. So $f \in \Gamma_1$.

The remaining part of the proof follows the last paragraph of the proof of Theorem 3.2 (with some obvious changes: we set $A = f(Q)$ instead of $f([0, 1])$ and observe that $\alpha_s \circ f \in \Gamma_1$ because $\alpha_s \circ f \approx f$ in $I_{c-\varepsilon/2}$).

We still have to show that $c \geqq \alpha$. The argument can be found in [*24*, proof of Theorem 4.1] but for the sake of completeness we include it here. Since $I|_{\partial B_\rho \cap X_2} \geqq \alpha$, it suffices to prove that for each $f \in \Gamma$, $f(Q) \cap \partial B_\rho \cap X_2 \neq \phi$. In other words, we must find an $x \in Q$ such that $f(x) \in \partial B_\rho \cap X_2$. Denote by P_1 and P_2 the projections from X to X_1 and X_2 associated with the decomposition $X = X_1 \oplus X_2$. For $x \in Y \equiv X_1 \oplus \mathrm{span}\{e\}$ write $x = x_1 + re$ with $x_1 \in X_1$ and $r \in \mathbb{R}$. Define a mapping $h : Y \to Y$ by the formula

$$h(x_1 + re) = P_1 f(x_1 + re) + \| P_2 f(x_1 + re) \| e.$$

Since $h|_{\partial Q} = \mathrm{id}_{\partial Q}$, it follows from the properties of Brouwer's degree [*25*] that

$$\deg(h, \mathring{Q}, \rho e) = \deg(\mathrm{id}, \mathring{Q}, \rho e) = 1$$

($\mathring{Q}$ denotes the interior of Q in Y). Consequently, there exists an $x \in \mathring{Q}$ such that $P_1 f(x) = 0$ and $\| P_2 f(x) \| = \rho$. So $f(x) \in \partial B_\rho \cap X_2$ as required. □

3.5. **Theorem** (Saddle Point Theorem). — Suppose that

$$I : X \to (-\infty, +\infty]$$

satisfies (H) and (PS). Let $X = X_1 \oplus X_2$, $\dim X_1 < \infty$, and suppose that

i) there exist constants $\rho > 0$ and α_1 such that $I|_{\partial B_\rho \cap X_1} \leqq \alpha_1$,

ii) there is a constant $\alpha_2 > \alpha_1$ such that $I|_{X_2} \geqq \alpha_2$.

Then I has a critical point $c \geqq \alpha_2$ which may be characterized by

$$c = \inf_{f \in \Gamma} \sup_{x \in D} I(f(x)),$$

where $D = \overline{B}_\rho \cap X_1$ and $\Gamma = \{ f \in C(D, X) : f|_{\partial D} = \mathrm{id}_{\partial D} \}$.

Proof (outline). — It follows from a degree-theoretic argument [*22*, proof of Theorem 1.2; *24*, proof of Theorem 3.1] that $f(D) \cap X_2 \neq \phi$. So $c \geqq \alpha_2$. By repeating the reasoning used in the proof of Theorem 3.4 (with D and ∂D replacing Q and ∂Q) it can be shown that $K_c \neq \phi$. □

4. CRITICAL POINTS OF EVEN FUNCTIONS

Let X be a real Banach space and Σ the collection of all symmetric subsets of $X - \{0\}$ which are closed in X. A nonempty set $A \in \Sigma$ is said to have *genus* k (denoted $\gamma(A) = k$) if k is the smallest integer with the property that there exists an odd continuous mapping $h : A \to \mathbb{R}^k - \{0\}$. If such an integer does not exist, $\gamma(A) = \infty$. For the empty set ϕ we define $\gamma(\phi) = 0$. Properties of genus are summarized below. Denote by $d(u, A)$ the distance from u to the set A and let

$$N_\delta(A) = \{ u \in X : d(u, A) \leqq \delta \} .$$

4.1. Proposition. — Let $A, B \in \Sigma$.

i) If there exists an odd continuous mapping $f : A \to B$, then $\gamma(A) \leqq \gamma(B)$.

ii) If $A \subset B$, then $\gamma(A) \leqq \gamma(B)$.

iii) $\gamma(A \cup B) \leqq \gamma(A) + \gamma(B)$.

iv) If $\gamma(B) < \infty$, $\gamma(\overline{A - B}) \geqq \gamma(A) - \gamma(B)$.

v) If A is homeomorphic to S^{k-1} by an odd homeomorphism, $\gamma(A) = k$.

vi) If A is compact, then $\gamma(A) < \infty$ and $\gamma(N_\delta(A)) = \gamma(A)$ for all sufficiently small $\delta > 0$.

vii) If $U \subset \mathbb{R}^k$ is an open, bounded and symmetric neighbourhood of 0, then $\gamma(\partial U) = k$.

Proofs and a more detailed discussion of the notion of genus can be found e. g. in [*21*] [*24*].

Let $\mathscr{S}$ be the collection of all nonempty closed and bounded subsets of X. In $\mathscr{S}$ we introduce the Hausdorff metric *dist* [*18*, § 15, VII], given by

$$\operatorname{dist}(A, B) = \max \{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \} .$$

The space $(\mathscr{S}, \operatorname{dist})$ is complete [*18*, § 29, IV]. Denote by Γ the subcollection of $\mathscr{S}$ consisting of all nonempty compact symmetric subsets of X and let

$$\Gamma_j = \operatorname{cl} \{ A \in \Gamma : 0 \notin A, \gamma(A) \geqq j \}$$

(cl is the closure in Γ). It is easy to verify that Γ is closed in $\mathscr{S}$, so $(\Gamma, \operatorname{dist})$ and $(\Gamma_j, \operatorname{dist})$ are complete metric spaces.

4.2. Lemma. — If $A \in \Gamma_j$ and $0 \notin A$, $\gamma(A) \geqq j$.

Proof. — Let (A_n) be a sequence in Γ_j such that $A_n \to A$, $0 \notin A_n$ and $\gamma(A_n) \geqq j$. By (*vi*) of Proposition 4.1, there is a $\delta > 0$ such that $\gamma(A) = \gamma(N_\delta(A))$. Since $A_n \to A$, $A_n \subset N_\delta(A)$ for almost all n. So $j \leqq \gamma(A_n) \leqq \gamma(N_\delta(A)) = \gamma(A)$. □

Observe that if $I = \Phi + \psi$ satisfies (H) and Φ, ψ are even, then $\Phi'(0) = 0$ and $\psi(0)$ is a (global) minimum of ψ. So $u = 0$ is necessarily a critical point of I.

4.3. THEOREM. — Suppose that $I : X \to (-\infty, +\infty]$ satisfies (H) and (PS), $I(0) = 0$ and Φ, ψ are even. Define

$$c_j = \inf_{A \in \Gamma_j} \sup_{u \in A} I(u).$$

If $-\infty < c_j < 0$ for $j = 1, \ldots, k$, then I has at least k distinct pairs of nontrivial critical points.

Proof. — Given j, $1 \leqq j \leqq k$, suppose that $c_j = \ldots = c_{j+p} \equiv c$ for some $p \geqq 0$. Note that $0 \notin K_c$ because $c < 0$. We shall show that $\gamma(K_c) \geqq p + 1$. Arguing indirectly, assume $\gamma(K_c) \leqq p$. Let $\rho > 0$ be such that $\gamma(N_{2\rho}(K_c)) = \gamma(K_c)$. Define

$$\Pi(A) = \sup_{u \in A} I(u).$$

Then Π is a function on Γ_j and Π is l. s. c. by an argument similar to that of Lemma 3.1 (note that $\forall u \in A$ there is a sequence $u_n \to u$ with $u_n \in A_n$). Let $N = N_\rho(K_c)$ and $\bar{\varepsilon} = \min\{1, \rho, -c\}$. Apply Proposition 2.3 to obtain an $\varepsilon < \bar{\varepsilon}$. Choose $A_1 \in \Gamma_{j+p}$ such that $\Pi(A_1) \leqq c + \varepsilon^2$. Since $c+\varepsilon^2 \leqq c+\varepsilon < 0$, $0 \notin A_1$ and it follows from Lemma 4.2 that $\gamma(A_1) \geqq j+p$. Let $A_2 = \overline{A_1 - N_{2\rho}(K_c)}$. Then $\Pi(A_2) \leqq c + \varepsilon^2$ and, according to (*iv*) of Proposition 4.1, $\gamma(A_2) \geqq \gamma(A_1) - \gamma(N_{2\rho}(K_c)) \geqq j + p - p = j$. By Proposition 1.6 with $\delta = \varepsilon^2$ and $\lambda = 1/\varepsilon$, there is an $A \in \Gamma_j$ with

$$\Pi(A) \leqq c + \varepsilon^2, \qquad \operatorname{dist}(A, A_2) \leqq \varepsilon$$

and

$$\Pi(B) - \Pi(A) \geqq -\varepsilon \operatorname{dist}(A, B) \qquad \forall B \in \Gamma_j. \tag{23}$$

Since $\varepsilon < \rho$ and $A \in \Gamma_j$, $A \cap N_\rho(K_c) = \phi$ and $\Pi(A) \geqq c$. A satisfies therefore the hypotheses of Proposition 2.3 and Corollary 2.4 and we obtain an odd deformation $\alpha_s : A \to X$. Since $c + \varepsilon^2 < 0$, $0 \notin A$ and $\gamma(A) \geqq j$. Let $B = \alpha_s(A)$ with s small. Then $\gamma(B) \geqq \gamma(A) \geqq j$ according to (*i*) of Proposition 4.1. So $B \in \Gamma_j$. By (8), $\operatorname{dist}(A, B) \leqq s$. It follows therefore from (23) and (11) that

$$-2\varepsilon s \geqq \Pi(B) - \Pi(A) \geqq -\varepsilon \operatorname{dist}(A, B) \geqq -\varepsilon s.$$

This is the desired contradiction.

We have shown that $\gamma(K_c) \geqq p + 1$. In particular, $\gamma(K_{c_j}) \geqq 1$, so each K_{c_j} has at least two points, u_j and $-u_j$. This gives the required number of critical points if all c_j are distinct. If they are not, $p > 0$ for some j. Hence $\gamma(K_{c_j}) \geqq 2$ and K_{c_j} is an infinite set. □

4.4. THEOREM. — Suppose that $I : X \to (-\infty, +\infty]$ satisfies (H) and (PS), $I(0) = 0$ and Φ, ψ are even. Assume also that

i) there exists a subspace X_1 of X of finite codimension and numbers α, $\rho > 0$ such that $I|_{\partial B_\rho \cap X_1} \geqq \alpha$,

ii) there is a finite dimensional subspace X_2 of X, $\dim X_2 > \operatorname{codim} X_1$, such that $I(u) \to -\infty$ as $\|u\| \to \infty$, $u \in X_2$.

Then I has at least $\dim X_2 - \operatorname{codim} X_1$ distinct pairs of nontrivial critical points.

Proof. — Assume that I has no critical points in I_{-d} for some $d > 0$ (otherwise there are infinitely many critical points and there is nothing to prove). Set $m = \operatorname{codim} X_1$, $k = \dim X_2$, $Q = \overline{B}_R \cap X_2$, where $R > \rho$ is chosen so that $I|_{\partial Q} \leqq -d$. Define for $1 \leqq j \leqq k$

$$\mathcal{F} = \{ f \in C(Q, X) : f \text{ is odd and } f|_{\partial Q} \approx \mathrm{id}_{\partial Q} \text{ in } I_{-d} \text{ by an odd homotopy} \},$$

$$\Lambda'_j = \{ f(Q - V) : f \in \mathcal{F},\ V \text{ is open in } Q \text{ and symmetric, } V \cap \partial Q = \phi \text{ and for each } Y \subset V \text{ such that } Y \in \Sigma,\ \gamma(Y) \leqq k - j \},$$

$$\Lambda_j = \{ A \subset X : A \text{ is compact, symmetric and for each open set } U \supset A \text{ there is an } A_0 \in \Lambda'_j \text{ such that } A_0 \subset U \}$$

and

$$c_j = \inf_{A \in \Lambda_j} \sup_{u \in A} I(u).$$

Since $\mathrm{id}_Q \in \mathcal{F}$, $Q \in \Lambda_j$ and $\Lambda_j \neq \phi$ for $j = 1, \ldots, k$. It is easy to see that Λ_j is a closed subset of $\mathcal{S}$ (and therefore a complete metric space). Indeed, suppose $A_n \in \Lambda_j$ and $A_n \to A$. Let U be an arbitrary open set containing A. Then $U \supset A_n$ for almost all n and, since $A_n \in \Lambda_j$, there exists an $A_0 \in \Lambda'_j$ such that $A_0 \subset U$. Hence $A \in \Lambda_j$.

In order to continue the proof we need two results which we state separately.

4.5. LEMMA. — For $m + 1 \leqq j \leqq k$, $c_j \geqq \alpha$.

Proof. — Suppose $c_j < \alpha$. Then $A \cap X_1 \cap \partial B_\rho = \phi$ for some $A \in \Lambda_j$. Since $X - (X_1 \cap \partial B_\rho)$ is an open set containing A, we can first find an $A_0 = f(Q - V) \in \Lambda'_j$ which does not intersect $X_1 \cap \partial B_\rho$ and then a $Y \subset V$ such that $Y \in \Sigma$, $\gamma(Y) \leqq k - j$ and $f(\overline{Q - Y}) \cap X_1 \cap \partial B_\rho = \phi$. Let $F(y, t)$, $y \in \partial Q$, $0 \leqq t \leqq 1$, be an odd homotopy joining $f|_{\partial Q}$ to $\mathrm{id}_{\partial Q}$ in I_{-d}. Define $Y_1 = \frac{1}{2} Y$ and

$$f_1(y, s) = \begin{cases} f(y, 2s) & \text{for } 0 \leqq s \leqq \frac{1}{2} R \\ F(y, 2s/R - 1) & \text{for } \frac{1}{2} R \leqq s \leqq R, \end{cases}$$

where $(y, s) \in \partial Q \times [0, R]$ are polar coordinates of $x \in Q$. Since $f_1(y, s) \in I_{-d}$ for $s \geqq \frac{1}{2} R$ and $I|_{\partial B_\rho \cap X_1} \geqq \alpha > 0$, $f_1(\overline{Q - Y_1}) \cap X_1 \cap \partial B_\rho = \phi$. Now we use a standard argument (see e. g. [*24*, Proposition 6.11]). Let W be the component of $f_1^{-1}(B_\rho)$ containing the origin. For $x \in \partial Q$, $f_1(x) = x \notin B_\rho$. So $W \cap \partial Q = \phi$ and consequently, W is open and bounded in X_2. According to (*vii*) of Proposition 4.1, $\gamma(\partial W) = k$. Set $C = f_1^{-1}(\partial B_\rho)$. Then $C \supset \partial W$ and $\gamma(C) = k$. Furthermore, by (*iv*) of Proposition 4.1,

$$\gamma(\overline{C - Y_1}) \geqq \gamma(C) - \gamma(Y_1) \geqq k - (k - j) = j.$$

Let $X = X_1 \oplus \tilde{X}_1$ and denote by P the projection from X to $\tilde{X}_1$ along X_1. Since $f_1(\overline{Q - Y_1}) \cap X_1 \cap \partial B_\rho = \phi$ and $f_1(\overline{C - Y_1}) \subset \partial B_\rho$,

$$Z \equiv P f_1(\overline{C - Y_1}) \subset \tilde{X}_1 - \{0\}.$$

Hence $Z \in \Sigma$ and by (*i*) of Proposition 4.1, $\gamma(Z) \geqq \gamma(\overline{C - Y_1}) \geqq j$. On the other hand, $\dim \tilde{X}_1 = m < j$, so $\gamma(Z) < j$. □

Denote by $\mathring{N}_\delta(Z)$ the interior of $N_\delta(Z)$ in X.

4.6. LEMMA. — The sets Λ_j have the following properties:

i) $\Lambda_{j+1} \subset \Lambda_j$.

ii) If $A \in \Lambda_j$, W is a closed and symmetric set containing A in its interior and $\alpha : W \to X$ an odd mapping such that $\alpha|_{W \cap I_{-d}} \approx \mathrm{id}_{W \cap I_{-d}}$ in I_{-d} by an odd homotopy, then $\alpha(A) \in \Lambda_j$.

iii) If $Z \in \Sigma$ is compact, $\gamma(Z) \leqq p$ and $I|_Z > -d$, then there exists a $\delta > 0$ such that for each $A \in \Lambda_{j+p}$, $A - \mathring{N}_\delta(Z) \in \Lambda_j$.

Proof. — *i*) Let $A \in \Lambda_{j+1}$ and choose an open set $U \supset A$. There exists an $A_0 = f(Q - V) \in \Lambda'_{j+1}$ such that $A_0 \subset U$. For each $Y \subset V$ such that $Y \in \Sigma$, $\gamma(Y) \leqq k - (j+1) < k - j$. So $A_0 \in \Lambda'_j$ and $A \in \Lambda_j$.

ii) Let $U \supset \alpha(A)$ be open. Let W_1 be an open set such that $A \subset W_1 \subset W$ and $\alpha(W_1) \subset U$. Since $A \in \Lambda_j$, there is an $A_0 = f(Q - V) \in \Lambda'_j$ such that $A_0 \subset W_1$. Extend α to an odd mapping $\tilde{\alpha} : f(Q) \cup W \to X$. Since $f(\partial Q) \subset W \cap I_{-d}$, $\tilde{\alpha} \circ f|_{\partial Q} \approx f|_{\partial Q} \approx \mathrm{id}_{\partial Q}$ in I_{-d}. Consequently, $\tilde{\alpha} \circ f \in \mathcal{F}$. Furthermore, $\tilde{\alpha} \circ f(Q - V) = \alpha \circ f(Q - V) = \alpha(A_0) \subset \alpha(W_1) \subset U$, so $\alpha(A_0) \in \Lambda'_j$ and $\alpha(A) \in \Lambda_j$.

iii) By *vi*) of Proposition 4.1, we may choose a $\delta > 0$ such that $\gamma(N_\delta(Z)) = \gamma(Z)$. Denote $Z_0 = N_\delta(Z)$ and $\mathring{Z}_0 = \mathring{N}_\delta(Z)$. Let $U \supset A - \mathring{Z}_0$ be open and set $U_0 = U \cup \mathring{Z}_0$. Then $A \subset U_0$. Since $A \in \Lambda_{j+p}$, there is an $A_0 = f(Q - V) \in \Lambda'_{j+p}$, $A_0 \subset U_0$. Note that

$$\begin{aligned}(24)\quad A_0 - \mathring{Z}_0 = f(Q - V) - \mathring{Z}_0 &= f((Q - V) \cap (Q - f^{-1}(\mathring{Z}_0))) \\ &= f(Q - (V \cup f^{-1}(\mathring{Z}_0))).\end{aligned}$$

Since $I|_Z > -d$ and $X - I_{-d}$ is an open set, $I|_{Z_0} > -d$ provided δ is sufficiently small. It follows that $f^{-1}(\overset{\circ}{Z}_0) \cap \partial Q = \phi$ because $I|_{f(\partial Q)} \leqq -d$. Consequently, the set $V \cup f^{-1}(\overset{\circ}{Z}_0)$ is open in Q, symmetric and does not intersect ∂Q. Furthermore, if $Y \subset V \cup f^{-1}(\overset{\circ}{Z}_0)$ and $Y \in \Sigma$, there exist Y_1, $Y_2 \in \Sigma$ such that $Y = Y_1 \cup Y_2$ and $Y_1 \subset V$, $Y_2 \subset f^{-1}(\overset{\circ}{Z}_0)$ (e. g., $Y_1 = \{ x \in Y : d(x, Q - V) \geqq d(x, Q - f^{-1}(Z_0)) \}$ and Y_2 is obtained by reversing the inequality). Since $A_0 \in \Lambda'_{j+p}$, $\gamma(Y_1) \leqq k - (j + p)$. By *i)-iii)* of Proposition 4.1, $\gamma(Y_2) \leqq \gamma(f^{-1}(Z_0)) \leqq \gamma(Z_0) \leqq p$ and $\gamma(Y) \leqq \gamma(Y_1) + \gamma(Y_2) \leqq k - (j + p) + p = k - j$. Now it follows from (24) that $A_0 - \overset{\circ}{Z}_0 \in \Lambda'_j$. Since $A_0 - \overset{\circ}{Z}_0 \subset U$ and U was chosen arbitrarily, $A - \overset{\circ}{Z}_0 \in \Lambda_j$. □

Proof of Theorem 4.4 continued. — By Lemma 4.5 and *i)* of Lemma 4.6,

$$\alpha \leqq c_{m+1} \leqq \dots \leqq c_k .$$

Suppose that $c_j = \dots = c_{j+p} \equiv c$ for some j, $m + 1 \leqq j \leqq k$, and $p \geqq 0$. Since $c > 0$, $0 \notin K_c$. We complete the proof by demonstrating that $\gamma(K_c) \geqq p + 1$. To obtain a contradiction, assume $\gamma(K_c) \leqq p$. Choose $\rho > 0$ so that $\gamma(N_{2\rho}(K_c)) = \gamma(K_c)$. Let $N = N_\rho(K_c)$ and $\bar{\varepsilon} = \min \{ 1, \rho \}$. Let $\varepsilon < \bar{\varepsilon}$ be the number given by Proposition 2.3. Recall that Λ_j is a complete metric space and $\Pi(A) = \sup_{u \in A} I(u)$. There exists an $A_1 \in \Lambda_{j+p}$ such that $\Pi(A_1) \leqq c + \varepsilon^2$. Let $A_2 = A_1 - \overset{\circ}{N}_{2\rho}(K_c)$. If ρ is sufficiently small, it follows from *iii)* of Lemma 4.6 that $A_2 \in \Lambda_j$. According to Proposition 1.6 (with $\delta = \varepsilon^2$ and $\lambda = 1/\varepsilon$), we can find an $A \in \Lambda_j$ such that

$$\Pi(A) \leqq c + \varepsilon^2, \qquad \operatorname{dist}(A, A_2) \leqq \varepsilon$$

and

$$\Pi(B) - \Pi(A) \geqq -\varepsilon \operatorname{dist}(A, B) \qquad \forall B \in \Lambda_j . \tag{25}$$

Since $\varepsilon < \rho$, $A \cap N_\rho(K_c) = \phi$. Moreover, $c \leqq \Pi(A) \leqq c + \varepsilon^2 \leqq c + \varepsilon$. So according to Proposition 2.3 and Corollary 2.4, there exists an odd deformation $\alpha_s : W \to X$ satisfying (8)-(12). Let $B = \alpha_s(A)$ with s small. It follows from (12) (with $W_0 = I_{-d}$) that α_s satisfies the hypotheses of *ii)* of Lemma 4.6. Hence $B \in \Lambda_j$. Using (8), (11) and (25) we obtain

$$-2\varepsilon s \geqq \Pi(B) - \Pi(A) \geqq -\varepsilon \operatorname{dist}(A, B) \geqq -\varepsilon s ,$$

a contradiction. □

4.7. REMARK. — The minimax characterization of critical values obtained in the proof of Theorem 4.4 is not fully satisfactory because it depends on the a priori assumption that the function I does not have critical values below a certain level.

4.8. COROLLARY. — Suppose that the hypotheses of Theorem 4.4 are satisfied with *ii*) replaced by

ii') for any positive integer k there is a k-dimensional subspace X_2 of X such that $I(u) \to -\infty$ as $\|u\| \to \infty$, $u \in X_2$.

Then I has infinitely many distinct pairs of nontrivial critical points.

Proof. — Obvious. $\square$

5. APPLICATIONS

Throughout this section we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary Γ. $H^m(\Omega) \equiv H^m$ and $H_0^m(\Omega) \equiv H_0^m$ are the usual Sobolev spaces of real-valued functions in Ω. In H^m we shall use the inner product $(u, v) = \int_\Omega (\nabla^m u . \nabla^m v + uv)dx$, and in H_0^m, $(u, v) = \int_\Omega \nabla^m u . \nabla^m v dx$. The corresponding norm will be denoted by $\| \quad \|$. Standard results in Sobolev spaces, in a form suitable for our purposes, can be found in [*21*] [*24*] and in references given there. Let p^* be the critical exponent for the Sobolev embedding $H^m \hookrightarrow L^p$. Recall that

$$p^* = \begin{cases} 2N/(N-2m) & \text{if } \ 2m < N \\ +\infty & \text{otherwise} \end{cases}$$

and the embedding is compact if $p < p^*$.

We shall employ some results from the theory of maximal monotone operators [*3*] [*4*]. A mapping A from X to X* is said to be *a multivalued operator* if it maps each $u \in X$ onto an element $Au \in 2^{X^*}$. *The domain* of A is defined by $D(A) = \{ u \in X : Au \neq \phi \}$. A is called *monotone* if

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geqq 0 \qquad \forall u_1, u_2 \in D(A),\ v_1 \in Au_1,\ v_2 \in Au_2$$

and *maximal monotone* if it admits no proper monotone extension, i. e., if there is no monotone operator B such the graph of B properly contains the graph of A. The subdifferential $\partial\psi$ of a proper convex l. s. c. function $\psi : X \to (-\infty, +\infty]$ is maximal monotone [*3*, Theorem II.2.1].

Below we shall show in a number of examples how the results of Sections 3 and 4 can be applied to boundary value problems for semilinear elliptic operators. For simplicity, we choose the linear part of the operator to be $-\Delta +$ const. or $(-\Delta)^m +$ const. It will, however, be clear from the proofs that other uniformly elliptic operators which induce symmetric bilinear forms and have reasonably smooth coefficients could be chosen as well. We also prefer to work with nonlinearities not explicitly depending on $x \in \Omega$, although more general results could easily be obtained. In Theorems 5.1, 5.8 and Corollaries 5.9, 5.10 we could also replace $|u|^{p-2}u$

by a more general odd function satisfying suitable superlinearity conditions at 0 and ∞ [*1*, $(p_3) - (p_5)$ on p. 362-363, see also *21*, *24*].

Our first example is concerned with a variational inequality on a convex set $\mathbb{K} \subset \mathrm{H}_0^1$.

5.1. THEOREM. — Let $\lambda \in \mathbb{R}$, $g \in \mathrm{L}^2$, $g < 0$ a. e. in Ω, $2 < p < p^*$ and

$$\mathbb{K} = \{ u \in \mathrm{H}_0^1 : u \geqq 0 \text{ a. e. in } \Omega \} .$$

Then the variational inequality

$$u \in \mathbb{K} : \int_\Omega \nabla u . \nabla(v - u)dx - \lambda \int_\Omega u(v - u)dx - \int_\Omega u^{p-1}(v - u)dx \geq \int_\Omega g(v - u)dx \qquad \forall v \in \mathbb{K}$$

has a nontrivial solution (in addition to the trivial one $u = 0$).

Proof. — Let $\mathrm{I} = \Phi + \psi$ with

$$\Phi(u) = \frac{1}{2} \| u \|^2 - \frac{1}{2} \lambda \int_\Omega u^2 dx - p^{-1} \int_\Omega u^p dx - \int_\Omega gudx$$

and ψ the indicator function of $\mathbb{K}$. It is easy to verify that $\Phi \in \mathrm{C}^1$ with Φ' given by

$$(\Phi'(u), v) = (u, v) - \lambda \int_\Omega uvdx - \int_\Omega u^{p-1} vdx - \int_\Omega gvdx .$$

So I satisfies (H) and u is a solution of the variational inequality if and only if u is a critical point of I, i. e., if $u \in \mathbb{K}$ and

$$(\Phi'(u), v - u) \geqq 0 \qquad \forall v \in \mathbb{K} .$$

We shall show that I satisfies (PS). Choose a constant $d \in \left(p^{-1}, \frac{1}{2} \right)$. Let (u_n) be a sequence in $\mathbb{K}$ such that $\Phi(u_n) \to c \in \mathbb{R}$, $\varepsilon_n \to 0$ and

$$(\Phi'(u_n), v - u_n) \geqq - \varepsilon_n \| v - u_n \| \qquad \forall v \in \mathbb{K} . \tag{26}$$

Set $v = 2u_n$. Then $(\Phi'(u_n), u_n) \geqq - \varepsilon_n \| u_n \|$. Since $\Phi(u_n) \leqq c + 1$ for almost all n,

$$\begin{aligned} c + 1 + \| u_n \| &\geqq \Phi(u_n) - d(\Phi'(u_n), u_n) \\ &= \left(\frac{1}{2} - d \right) \| u_n \|^2 + \int_\Omega \left\{ (d - p^{-1}) u_n^p + \lambda \left(d - \frac{1}{2} \right) u_n^2 + (d - 1) g u_n \right\} dx . \end{aligned}$$

Since $d > p^{-1}$, $p > 2$ and $(d-1)gu_n \geqq 0$ a. e. in Ω, there exists an $\mathrm{R} > 0$ such that the integrand is positive for $u_n > \mathrm{R}$. Hence

$$c + 1 + \| u_n \| \geqq \left(\frac{1}{2} - d\right) \| u_n \|^2 - \mathrm{C},$$

where C is a constant. So the sequence (u_n) is bounded. Denote $\Phi_1(u) = \Phi(u) - \frac{1}{2}\| u \|^2$. Using standard results in Sobolev spaces we see that after passing to a subsequence, $u_n \to \bar{u}$ weakly, $\| u_n \| \to a$ and $\Phi_1'(u_n) \to \Phi_1'(\bar{u})$ strongly (because Φ_1' is compact as $p < p^*$). It follows from (26) with $v = \bar{u}$ that

$$(u_n, \bar{u} - u_n) + (\Phi_1'(u_n), \bar{u} - u_n) \geqq -\varepsilon_n \| \bar{u} - u_n \| .$$

Passing to the limit we obtain $\| \bar{u} \|^2 - a^2 \geqq 0$, i. e., $\lim \| u_n \| \leqq \| \bar{u} \|$. Thus $u_n \to \bar{u}$ strongly.

In order to obtain a nontrivial critical point we shall use Theorem 3.2. Let $u \in \mathbb{K} - \{0\}$. It is easy to see that $\mathrm{I}(tu) \to -\infty$ as $t \to \infty$. So hypothesis *ii)* is satisfied. To verify *i)*, suppose that no $\alpha, \rho > 0$ with $\mathrm{I}|_{\partial \mathrm{B}_\rho} \geqq \alpha$ exist. This implies that we can find a sequence (u_n) in $\mathbb{K}$ such that $u_n \to 0$ and $\Phi(u_n) \leqq \| u_n \|^2/n$. Let $z_n = u_n/\| u_n \|$. Then

$$\Phi(u_n)\| u_n \|^{-2} = \frac{1}{2} - \frac{1}{2}\lambda \int_\Omega z_n^2 dx - p^{-1}\| u_n \|^{p-2} \int_\Omega z_n^p dx - \| u_n \|^{-1} \int_\Omega g z_n dx \leqq 1/n.$$

Assume after passing to a subsequence that $z_n \to \bar{z}$ weakly in H_0^1 and strongly in L^2. If $\bar{z} = 0$, $\liminf \Phi(u_n) \| u_n \|^{-2} \geqq \frac{1}{2}$. So $\bar{z} \neq 0$. But then $\int_\Omega g\bar{z}dx < 0$ and consequently, $\Phi(u_n) \| u_n \|^{-2} \to +\infty$. This contradiction shows that *i)* is satisfied for some $\alpha, \rho > 0$. $\square$

Consider now a functional $\mathrm{I} = \Phi + \psi : \mathrm{H}_0^1 \to (-\infty, +\infty]$ with

$$\Phi(u) = -\frac{1}{2}\lambda \int_\Omega u^2 dx,$$

$$\psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla u| \, dx \equiv \frac{1}{2}\| u \|^2 + \int_\Omega |\nabla u| \, dx .$$

Observe that $\mathrm{D}(\psi) = \mathrm{H}_0^1$. Denote by λ_j the j-th eigenvalue of $-\Delta$ in H_0^1 (counted according to its multiplicity) and by e_j a corresponding eigenfunction chosen so that $(e_i, e_j) = \delta_{ij}$ (Kronecker's δ).

5.2. THEOREM. — *Suppose that $\lambda_k < \lambda < \lambda_{k+1}$. Then* I *has at least k distinct pairs of nontrivial critical points.*

Proof. — It is easy to see that I satisfies (H). To verify (PS), let $I(u_n) \to c$, $\varepsilon_n \to 0$ and

$$(27) \qquad (\Phi'(u_n), v - u_n) + \psi(v) - \psi(u_n) \geqq - \varepsilon_n \| v - u_n \| \qquad \forall v \in H_0^1 .$$

First we show that the sequence (u_n) is bounded. Suppose $\| u_n \| \to \infty$, let $z_n = u_n / \| u_n \|$ and assume after passing to a subsequence that $z_n \to \bar{z}$ weakly in H_0^1 and strongly in L^2. Since

$$I(u_n) \| u_n \|^{-2} = \frac{1}{2} + \| u_n \|^{-1} \int_\Omega | \nabla z_n | \, dx - \frac{1}{2} \lambda \int_\Omega z_n^2 dx \to 0 ,$$

$\bar{z} \neq 0$. Take $v = u_n + \varphi$ in (27). Then

$$\frac{1}{2} \| \varphi \|^2 + (u_n, \varphi) + \int_\Omega (| \nabla (u_n + \varphi) | - | \nabla u_n |) dx - \lambda \int_\Omega u_n \varphi dx \geqq - \varepsilon_n \| \varphi \| .$$

After dividing by $\| u_n \|$ and taking limits we obtain

$$\int_\Omega \nabla \bar{z} . \nabla \varphi dx - \lambda \int_\Omega \bar{z} \varphi dx \geqq 0 .$$

Since φ was chosen arbitrarily, the left-hand side is equal to zero for all $\varphi \in H_0^1$. This contradicts the fact that $\bar{z} \neq 0$ and λ is not an eigenvalue. So the sequence (u_n) is bounded. We may therefore assume that $u_n \to \bar{u}$ weakly in H_0^1, strongly in L^2 and $\| u_n \| \to a$. Set $v = \bar{u}$ in (27). Then

$$\frac{1}{2} \| \bar{u} \|^2 - \frac{1}{2} \| u_n \|^2 + \int_\Omega | \nabla \bar{u} | dx - \int_\Omega | \nabla u_n | dx - \lambda \int_\Omega u_n (\bar{u} - u_n) dx \geqq - \varepsilon \| \bar{u} - u_n \| .$$

Letting $n \to \infty$ and using the fact that $\liminf \int_\Omega | \nabla u_n | dx \geqq \int_\Omega | \nabla \bar{u} | dx$, we obtain $a = \lim \| u_n \| \leqq \| \bar{u} \|$. Hence $u_n \to \bar{u}$ strongly in H_0^1.

We shall show that *i)* and *ii)* of Theorem 4.4 are satisfied. Suppose that no $\alpha, \rho > 0$ such that $I |_{\partial B_\rho} \geqq \alpha$ exist. We may then find a sequence (u_n) such that $u_n \to 0$ and $I(u_n) \leqq \| u_n \|^2 / n$. Let $z_n = u_n / \| u_n \|$. It follows that

$$I(u_n) \| u_n \|^{-2} = \frac{1}{2} + \| u_n \|^{-1} \int_\Omega | \nabla z_n | \, dx - \frac{1}{2} \lambda \int_\Omega z_n^2 dx \leqq 1/n .$$

After passing to a subsequence, $z_n \to \bar{z}$ weakly in H_0^1, strongly in L^2. Now it is easy to see that $\liminf I(u_n) \| u_n \|^{-2} \geqq \frac{1}{2}$ if $\bar{z} = 0$ and $\lim I(u_n) \| u_n \|^{-2} = + \infty$ if $\bar{z} \neq 0$. So *i)* is satisfied with $X_1 = H_0^1$.

To verify *ii*), let $X_2 = \text{span}\{e_1, \ldots, e_k\}$ and $u = \alpha_1 e_1 + \ldots + \alpha_k e_k$. Since $\int_\Omega e_i^2 dx = \lambda_i^{-1} \int_\Omega |\nabla e_i|^2 dx = \lambda_i^{-1}$,

$$I(u) = \frac{1}{2} \sum_{i=1}^{k} (1 - \lambda/\lambda_i)\alpha_i^2 + \int_\Omega |\nabla(\alpha_1 e_1 + \ldots + \alpha_k e_k)| \, dx$$

$$\leqq \frac{1}{2}(1-\lambda/\lambda_k)(\alpha_1^2 + \ldots + \alpha_k^2) + C(\alpha_1^2 + \ldots + \alpha_k^2)^{\frac{1}{2}} = \frac{1}{2}(1-\lambda/\lambda_k)\|u\|^2 + C\|u\|,$$

where C is a constant. Since $\lambda > \lambda_k$, $I(u) \to -\infty$ as $\|u\| \to \infty$, $u \in X_2$. Now it follows from Theorem 4.4 that I has at least $\dim X_2 - \operatorname{codim} X_1 = k$ pairs of nontrivial critical points. □

5.3. Corollary. — If $\lambda_k < \lambda < \lambda_{k+1}$, there exist at least k distinct pairs of nontrivial solutions of the inequality

$$(28) \quad u \in H^2 \cap H_0^1 : \int_\Omega (-\Delta u)(v - u)dx + \int_\Omega |\nabla v| \, dx - \int_\Omega |\nabla u| \, dx \geqq \lambda \int_\Omega u(v - u)dx \qquad \forall v \in H_0^1.$$

Proof. — Extend ψ to a functional $\tilde{\psi}: L^2 \to (-\infty, +\infty]$ by setting $\tilde{\psi}(u) = \psi(u)$ for $u \in H_0^1$ and $\tilde{\psi}(u) = +\infty$ otherwise. Since $D(\tilde{\psi}) = D(\psi)$, $u \in H_0^1$ is a critical point of I if and only if

$$\tilde{\psi}(v) - \tilde{\psi}(u) \geqq \lambda \int_\Omega u(v - u)dx \qquad \forall v \in L^2,$$

or equivalently, if $\lambda u \in \partial\tilde{\psi}(u)$. On the other hand, according to [5, Theorem 15], $\lambda u \in \partial\tilde{\psi}(u)$ if and only if u satisfies (28). So the result follows from Theorem 5.2. □

Let F and G be two functions satisfying the following assumptions. $F : \mathbb{R} \to [0, +\infty]$ is even, l. s. c., convex and $F(0) = 0$; $G : \mathbb{R} \to \mathbb{R}$ is even, of class C^1, $G(0) = 0$, $G'(t) = g(t)$ and $|g(t)| \leqq c_1 + c_2 |t|^{p-1}$ $\forall t \in \mathbb{R}$, where $2 \leqq p < p^*$ and c_1, c_2 are positive constants. Let I be a functional on H_0^m such that $I = \Phi + \psi$ and

$$\Phi(u) = -\int_\Omega G(u)dx,$$

$$\psi(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla^m u|^2 dx + \int_\Omega F(u)dx & \text{if} \quad F(u) \in L^1 \\ +\infty & \text{otherwise}. \end{cases}$$

Denote by λ_j the j-th eigenvalue of $(-\Delta)^m$ in H_0^m (counted according to its multiplicity) and by e_j a corresponding eigenfunction satisfying $(e_i, e_j) = \delta_{ij}$.

5.4. THEOREM. — Suppose that $\liminf_{|t|\to\infty} (F(t) - G(t))/t^2 > -\frac{1}{2}\lambda_1$ and $\limsup_{t\to 0} (F(t) - G(t))/t^2 < -\frac{1}{2}\lambda_k$. Then I has at least k distinct pairs of nontrivial critical points.

Proof. — We employ Theorem 4.3. It follows from the growth restriction on g that Φ is of class C^1 and from [3, Proposition II.2.8] that ψ is l. s. c. and convex. So (H) is satisfied. Now we proceed to verify (PS). Choose $R > 0$ and $\lambda < \lambda_1$ such that $(F(t) - G(t))/t^2 \geqq -\frac{1}{2}\lambda$ for $|t| > R$. Then

$$I(u) = \frac{1}{2}\|u\|^2 + \int_{|u|>R} (F(u) - G(u))dx + \int_{|u|\leqq R} (F(u) - G(u))dx$$
$$\geqq \frac{1}{2}\|u\|^2 - \frac{1}{2}\lambda\int_\Omega u^2 dx - C \geqq \frac{1}{2}(1 - \lambda/\lambda_1)\|u\|^2 - C.$$

Since $\lambda < \lambda_1$, $I(u) \to +\infty$ as $\|u\| \to \infty$. It follows that if $I(u_n) \to c \in \mathbb{R}$, (u_n) is a bounded sequence. We may therefore assume that $u_n \to \bar{u}$ weakly, a. e. in Ω, $\Phi'(u_n) \to \Phi'(\bar{u})$ strongly and $\|u_n\| \to a$. Set $v = \bar{u}$ in (2). Then

$$\frac{1}{2}\|\bar{u}\|^2 - \frac{1}{2}\|u_n\|^2 + \int_\Omega F(\bar{u})dx - \int_\Omega F(u_n)dx + (\Phi'(u_n), \bar{u} - u_n) \geqq -\varepsilon_n\|\bar{u} - u_n\|.$$

Passing to the limit and using Fatou's lemma, we see as in the proof of Theorem 5.2 that $u_n \to \bar{u}$ strongly.

We complete the proof by demonstrating that

$$c_j = \inf_{A\in\Gamma_j} \sup_{u\in A} I(u)$$

satisfies $-\infty < c_j < 0$ for $1 \leqq j \leqq k$. Since Φ is weakly continuous, $\psi \geqq 0$ and $I(u) \to +\infty$ as $\|u\| \to \infty$, I is bounded below. Thus $c_j > -\infty$. Let

$$A = \{ u = \alpha_1 e_1 + \ldots + \alpha_j e_j : \|u\|^2 = \alpha_1^2 + \ldots + \alpha_j^2 = \rho^2 \}.$$

Then $A \in \Gamma_j$ because $\gamma(A) = j$ according to $v)$ of Proposition 4.1. Choose $r > 0$ and $\lambda > \lambda_k$ so that $(F(t) - G(t))/t^2 \leqq -\frac{1}{2}\lambda$ as $|t| \leqq r$. Let ρ in the definition of A be so small that $\| u \|_{L^\infty} \leqq r$ whenever $u \in A$. Then

$$I(u) = \frac{1}{2}\| u \|^2 + \int_\Omega (F(u) - G(u))dx \leqq \frac{1}{2}\| u \|^2 - \frac{1}{2}\lambda \int_\Omega u^2 dx$$
$$= \frac{1}{2}\sum_{i=1}^{j}(1 - \lambda/\lambda_i)\alpha_i^2 \leqq \frac{1}{2}(1 - \lambda/\lambda_j)\| u \|^2 < 0 .$$

It follows that $c_j < 0$. □

5.5. COROLLARY. — Let F be as above and let $m = 1$. Denote the subdifferential of F by f and let $\lambda \in \mathbb{R}$. Suppose that $\liminf_{|t|\to\infty} F(t)/t^2 > \frac{1}{2}(\lambda - \lambda_1)$ and $\lim_{t\to 0} F(t)/t^2 = 0$. If $\lambda > \lambda_k$, the boundary value problem

$$\begin{cases} -\Delta u + f(u) \ni \lambda u & \text{in } \Omega \\ u \in H_0^1 \end{cases}$$

has at least k distinct pairs of nontrivial solutions $u \in H^2 \cap H_0^1$.

Proof. — Let $G(t) = \frac{1}{2}\lambda t^2$. Then F and G satisfy the hypotheses of Theorem 5.4, so that I has k pairs of nontrivial critical points. Extend ψ to a functional $\tilde{\psi} : L^2 \to (-\infty, +\infty]$ by setting $\tilde{\psi}(u) = \psi(u)$ if $u \in D(\psi)$, $\tilde{\psi}(u) = +\infty$ otherwise. Then u is a critical point of I if and only if $\lambda u \in \partial\tilde{\psi}(u)$. Since $\lambda u \in \partial\tilde{\psi}(u)$ is equivalent to $u \in H^2 \cap H_0^1$ and $\lambda u \in -\Delta u + f(u)$ a. e. in Ω [3, Proposition II.3.8], the result follows. □

5.6. COROLLARY. — Let $f(t)$ and $g(t)$ be two odd C^1 functions on $\mathbb{R}$ such that $f(0) = g(0) = 0$, f is nondecreasing, $| g(t) | \leqq c_1 + c_2 | t |^{p-1}$ for some $p \in [2, p^*)$ and $\liminf_{|t|\to\infty}(f'(t) - g'(t)) > -\lambda_1$. If $f'(0) - g'(0) < -\lambda_k$, the boundary value problem

$$\begin{cases} (-\Delta)^m u + f(u) = g(u) & \text{in } \Omega \\ u \in H_0^m \end{cases}$$

has at least k distinct pairs of nontrivial solutions u such that $uf(u) \in L^1$.

Proof. — Setting $F(t) = \int_0^t f(s)ds$ and $G(t) = \int_0^t g(s)ds$ it is easy to see that F and G satisfy the hypotheses of Theorem 5.4. So I has k distinct pairs of nontrivial critical points. Let $\tilde{\psi} : L^p \to (-\infty, +\infty]$ be given by $\tilde{\psi}(u) = \psi(u)$ if $u \in D(\psi)$ and $\tilde{\psi}(u) = +\infty$ otherwise. Set $q = p/(p-1)$. Since $g(u) \in L^q$ whenever $u \in L^p$, u is a critical point of I if and only if

$g(u) \in \partial\tilde{\psi}(u)$. Define an operator $A : D(A) \subset L^p \to L^q$ by $Au = (-\Delta)^m u + f(u)$ with

$$D(A) = \{ u \in H_0^m : (-\Delta)^m u + f(u) \in L^q, uf(u) \in L^1 \}.$$

It is easy to see from the definitions that A is monotone (cf. the proof of Corollary IV.3 in [7]) and $A \subset \partial\tilde{\psi}$ (in the sense of graph inclusion). We complete the proof by showing that A is maximal monotone. It will then follow that $A = \partial\tilde{\psi}$, so for each critical point u, $g(u) = Au$ and, since $u \in D(A)$, $uf(u) \in L^1$.

The equation $Au + u = h$ has a solution $u \in D(A)$ for each $h \in L^q$ [7, Proposition IV.2 and Remark IV.1]. Note that $L^p \subset L^q$ because $p \geqq 2$. We shall use an argument similar to that of [4, Proposition 2.2]. If $A \subset B$ with B monotone and if $h \in Bu$, then $Av + v = h + u$ for some $v \in D(A)$ (because $h + u \in L^q$). Consequently, $h + u - v = Av \in Bv$ and by monotonocity of B, $\langle (h + u - v) - h, v - u \rangle \geqq 0$, where $\langle \ , \ \rangle$ denotes the duality pairing between L^q and L^p. Hence $\langle u - v, u - v \rangle \leqq 0$, so $u = v$ and $h = Au$. Since u was chosen arbitrarily, $A = B$ and A is maximal monotone. $\square$

5.7. REMARK. — If $m = 1$, the conclusion of Corollary 5.6 is essentially contained in [21, Theorem 3.4 and 24, Theorem 5.23]. The proof given there uses a truncation argument based on the maximum principle, so it does not extend to the case of $m > 1$.

Let $B : \mathbb{R} \to [0, +\infty]$ be an even, l. s. c. and convex function with subdifferential $\partial B = \beta$ and let $p \in (2, p^*)$. Suppose that $B(0) = 0$ and β and B satisfy the following growth restrictions: there is a constant $c > 0$ such that

$$B(2t) \leqq cB(t) \qquad \forall t \in \mathbb{R}, \tag{29}$$

and for some $q \in (2, p)$,

$$st \leqq qB(t) \qquad \forall s \in \beta(t), \quad t \in \mathbb{R}. \tag{30}$$

Consider the functional $I = \Phi + \psi : H^1 \to (-\infty, +\infty]$ with

$$\Phi(u) = -\frac{1}{2}(\lambda + 1)\int_\Omega u^2 dx - p^{-1}\int_\Omega |u|^p dx,$$

$$\psi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega (|\nabla u|^2 + u^2)dx + \int_\Gamma B(u)d\sigma \equiv \frac{1}{2}\|u\|^2 + \int_\Gamma B(u)d\sigma & \text{if } B(u) \in L^1(\Gamma) \\ +\infty & \text{otherwise.} \end{cases}$$

It follows from (29) that $D(B) = \mathbb{R}$ and if $u \in D(\psi)$, then also $ku \in D(\psi)$ for any $k \in \mathbb{R}$. Note that $B(t) = r^{-1}|t|^r$, $1 \leqq r < p$, satisfies the restrictions (29) and (30).

5.8. THEOREM. — The functional I has infinitely many distinct pairs of nontrivial critical points.

Proof. — It is easy to see that I satisfies (H) (cf. [*3*, p. 63]). Let (u_n) be a sequence such that $I(u_n) \to c$ and $(\Phi'(u_n), v-u_n)+\psi(v)-\psi(u_n) \geqq -\varepsilon_n \|v-u_n\|$ $\forall v \in H^1$. Set $v = u_n + tu_n$, $t > 0$, divide by t and let $t \to 0$. This gives

$$\|u_n\|^2 + \lim_{t\to 0}\int_\Gamma B(u_n+tu_n)-B(u_n))t^{-1}d\sigma - \lambda\int_\Omega u_n^2dx - \int_\Omega |u_n|^p dx \geqq -\varepsilon_n \|u_n\|.$$

By Lebesgue's monotone convergence theorem, we can take the limit under the integral sign. It follows that

$$\|u_n\|^2 + \int_\Gamma w_n u_n d\sigma - \lambda\int_\Omega u_n^2 dx - \int_\Omega |u_n|^p dx \geqq -\varepsilon_n\|u_n\|, \tag{31}$$

where $w_n(x) \in \beta(u_n(x))$ a. e. on Γ. Multiplying (31) by $-q^{-1}$ and adding it to the inequality $I(u_n) \leqq c + 1$ (which holds for almost all n) gives

$$c + 1 + \|u_n\| \geqq \left(\frac{1}{2} - q^{-1}\right)\|u_n\|^2 + \int_\Gamma (B(u_n) - q^{-1}w_n u_n)d\sigma$$
$$+ \int_\Omega \left\{(q^{-1} - p^{-1})|u_n|^p + \lambda\left(q^{-1} - \frac{1}{2}\right)u_n^2\right\}dx.$$

Note that the first integral is nonnegative according to (30). Since $q^{-1} > p^{-1}$ and $p > 2$, the second integrand is positive for large $|u_n|$. We can therefore find a constant C such that

$$c + 1 + \|u_n\| \geqq \left(\frac{1}{2} - q^{-1}\right)\|u_n\|^2 - C.$$

It follows that the sequence (u_n) is bounded. Using Fatou's lemma in the same way we did before we deduce that (u_n) possesses a convergent subsequence. Hence I satisfies (PS).

We complete the proof by showing that the hypotheses *i)* and *ii')* of Corollary 4.8 are satisfied. Suppose $\lambda < \lambda_m$ and let $X_1 = \text{span}\{e_1, \ldots, e_m\}^\perp$. If $u \in X_1$, $\frac{1}{2}\|u\|^2 - \frac{1}{2}\lambda\int_\Omega u^2dx \geqq \alpha_1\|u\|^2$ for some constant $\alpha_1 > 0$. It follows from a well known argument [*1*, proof of Lemma 3.3; *21*, proof of Theorem 3.19] that $\int_\Omega |u|^p dx = o(\|u\|^2)$ as $u \to 0$. Hence we can find $\alpha, \rho > 0$ such that $I|_{\partial B_\rho \cap X_1} \geqq \alpha$. Finally, let k be an arbitrary positive integer, $\varphi_1, \ldots, \varphi_k$ linearly independent functions in $C_0^\infty(\Omega)$ and $X_2 = \text{span}\{\varphi_1, \ldots, \varphi_k\}$. Note that $\int_\Gamma B(u)d\sigma = 0$ $\forall u \in X_2$. Since $\partial B_1 \cap X_2$

is compact, there is a constant $\alpha_2 > 0$ such that $p^{-1}\int_\Omega |v|^p dx \geqq \alpha_2$ $\forall v \in \partial B_1 \cap X_2$. Let $u = tv$, where $t > 0$ and $v \in \partial B_1 \cap X_2$. Then

$$I(u) \leqq \frac{1}{2}\|u\|^2 - p^{-1}\int_\Omega |u|^p dx \leqq \frac{1}{2}t^2 - \alpha_2 t^p .$$

Hence $I(u) \to -\infty$ as $\|u\| \to \infty$, $u \in X_2$. □

Denote by $\partial/\partial n$ the outward normal derivative.

5.9. Corollary. — In addition to the above hypotheses, suppose that $2 < p \leqq (2N - 2)/(N - 2)$ if $N > 2$ and $2 < p < \infty$ otherwise. Then the boundary value problem

$$\begin{cases} -\Delta u - \lambda u - |u|^{p-2}u = 0 & \text{in } \Omega \\ -\partial u/\partial n \in \beta(u) & \text{on } \Gamma \end{cases}$$

has infinitely many distinct pairs of nontrivial solutions $u \in H^2$.

Proof. — Let $\tilde{\psi}$ be the extension of ψ to L^2 defined by $\tilde{\psi}(u) = \psi(u)$ if $u \in D(\psi)$, $\tilde{\psi}(u) = +\infty$ otherwise. By Sobolev's embedding theorem, each element of H^1 is in L^{p^*}. So if $u \in H^1$, $|u|^{p-2}u \in L^{p^*/(p-1)}$. Since $p - 1 \leqq N/(N-2)$, $p^*/(p-1) \geqq 2$. Hence $|u|^{p-2}u \in L^2$. It follows that u is a critical point of I if and only if $\lambda u + |u|^{p-2}u \in \partial\tilde{\psi}(u)$. By [*3*, Proposition II.2.9 or *5*, Theorem 12], $\partial\tilde{\psi}(u) = -\Delta u$ with $D(\partial\tilde{\psi}) = \{ u \in H^2 : -\partial u/\partial n \in \beta(u) \text{ a.e. on } \Gamma \}$. So each critical point of I is a solution of the boundary value problem. Now the result follows from Theorem 5.8. □

Observe that the growth restrictions (29) and (30) were used only in order to verify (PS). So if one removes them, the conclusions of Theorem 5.8 and Corollary 5.9 remain true as long as (PS) is satisfied.

5.10. Corollary. — Suppose that the hypotheses of Theorem 5.8 and Corollary 5.9, with possible exception of (29) and (30), are satisfied. If the domain of B, D(B), is a proper subset of $\mathbb{R}$, the conclusions remain true.

Proof. — As we have already observed, we need only verify (PS). An argument similar to that of [*15*, p. 75] shows that if

$$D = \{ v \in H^1 : -\Delta v + v = 0 \text{ in } \Omega \text{ in the sense of distributions} \},$$

then $H^1 = H_0^1 \oplus D$ and D is orthogonal to H_0^1. (Given $u \in H^1$, let v be the minimizer of $\|w\|^2$ on the closed convex set $\{w \in H^1 : u - w \in H_0^1\}$. Then $(v, \varphi) = 0$ $\forall \varphi \in H_0^1$. Thus $v \in D$, $u - v \in H_0^1$ and $(v, u - v) = 0$.) Since D(B) is properly contained in $\mathbb{R}$ and B is even, there exists a constant α such that $\|u\|_{L^\infty(\Gamma)} \leqq \alpha$ whenever $u \in D(\psi)$. Let $u = u' + u''$ with $u' \in H_0^1$,

$u'' \in \mathrm{D}$. It follows from the maximum principle [*17*, Theorem II.5.5] that

(32) $$\| u'' \|_{\mathrm{L}^\infty(\Omega)} \leqq \| u'' \|_{\mathrm{L}^\infty(\Gamma)} = \| u \|_{\mathrm{L}^\infty(\Gamma)} \leqq \alpha$$

(we have used the fact that $\sup_\Gamma |u| = \| u \|_{\mathrm{L}^\infty(\Gamma)}$ for Γ smooth).

Let now (u_n) be a sequence satisfying the hypotheses of (PS). Write $u_n = u_n' + u_n''$ with $u_n' \in \mathrm{H}_0^1$, $u_n'' \in \mathrm{D}$ and set $v = u_n + tu_n'$, $t > 0$. Then $t(\Phi'(u_n), u_n') + \psi(u_n + tu_n') - \psi(u_n) \geqq -\varepsilon_n t \| u_n' \|$. Divide by t and let $t \to 0$. Since $\int_\Gamma \mathrm{B}(u_n + tu_n')d\sigma = \int_\Gamma \mathrm{B}(u_n)d\sigma$ and $(u_n', u_n'') = 0$,

$$\| u_n' \|^2 - \lambda \int_\Omega u_n u_n' dx - \int_\Omega | u_n |^{p-2} u_n u_n' dx \geqq -\varepsilon_n \| u_n' \| .$$

Let $d \in \left(p^{-1}, \frac{1}{2} \right)$, multiply the above inequality by $-d$ and add to $\mathrm{I}(u_n) \leqq c + 1$. Then

$$c + 1 + \| u_n' \| \geqq \left(\frac{1}{2} - d \right) \| u_n' \|^2 + \frac{1}{2} \| u_n'' \|^2 + \int_\Omega \left\{ d \, | u_n |^{p-2} u_n (u_n' - d^{-1} p^{-1} u_n) + \lambda u_n \left(d u_n' - \frac{1}{2} u_n \right) \right\} dx .$$

Since $| u_n - u_n' | \leqq \alpha$ according to (32) and $d^{-1} p^{-1} < 1$, the integrand is positive if $| u_n | \geqq \mathrm{R}$ and R is sufficiently large. Hence

$$c + 1 + \| u_n' \| \geqq \left(\frac{1}{2} - d \right) \| u_n' \|^2 + \frac{1}{2} \| u_n'' \|^2 - \mathrm{C} .$$

So the sequence (u_n) is bounded and a familiar argument shows that it possesses a strongly convergent subsequence. □

Note that Corollaries 5.9 and 5.10 partially generalize a result of Ambrosetti and Rabinowitz [*1*, Theorem 3.32, see also *21*, *24*].

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