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The global Cauchy problem for the non linear Schrödinger equation revisited

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ABSTRACT. — We study the Cauchy problem for a class of non-linear Schrödinger equations. We prove the existence of global weak solutions by a compactness method and, under stronger assumptions, the uniqueness of those solutions, thereby generalizing previous results.

RÉSUMÉ. — On étudie le problème de Cauchy pour une classe d'équations de Schrödinger non linéaires. On démontre l'existence de solutions faibles globales par une méthode de compacité, et sous des hypothèses plus fortes, l'unicité de ces solutions, généralisant ainsi les résultats connus précédemment.

1. INTRODUCTION

A large amount of work has been devoted in the last few years to the study of the Cauchy problem for the non-linear Schrödinger equation

$$i\dot{\varphi} \equiv i\frac{d\varphi}{dt} = -\frac{1}{2}\Delta\varphi + f(\varphi) \tag{1.1}$$

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where φ is a complex function defined in space time $\mathbb{R}^{n+1}$, Δ is the Laplace operator in $\mathbb{R}^n$ and f a non-linear complex valued function [1] [3] [4] [6] [7] [11] [15] [20]. That problem has been studied mostly by the use of contraction methods. Under suitable assumptions it has been shown that the Cauchy problem has a unique solution which is a bounded continuous function of time with values in the energy space, namely in the Sobolev space $H^1 \equiv H^1(\mathbb{R}^n)$, for initial data $\varphi_0 \in H^1$ [6]. On the other hand the Cauchy problem for a large number of semi-linear evolution equations has been studied by compactness methods (see [12] for a general survey, and also [9] [16] [17] [19] for the special case of the non-linear Klein-Gordon equation). Here we apply the compactness method to the equation (1.1). We prove the existence of weak global solutions for slowly increasing or repulsive rapidly increasing interactions (Section 2) and the existence and uniqueness of more regular solutions under stronger assumptions (Section 3). The latter results improve previous ones [6] as regards the assumptions on the interaction. The method of the proof of existence follows closely [12], while the proof of uniqueness combines a variation of the previously used contraction method with suitable space time integrability properties of the solutions. Under additional repulsivity conditions on the interaction one can show that all the solutions obtained here are dispersive ([8] especially Proposition 5.1).

We conclude this introduction by giving the main notation used in this paper and some elementary results on the free evolution. We restrict our attention to $n \geqslant 2$ since the special simpler case $n = 1$ would require slightly modified statements. We use the notation $2^* = 2n/(n-2)$. We denote by $\| \, . \, \|_r$ the norm in $L^r \equiv L^r(\mathbb{R}^n)$. Pairs of conjugate indices are written as $r, \bar{r}$, where $2 \leqslant r \leqslant \infty$ and $r^{-1} + \bar{r}^{-1} = 1$. For any integer k, we denote by $H^k \equiv H^k(\mathbb{R}^n)$ the usual Sobolev spaces. For any interval I of $\mathbb{R}$ we denote by $\bar{I}$ the closure of I. For any Banach space B, we denote by $\mathscr{C}(I, B)$ (respectively $\mathscr{C}_b(I, B)$) the space of strongly continuous (respectively bounded strongly continuous) functions from I to B, by $\mathscr{C}_w(I, B)$ the space of weakly continuous functions from I to B, by $\mathscr{C}^\alpha(I, B)$ $(0 < \alpha < 1)$ the space of uniformly strongly Hölder continuous functions from I to B with exponent α, namely functions φ from I to B such that

$$\| \varphi(t) - \varphi(s) \|_B \leqslant C \, | t - s |^\alpha$$

for all s and $t \in I$, and by $\mathscr{C}_0^\infty(I, B)$ the space of infinitely differentiable functions from I to B with compact support. For any q, $1 \leqslant q \leqslant \infty$, we denote by $L^q(I, B)$ (respectively $L^q_{loc}(I, B)$) the space of measurable functions φ from I to B such that $\| \varphi(\, . \,) \|_B \in L^q(I)$ (respectively $\in L^q_{loc}(I)$) [23]. If $B = L^r$, we denote by $| \, . \, |_{r,q,I}$ the norm in $L^q(I, L^r)$. If I is open we denote by $\mathscr{D}'(I, B)$ the space of vector valued distributions from I to B [13].

We shall use the one parameter group $U(t) = \exp\left(\frac{i}{2}t\Delta\right)$ generated by the free equation and the fact that, for $t \neq 0$ and for any $r \geq 2$, $U(t)$ is bounded and strongly continuous from $L^{\bar{r}}$ to L^r with

$$\| U(t)\varphi \|_r \leq (2\pi \mid t \mid)^{-\delta(r)} \| \varphi \|_{\bar{r}} \tag{1.2}$$

for all $\varphi \in L^{\bar{r}}$, where $\delta(r) = n/2 - n/r$. (See for instance [*6*], Lemma 1.2).

2. EXISTENCE OF SOLUTIONS

Before discussing the existence problem we first give some preliminary properties of the equation (1.1) for a very general class of interactions. We introduce the following assumptions on f:

(H1) $f \in \mathscr{C}(\mathbb{C}, \mathbb{C})$ and for some p, $1 \leq p < \infty$, and for all $z \in \mathbb{C}$

$$| f(z) | \leq C(| z | + | z |^p). \tag{2.1}$$

(H2) (*a*). There exists a function $V \in \mathscr{C}^1(\mathbb{C}, \mathbb{R})$ such that $V(0) = 0$, $V(z) = V(| z |)$ for all $z \in \mathbb{C}$ and $f(z) = \partial V/\partial \bar{z}$.

(*b*). For all $\rho \in \mathbb{R}^+$, V satisfies the estimate

$$V(\rho) \geq -C(\rho^2 + \rho^{p_3+1}) \quad \text{with} \quad 1 \leq p_3 < 1 + 4/n. \tag{2.2}$$

For $\varphi \in H^1$ such that $V(\varphi) \in L^1$, the energy is defined by

$$E(\varphi) \equiv \frac{1}{2}\| \nabla\varphi \|_2^2 + \int dx V(\varphi(x)) \tag{2.3}$$

The assumption (H2), part (*a*) formally implies the conservation of the L^2-norm and of the energy for the equation (1.1).

We set $q = p + 1$ and we introduce the Banach space $X = H^1 \cap L^q$. The dual space is $X' = H^{-1} + L^{\bar{q}}$, the duality being realized through the scalar product $\langle \varphi, \psi \rangle$ in L^2, linear in ψ and antilinear in φ.

LEMMA 2.1. — Let f satisfy (H1), let I be a bounded open interval and let $\varphi \in L^\infty(I, X)$. Then

(1) $$f(\varphi) \in L^\infty(I, L^2 + L^{\bar{q}}).$$

Let φ satisfy the equation (1.1) in $\mathscr{D}'(I, X')$. Then

(2) $$\dot{\varphi} \in L^\infty(I, X') \quad \text{and} \quad \varphi \in \mathscr{C}_w(\bar{I}, X) \cap \bigcap_{2 \leq r < \mathrm{Max}(q, 2^*)} \mathscr{C}^{\alpha(r)}(\bar{I}, L^r)$$

where

$$\alpha(r) = (1/2)\{ 1 - \delta(r) \mathrm{Min}(1. \delta(q)^{-1}) \}. \tag{2.4}$$

(3) For any t, $s \in \bar{\mathrm{I}}$, φ satisfies the integral equation

$$\varphi(t) = \mathrm{U}(t-s)\varphi(s) - i \int_s^t d\tau \mathrm{U}(t-\tau) f(\varphi(\tau)) \tag{2.5}$$

where the integral is a Bochner integral in H^{-k}, $k \geqslant \mathrm{Max}\,(1, \delta(q))$.

(4) Let in addition f satisfy (H2), part (a). Then $\| \varphi(t) \|_2$ is constant in $\bar{\mathrm{I}}$.

Proof. — PART 1. — We decompose f as $f = f_1 + f_2$ where $f_j \in \mathscr{C}(\mathbb{C}, \mathbb{C})$ and

$$| f_j(z) | \leqslant \mathrm{C} \, | z |^{p_j} \tag{2.6}$$

with $p_1 = 1$ and $p_2 = p$. The result follows from standard measurability arguments and from the estimate

$$| f_j(\varphi) |^{\bar{q}_j} \leqslant \mathrm{C} \, | \varphi |^{q_j}, \tag{2.7}$$

where $q_j = p_j + 1$.

PART 2. — Under the assumptions made, the right hand side of (1.1) is in $\mathrm{L}^\infty(\mathrm{I}, \mathrm{X}')$ and the left hand side is in $\mathscr{D}'(\mathrm{I}, \mathrm{X})$. Therefore the equation makes sense in $\mathscr{D}'(\mathrm{I}, \mathrm{X}')$ and implies that $\dot{\varphi} \in \mathrm{L}^\infty(\mathrm{I}, \mathrm{X}')$. By integration and after extension to $\bar{\mathrm{I}}$ by continuity, $\varphi \in \mathscr{C}_w(\bar{\mathrm{I}}, \mathrm{X}')$, so that by Lemma 8.1 in [*13*], $\varphi \in \mathscr{C}_w(\bar{\mathrm{I}}, \mathrm{X})$ and the L^∞-bound on $\| \varphi(.) \|_{\mathrm{X}}$ extends to all $t \in \bar{\mathrm{I}}$. On the other hand, one can approximate φ by a sequence $\{ \varphi_j \}$ in $\mathscr{C}_0^\infty(\mathbb{R}, \mathrm{X})$ such that, after restriction to I, φ_j converges to φ in $\mathrm{L}^2(\mathrm{I}, \mathrm{X})$ and $\dot{\varphi}_j$ converges to $\dot{\varphi}$ in $\mathrm{L}^2(\mathrm{I}, \mathrm{X}')$. This implies that $\varphi \in \mathscr{C}(\bar{\mathrm{I}}, \mathrm{L}^2)$ and that φ_j tends to φ in $\mathscr{C}(\bar{\mathrm{I}}, \mathrm{L}^2)$ (see [*22*], exercise 40.1). Taking the limit $j \to \infty$ in the identity

$$\| \varphi_j(t) - \varphi_j(s) \|_2^2 = 2 \int_s^t d\tau \, \mathrm{Re} \, \langle \dot{\varphi}_j(\tau), \varphi_j(\tau) - \varphi_j(s) \rangle,$$

we obtain

$$\| \varphi(t) - \varphi(s) \|_2^2 \leqslant 4 \, | t-s | \, \| \varphi \|_{\mathrm{L}^\infty(\mathrm{I},\mathrm{X})} \, \| \dot{\varphi} \|_{\mathrm{L}^\infty(\mathrm{I},\mathrm{X}')} \tag{2.8}$$

so that $\varphi \in \mathscr{C}^{1/2}(\bar{\mathrm{I}}, \mathrm{L}^2)$. The remaining Hölder continuity properties of φ follow by interpolation between L^2 and X.

PART 3. — We again approximate φ by a regularized sequence $\{ \varphi_j \}$ as in Part (2). The functions φ_j satisfy

$$\varphi_j(t) - \mathrm{U}(t-s)\varphi_j(s) = \int_s^t d\tau \mathrm{U}(t-\tau)(\dot{\varphi}_j(\tau) - \frac{i}{2} \Delta \varphi_j(\tau)) \tag{2.9}$$

where t, $s \in \bar{\mathrm{I}}$ and the integral is in H^{-k}. We now take the limit $j \to \infty$

in (2.9). For fixed t and s, the left hand side of (2.9) tends to $\varphi(t) - U(t-s)\varphi(s)$ in L^2, while the right hand side tends to

$$\int_s^t d\tau U(t-\tau)\left(\dot{\varphi}(\tau) - \frac{i}{2}\Delta\varphi(\tau)\right)$$

in H^{-k}. Then (2.5) follows from (1.1).

PART 4. — We again approximate φ by a regularized sequence $\{ \varphi_j \}$ as in Part (2). Taking the limit $j \to \infty$ in the identity

$$\| \varphi_j(t) \|_2^2 - \| \varphi_j(s) \|_2^2 = \int_s^t d\tau 2 \operatorname{Re} < \varphi_j(\tau), \dot{\varphi}_j(\tau) >$$

and using the fact that $\operatorname{Re} \langle \varphi, \dot{\varphi} \rangle = 0$ as a consequence of the equation (1.1) yields the result. Q. E. D.

The continuity properties of φ obtained in Lemma 2.1 make it meaningful to study the Cauchy problem for the equation (1.1) for $\varphi \in L^\infty(I, X)$ with given initial data $\varphi_0 \in X$ at $t_0 \in \bar{I}$. We first prove the existence of solutions.

PROPOSITION 2.1. — Let f satisfy (H1) and (H2). If $p + 1 > 2^*$, assume in addition that

$$V(\rho) \geqslant - C_1\rho^2 + C_2\rho^{p+1} \tag{2.10}$$

for some $C_2 > 0$ and all $\rho \in \mathbb{R}^+$. Let $t_0 \in \mathbb{R}$ and let $\varphi_0 \in X$. Then the equation (1.1) has a solution $\varphi \in L^\infty(\mathbb{R}, X) \cap \mathscr{C}_w(\mathbb{R}, X) \cap \bigcap_{2 \leqslant r < \operatorname{Max}(p+1, 2^*)} \mathscr{C}^{\alpha(r)}(\mathbb{R}, L^r)$ with $\alpha(r)$ defined by (2.4), and with $\varphi(t_0) = \varphi_0$. Furthermore

$$\| \varphi(t) \|_2 = \| \varphi_0 \|_2 \tag{2.11}$$

for all $t \in \mathbb{R}$.

If $p + 1 \geqslant 2^*$, assume in addition that V can be decomposed as $V = V_1 + V_2$ where V_1 satisfies the estimate

$$| V_1(\rho) | \leqslant C(\rho^2 + \rho^{p'+1}) \tag{2.12}$$

for some p', $1 \leqslant p' < p$, and for all $\rho \in \mathbb{R}^+$, and where the map $\varphi \to \int dx V_2(\varphi)$ is weakly lower semi-continuous from X to $\mathbb{R}$ on the bounded sets of X. Then for all $t \in \mathbb{R}$, φ satisfies the energy inequality

$$E(\varphi(t)) \leqslant E(\varphi_0) . \tag{2.13}$$

REMARK 2.1. — A sufficient condition to ensure the required lower semi-continuity of the map $\varphi \to \int dx V_2(\varphi)$ under the assumptions (H1) and (H2) is that that map be convex from X to $\mathbb{R}$ (see for instance [5], Corollary 2.2). For that purpose, it is sufficient that V_2 be convex from $\mathbb{C}$ to $\mathbb{R}$, or equivalently that V_2 be increasing and convex from $\mathbb{R}^+$ to $\mathbb{R}$. If $p + 1 > 2^*$ and under the assumptions (H1), (H2 *a*) and (2.10), there is no loss of generality in assuming that

$$C_3 \rho^{p+1} \leqslant V_2(\rho) \geqslant C_4 \rho^{p+1} .$$

In particular the required lower semi-continuity is satisfied by $V_2(\rho) = C\rho^{p+1}$.

Proof of Proposition 2.1. — The proof proceeds in several steps. One first solves a finite dimensional approximation of the equation (1.1), one then estimates the solutions uniformly with respect to the approximation, and one finally removes the approximation by a compactness argument.

STEP 1. — Finite dimensional approximation.

Let $\{ w_j \}$, $j \in \mathbb{Z}^+$, be a basis in X, namely a set of linearly independent vectors, the finite linear combinations of which are dense in X. For any $m \in \mathbb{Z}^+$ we look for an approximate solution of (1.1) of the form

$$\varphi_m(t) = \sum_{1 \leqslant k \leqslant m} g_{mk}(t) w_k \tag{2.14}$$

by requiring that φ_m satisfies the equation

$$\langle w_j, i\dot{\varphi}_m + (1/2)\Delta\varphi_m - f(\varphi_m) \rangle = 0 \tag{2.15}$$

for $1 \leqslant j \leqslant m$, and the initial condition

$$\varphi_m(t_0) = \varphi_{m0} \equiv \sum_{1 \leqslant k \leqslant m} c_{mk} w_k \tag{2.16}$$

with the c_{mk} chosen in such a way that φ_{m0} tends to φ_0 strongly in X when $m \to \infty$. By the linear independence of the w_j's, the equation (2.15) can be put in normal form and by Peano's theorem, it has a solution in some interval $[t_0 - T_m, t_0 + T_m]$ with $T_m > 0$ [*10*]. In order to prove that T_m can be taken infinite, we next derive an *a priori* estimate on the solutions of (2.15). Multiplying by $\bar{g}_{mj}$ and summing over j, we obtain

$$\langle \varphi_m, i\dot{\varphi}_m + (1/2)\Delta\varphi_m - f(\varphi_m) \rangle = 0 , \tag{2.17}$$

the imaginary part of which yields

$$(d/dt) \| \varphi_m \|_2^2 = 0 , \tag{2.18}$$

so that

$$\| \varphi_m(t) \|_2 = \| \varphi_{m0} \|_2 . \tag{2.19}$$

Now

$$\sum_{1 \leqslant k \leqslant m} | g_{mk}(t) |^2 \leqslant C_m \| \varphi_m(t) \|_2^2 . \tag{2.20}$$

By standard arguments, (2.19) and (2.20) imply the existence of global solutions of (2.15), namely one can take $T_m = \infty$.

STEP 2. — Uniform estimates on φ_m.

In order to take the limit $m \to \infty$, we need stronger uniform estimates on φ_m, which we derive from the energy conservation. Multiplying (2.15) by $(d/dt)\bar{g}_{mj}$ and summing over j, we obtain

$$\langle \dot{\varphi}_m, i\dot{\varphi}_m + (1/2)\Delta\varphi_m - f(\varphi_m) \rangle = 0, \tag{2.21}$$

the real part of which yields

$$\frac{d}{dt}\frac{1}{2} \| \nabla\varphi_m \|_2^2 + 2 \operatorname{Re} \langle \dot{\varphi}_m, f(\varphi_m) \rangle = 0. \tag{2.22}$$

We now prove that

$$2 \operatorname{Re} \langle \dot{\varphi}_m, f(\varphi_m) \rangle = (d/dt) \int dx V(\varphi_m). \tag{2.23}$$

For that purpose we consider for fixed t the quantity

$$\begin{aligned} J(\tau) &\equiv \tau^{-1} \int dx \{ V(\varphi_m(t + \tau, x)) - V(\varphi_m(t, x)) \} - 2\operatorname{Re} \langle f(\varphi_m(t)), \dot{\varphi}_m(t) \rangle \\ &= 2 \operatorname{Re} \int_0^1 d\sigma \{ \langle f(\sigma\varphi_m(t+\tau)+(1-\sigma)\varphi_m(t)), \tau^{-1}(\varphi_m(t+\tau)-\varphi_m(t)) \rangle \\ &\qquad - \langle f(\varphi_m(t)), \dot{\varphi}_m(t) \rangle \}. \end{aligned} \tag{2.24}$$

We estimate $J(\tau)$ as

$$\begin{aligned} | J(\tau) | \leqslant 2 \int_0^1 d\sigma \sum_{j=1,2} \{ & \| f_j(\sigma\varphi_m(t+\tau)+(1-\sigma)\varphi_m(t)) \|_{\bar{q}_j} \\ & \times \| \tau^{-1}(\varphi_m(t + \tau) - \varphi_m(t)) - \dot{\varphi}_m(t) \|_{q_j} \\ & + \| f_j(\sigma\varphi_m(t + \tau) + (1 - \sigma)\varphi_m(t)) - f_j(\varphi_m(t)) \|_{\bar{q}_j} \| \dot{\varphi}_m(t) \|_{q_j} \} \end{aligned} \tag{2.25}$$

where f_j, $j = 1, 2$, are defined as in the proof of Lemma 2.1, part (1). Estimating the first norm in the right hand side of (2.25) by the use of (2.7), taking the limit $\tau \to 0$, and applying the Lebesgue theorem to the integral over σ, we obtain (2.23). From (2.22) and (2.23), it follows that

$$E(\varphi_m(t)) = E(\varphi_{m0}) \tag{2.26}$$

for all $t \in \mathbb{R}$, with $E(.)$ defined by (2.3).

Under the assumptions (H2) and possibly (2.10) made on V, the conservation laws (2.19) and (2.26) imply

$$\sup_{t\in\mathbb{R}} \| \varphi_m(t) \|_X \leqslant M(\| \varphi_{m0} \|_2, E(\varphi_{m0})) \tag{2.27}$$

for some locally bounded real function M. That result follows from a simple computation if $p + 1 \leqslant 2^*$ (see for instance Lemma 3.2 in [6]) and directly from (2.10) if $p + 1 > 2^*$. Since $\| \varphi \|_2$ and $E(\varphi)$ are continuous functions of φ in X and since φ_{m0} tends to φ_0 in X when $m \to \infty$, (2.27) yields an estimate uniform in m

$$\operatorname*{Sup}_m \operatorname*{Sup}_t \| \varphi_m(t) \|_X < \infty , \tag{2.28}$$

namely φ_m is uniformly bounded in $L^\infty(\mathbb{R}, X)$. Since f is bounded from $L^2 \cap L^q$ to $L^2 + L^{\bar{q}}$, (2.28) implies that $f(\varphi_m)$ is uniformly bounded in $L^\infty(\mathbb{R}, L^2 + L^{\bar{q}})$ and therefore that ξ_m, defined by

$$\xi_m = -(1/2)\Delta\varphi_m + f(\varphi_m) \tag{2.29}$$

is uniformly bounded in $L^\infty(\mathbb{R}, X')$. It then follows from the relation

$$\begin{aligned} \| \varphi_m(t) - \varphi_m(s) \|_2^2 &= -2 \int_s^t d\tau \operatorname{Im} \langle \varphi_m(s), \xi_m(\tau) \rangle \\ &\leqslant 2 | t - s | \; \| \varphi_m \|_{L^\infty(\mathbb{R}, X)} \| \xi_m \|_{L^\infty(\mathbb{R}, X')} \end{aligned} \tag{2.30}$$

that the sequence $\{ \varphi_m \}$ is uniformly (in m and t) Hölder continuous in L^2 with exponent 1/2, and by interpolation between (2.28) and (2.30), in L^r with exponent $\alpha(r)$ for $2 \leqslant r < \operatorname{Max}(2^*, q)$.

STEP 3. — Convergence of a subsequence.

We now take the limit $m \to \infty$ by using a compactness argument. By (2.28), the sequence $\{ \varphi_m \}$ is bounded in $L^\infty(\mathbb{R}, X)$, which is the dual of $L^1(\mathbb{R}, X')$, and is therefore relatively compact in the weak-* topology of $L^\infty(\mathbb{R}, X)$. One can then extract from that sequence a subsequence, still called $\{ \varphi_m \}$ for simplicity, which converges to some $\varphi \in L^\infty(\mathbb{R}, X)$ in the weak-* sense.

As a preparation to the proof of the fact that φ satisfies the equation (1.1) (Step 5 below), we now derive some easy consequences of the previous convergence. Let

$$\phi = i\dot{\varphi} + (1/2)\Delta\varphi \tag{2.31}$$

(so that $\phi \in \mathscr{D}'(\mathbb{R}, X) + L^\infty(\mathbb{R}, H^{-1})$). We first prove that $\phi \in L^\infty(\mathbb{R}, L^2 + L^{\bar{q}})$ and that $f(\varphi_m)$ tends to ϕ in the weak-* sense in $L^\infty(\mathbb{R}, L^2 + L^{\bar{q}})$. In fact, by (2.28) and the assumption (H1), the sequence $\{ f(\varphi_m) \}$ is bounded and therefore weak-* relatively compact in $L^\infty(\mathbb{R}, L^2 + L^{\bar{q}})$. On the other hand, from the equation (2.15), we obtain for $j \leqslant m$ and $\theta \in \mathscr{C}_0^\infty(\mathbb{R}, \mathbb{C})$

$$\int d\tau \theta(\tau) \langle w_j, f(\varphi_m(\tau)) \rangle = - \int d\tau \left\{ \dot{\theta}(\tau) \langle w_j, i\varphi_m(\tau) \rangle - \theta(\tau) \left\langle w_j, \frac{1}{2}\Delta\varphi_m(\tau) \right\rangle \right\} \tag{2.32}$$

$$\underset{m \to \infty}{\longrightarrow} - \int d\tau \left\{ \dot{\theta}(\tau) \langle w_j, i\varphi(\tau) \rangle - \theta(\tau) \left\langle w_j, \frac{1}{2}\Delta\varphi(\tau) \right\rangle \right\} = \int d\tau \theta(\tau) \langle w_j, \phi(\tau) \rangle \tag{2.33}$$

by the weak-* convergence of φ_m to φ in $L^\infty(\mathbb{R}, X)$, where the last expression in (2.33) has to be appropriately interpreted.

From (2.33) and the fact that the w_j's form a basis in X, it follows that any weak-* convergent subsequence of $\{ f(\varphi_m) \}$ in $L^\infty(\mathbb{R}, L^2 + L^{\bar{q}})$ converges to ϕ in $\mathscr{D}'(\mathbb{R}, X')$. Therefore $\phi \in L^\infty(\mathbb{R}, L^2 + L^{\bar{q}})$ and $f(\varphi_m)$ converges to ϕ in $L^\infty(\mathbb{R}, L^2 + L^{\bar{q}})$ in the weak-* sense.

From (2.31) it now follows that $\dot{\varphi} \in L^\infty(\mathbb{R}, X')$. Together with the fact that $\varphi \in L^\infty(\mathbb{R}, X)$, this implies, by the same argument as in Lemma 2.1, part (2), that $\varphi \in \mathscr{C}_w(\mathbb{R}, X) \cap \bigcap_{2 \leqslant r < \mathrm{Max}(2^*, q)} \mathscr{C}^{\alpha(r)}(\mathbb{R}, L^r)$ with $\alpha(r)$ given by (2.4).

We next prove that for all $t \in \mathbb{R}$, $\varphi_m(t)$ tends to $\varphi(t)$ weakly in X. Now for fixed t, the sequence $\{ \varphi_m(t) \}$ is bounded in X uniformly in m by (2.28) and therefore weakly relatively compact (X is reflexive). It is therefore sufficient to prove that it can have no other weak accumulation point than $\varphi(t)$ in X. Now assume that a subsequence (still denoted $\{ \varphi_m(t) \}$ for brevity) converges weakly to χ in X. By the Hölder continuity of φ and Hölder equicontinuity of φ_m in L^2, there exists a constant C such that

$$\left\{ \begin{aligned} &\| \varphi(\tau) - \varphi(t) \|_2 \leqslant C \,| t - \tau |^{1/2} \\ &\| \varphi_m(\tau) - \varphi_m(t) \|_2 \leqslant C \,| t - \tau |^{1/2} \end{aligned} \right. \tag{2.34}$$

for τ in a neighborhood of t (actually for all $\tau \in \mathbb{R}$). We can then estimate for $\gamma > 0$

$$\begin{aligned} &\| \varphi(t) - \chi \|_2^2 = \langle \varphi(t) - \chi, \varphi_m(t) - \chi \rangle + \\ &+ (2\gamma)^{-1} \int_{t-\gamma}^{t+\gamma} d\tau \langle \varphi(t) - \chi, \varphi(t) - \varphi(\tau) - (\varphi_m(t) - \varphi_m(\tau)) + \varphi(\tau) - \varphi_m(\tau) \rangle \\ &\leqslant 2 \| \varphi(t) - \chi \|_2 C \gamma^{1/2} + \langle \varphi(t) - \chi, \varphi_m(t) - \chi \rangle \\ &+ (2\gamma)^{-1} \int_{t-\gamma}^{t+\gamma} d\tau \langle \varphi(t) - \chi, \varphi(\tau) - \varphi_m(\tau) \rangle . \end{aligned} \tag{2.35}$$

Now the second term in the last member of (2.35) tends to zero when $m \to \infty$ by the weak convergence of $\varphi_m(t)$ to χ in X (actually L^2 would be sufficient), the third term tends to zero when $m \to \infty$ for fixed γ by the weak-* convergence of φ_m to φ in $L^\infty(\mathbb{R}, X)$, and the first term tends to zero with γ, so that $\chi = \varphi(t)$.

STEP 4. — Initial condition.

We next prove that φ satisfies the initial condition $\varphi(t_0) = \varphi_0$. For that purpose we use again (2.32) for some $\theta \in \mathscr{C}_0^\infty(\mathbb{R}, \mathbb{C})$ with $\theta(t_0) = 1$, and with the integration over τ now running from t_0 to infinity. Integrating by parts, we obtain

$$\int_{t_0}^{\infty} d\tau \langle w_j, i\varphi_m(\tau)\dot{\theta}(\tau) + \xi_m(\tau)\theta(\tau) \rangle = - \langle w_j, i\varphi_{m0} \rangle . \tag{2.36}$$

Taking the limit $m \to \infty$, using (2.33) and the convergence of φ_{m0} to φ_0 in X, we obtain

$$\int_{t_0}^{\infty} d\tau(i\varphi(\tau)\dot{\theta}(\tau) + i\dot{\varphi}(\tau)\theta(\tau)) = -\, i\varphi_0 \tag{2.37}$$

where the integral is taken in X'. This implies $\varphi(t_0) = \varphi_0$.

STEP 5. — Differential equation.

We next prove that φ satisfies the equation (1.1). In view of (2.31), it remains only to be shown that $f(\varphi) = \phi$. Now from weak-* convergence of $f(\varphi_m)$ to ϕ in $L^\infty(\mathbb{R}, L^2 + L^{\bar{q}})$, it follows that for any compact interval I and any bounded open set $\Omega \subset \mathbb{R}^n$, $f(\varphi_m)$ tends to ϕ weakly in $L^{\bar{q}}(I \times \Omega)$. On the other hand by (2.30), the restrictions of the functions φ_m to $I \times \Omega$ are equicontinuous from I to $L^2(\Omega)$, and by (2.28), are uniformly bounded in $H^1(\Omega)$, so that, by the compactness of the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ and by the Ascoli theorem, the sequence $\{ \varphi_m \}$ is relatively compact in $\mathscr{C}(I, L^2(\Omega))$. Together with the fact that φ_m tends to φ in the weak-* topology of $L^\infty(\mathbb{R}, X)$ and *a fortiori* of $L^\infty(I, L^2(\Omega))$, that compactness implies that φ_m converges strongly to φ in $\mathscr{C}(I, L^2(\Omega))$ and *a fortiori* in $L^2(I \times \Omega)$. We can therefore extract from the sequence $\{ \varphi_m \}$ a subsequence which converges to φ almost everywhere in $I \times \Omega$. Along that subsequence, $f(\varphi_m)$ tends to $f(\varphi)$ almost everywhere in $I \times \Omega$. Now $f(\varphi_m)$ is bounded uniformly with respect to m in $L^\infty(\mathbb{R}, L^2 + L^{\bar{q}})$ and therefore in $L^{\bar{q}}(I \times \Omega)$. The last two facts imply (see Lemma 1.3, p. 12, in [*12*]) that $f(\varphi_m)$ tends to $f(\varphi)$ weakly in $L^{\bar{q}}(I \times \Omega)$ along the previous subsequence. Therefore $f(\varphi) = \phi$ in $I \times \Omega$, so that $f(\varphi) = \phi$ since I and Ω are arbitrary.

STEP 6. — Conservation laws.

Since φ satisfies the equations (1.1), the conservation of the L^2-norm (2.11) follows from Lemma 2.1, part (4). In order to prove the energy inequality (2.13), we need some additional convergence properties of φ_m to φ. We now prove that φ_m tends to φ strongly in $\mathscr{C}(I, L^r)$ for all compact I and all r, $2 \leqslant r < \text{Max}(2^*, q)$. In fact for each t, weak convergence of $\varphi_m(t)$ to $\varphi(t)$ in X implies weak convergence in L^2. On the other hand,

$$\| \varphi_m(t) \|_2 = \| \varphi_{m0} \|_2 \underset{m\to\infty}{\longrightarrow} \| \varphi_0 \|_2 = \| \varphi(t) \|_2 \tag{2.38}$$

by the conservation of the L^2-norm and the fulfillment of the initial condition. When combined with weak convergence, (2.38) implies strong convergence in L^2 for each t, which together with the strong equicontinuity (2.30) implies uniform strong convergence in L^2 in compact intervals. Convergence in $\mathscr{C}(I, L^r)$ for other values of r follows as usual by interpolation between L^2 and X.

We finally prove the energy inequality (2.13). For that purpose, we take the limit $m \to \infty$ in (2.26). The right hand side tends to $E(\varphi_0)$ since φ_{m0} tends to φ_0 strongly in X. In the left hand side, the contribution of V_1 (if $p + 1 \geqslant 2^*$) or of V (if $p + 1 < 2^*$) converges to its value for $\varphi(t)$ by the strong convergence of $\varphi_m(t)$ to $\varphi(t)$ in $L^2 \cap L^{p'+1}$ (if $p + 1 \geqslant 2^*$) or in $L^2 \cap L^{p+1}$ (if $p + 1 < 2^*$), while the contribution of V_2 (if any) and the kinetic term are weakly lower semi-continuous in X. The result then follows from the weak convergence of $\varphi_m(t)$ to $\varphi(t)$ in X. Q. E. D.

3. UNIQUENESS OF SOLUTIONS

In this section, we prove that, under stronger assumptions on the interaction, the solutions obtained in Section 2 are unique. We replace (H1) by the following stronger assumptions.

(H3) $f \in \mathscr{C}^1(\mathbb{C}, \mathbb{C})$, $f(0) = 0$, and for $n \geqslant 2$,

$$|f'(z)| \equiv \operatorname{Max}\left(\left|\frac{\partial f}{\partial z}\right|, \left|\frac{\partial f}{\partial \bar{z}}\right|\right) \leqslant C(|z|^{p_1-1} + |z|^{p_2-1}), \qquad (3.1)$$

where p_1 and p_2 satisfy the following conditions:

If $n = 2$: $1 < p_1 \leqslant p_2 < \infty$,
If $n \geqslant 3$:

$$0 \leqslant (n-2)(p_2-1)/(np_2) \leqslant p_1 - 1 \leqslant p_2 - 1 < 4/(n-2) \qquad (3.2)$$

and in addition, if $n \geqslant 7$ and $p_2 - 1 \geqslant 2/(n-4)$:

$$(p_2 - 1)(n-4)/n < p_1 - 1 . \qquad (3.3)$$

REMARK 3.1. — The maximum value of $p_2 - 1$ for $n \geqslant 3$ is $4/(n-2)$ as expected. The minimum value of $p_1 - 1$ compatible with it is $p_1 - 1 = 4(n-2)/(n(n+2))$ for $3 \leqslant n \leqslant 6$ and $p_1 - 1 = 4(n-4)/(n(n-2))$ for $n \geqslant 6$. Clearly (H3) implies (H1) with $p = p_2$ and the condition $p - 1 < 4/(n-2)$ contained in (H3) implies that $H^1 \subset L^q$ so that $X = H^1$ and $X' = H^{-1}$.

The proof of uniqueness proceeds by a contraction method. In dimension $n \geqslant 3$, it requires some space time integrability properties of the solutions, which we derive below. We first recall the corresponding properties for the solutions of the free equation. That result is an extension of a result of Strichartz along the line followed by Pecher in the case of the Klein-Gordon equation ([*21*] [*14*]). We use the notation $\omega = (-\Delta)^{1/2}$.

LEMMA 3.1. — Let $n \geqslant 3$, $0 \leqslant \rho \leqslant 1$, $0 \leqslant \delta(r) < 2 - \rho$. $\delta(r) \leqslant n/2$, and let q satisfy

$$\text{Max}\,(0, \delta(r) + \rho - 1) \leqslant 2/q \leqslant \delta(r),\ q > 2\,. \tag{3.4}$$

Let $\varphi \in \mathrm{H}^1$. Then $\omega^\rho\, \mathrm{U}(.)\, \varphi \in \mathrm{L}^q(\mathbb{R}, \mathrm{L}^r)$ and

$$|\, \omega^\rho \mathrm{U}(.)\varphi\, |_{r,q,\mathbb{R}} \leqslant \mathrm{C}\, \|\, \varphi\, \|_{\mathrm{H}^1}\,. \tag{3.5}$$

Proof. — By the Sobolev inequalities, it is sufficient to prove that for $\varphi \in \mathrm{L}^2$, $0 \leqslant \delta(r) < 1$ and $2/q = \delta(r)$, $\mathrm{U}(.)\varphi \in \mathrm{L}^q(\mathbb{R}, \mathrm{L}^r)$ and

$$|\, \mathrm{U}(.)\varphi\, |_{r,q,\mathbb{R}} \leqslant \mathrm{C}\, \|\, \varphi\, \|_2\,.$$

By density, it is sufficient to prove that result for $\varphi \in \mathscr{S}(\mathbb{R}^n)$, and by duality and density again, it is sufficient to prove that

$$|\, \langle\, \theta, \mathrm{U}(.)\varphi\, \rangle_{n+1}\, | \leqslant \mathrm{C}\, \|\, \varphi\, \|_2\, |\, \theta\, |_{\bar{r},\bar{q},\mathbb{R}} \tag{3.6}$$

for any $\theta \in \mathscr{S}(\mathbb{R}^{n+1})$, where $\langle\, .,\, .\, \rangle_{n+1}$ denotes the standard duality in $n+1$ variables. Now

$$\begin{aligned} |\, \langle\, \theta, \mathrm{U}(.)\varphi\, \rangle_{n+1}\, | &= \left| \left\langle \int d\tau \mathrm{U}(-\tau)\theta(\tau, .), \varphi \right\rangle \right| \\ &\leqslant \|\, \varphi\, \|_2 \left\| \int d\tau \mathrm{U}(-\tau)\theta(\tau,.) \right\|_2 . \end{aligned} \tag{3.7}$$

The last term in (3.7) is estimated by

$$\begin{aligned} \left\| \int d\tau \mathrm{U}(-\tau)\theta(\tau, .) \right\|_2^2 &= \int dt \left\langle \theta(t, .), \int d\tau \mathrm{U}(t - \tau)\theta(\tau, .) \right\rangle_n \\ &\leqslant |\, \theta\, |_{\bar{r},\bar{q},\mathbb{R}} \left| \int d\tau \mathrm{U}(. - \tau)\theta(\tau, .) \right|_{r,q,\mathbb{R}} . \end{aligned} \tag{3.8}$$

Using (1.2) and the Hardy-Littlewood-Sobolev inequality in the time variable ([*18*], p. 117) we can continue (3.8) as

$$\ldots \leqslant \mathrm{C}\, |\, \theta\, |^2_{\bar{r},\bar{q},\mathbb{R}}\,.$$

which implies (3.6) and thereby completes the proof. Q. E. D.

We shall need the following estimate on the interaction term in the equation (1.1).

LEMMA 3.2. — Let $f \in \mathscr{C}^1(\mathbb{C}, \mathbb{C})$ with $f(0) = 0$ and $|\, f'(z)\, | \leqslant \mathrm{C}\, |\, z\, |^{p-1}$ for some p, $1 \leqslant p < \infty$. Let $1 < \bar{l} \leqslant 2 \leqslant r < \infty$ and $0 < \lambda < 1$. Let either $\rho = 0$ or $0 < \rho < \lambda$. Then

$$\|\, \omega^\lambda f(\varphi)\, \|_{\bar{l}} \leqslant \mathrm{C}\, \|\, \nabla\varphi\, \|_2^\mu\, \|\, \omega^\rho \varphi\, \|_r^{1-\mu} \prod_{s=s_\pm} \|\, |\, \varphi\, |^{p-1}\, \|_s^{1/2} \tag{3.9}$$

for all $\varphi \in H^1 \cap H_r^\rho \cap L^{(p-1)s}$, with

$$\mu = (\lambda - \rho)(1 - \rho)^{-1}, \tag{3.10}$$

$$n/s_\pm = \delta(l) + (1 - \theta_\pm)\delta(r), \qquad 0 \leqslant \theta_\pm \leqslant 1 \tag{3.11}$$

$$(1 - \rho)(\theta_\pm - \mu) = \pm \eta, \qquad \eta > 0. \tag{3.12}$$

The constant C is uniform in $\bar{l}, r, \lambda, \eta$ (and ρ if $\rho \neq 0$) provided those variables stay away by a finite amount from the limiting cases in the inequalities $\bar{l} > 1$, $\bar{r} > 1$, $0\,(< \rho) < \lambda < 1$ and $\eta > 0$.

Proof. — The proof requires the use of Besov spaces. We refer to [2] for various equivalent definitions and basic properties thereof. We shall use the following definition. Let $1 \leqslant l, m \leqslant \infty$ and $0 < \lambda < 1$. The Besov space $B_{l,m}^\lambda$ is the space of all $\varphi \in L^l$ for which the following quantity (taken as the definition of the norm of φ) is finite:

$$\| \varphi \|_{B_{l,m}^\lambda} \equiv \| \varphi \|_l + \left\{ \int_0^\infty \frac{dt}{t} \left[t^{-\lambda} \operatorname*{Sup}_{|y| \leqslant t} \| \tau_y \varphi - \varphi \|_l \right]^m \right\}^{1/m}, \tag{3.13}$$

where τ_y denotes the space translation by y in $\mathbb{R}^n$.

We shall also use the Sobolev space of non integer order H_l^λ defined as the space of all $\varphi \in L^l$ for which

$$\| \varphi \|_{H_l^\lambda} \equiv \| (1 + \omega^2)^{\lambda/2} \varphi \|_l < \infty .$$

We need the continuous inclusions

$$B_{l,2}^\lambda \subset H_l^\lambda \subset B_{l,l}^\lambda, \qquad B_{\bar{l},\bar{l}}^\lambda \subset H_{\bar{l}}^\lambda \subset B_{\bar{l},2}^\lambda . \tag{3.14}$$

From (3.14) and the Mikhlin theorem, we obtain

$$\| \omega^\lambda f(\varphi) \|_{\bar{l}} \leqslant C \| f(\varphi) \|_{B_{\bar{l},\bar{l}}} . \tag{3.15}$$

We next estimate $f(\varphi)$ in $B_{\bar{l},\bar{l}}^\lambda$ by using (3.13) and omitting the first norm, which is easily eliminated by an homogeneity argument. In order to estimate the remaining integral, we note that for any θ, $0 \leqslant \theta \leqslant 1$,

$$| \tau_y f(\varphi) - f(\varphi) | \leqslant \int_0^1 d\alpha \, | f'(\alpha \tau_y \varphi + (1-\alpha)\varphi) | \times | y |^\theta \left(\int_0^1 d\beta \, | \tau_{\beta y} \nabla \varphi | \right)^\theta | \tau_y \varphi - \varphi |^{1-\theta},$$

so that

$$| \tau_y f(\varphi) - f(\varphi) \|_{\bar{l}} \leqslant C \| \, | \varphi |^{p-1} \|_s \, | y |^\theta \, \| \nabla \varphi \|_2^\theta \, \| \tau_y \varphi - \varphi \|_r^{1-\theta} \tag{3.16}$$

with s and θ related by (3.11). We now split the t integration in (3.13) with φ replaced by $f(\varphi)$ into the contributions of the two subregions $t \geqslant a$ and $t \leqslant a$ for some $\alpha > 0$ to be chosen later, and we estimate the norm of the integrand by (3.16) with $\theta = \theta_+ > \mu$ for $t \leqslant a$ and $\theta = \theta_- < \mu$ for $t \geqslant a$, where μ is defined by (3.10), thereby obtaining

$$\left\{ \int_0^\infty \frac{dt}{t} \{ t^{-\lambda} \operatorname*{Sup}_{|y| \leqslant t} \| \tau_y f(\varphi) - f(\varphi) \|_{\bar{l}} \}^{\bar{l}} \right\}^{1/\bar{l}} \leqslant$$
$$\leqslant C \sum_{\theta = \theta_\pm} \| \nabla\varphi \|_2^\theta \| | \varphi |^{p-1} \|_s \left\{ \int_{t \gtrless a} \frac{dt}{t} \{ \operatorname*{Sup}_{|y| \leqslant t} t^{-\lambda+\theta} \| \tau_y \varphi - \varphi \|_r^{1-\theta} \}^{\bar{l}} \right\}^{1/\bar{l}} \tag{3.17}$$

with the range of integration appropriately coupled with θ. We continue the estimate for $\rho = 0$ by

$$\ldots \leqslant C \sum_\theta \| \nabla\varphi \|_2^\theta \| | \varphi |^{p-1} \|_s \| \varphi \|_r (| \theta - \lambda | \bar{l})^{-1/\bar{l}} a^{\theta - \lambda}, \tag{3.18}$$

and for $\rho > 0$ by

$$\ldots \leqslant C \sum_\theta \| \nabla\varphi \|_2^\theta \| | \varphi |^{p-1} \|_s \| \varphi \|_{B^\rho_{r,r}}^{1-\theta} ((1-\rho) | \theta - \mu | k)^{-1/k} a^{(1-\rho)(\theta-\mu)} \tag{3.19}$$

where we have applied Hölder's inequality to the t integration with $1/k = 1/\bar{l} - (1 - \theta)/r$. We then optimize either (3.18) or (3.19) with respect to a, making for $\theta_\pm$ the symmetric choice (3.12). If $\rho > 0$, we use in addition the inclusion (3.14), and use again an homogeneity argument to replace the norm of φ in H^ρ_r by $\| \omega^\rho \varphi \|_r$. We then obtain for $\rho > 0$

$$\ldots \leqslant C(\eta k)^{-1/k} \| \nabla\varphi \|_2^\mu \| \omega^\rho \varphi \|_r^{1-\mu} \prod_{s = s_\pm} \| | \varphi |^{p-1} \|_s^{1/2} \tag{3.20}$$

and for $\rho = 0$, a similar expression with k replaced by $\bar{l}$, thereby proving (3.9). The stated uniformity of C in (3.9) with respect to r and $\bar{l}$ follows from that of the inclusions (3.14), the uniformity with respect to λ and ρ from the adequacy of the definition (3.13), and the uniformity with respect to η from the explicit dependence given by (3.20). Q. E. D.

We are now in a position to derive the relevant integrability properties of the solutions of the equation (1.1). We need an assumption stronger than (H1), but weaker than (H3).

LEMMA 3.3. — Let $n \geqslant 3$. Let $f \in \mathscr{C}^1(\mathbb{C}, \mathbb{C})$ with $f(0) = 0$ and let f satisfy (3.1) with

$$0 \leqslant p_1 - 1 \leqslant p_2 - 1 < 4/(n-2). \tag{3.21}$$

Let I be an interval of $\mathbb{R}$, let $t_0 \in I$, let $\varphi_0 \in H^1$ and let $\varphi \in L^\infty(I, H^1)$ be a solution of the equation (1.1) with $\varphi(t_0) = \varphi_0$. Let r and ρ satisfy $1 \leqslant \delta(r) < 2$, $\delta(r) \leqslant n/2$, $0 \leqslant \rho < 1$, $\delta(r) + \rho < 2$ and

$$\delta(r) + \rho \leqslant 1 + \frac{n}{2}(p_1 - 1). \tag{3.22}$$

Let q satisfy $2/q \geqslant \delta(r) + \rho - 1$. Then $\omega^\rho \varphi \in L^q_{loc}(I, L^r)$ and for any compact interval $J \subset I$, $\omega^\rho \varphi$ is estimated in $L^q(J, L^r)$ in terms of the norm of φ in $L^\infty(I, H^1)$.

Proof. — We first remark that under the assumptions (3.1) and (3.21), the results of Lemma 2.1 are available with $X = H^1$, so that it makes sense to consider solutions of (1.1) in $L^\infty(I, H^1)$.

By the Sobolev inequalities, it is sufficient to consider the case $\delta(r) = 1$. The proof is based on the integral equation (2.5) with $s = t_0$, namely

$$\varphi(t) = U(t - t_0)\varphi_0 - i \int_{t_0}^t d\tau U(t - \tau) f(\varphi(\tau)) \tag{3.23}$$

and on estimates on its integrand. We first take ρ' and ρ'' such that

$$0 \leqslant \rho' < \rho' + \varepsilon^2 = \rho'' \leqslant \operatorname{Min}\left(1 - 2\varepsilon, \frac{n}{2}(p_1 - 1)\right) \tag{3.24}$$

with ε sufficiently small, to be chosen later, and we prove that for any $\varphi \in H^1$ such that $\omega^{\rho'}\varphi \in L^{2^*}$ and for any $t \neq 0$, $\omega^{\rho''}U(t)\varphi$ belongs to L^{2^*} and satisfies the estimate

$$\| \omega^{\rho''} U(t) f(\varphi) \|_{2^*} \leqslant M(|t|^{-\rho''} + |t|^{-\delta} \| \omega^{\rho'}\varphi \|_{2^*}^{\nu}) \tag{3.25}$$

for some δ, ν such that $0 \leqslant \delta < 1$, $0 \leqslant \nu \leqslant 1$ and with M depending only on δ, ν and ε but not otherwise on ρ' and ρ'', and depending on φ through $\| \varphi \|_{H^1}$ only. By (3.1) and by (3.24) we can decompose f for each ρ'' as $f = f_1 + f_2$ with $| f_1'(z) | \leqslant C | z |^{p_1' - 1}$ and $| f_2'(z) | \leqslant C | z |^{p_2 - 1}$, $p_1' - 1 = 2\rho''/n$ and C independent of ρ'', and we estimate separately the contributions of f_1 and f_2 to (3.25). The contribution of f_1 is estimated by the Sobolev inequalities and by (1.2) as

$$\| \omega^{\rho''} U(t) f_1(\varphi) \|_{2^*} \leqslant C \| \nabla U(t) f_1(\varphi) \|_l \leqslant C | t |^{-\rho''} \| \nabla \varphi \|_2 \| \varphi \|_{(p_1' - 1)s}^{p_1' - 1} \tag{3.26}$$

with $n/s = \delta(l) = \rho''$, so that $(p_1' - 1)s = 2$ and both norms in the last member are estimated in terms of $\| \varphi \|_{H^1}$. Then (3.26) yields the first term in the right hand side of (3.25). The contribution of f_2 is estimated by the Sobolev inequalities, by (1.2) and by Lemma 3.2 as

$$\| \omega^{\rho''} U(t) f_2(\varphi) \|_{2^*} \leqslant C \| \omega^\lambda U(t) f_2(\varphi) \|_l \leqslant C | t |^{-\delta(l)} \| \omega^\lambda f_2(\varphi) \|_{\bar{l}} \leqslant$$

$$\leqslant C | t |^{-\delta(l)} \| \nabla \varphi \|_2^\mu \| \omega^{\rho'} \varphi \|_{2^*}^{1-\mu} \prod_{s = s_\pm} \| \varphi \|_{(p_2 - 1)s}^{(p_2 - 1)/2} \tag{3.27}$$

with $\rho'' < \lambda < 1$, so that $\delta(l) = 1 + \rho'' - \lambda < 1$, with $n/s_\pm = 1+\delta(l)-\theta_\pm$ and $\theta_\pm$ defined by (3.12) with ρ replaced by ρ'. We estimate the norm of φ in $L^{(p_2-1)s}$ by interpolation between the norms in L^2, L^{2^*} and $L^{r'}$, where $\delta(r') = 1 + \rho'$. The interpolation is possible provided $2 \leqslant (p_2-1)s \leqslant r'$. The condition $(p_2-1)s \geqslant 2$ is equivalent to $p_2 - 1 \geqslant (2/n)(\delta(l)+1-\theta)$ and is easily satisfied for p_2 sufficiently large within (3.21). The condition $(p_2-1)s \leqslant r'$ is easily rewritten as

$$p_2 - 1 \leqslant 2(n - 2 - 2\rho')^{-1}(\delta(l) + 1 - \theta) \tag{3.28}$$

and will be considered below. In order to minimize ν in (3.25), we interpolate $(p_2 - 1)s$ between 2 and 2^* if $2 \leqslant (p_2 - 1)s \leqslant 2^*$, thereby obtaining the second term in the right hand side of (3.25) with $\nu = 1 - \mu$, and between 2^* and r' if $2^* \leqslant (p_2 - 1)s \leqslant r'$, thereby continuing (3.27) as

$$\ldots \leqslant C\,|\,t\,|^{-\delta(l)}\,\|\,\nabla\varphi\,\|_2^{p_2-\nu}\,\|\,\omega^{\rho'}\varphi\,\|_{2^*}^{\nu} \tag{3.29}$$

after an additional use of the Sobolev inequalities, and with

$$p_2 - 1 = 2(n - 2)^{-1}(\delta(l) + 1 - \lambda + \nu\rho'), \tag{3.30}$$

so that the condition $\nu \leqslant 1$ is equivalent to

$$p_2 - 1 \leqslant 2(n - 2)^{-1}(2\delta(l) - \rho'' + \rho') \tag{3.31}$$

after elimination of λ. It remains only to show that the conditions (3.31) and (3.28) with (3.12) for ρ' can be satisfied uniformly for ρ' and ρ'' satisfying (3.24). For that purpose, we finally choose $\delta(l) = 1 - \varepsilon^2$ so that $\lambda = \rho'' + \varepsilon^2$ and $\mu(1 - \rho') = \lambda - \rho' = 2\varepsilon^2$ (see (3.10)) and we choose $\eta = 2\varepsilon^2$ in (3.12) so that $\theta_- = 0$ and $\theta_+ = 4\varepsilon^2(1 - \rho')^{-1} \leqslant 2\varepsilon$ by (3.24). The conditions (3.28) and (3.31) are then implied respectively by

$$p_2 - 1 \leqslant 2(n - 2)^{-1}(2 - \varepsilon^2 - 2\varepsilon)$$

and

$$p_2 - 1 \leqslant 2(n - 2)^{-1}(2 - 3\varepsilon^2)$$

which are satisfied for ε sufficiently small under the condition (3.21). This completes the proof of (3.25) with the uniformity thereafter stated.

We now use (3.25) to complete the proof of Lemma 3.3. We continue to take $\delta(r) = 1$. Let $\varphi \in L^\infty(I, H^1)$ be solution of (1.1). We estimate $\omega^\rho\varphi$ in $L^q_{loc}(I, L^{2^*})$ by applying $\omega^{\rho''}$ to the integral equation (3.23) and estimating iteratively the right hand side in $L^q_{loc}(I, L^{2^*})$ for successive values of ρ'' increasing from 0 to ρ in steps of ε^2, with ε chosen as explained above. The free term $\omega^{\rho''}U(t - t_0)\varphi_0$ is in $L^q(I, L^{2^*})$ by Lemma 3.1 and the integrand is estimated at each step by using (3.25) and applying the Young inequality to the time integration in bounded intervals. For any compact interval $J \subset I$, the norm of $\omega^\rho\varphi$ in $L^q(J, L^{2^*})$ depends on φ only through the constant M in (3.25) and the norm of the free term, both of which are estimated in terms of the norm of φ in $L^\infty(I, H^1)$. Q. E. D.

We are now in a position to prove the uniqueness of solutions of (1.1) under the assumption (H3).

PROPOSITION 3.1. — Let f satisfy (H3). Let I be an open interval, let $t_0 \in \bar{I}$ and let $\varphi_0 \in H^1$. Then there exists at most one $\varphi \in L^\infty(I, H^1)$ which satisfies the equation (1.1) in $\mathscr{D}'(I, H^1)$ and the condition $\varphi(t_0) = \varphi_0$.

Proof. — Let φ_1 and φ_2 be two solutions of the equation (1.1) in $L^\infty(I, H^1)$ with the same initial data φ_0 at $t = t_0$. From Lemma 2.1, part (3), we obtain

$$\varphi_1(t) - \varphi_2(t) = -i \int_{t_0}^{t} d\tau U(t-\tau)(f(\varphi_1(\tau)) - f(\varphi_2(\tau))) \tag{3.32}$$

where the integral is taken in H^{-1}. We estimate (3.32) in $L^{r'}$ for some r' satisfying $0 \leqslant \delta' \equiv \delta(r') < 1$ to be chosen more precisely below. We obtain

$$\begin{aligned} \| U(t-\tau)(f(\varphi_1(\tau)) - f(\varphi_2(\tau))) \|_{r'} &\leqslant C \,|\, t-\tau \,|^{-\delta'} \| f(\varphi_1(\tau)) - f(\varphi_2(\tau)) \|_{\bar{r}'} \\ &\leqslant C \,|\, t-\tau \,|^{-\delta'} \| \varphi_1(\tau) - \varphi_2(\tau) \|_{r'} \sum_{i,j=1,2} \| \varphi_i(\tau) \|_{(p_j-1)l'}^{p_j-1} \end{aligned} \tag{3.33}$$

where we have used (1.2) and (3.1), and where $1/l' = 2\delta'/n$. For $n = 2$, the last norms in (3.33) are estimated in terms of the norms of φ_i in $L^\infty(I, H^1)$ provided $2 \leqslant (p_j - 1)l' < \infty$ which is achieved under (H3) provided $0 < \delta' \leqslant (n/4)(p_1 - 1)$, and we obtain from (3.32) and (3.33)

$$\| \varphi_1(t) - \varphi_2(t) \|_{r'} \leqslant M \int_{t_0}^{t} d\tau \,|\, t - \tau \,|^{-\delta'} \| \varphi_1(\tau) - \varphi_2(\tau) \|_{r'} . \tag{3.34}$$

By an elementary argument, this implies that $\varphi_1 = \varphi_2$.

For $n \geqslant 3$, we substitute (3.33) into (3.32) and we estimate the time integral by Hölder's inequality, so that

$$\begin{aligned} \| \varphi_1(t) - \varphi_2(t) \|_{r'} \leqslant C \sup_{t_0 \leqslant \tau \leqslant t} \| \varphi_1(\tau) - \varphi_2(\tau) \|_{r'} \\ \times \,|\, t - t_0 \,|^{1-\delta'-1/m'} \sum_{i,j=1,2} |\, \varphi_i \,|_{(p_j-1)l',(p_j-1)m',[t_0,t]}^{p_j-1} \end{aligned} \tag{3.35}$$

with

$$1/m' < 1 - \delta' . \tag{3.36}$$

We estimate the last norms in (3.35) by using either the boundedness of φ in H^1 or Lemma 3.3 where we now take $\rho = 0$. Those norms are estimated in terms of t and of the norms of φ_i in $L^\infty(I, H^1)$ provided there exists r such that $1 \leqslant \delta(r) < 2$ and

$$\delta(r) - 1 \leqslant (n/2)(p_1 - 1), \tag{3.37}$$

$$2 \leqslant (p_j - 1)l' \leqslant r \tag{3.38}$$

and

$$2[(p_j - 1)m']^{-1} \geqslant \delta((p_j - 1)l') - 1 . \tag{3.39}$$

The last two conditions are equivalent to

$$4\delta'/n \leqslant p_1 - 1 \leqslant p_2 - 1 \leqslant 4\delta'(n - 2\delta(r))^{-1} \tag{3.40}$$

and

$$2/m' \geqslant (p_j - 1)(n/2 - 1) - 2\delta' \tag{3.41}$$

The existence of m' satisfying (3.36) and (3.41) is ensured by (3.21) which is part of (H3), and it remains only to ensure (3.37) and (3.40) with $1 \leqslant \delta(r) < 2$ and $0 \leqslant \delta' < 1$ under (H3). Now (3.37) and (3.40) follow from (3.2) by choosing $\delta' = (1/2)(\delta(r) - 1)$ and eliminating $\delta(r)$, or from (3.3) whenever relevant by choosing $\delta(r)$ close to 2 and eliminating δ'. Taking in (3.35) the Supremum over t in a sufficiently small interval containing t_0, one obtains a linear inequality which implies that $\varphi_1 = \varphi_2$ in that interval. Iterating the process yields $\varphi_1 = \varphi_2$ everywhere in I. Q. E. D.

Under the assumptions of both Propositions 2.1 and 3.1 we obtain existence and uniqueness of the solution of the Cauchy problem for the equation (1.1). In addition the solutions satisfy the conservation of the energy, which implies a regularity in time stronger than the previous one.

PROPOSITION 3.2. — Let f satisfy (H2) and (H3), let $t_0 \in \mathbb{R}$ and let $\varphi_0 \in H^1$. Then the equation (1.1) has a unique solution in $L^\infty_{loc}(\mathbb{R}, H^1)$ with $\varphi(t_0) = \varphi_0$. The solution φ belongs to $\mathscr{C}_b(\mathbb{R}, H^1)$ and satisfies the conservation of the L^2-norm (2.11) and the conservation of the energy

$$E(\varphi(t)) = E(\varphi_0)\,. \tag{3.42}$$

Proof. — Propositions 2.1 and 3.1 assert the existence and uniqueness of the solution φ of the equation (1.1) in $L^\infty_{loc}(\mathbb{R}, H^1)$ and the fact that $\varphi \in L^\infty(\mathbb{R}, H^1) \cap \mathscr{C}_w(\mathbb{R}, H^1) \cap \bigcap_{2 \leqslant r < 2^*} \mathscr{C}^{\alpha(r)}(\mathbb{R}, L^r)$. In addition, by uniqueness, we can apply time reversal to the equation (1.1) and therefore to the energy inequality (2.13), thereby obtaining the energy conservation (3.42). It remains to prove the strong continuity of φ in H^1. Since $\varphi \in \bigcap_{2 \leqslant r < 2^*} \mathscr{C}^{\alpha(r)}(\mathbb{R}, L^r)$, the function $(1/2)\,\|\nabla\varphi(t)\|_2^2 = E(\varphi_0) - \int dx V(\varphi(t))$ is a continuous function of time, so that the H^1-norm of φ is a continuous function of time. The strong continuity of φ in H^1 follows now from the weak continuity and the continuity of the norm. Q. E. D.

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REFERENCES

[1] J. B. BAILLON, T. CAZENAVE, M. FIGUERA, *C. R. Acad. Sci. Paris*, t. **284**, 1977, p. 869-872.
[2] J. BERGH, J. LÖFSTRÖM, *Interpolation Spaces*, Springer, Berlin-Heidelberg-New York, 1976.
[3] T. CAZENAVE, *Proc. Roy. Soc. Edinburgh*, t. **84**, 1979, p. 327-346.
[4] T. CAZENAVE, A. HARAUX, *Ann. Fac. Sc. Toulouse*, t. **2**, 1980, p. 21-25.
[5] I. EKELAND, R. TEMAM, *Analyse convexe et problèmes variationnels*, Dunod, Paris, 1972.
[6] J. GINIBRE, G. VELO, *J. Funct. Anal.*, t. **32**, 1979, p. 1-32 and *Ann. IHP*, t. **A 28**, 1978, p. 287-316.
[7] J. GINIBRE, G. VELO, *Ann. IHP*, t. **C1**, 1984, p. 309-323.
[8] J. GINIBRE, G. VELO, *C.R. Acad. Sci. Paris*, t. **298**, 1984, p. 137-140, and *J. Math. Pur. Appl.*, in press.
[9] J. GINIBRE, G. VELO, *Math. Z.*, 1985, in press.
[10] P. HARTMAN, *Ordinary differential equations*, Birkhäuser, Boston-Basel-Stuttgart, 1982.
[11] J. E. LIN, W. STRAUSS, *J. Funct. Anal.*, t. **30**, 1978, p. 245-263.
[12] J. L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod and Gauthier-Villars, Paris, 1969.
[13] J. L. LIONS, E. MAGENES, *Problèmes aux limites non homogènes et applications*, Vol. 1, Dunod, Paris, 1968.
[14] H. PECHER, *Math. Z.*, t. **185**, 1984, p. 261-270.
[15] H. PECHER, W. VON WAHL, *Manuscripta Math.*, t. **27**, 1979, p. 125-157.
[16] M. REED, *Abstract non-linear wave equations*, Springer, Berlin-Heidelberg-New York, 1976.
[17] I. E. SEGAL, *Bull. Soc. Math. France*, t. **91**, 1963, p. 129-135.
[18] E. M. STEIN, *Singular Integrals and differentiability properties of functions*, Princeton University Press, Princeton, 1970.
[19] W. STRAUSS, *Anais Acad. Brazil. Ciencias*, t. **42**, 1970, p. 645-651.
[20] W. STRAUSS, *J. Funct. Anal.*, t. **41**, 1981, p. 110-133. *Ibid.*, t. **43**, 1981, p. 281-293.
[21] R. STRICHARTZ, *Duke Math. J.*, t. **44**, 1977, p. 705-714.
[22] F. TREVES, *Basic linear differential equations*, Academic Press, New York-London, 1975.
[23] K. YOSIDA, *Functional analysis*, Springer, Berlin-Heidelberg-New York, 1978.

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Note added in proof.

The proof of Lemma 3.3 can be somewhat simplified by using systematically the more convenient Besov spaces instead of the Sobolev spaces and in particular the simpler and more efficient Lemma 3.2 of [9] instead of Lemma 3.2 of this paper.