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Radially symmetric cavitation for hyperelastic materials

by

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ABSTRACT. — The question of radially symmetric cavitation for a ball of hyperelastic material is considered. It reduces to a non-linear boundary value problem for a singular second order differential equation. For a broad class of stored-energy densities, the shooting method is used to determine whether or not cavitation occurs under various conditions on the boundary of the ball.

RÉSUMÉ. — On considère la question de cavitation avec symétrie radiale d'une boule d'un milieu hyperélastique. Elle est ramenée à un problème aux limites non-linéaire pour une équation différentielle singulière du deuxième ordre. Pour une grande classe de densités d'énergie, la méthode du tir permet de déterminer si oui ou non il y aura cavitation sous des conditions diverses sur le bord de la boule.

CONTENTS

1. INTRODUCTION

Let $\Omega = \{ x \in \mathbb{R}^N : |x| < 1 \}$ where $N \geqslant 2$ and consider a piece of homogeneous isotropic material occupying the region Ω. Radial deformations of this body are given by functions $u : \Omega \to \mathbb{R}^N$ which have the form,

$$u(x) = \frac{U(r)}{r} x \qquad \text{where} \qquad 0 < |x| = r < 1$$

and $U : (0, 1) \to (0, \infty)$. A radial deformation is in equilibrium if u satisfies the equations of elastostatics and these reduce to an ordinary differential equation for U which is given in section 2. In order to avoid self-penetration of the body, it is natural to require U to be strictly increasing on (0, 1). If $U(0) \equiv \lim_{r \to 0+} U(r) = 0$, the deformed body corresponding to U is again a ball of radius $U(1) \equiv \lim_{r \to 1-} U(r)$. If $U(0) > 0$ the deformed body is a ball with a spherical hole in the middle. In this case the original solid ball has ruptured and a spherical cavity of radius $U(0) > 0$ has formed. The basic problem is to establish the existence of radial equilibrium deformations with cavities and to discuss their stability. These issues are the subject of a fundamental paper by Ball [*1*].

The contribution which we offer differs from Ball's work in two respects. Firstly we deal directly with the ordinary differential equation for U corresponding to the equilibrium equations. Our results are obtained by a version of the « shooting method » and so involve only elementary arguments for differential equations as opposed to the combination of variational and differential equation techniques employed by Ball. Since we deal only with solutions of the equilibrium equations our discussion cannot yield a complete analysis of the stability of the solutions. This involves the study of the energy in a full neighbourhood of a solution in an appropriate function space. In this respect our analysis of the problem is less complete then Ball's. On the other hand we deal with the general form of the constitutive assumption for nonlinear hyperelasticity rather than the special form (4.4) treated by Ball. To carry through our analysis we make a number of assumptions concerning the function which gives the stored-energy per unit volume in terms of the deformation. When we interpret these assumptions in the special case treated by Ball, we find that they reduce to conditions which are rather similar to (but in some respects less restrictive than) those introduced by Ball.

Having stressed the differences between the present approach and that used by Ball, let me close this introduction by acknowledging the extent to which I have benefitted from the numerous insights contained in Ball's paper.

The rest of this article is set out as follows. In section 2 the equilibrium equations are given and the problem is formulated as a nonlinear boundary value problem. The main results are then stated informally in section 3 and the method of proof is outlined. Section 4 contains the exact hypotheses concerning the stored-energy function which are used to obtain these results. The proofs of the main results are given in section 5, together with some additional qualitative information about the behaviour of solutions. Finally in section 6, the energies of the various solutions are compared.

2. FORMULATION OF THE BOUNDARY VALUE PROBLEM

Let $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N$ be a sufficiently smooth deformation and let $S(x)$ be the corresponding Piola-Kirchhoff stress matrix at $x \in \Omega$, [2, Chapter 7]. In the absence of body forces, the conditions for equilibrium are that,

$$\sum_{j=1}^{N} \frac{\partial}{\partial x_j} \{ S_{ij}(x) \} = 0 \qquad \text{for} \quad 1 \leqslant i \leqslant N \tag{2.1}$$

and

$$S(x)F(x)^t = F(x)S(x)^t, \tag{2.2}$$

for $x \in \Omega$, where $F_{ij}(x) = \dfrac{\partial u_i}{\partial x_j}(x)$ for $1 \leqslant i, j \leqslant N$ and t denotes the transpose of a matrix. The matrix $F(x) = \nabla u(x)$ is referred to as the deformation gradient at x. Physical deformations are one to one and are subject to the restriction

$$\det \nabla u(x) > 0 \qquad \text{for all} \quad x \in \Omega \tag{2.3}$$

[2, Chapter 2].

A material is said to be hyperelastic if there exists a function $W : M \to \mathbb{R}$ such that

$$S(x) = T(\nabla u(x)) \qquad \text{for} \quad x \in \Omega, \tag{2.4}$$

for all physical (sufficiently smooth) deformations where M is the set of $(N \times N)$-matrices having positive determinant and

$$T(F) \equiv \frac{\partial W}{\partial F}(F) \qquad \text{for} \quad F \in M \tag{2.5}$$

is called the Piola-Kirchhoff stress at F, [2, Chapter 8]. The function W is known as the stored-energy function for the material and the Cauchy stress at $F \in M$ is defined by

$$\tilde{T}(F) = \frac{T(F)F^t}{\det F}. \tag{2.6}$$

The assumptions of frame indifference and isotropy of the material imply [1; section 3] that W can be expressed in the following way:

$$W(F) = \Phi(v_1, \ldots, v_N) \qquad \text{for} \quad F \in M, \tag{2.7}$$

where $\Phi: (0, \infty)^N \to \mathbb{R}$ is a symmetric function and $v_1, v_2, \ldots, v_N$ are the eigenvalues of $(F^tF)^{\frac{1}{2}}$. It follows from this that

$$T(F)F^t = FT(F)^t \qquad \forall F \in M \tag{2.8}$$

and so the conditions (2.2) are satisfied by every physical (sufficiently smooth) deformation of a hyperelastic material. Thus the conditions for equilibrium reduce to,

$$\sum_{j=1}^{N} \frac{\partial}{\partial x_j} \{ T_{ij}(\nabla u(x)) \} \qquad \text{for} \quad 1 \leqslant i \leqslant N, \tag{2.9}$$

for $x \in \Omega$, where T is given by (2.5).

For a hyperelastic material, the total stored energy of a deformation $u: \Omega \to \mathbb{R}^N$ is given by

$$E(u) \equiv \int_\Omega W(\nabla u(x))dx \tag{2.10}$$

and the equilibrium equations (2.9) are seen to be conditions that u be a stationary point of E in some suitable function space.

Henceforth $\Omega = \{ x \in \mathbb{R}^N : |x| < 1 \}$ and we consider only radial deformations:

$$u(x) = \frac{U(r)}{r} x \qquad \text{for} \quad 0 < |x| = r < 1 \tag{2.11}$$

where $U : (0, 1) \to (0, \infty)$. Thus,

$$\nabla u(x) = \frac{U(r)}{r} I + \left[U'(r) - \frac{U(r)}{r} \right] \frac{x \otimes x}{r^2} \tag{2.12}$$

is a symmetric matrix with eigenvalues,

$$v_1 = U'(r) \quad \text{and} \quad v_i = \frac{U(r)}{r} \qquad \text{for} \quad 2 \leqslant i \leqslant N.$$

Hence,

$$\det \nabla u(x) = U'(r) \left[\frac{U(r)}{r} \right]^{N-1} \in M$$

if and only if $U'(r) > 0$. In keeping with (2.3) we require that

$$U'(r) > 0 \qquad \text{for} \quad 0 < r < 1 \tag{2.13}$$

and we note that this excludes self-penetration of the body. Furthermore for a radial deformation,

$$T(\nabla u(x)) = \Phi_2 I + [\Phi_1 - \Phi_2]\frac{x \otimes x}{r^2} \tag{2.14}$$

and

$$\tilde{T}(\nabla u(x)) = \frac{1}{U'(r)}\left[\frac{U(r)}{r}\right]^{1-N}\left\{\frac{U(r)}{r}\Phi_2 I + \left[U'(r)\Phi_1 - \frac{U(r)}{r}\Phi_2\right]\frac{x \otimes x}{r^2}\right\} \tag{2.15}$$

where Φ_i denotes the i-th partial derivative of Φ evaluated at the argument $\left(U'(r), \frac{U(r)}{r}, \ldots, \frac{U(r)}{r}\right)$. With this notation, the equations (2.9) for equilibrium for a radial deformation reduce to,

$$r\frac{d}{dr}\Phi_1 = (N - 1)[\Phi_2 - \Phi_1] \tag{2.16}$$

which is a second order ordinary differential equation for $U : (0, 1) \to (0, \infty)$.

In a displacement boundary value problem [2, Chapter 10] the value of u is prescribed for all $x \in \partial\Omega$. For a radial deformation, this amounts to specifying the value of U at $r = 1$. Thus

$$U(1) = \lambda \qquad \text{where} \quad \lambda > 0 \quad \text{is given}. \tag{2.17}$$

For a radial deformation without a cavity we have

$$U(0) \equiv \lim_{r \to 0+} U(r) = 0, \tag{2.18}$$

whereas, if there is a cavity,

$$U(0) > 0. \tag{2.19}$$

In the case of a cavity (vacuous), the Cauchy stress on the boundary of the cavity should be zero. Thus we require

$$\lim_{r \to 0+} \tilde{T}(\nabla u(x))\frac{x}{r} = \lim_{r \to 0+}\left[\frac{U(r)}{r}\right]^{1-N}\Phi_1\frac{x}{r} = 0 \tag{2.20}$$

where Φ_1 is evaluated at $\left(U'(r), \frac{U(r)}{r}, \ldots, \frac{U(r)}{r}\right)$. (More generally, if the cavity is filled by material with hydrostatic pressure γ, then (2.20) is replaced by

$$\lim_{r \to 0+}\left(\left[\frac{U(r)}{r}\right]^{1-N}\Phi_1 = \gamma\right) \tag{2.21}$$

Summarising these formulae, the problem of radially symmetric cavitation for a displacement boundary value problem for a homogeneous isotropic hyperelastic material can be formulated as follows. Find

$$U \in C([0, 1]) \cap C^2((0, 1))$$

such that

$$U(r) > 0 \quad \text{and} \quad U'(r) > 0 \quad \text{for} \quad 0 < r < 1, \tag{2.22}$$

$$r\frac{d}{dr}\Phi_1 = (N-1)[\Phi_2 - \Phi_1] \quad \text{for} \quad 0 < r < 1, \tag{2.23}$$

$$U(1) = \lambda > 0 \tag{2.24}$$

and either

$$U(0) = 0 \tag{2.25}$$

or

$$U(0) > 0 \quad \text{and} \quad \lim_{r \to 0+} \tilde{T}(r) = 0 \tag{2.26}$$

where Φ_i denotes the i-th partial derivative of Φ evaluated at

$$\left(U'(r), \frac{U(r)}{r}, \ldots, \frac{U(r)}{r}\right)$$

and

$$\tilde{T}(r) = \left[\frac{U(r)}{r}\right]^{1-N} \Phi_1 . \tag{2.27}$$

Using our Lemma 8 and the fact that (2.23) can be written as

$$\frac{d}{dr}\{ r^{N-1}\Phi_1 \} = (N-1)r^{N-2}\Phi_2 ,$$

it follows from Ball's Theorem 4.2 that if U satisfies (2.22) to (2.26) then the function u defined by (2.11) belongs to $W^{1,p}(\Omega)$ for all $p \in [1, N)$ and is a weak solution of the system (2.9) in the usual sense.

Cavitation is also of interest for other types of boundary value problem. In a Cauchy traction boundary value problem the Cauchy stress is prescribed for all $x \in \partial\Omega$. Thus

$$\tilde{T}(\nabla u(x))n(x) = \gamma(x) \qquad \text{for} \quad x \in \partial\Omega$$

where $\gamma : \partial\Omega \to \mathbb{R}^N$ is a given function and $n(x)$ denotes the unit outward normal to the deformed boundary at the point $u(x)$. For a radial deformation (2.11), we have $n(x) = \frac{x}{r} = x$ since $\partial\Omega = \{ x \in \mathbb{R}^N : |x| = 1 \}$ and

$$\tilde{T}(\nabla u(x))x = [U(1)]^{1-N}\Phi_1(U'(1), U(1), \ldots, U(1))x$$

for $x \in \partial\Omega$. Thus the function γ must have the form, $\gamma(x) = Px$ for $x \in \partial\Omega$ where $P \in \mathbb{R}$ is a given constant. For radial deformations the problem for the Cauchy traction problem reduces to the system (2.22) to (2.26) with the condition (2.24) replaced by

$$[U(1)]^{1-N}\Phi(U'(1), U(1), \ldots, U(1)) = P \tag{2.28}$$

where $P \in \mathbb{R}$ is a given constant.

In a dead-load traction boundary value problem, it is the Piola-Kirchhoff stress which is prescribed at for all $x \in \partial\Omega$. Thus

$$T(\nabla u(x))N(x) = \gamma(x) \qquad \text{for} \quad x \in \partial\Omega\,,$$

where $\gamma : \partial\Omega \to \mathbb{R}^N$ is a given function and $N(x)$ is the unit outward normal to $\partial\Omega$ at the point x. For a radial deformation, $N(x) = x$ for $x \in \partial\Omega$ and

$$T(\nabla u(x))x = \Phi_1(U'(1), U(1), \ldots, U(1))x \qquad \text{for} \quad x \in \partial\Omega\,.$$

Thus the fonction γ must have the form $\gamma(x) = px$ for $x \in \partial\Omega$, where $p \in \mathbb{R}$ is a given constant. For radial deformations (2.11), the problem of cavitation for the dead-load traction boundary value problem reduces to the (2.22) to (2.26) with the condition (2.24) replaced by

$$\Phi_1(U'(1), U(1), \ldots, U(1)) = p \tag{2.29}$$

where $p \in \mathbb{R}$ is a given constant.

Our results are obtained primarily for the displacement boundary value problem. However they do yield some information about both the Cauchy and dead-load traction problems. Observe that if we have a radial solution of a Cauchy traction problem there exists a value $\lambda > 0$ (depending on P and denoted $\lambda(P)$) for which this solution satisfies the displacement boundary condition,

$$U(1) = \lambda(P)\,.$$

Conversely, by considering all radial solutions of the displacement problem for all positive values of λ we obtain the solutions to all Cauchy traction problems for all possible values of P. If U satisfies the displacement problem with $U(1) = \lambda$, it satisfies the Cauchy traction problem for

$$P = \lambda^{1-N}\Phi_1(U'(1), \lambda, \ldots, \lambda) \tag{2.30}$$

Similarly, we obtain solutions of the dead-load traction problem for

$$p = \Phi_1(U'(1), \lambda, \ldots, \lambda)\,. \tag{2.31}$$

Finally we note the expression for the total energy of a radial deformation associated with each of these boundary value problems:

$$EC(u) = ED(u) - \frac{PU(1)^N}{N}\omega_N\,, \tag{2.33}$$

$$ET(u) = ED(u) - pU(1)\omega_N\,, \tag{2.34}$$

where ω_N is the surface area of the unit sphere in $\mathbb{R}^N$, and ED, EC and ET refer to the displacement, Cauchy traction and dead-load traction problems respectively.

3. DESCRIPTION OF THE RESULTS

The analysis of the boundary value problem (2.22) to (2.26) is based upon the following observations.

For $\alpha > 0$, replace the conditions (2.25) and (2.26) by the condition,

$$U'(1) = \alpha . \tag{3.1}$$

Let $U(\lambda, \alpha)$ denote the (unique) solution of the initial value problem posed by (2.22) to (2.24) and (3.1). Solving the original boundary value problem (2.22) to (2.26) then amounts to identifying, for each $\lambda > 0$, those values of $\alpha > 0$ such that $U(\lambda, \alpha)$ satisfies either (2.25) or (2.26).

For $\alpha > \lambda > 0$, we show that $U(\lambda, \alpha)$ cannot be defined on all of the interval $(0, 1]$ and so such solutions cannot lead to solutions of (2.25) or (2.26). For $\alpha = \lambda > 0$, it is easily seen that

$$U(\lambda, \lambda)(r) = \lambda r \qquad \text{for} \quad r > 0$$

and so $U(\lambda, \lambda)$ satisfies (2.22) to (2.25). This is a homogeneous radial deformation without cavity. For $0 < \alpha < \lambda$, we show that $U(\lambda, \alpha)$ is defined on $(0, 1]$ and that $U''(r) > 0$ on $(0, 1)$. Hence we have

$$0 < \lambda - \alpha < U(\lambda, \alpha)(0) < \lambda .$$

Furthermore we find that for $0 < \alpha < \lambda$,

$$\tau(\lambda, \alpha) \equiv \lim_{r \to 0+} \tilde{T}(\lambda, \alpha)(r) \quad \text{exists}$$

where
$$\tilde{T}(\lambda, \alpha)(r) = \left[\frac{U(r)}{r}\right]^{1-N} \Phi_1\left(U'(r), \frac{U(r)}{r}, \ldots, \frac{U(r)}{r}\right).$$

and $U = U(\lambda, \alpha)$. Thus, for $0 < \alpha < \lambda$, solving the problem (2.22) to (2.26) amounts to finding α such that $\tau(\lambda, \alpha) = 0$.

Let $\Delta \equiv \{ (\lambda, \alpha) \in \mathbb{R}^2 : 0 < \alpha < \lambda \}$. We show that $\tau : \Delta \to \mathbb{R}$ is continuous,

$$\tau(\lambda, .) : (0, \lambda) \to \mathbb{R} \qquad \text{is strictly increasing},$$

$$\tau(., \alpha) : (\alpha, \infty) \to \mathbb{R} \qquad \text{is strictly increasing},$$

$$\lim_{\alpha \to 0+} \tau(\lambda, \alpha) = -\infty \qquad \text{and} \qquad \lim_{\alpha \to \lambda-} \tau(\lambda, \alpha) = g(\lambda)$$

where a formula for $g(\lambda)$ is given. The function $g : (0, \infty) \to \mathbb{R}$ is continuous and strictly increasing with $\lim_{\lambda \to 0+} g(\lambda) = -\infty$ and $\lim_{\lambda \to \infty} g(\lambda) = \infty$.

[It is important to realise that

$$g(\lambda) = \tau(\lambda, \lambda-) \neq \tau(\lambda, \lambda) = \lambda^{1-N}\Phi_1(\lambda, \lambda, \ldots, \lambda).$$

In fact, $\quad g(\lambda) = \tau(\lambda, \lambda^-) < \tau(\lambda, \lambda) \qquad \forall \lambda > 0.$]

From the properties of τ and g which have just been described, it follows that there exist a value $\lambda^* > 0$ and a continuous function $w : (\lambda^*, \infty) \to (0, \infty)$ such that $w(\lambda) \in (0, \lambda)$ for all $\lambda > \lambda^*$ and

$$\{ (\lambda, \alpha) \in \Delta : \tau(\lambda, \alpha) = 0 \} = \{ (\lambda, w(\lambda)) : \lambda > \lambda^* \} .$$

Thus, for $0 < \lambda \leqslant \lambda^*$, the problem (2.22) to (2.26) has only one solution, namely $U(\lambda, \lambda)$ and this has no cavity. For $\lambda > \lambda^*$, the problem has exactly two solutions, namely $U(\lambda, \lambda)$ and $U(\lambda, w(\lambda))$. The deformation corresponding to $U(\lambda, w(\lambda))$ has a cavity of radius $R(\lambda) \equiv U(\lambda, w(\lambda))(0) > \lambda - w(\lambda) > 0$. We show that R is a strictly increasing, concave continuous function with $\lim_{\lambda \to \lambda^*+} R(\lambda) = 0$ and $\lim_{\lambda \to \infty} R(\lambda) = \infty$.

In fact, the thickness of the shell is $\lambda - R(\lambda) \leqslant w(\lambda)$ and $\lim_{\lambda \to \infty} w(\lambda) = 0$. Furthermore, $U(\lambda, w(\lambda))$ converges to the homogeneous deformation $U(\lambda^*, \lambda^*)$ uniformly on compact subsets of $(0, 1]$ as $\lambda \to \lambda^*+$. In this sense there is a bifurcation from the homogeneous solutions to solutions with cavities at $\lambda = \lambda^*$. See Figures 1 and 2.

We now discuss the location of the critical value λ^*. A value of λ such that $\Phi_1(\lambda, \lambda, \ldots, \lambda) = 0$ is called a natural radius for the body because, for such values of λ, the Piola-Kirchhoff (equivalently Cauchy) stress associated with the homogeneous deformation $U(\lambda, \lambda)(r) = \lambda r$ is everywhere zero in Ω. Our assumptions imply the existence of at least one natural radius, but since we make no assumption about the monotonicity of $\lambda^{1-N}\Phi_1(\lambda, \ldots, \lambda)$, there may be several such values. In any case, if λ_{nat} is any natural radius then $\lambda^* > \lambda_{nat}$. This follows from the fact that

$$g(\lambda) < \lambda^{1-N}\Phi_1(\lambda, \lambda, \ldots, \lambda) \qquad \forall \lambda > 0 .$$

The exact value of λ^* can only be obtained in very special cases because the formula which gives $g(\lambda)$ involves the integration of a function $Q(\lambda)$ which is defined as the (unique) solution of a first order ordinary differential equation satisfying the initial condition, $Q(\lambda)(t) = \lambda$ at $t = \lambda$. In general, this solution $Q(\lambda)$ is not known explicitly and so only estimates for $g(\lambda)$ can be found by using approximations to $Q(\lambda)$. This implicit character is common to our bifurcation equation, $g(\lambda) = 0$, and to Ball's bifurcation equation [*1*, (7.31)] which also involves the solution of a differential equation. However our equation is obtained in a more direct way from the equilibrium equation. For $\lambda > 0$, let $ED_t(\lambda)$ be the total energy of the homogeneous displacement $U(\lambda, \lambda)$ and, for $\lambda > \lambda^*$, let $ED_c(\lambda)$ be the total energy of the solution $U(\lambda, w(\lambda))$ having a cavity. It turns out that

$$ED_t(\lambda) = \frac{\omega_N}{N} \Phi(\lambda, \lambda, \ldots, \lambda) \qquad \forall \lambda > 0$$

and

$$ED_c(\lambda) = \frac{\omega_N}{N} \{ \Phi(w(\lambda), \lambda, \ldots, \lambda) + [\lambda - w(\lambda)]\Phi_1(w(\lambda), \lambda, \ldots, \lambda) \} \qquad \forall \lambda > \lambda^* . \quad (3.2)$$

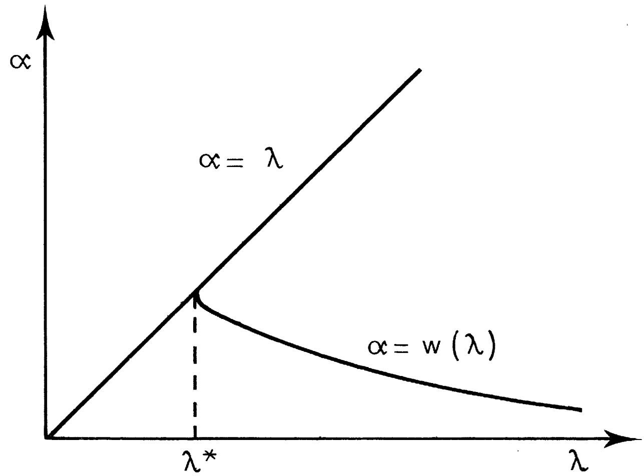

FIG. 1. — The solutions corresponding to the branch $\alpha = \lambda$ are the homogeneous deformations $U(\lambda, \lambda)(r) = \lambda r$ and have no cavities. The solutions corresponding to the branch $\alpha = w(\lambda)$ have cavities of radius $R(\lambda)$. There are no other solutions.

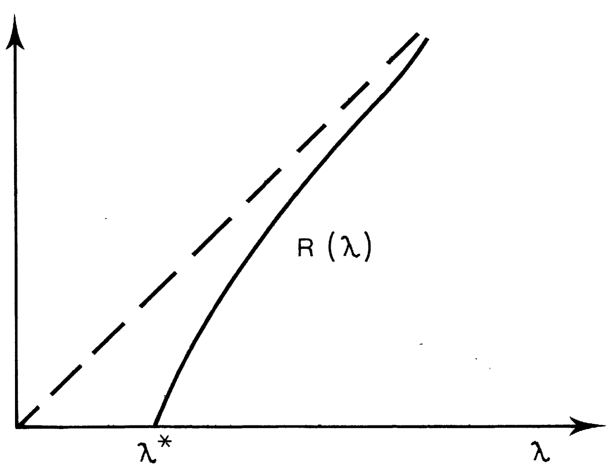

FIG. 2. — The cavity radius, $R(\lambda)$, is an increasing, strictly concave function of λ with

$$0 = \lim_{\lambda \to \lambda^*} R(\lambda) = \lim_{\lambda \to \infty} \{ \lambda - R(\lambda) \} \quad \text{and} \quad \lim_{\lambda \to \infty} \frac{dR(\lambda)}{d\lambda} = 1 .$$

For the displacement problem, the energy $ED_c(\lambda)$ of the solution $U(\lambda, w(\lambda))$ increases continuously from $ED_t(\lambda^*)$ to $+\infty$ as λ increases from λ^* to $+\infty$, where $ED_t(\lambda^*)$ is the energy of the homogeneous deformation $U(\lambda^*, \lambda^*)(r) = \lambda^* r$. Furthermore $ED_c(\lambda) < ED_t(\lambda)$ for all $\lambda > \lambda^*$.

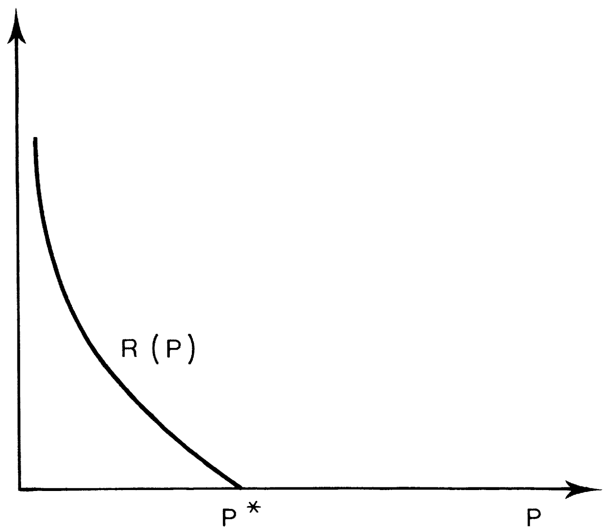

FIG. 3. — For the Cauchy traction problem, let $R(P) = R(\lambda_P)$ be the cavity radius for $P \in (0, P^*)$ where λ_P is the unique solution of $P = \lambda_P^{1-N}\Phi(w(\lambda_P), \lambda_P, \ldots, \lambda_P)$. As P increases from 0 to P^*, λ_P decreases continuously from $+\infty$ to λ^* and the energy, $EC_c(P)$, decreases continuously from $+\infty$ to $EC_t(P^*)$ (the energy of the homogeneous deformation $r \to \lambda^* r$).

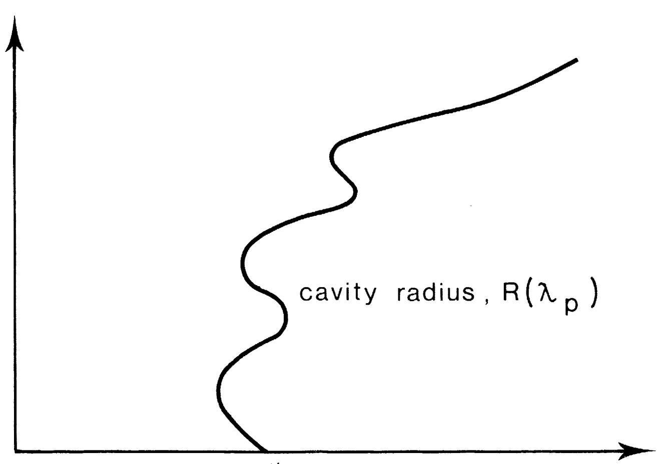

FIG. 4. — In the dead-load traction problem, there may be several values, λ_P, such that $p = \Phi_1(w(\lambda_P), \lambda_P, \ldots, \lambda_P)$. In any case, as p tends to $+\infty$, the radii of cavities tend to infinity and the energies of solutions also tend to infinity.

Our hypotheses imply that,

$$\mathrm{ED}_c(\lambda) < \mathrm{ED}_t(\lambda) \qquad \forall \lambda > \lambda^*$$

indicating that cavitation is energetically favourable for the displacement problem when $\lambda > \lambda^*$. We also have that $\mathrm{ED}_c(\lambda)$ is strictly increasing on (λ^*, ∞) with

$$\lim_{\lambda \to \lambda^*+} \mathrm{ED}_c(\lambda) = \mathrm{ED}_t(\lambda^*) \qquad \text{and} \qquad \lim_{\lambda \to \infty} \mathrm{ED}_c(\lambda) = +\infty$$

The Cauchy stress on the boundary of Ω for the solution $\mathrm{U}(\lambda, w(\lambda))$ is,

$$\mathrm{V}(\lambda) \equiv \lambda^{1-\mathrm{N}} \Phi_1(w(\lambda), \lambda, \ldots, \lambda) \tag{3.3}$$

and it turns out that this quantity is strictly decreasing as λ varies from λ^* to ∞, with

$$\lim_{\lambda \to \lambda^*} \mathrm{V}(\lambda) = [\lambda^*]^{1-\mathrm{N}} \Phi_1(\lambda^*, \lambda^*, \ldots, \lambda^*)$$

(the Cauchy stress for the homogeneous solution $\mathrm{U}(\lambda^*, \lambda^*)$) and $\lim_{\lambda \to \infty} \mathrm{V}(\lambda) = 0$. Thus the Cauchy traction problem has a solution with a cavity if and only if the constant P in (2.28) satisfies $\mathrm{P} \in (0, \mathrm{P}^*)$ where $\mathrm{P}^* = [\lambda^*]^{1-\mathrm{N}} \Phi_1(\lambda^*, \lambda^*, \ldots, \lambda^*)$. For each $\mathrm{P} \in (0, \mathrm{P}^*)$, there is a unique solution of the Cauchy traction problem having a cavity and this solution is given by $\mathrm{U}(\lambda_\mathrm{P}, w(\lambda_\mathrm{P}))$ where λ_P is the unique value of $\lambda > \lambda^*$ such that $\lambda^{1-\mathrm{N}} \Phi_1(w(\lambda), \lambda, \ldots, \lambda) = \mathrm{P}$. We observe that $\lim_{\mathrm{P} \to 0+} \mathrm{R}(\lambda_\mathrm{P}) = \infty$ and $\lim_{\mathrm{P} \to \mathrm{P}^*-} \mathrm{R}(\lambda_\mathrm{P}) = 0$. The total energy associated with the deformation $\mathrm{U}(\lambda_\mathrm{P}, w(\lambda_\mathrm{P}))$ for the Cauchy traction problem is, by (2.33) and (3.2),

$$\begin{aligned} \mathrm{EC}_c(\mathrm{P}) &\equiv \mathrm{ED}_c(\lambda_\mathrm{P}) - \mathrm{P} \lambda_\mathrm{P}^\mathrm{N} \frac{\omega_\mathrm{N}}{\mathrm{N}} \\ &= \frac{\omega_\mathrm{N}}{\mathrm{N}} \{ \Phi(w(\lambda_\mathrm{P}), \lambda_\mathrm{P}, \ldots, \lambda_\mathrm{P}) - w(\lambda_\mathrm{P}) \Phi_1(w(\lambda_\mathrm{P}), \lambda_\mathrm{P}, \ldots, \lambda_\mathrm{P}) \} \end{aligned} \tag{3.4}$$

and our hypotheses imply that $\lim_{\mathrm{P} \to 0+} \mathrm{EC}_c(\mathrm{P}) = +\infty$.

For the dead-load traction problem, we denote by $\mathrm{D}(\lambda)$ the Piola-Kirchhoff stress on $\partial\Omega$ for the solution $\mathrm{U}(\lambda, w(\lambda))$. Thus, for $\lambda > \lambda^*$,

$$\mathrm{D}(\lambda) = \Phi_1(w(\lambda), \lambda, \ldots, \lambda). \tag{3.5}$$

Our hypotheses do not ensure the monotonicity in λ of this quantity, but we do show that $\lim_{\lambda \to \lambda^*+} \mathrm{D}(\lambda) = \Phi_1(\lambda^*, \lambda^*, \ldots, \lambda^*)$ (the Piola-Kirchnoff stress p^* for the homogeneous solution $\mathrm{U}(\lambda^*, \lambda^*)$) and $\lim_{\lambda \to \infty} \mathrm{D}(\lambda) = \infty$. From this we can conclude that the dead-load traction problem has a solution with a cavity if and only if the constant p in (2.29) belongs to a semi-infinite interval G and $(p^*, \infty) \subset \mathrm{G}$. For $p \in \mathrm{G}$ there is at least one value of $\lambda > \lambda^*$ such that $\Phi_1(w(\lambda), \lambda, \ldots, \lambda) = p$ and we denote such a value by λ_P. The total energy associated with the deformation $U(\lambda_P, w(\lambda_P))$ for the dead-load traction problem is, by (2.34) and (3.2),

$$\begin{aligned} ET_c(p) &\equiv ED_c(\lambda_p) - p\lambda_p\omega_N \\ &= \frac{\omega_N}{N}\{\Phi(w(\lambda_p), \lambda_p, \ldots, \lambda_p) - [(N-1)\lambda_p + w(\lambda_p)]\Phi_1(w(\lambda_p), \lambda_p, \ldots, \lambda_p)\}. \end{aligned} \tag{3.6}$$

See Figure 4.

4. ASSUMPTIONS ON THE STORED-ENERGY FUNCTION

We consider the function introduced in (2.7).

A1) $\Phi : (0, \infty)^N \to \mathbb{R}$ is of class C^3 and symmetric. Thus

$$\Phi(\sigma(v_1, \ldots, v_N)) = \Phi(v_1, \ldots, v_N)$$

whenever σ is a permutation of the N variables $(v_1, \ldots, v_N)$. It follows that

$$\Phi_1(v_1, \ldots, v_N) = \Phi_2(v_1, \ldots, v_N) \quad \text{if} \quad v_1 = v_2$$

and $\Phi_2(v_1, \ldots, v_N) = \Phi_i(v_1, \ldots, v_N)$ for $i \geqslant 2$ if $v_2 = v_3 \ldots = v_N$.

Since we are concerned only with radial deformations our considerations only involve Φ and its partial derivatives evaluated at arguments of the form $v_2 = v_3 \ldots = v_N$ and this will be indicated by writing

$$\Phi_i(q, t, \ldots, t) \quad \text{for} \quad q, t \in (0, \infty).$$

A2) $\Phi_{11}(q, t, \ldots, t) > 0$ $\forall q, t \in (0, \infty)$ and $\exists$ constants $C > 0$ and $t_0 > 0$ such that $\Phi_{11}(q, t, \ldots, t) \geqslant Ct^{2(N-1)}$ whenever $0 < q < t$ and $t \geqslant t_0$.

A3) $\forall b > 0$ we have

$$\lim_{(q,t)\to(0,b)} t^{1-N}\Phi_1(q, t, \ldots, t) = -\infty$$

and

$$\lim_{(q,t)\to(b,\infty)} t^{1-N}\Phi_1(q, t, \ldots, t) = +\infty.$$

A4) $$\lim_{t\to 0+} t^{1-N}\Phi_1(t, t, \ldots, t) = -\infty$$

and $$\lim_{t\to\infty} t^{1-N}\Phi_1(t, t, \ldots, t) = +\infty$$

A5) $\inf \Phi > -\infty$ and $\dfrac{\Phi_2 - \Phi_1}{q - t} - \Phi_{12} < 0 \quad \forall q, t \in (0, \infty)$ with $q \neq t$

where the partial derivatives of Φ are evaluated at $(q, t, \ldots, t)$.

Let $P : (0, \infty)^2 \to \mathbb{R}$ and $R : (0, \infty)^2 \to \mathbb{R}$ be the functions defined as follows:

$$P(q, t) = \begin{cases} \dfrac{\Phi_2(q, t, \ldots, t) - \Phi_1(q, t, \ldots, t)}{q - t} & \text{for} \quad q \neq t \\ \Phi_{12}(t, t, \ldots, t) - \Phi_{11}(t, t, \ldots, t) & \text{for} \quad q = t \end{cases} \tag{4.1}$$

$$R(q, t) = \begin{cases} \dfrac{q\Phi_1(q, t, \ldots, t) - t\Phi_2(q, t, \ldots, t)}{q - t} & \text{if} \quad q \neq t \\ \Phi_1(t, t, \ldots, t) + t[\Phi_{11}(t, t, \ldots, t) - \Phi_{12}(t, t, \ldots, t)] & \text{if } q = t \end{cases} \tag{4.2}$$

Since $\Phi_1(t, t, \ldots, t) = \Phi_2(t, t, \ldots, t)$ it is easily seen that P and R are of class C^1 on $(0, \infty)^2$.

A6) $\exists$ constants $A > 0$, $B > 0$ and $0 < \beta < N - 1$ such that

$$0 < R(q, t) \leqslant A + Bt^{\beta} \qquad \text{for} \qquad 0 < q \leqslant t.$$

A7) $\exists$ constants $\varepsilon > 0$, $t_0 > 0$, $K > 0$ and $0 < \gamma < 2(N - 1)$ such that

$$| R(q, t) - R(q_1, t) | \leqslant Kt^{\gamma} | q - q_1 | \qquad \text{for} \qquad 0 < q, q_1 \leqslant \varepsilon \quad \text{and} \quad t \geqslant t_0.$$

Clearly (A7) is implied by the assumption that $\exists$ constants $\varepsilon > 0$, $t_0 > 0$, $K > 0$ and $\gamma < 2(N - 1)$, such that

$$\left| \frac{\partial R}{\partial q}(q, t) \right| \leqslant Kt^{\gamma} \qquad \text{for} \qquad 0 < q \leqslant \varepsilon \quad \text{and} \quad t \geqslant t_0. \tag{4.3}$$

In his article on cavitation Ball considers stored-energy functions which have the following special form:

$$\Phi(v_1, \ldots, v_N) = \sum_{i=1}^{N} \phi(v_i) + h\left(\prod_{i=1}^{N} v_i\right) \tag{4.4}$$

where ϕ and h are real-valued functions defined on $(0, \infty)$. We now introduce a series of hypotheses about ϕ and h which will imply that Φ satisfies the conditions (A1) to (A7).

B1) $\phi : (0, \infty) \to \mathbb{R}$ and $h : (0, \infty) \to \mathbb{R}$ are both of class C^3 and

$$\phi''(s) > 0, \qquad h''(s) \geqslant \delta > 0 \qquad \forall s > 0.$$

Thus ϕ and h are strictly convex on $(0, \infty)$.

(B2) $$\phi'(0+) \geqslant 0.$$

From (B1) and (B2) it follows that $\phi'(s) > 0$ $\forall s > 0$ and that ϕ and ϕ' are bounded on $(0, b]$, $\forall b > 0$. Furthermore $t \to t\phi'(t)$ is strictly increasing on $(0, \infty)$ and Φ is bounded below. (4.5).

B3) $\exists$ constants $A > 0$, $B > 0$ and $0 < \beta < N - 1$ such that

$$\phi'(t) \leqslant A + Bt^{\beta}$$

and

$$t\phi''(t) \leqslant A + Bt^{\beta} \qquad \forall t > 0 .$$

B4) $\exists$ constants $A > 0$ and $s_0 > 0$ such that

$$sh'(s) < -A \qquad \text{for} \qquad 0 < s < s_0 .$$

Hence we have that $\lim_{s \to 0+} h'(s) = -\infty$ and $\lim_{s \to 0+} h(s) = +\infty$.

B5)

$$\lim_{s \to \infty} h'(s) = +\infty .$$

The hypotheses for Ball's work on cavitation are given on pages 593 and 600 of his paper and it is rather easy to compare them with (B1) to (B5). In particular, for functions ϕ of the form,

$$\phi(s) = \mu s^{\gamma} \qquad \text{where} \qquad \mu > 0 \quad \text{and} \quad \gamma > 0 ,$$

we see that Ball's hypotheses require $2 \leqslant \gamma < N$ whereas as (B1) to (B5) are satisfied provided that $1 < \gamma < N$. On the other hand the assumptions (B1) and (B5) on h are a shade more restrictive than those required by Ball.

Let us check that when Φ has the form (4.4), the assumptions (B1) to (B5) do indeed imply that Φ satisfies the conditions (A1) to (A7).

Clearly (B1) implies (A1) and, setting $p = qt^{N-1}$,

$$\begin{aligned}
\Phi_1(q, t, \ldots, t) &= \phi'(q) + t^{N-1}h'(p) \\
\Phi_2(q, t, \ldots, t) &= \phi'(t) + qt^{N-2}h'(p) \\
\Phi_{11}(q, t, \ldots, t) &= \phi''(q) + t^{2(N-1)}h''(p) \\
\Phi_{12}(q, t, \ldots, t) &= t^{N-2}h'(p) + qt^{2N-3}h''(p) .
\end{aligned}$$

Thus (A2) also follows from (B1) and it is easy to verify (A3) and (A4) using the assumptions (B1), (B2), (B4) and (B5).

For (A5) we note that

$$\frac{\Phi_2 - \Phi_1}{q - t} - \Phi_{12} = \frac{\phi'(t) - \phi'(q)}{q - t} - qt^{2N-3}h''(p) < 0$$

by (B1).

For Φ of the form (4.4) we find that

$$R(q, t) = \begin{cases} \dfrac{q\phi'(q) - t\phi'(t)}{q - t} & \text{for} \quad q \neq t \\ \phi'(t) + t\phi''(t) & \text{for} \quad q = t . \end{cases}$$

Since by (4.5), $t \to t\phi'(t)$ is strictly increasing on $(0, \infty)$, we have that $0 < R(q, t)$, $\forall q, t > 0$.

Furthermore $R(q,t) \leqslant \sup_{q\leqslant s\leqslant t} \{ s\phi'(s) \}'$ for $q \leqslant t$

$$= \sup_{q\leqslant s\leqslant t} \{ \phi'(s) + s\phi''(s) \} .$$

Hence, by (B3), $R(q,t) \leqslant 2(A + Bt^{\beta})$ for $0 < q \leqslant t$. Thus we see that (A6) is satisfied.

For (A7), we note that for $q \neq t$,

$$\frac{\partial R}{\partial q}(q,t) = \frac{t\{ \phi'(t) - \phi'(q) \}}{(q-t)^2} + \frac{q\phi''(q)}{q-t} .$$

Thus for $0 < q < t$, since $\phi''(q) > 0$ and $\phi'(q) > 0$ we have

$$\frac{q\phi''(q)}{q-t} < \frac{\partial R}{\partial q} < \frac{t\phi'(t)}{(q-t)^2} .$$

By (B2) we can choose $\varepsilon > 0$ and $t_0 > 0$ such that

$$0 < q\phi''(q) \leqslant 2A \quad \text{and} \quad t - q > \frac{1}{2}t \qquad \text{for} \qquad 0 < q \leqslant \varepsilon \quad \text{and} \quad t \geqslant t_0 .$$

Then

$$\left| \frac{\partial R}{\partial q}(q,t) \right| \leqslant \max \left\{ \frac{q\phi''(q)}{t-q}, \frac{t\phi'(t)}{(q-t)^2} \right\}$$
$$\leqslant \max \left\{ \frac{4A}{t}, \frac{4\phi'(t)}{t} \right\} .$$

Using (B3) we now see that (4.3) is satisfied and consequently (A7) is verified.

We close this section with a few remarks about our hypotheses.

Remark. — 1. We have shown above how to verify our hypotheses when the function Φ has the special form (4.4). Although this form is consistent with the usual axiom *s* for the constitutive assumption in hyperelasticity, it is by no means implied by them. In fact for $N = 3$, it is quite common to take the function Φ in the following form:

$$\Phi(v_1, v_2, v_3) = \sum_{i=1}^{3} \phi(v_i) + h(v_1 v_2 v_3) + \psi(v_1 v_2) + \psi(v_2 v_3) + \psi(v_3 v_1) \quad (4.6)$$

where ϕ, h and ψ are real-valued functions defined on $(0, \infty)$. In much the same way as we have done for the form (4.4), it is not hard to give conditions on ϕ, h and ψ which ensure that a function Φ of the form (4.6) satisfies (A1) to (A7).

2. In (A5) it is assumed that Φ is bounded below, but this part of the assumption is only used to discuss the energy of the solutions in section 6. The basic results on existence, given in section 5, do not require this assump-

tion. Clearly this requirement, that $\inf \Phi > -\infty$, can be replaced by $\inf \Phi = 0$ without loss of generality.

3. The inequalities $\Phi_{11} > 0$ and $R > 0$ are implied by the strict rank-one convexity of W and are referred to as the tension-extension inequality and Baker-Ericksen inequality respectively [1].

4. The assumptions (A3) and (A4) can be interpreted in terms of the Cauchy stresses corresponding to the homogeneous deformations $x \to \operatorname{diag}(q, t, \ldots, t)x$ and $x \to tx$ of a unit cube and unit ball respectively.

5. The inequality $\dfrac{\Phi_2 - \Phi_1}{q - t} - \Phi_{12} < 0$ in (A5) does not seem to have a physical interpretation, but is discussed on page 583 of [1]. Likewise the growth conditions on Φ_{11}, R and $\dfrac{\partial R}{\partial q}$ in (A2), (A6) and (A7) seem to be of a technical rather than a physical nature.

5. PROOFS OF THE RESULTS

We discuss the boundary value problem (2.22) to (2.26). As described in section 3, we approach this problem by considering the following initial value problem,

$$r\frac{d\Phi_1}{dr} = (N-1)[\Phi_2 - \Phi_1] \qquad \text{for} \quad 0 < r < 1 \tag{5.1}$$

$$U(1) = \lambda, \qquad U'(1) = \alpha, \tag{5.2}$$

where $$U(r) > 0 \quad \text{and} \quad U'(r) > 0 \qquad \text{for} \qquad 0 < r < 1, \tag{5.3}$$

and, as usual, Φ_i denotes the i-th partial derivative of Φ evaluated at the argument

$$\left(U'(r), \frac{U(r)}{r}, \ldots, \frac{U(r)}{r}\right) \in (0, \infty)^N \qquad \text{for} \quad 0 < r < 1.$$

With this notation, we observe that (5.1) is equivalent to the equation,

$$r\Phi_{11}U''(r) = (N-1)\left\{\Phi_2 - \Phi_1 - \left(U'(r) - \frac{U(r)}{r}\right)\Phi_{12}\right\} \tag{5.4}$$

and so, by (A1) and (A2), the classical Picard theorem establishes the existence of a unique maximal solution of the system (5.1) to (5.3) for each pair $(\lambda, \alpha) \in (0, \infty)^2$. This solution will be denoted by

$$U(\lambda, \alpha) : J(\lambda, \alpha) \to (0, \infty)$$

where $J(\lambda, \alpha)$ is an open sub-interval of $(0, \infty)$, containing $r = 1$. For $r \in J(\lambda, \alpha)$, let

$$\tilde{T}(\lambda, \alpha)(r) = \left[\frac{U(r)}{r}\right]^{1-N} \Phi_1\left(U'(r), \frac{U(r)}{r}, \ldots, \frac{U(r)}{r}\right)$$

where $U = U(\lambda, \alpha)$. Then $\tilde{T}(\lambda, \alpha)$ gives the Cauchy stress via (2.20). Noting that (5.1) can be written as,

$$\frac{d}{dr}\{ r^{N-1}\Phi_1 \} = (N-1)r^{N-2}\Phi_2 ,$$

we see that

$$\frac{d\tilde{T}(r)}{dr} = -(N-1)\left[\frac{U(r)}{r}\right]^{-N} R\left(U'(r), \frac{U(r)}{r}\right)\left(\frac{U(r)}{r}\right)' \qquad (5.5)$$

for $r \in J(\lambda, \alpha)$ where $\tilde{T} = T(\lambda, \alpha)$, $U = U(\lambda, \alpha)$ and R is defined by (4.2).

LEMMA 1. — *a)* $0 < \alpha < \lambda$ and $r \in J(\lambda, \alpha)$ we have

$$\left(\frac{U(r)}{r}\right)' < 0, \qquad U''(r) > 0, \qquad \left(\frac{U(r)}{r}\right)'' > 0 \quad \text{and} \quad \tilde{T}'(r) > 0$$

where $U = U(\lambda, \alpha)$ and $\tilde{T} = \tilde{T}(\lambda, \alpha)$.

b) For $0 < \alpha = \lambda$ and $r \in J(\lambda, \lambda)$, we have

$$U(r) = \lambda r, \qquad \tilde{T}(r) = \lambda^{1-N}\Phi_1(\lambda, \lambda, \ldots, \lambda) \quad \text{and} \quad J(\lambda, \lambda) = (0, \infty).$$

c) For $0 < \lambda < \alpha$ and $r \in J(\lambda, \alpha)$ we have

$$\left(\frac{U(r)}{r}\right)' > 0 \quad \text{and} \quad U''(r) < 0 .$$

Proof. — *a)* At $r = 1$, $\left(\frac{U(r)}{r}\right)' = \alpha - \lambda < 0$.

Suppose that $\exists r_0 \in J(\lambda, \alpha)$ such that $\left(\frac{U(r)}{r}\right)' = 0$ at $r = r_0$. Then

$$U'(r_0) = \frac{U(r_0)}{r_0} = t \quad \text{(say)} .$$

Thus U satisfies (5.1) and the initial conditions

$$U(r_0) = tr_0, \qquad U'(r_0) = t .$$

Now it is easily verified that $w(r) = tr$ is a solution of this initial value problem and so by the uniqueness of the solution we have that

$$U(r) = w(r) = tr \qquad \text{for all} \quad r \in J(\lambda, \alpha) .$$

Clearly this implies that $\alpha = \lambda = t$, contradicting the fact that $0 < \alpha < \lambda$.

Hence we see that $\left(\frac{U(r)}{r}\right)' < 0$ on $J(\lambda, \alpha)$.

From (A5) and (5.4) it now follows immediately that

$$U''(r) > 0 \qquad \text{for} \quad r \in J(\lambda, \alpha).$$

since $\left(\frac{U(r)}{r}\right)'' = \frac{U''(r)}{r} - \frac{2}{r}\left(\frac{U(r)}{r}\right)'$, we have that $\left(\frac{U(r)}{r}\right)'' > 0$ for $r \in J(\lambda, \alpha)$.

Finally from (A6) and equation (5.5), we see that $\tilde{T}'(r) > 0$ for $r \in J(\lambda, \alpha)$.

b) Clearly $U(r) = \lambda r$ satisfies (4.2) and (4.3) with $\alpha = \lambda$. Since $\lambda > 0$ we have $U(r) > 0$ and $U'(r) > 0$ for $r > 0$.

c) The proof is similar to part (*a*).

LEMMA 2. — *a*) For $0 < \alpha \leqslant \lambda$, $\inf J(\lambda, \alpha) = 0$.

b) For $0 < \lambda < \alpha$, $\inf J(\lambda, \alpha) \geqslant 1 - \frac{\lambda}{\alpha} > 0$.

Proof. — *a*) Suppose that $0 < \alpha < \lambda$ and that $l \equiv \inf J(\lambda, \alpha) > 0$. By the maximality of U, at least one of the following cases must occur:

i) $$\lim_{r \to l+} \frac{U(r)}{r} = \infty,$$

ii) $$\lim_{r \to l+} \frac{U(r)}{r} = 0,$$

iii) $$\lim_{r \to l+} U'(r) = \infty,$$

iv) $$\lim_{r \to l+} U'(r) = 0.$$

Since $U'(r) > 0$ on $J(\lambda, \alpha)$, we have that $0 < \frac{U(r)}{r} < \frac{U(1)}{l}$ for $l < r \leqslant 1$. Thus (*i*) cannot occur.

By Lemma 1(*a*), $U''(r) > 0$ on $J(\lambda, \alpha)$ and so

$$U(r) - U(1) \geqslant (r - 1)U'(1) \qquad \text{for} \quad r \in J(\lambda, \alpha).$$

Hence $U(r) \geqslant \lambda + (l - 1)\alpha > \lambda - \alpha$ for $r \in J(\lambda, \alpha)$ and so (*ii*) cannot occur.

Furthermore, $U'(r) < U'(1)$ for $l < r < 1$ and hence (*iii*) cannot occur.

For $p \in J(\lambda, \alpha)$, it follows from (5.5) that

$$\begin{aligned} \tilde{T}(1) - \tilde{T}(p) &= -(N-1)\int_p^1 \left[\frac{U(r)}{r}\right]^{-N} R\left(U'(r), \frac{U(r)}{r}\right)\left(\frac{U(r)}{r}\right)' dr \\ &\leqslant -(N-1)K\int_p^1 \left[\frac{U(r)}{r}\right]^{-N}\left(\frac{U(r)}{r}\right)' dr \end{aligned}$$

for $l < p < 1$, by (A6) since $\left(\frac{U(r)}{r}\right)' < 0$ and $\frac{U(r)}{r} \leqslant \frac{U(1)}{l}$ for $r \in [l, 1]$.

Thus $\tilde{T}(p) \geqslant \tilde{T}(1) + (N-1)K \int_c^\lambda t^{-N} dt$ where $t = \frac{U(r)}{r}$ and $c = \frac{U(p)}{p}$.
Hence, we have that for $l < p < 1$,

$$\tilde{T}(p) \geqslant \tilde{T}(1) + K\{c^{1-N} - \lambda^{1-N}\} > \tilde{T}(1) - K\lambda^{1-N}$$

and so $\lim_{p\to l+} \tilde{T}(p) > -\infty$.

But
$$\tilde{T}(r) = \left[\frac{U(r)}{r}\right]^{1-N} \Phi_1\left(U'(r), \frac{U(r)}{r}, \ldots, \frac{U(r)}{r}\right).$$

Setting $b = \lim_{r\to l+} \frac{U(r)}{r}$ we have shown that $\lambda < b < \infty$ and so, if $\lim_{r\to l+} U'(r) = 0$, it follows from (A3) that $\lim_{r\to l+} \tilde{T}(r) = -\infty$. Hence we must conclude that $\lim_{r\to l+} U'(r) > 0$ and (*iv*) does not occur.

This proves that $\inf J(\lambda, \alpha) = 0$ for $0 < \alpha < \lambda$. The case $\alpha = \lambda > 0$ is trivial since $U(\lambda, \lambda)(r) = \lambda r$.

b) By Lemma 1(*c*), $U''(r) < 0$ on $J(\lambda, \alpha)$. Therefore

$$\begin{aligned} U(r) &\leqslant U(1) + (r-1)U'(1) \qquad \text{for} \quad r \in J(\lambda, \alpha) \\ &= \lambda + (r-1)\alpha . \end{aligned}$$

Since $U(r) > 0$ for $r \in J(\lambda, \alpha)$, it follows that $\inf J(\lambda, \alpha) \geqslant 1 - \frac{\lambda}{\alpha} > 0$.

Remark. — From Lemma 2(*b*) we see that there cannot be a solution of the boundary value problem corresponding to a case where $0 < \lambda < \alpha$. The solutions $U(\lambda, \lambda)$ corresponding to $\alpha = \lambda > 0$, do indeed give solutions of the system (2.22) to (2.25). Henceforth we need only consider the case $0 < \alpha < \lambda$ and for this we set

$$\Delta = \{(\lambda, \alpha) \in \mathbb{R}^2 : 0 < \alpha < \lambda\}.$$

LEMMA 3. — For $(\lambda, \alpha) \in \Delta$ and $0 < r < 1$ we have the following inequalities :

a) $$0 < \lambda - \alpha < U(r) < \lambda$$

b) $$0 < U'(r) < \alpha$$

c) $$\tilde{T}(r) < \lambda^{1-N}\Phi_1(\alpha, \lambda, \ldots, \lambda) < \lambda^{1-N}\Phi_1(\lambda, \lambda, \ldots, \lambda)$$

where $U = U(\lambda, \alpha)$ and $\tilde{T} = \tilde{T}(\lambda, \alpha)$.

In particular

$$U(0) \equiv \lim_{r\to 0+} U(r) \geqslant \lambda - \alpha > 0 \quad \text{and} \quad \lim_{r\to 0+} \tilde{T}(r) < \lambda^{1-N}\Phi_1(\lambda, \lambda, \ldots, \lambda).$$

Proof. — This follows easily from Lemmas 1(*a*) and 2(*a*).

LEMMA 4. — For $(\lambda, \alpha) \in \Delta$, let $U(r) = U(\lambda, \alpha)(r)$.

Set $t = \dfrac{U(r)}{r}$ and $q(t) = U'(r)$.

Then $q = q(\lambda, \alpha)$ is defined on $[\lambda, \infty)$ and $0 < q(t) < t$ for $t \in (\lambda, \infty)$. Furthermore, $q(\lambda, \alpha)$ is the unique solution of the first order equation:

$$\Phi_{11}(q, t, \ldots, t)q' = (N-1)[P(q, t) - \Phi_{12}(q, t, \ldots, t)] \tag{5.6}$$

satisfying the initial condition

$$q(\lambda) = \alpha . \tag{5.7}$$

If $(\lambda, \alpha) \to (\lambda_0, \alpha_0)$ in Δ, then

$$q(\lambda, \alpha)(t) \to q(\lambda_0, \alpha_0)(t) \quad \text{pointwise on } (\lambda_0, \infty)$$

and, if $0 < \alpha < \alpha_1 < \lambda$, then

$$q(\lambda, \alpha)(t) < q(\lambda, \alpha_1)(t) \qquad \forall t \in [\lambda, \infty).$$

Let $Q(\lambda)$ be the unique solution of (5.6) satisfying the initial condition:

$$q(\lambda) = \lambda \tag{5.8}$$

Then $Q(\lambda)$ is defined on $[\lambda, \infty)$ and

$$0 < q(\lambda, \alpha)(t) < Q(\lambda)(t) \quad \text{for} \quad t \in [\lambda, \infty) \quad \text{and} \quad 0 < \alpha < \lambda .$$

Furthermore for $0 < \alpha < \lambda$,

$$q(\lambda, \alpha) \quad \text{and} \quad Q(\lambda) \quad \text{are strictly decreasing on } [\lambda, \infty)$$

and

$$\lim_{t \to \infty} q(\lambda, \alpha)(t) = \lim_{t \to \infty} Q(\lambda)(t) = 0 .$$

Remark. — Since $\lim_{t \to \infty} q(\lambda, \alpha)(t) = 0$, we have that

$$\lim_{r \to 0} U'(r) = 0 \quad \text{for} \quad U = U(\lambda, \alpha) \quad \text{with} \quad (\lambda, \alpha) \in \Delta .$$

Proof. — By Lemmas 1, 2 and 3, $\dfrac{U(r)}{r}$ is strictly decreasing on $(0, 1]$,

$$\lim_{r \to 0+} \frac{U(r)}{r} = +\infty, \qquad \lim_{r \to 1} \frac{U(r)}{r} = \lambda \quad \text{and} \quad 0 < q(t) < t \qquad \forall t \in [\lambda, \infty).$$

Furthermore $\quad r\dfrac{d}{dr} = r\dfrac{dt}{dr}\dfrac{d}{dt} = r\left(\dfrac{U(r)}{r}\right)'\dfrac{d}{dt} = (q - t)\dfrac{d}{dt}.$

Hence the equations (5.1) and (5.4) become

$$\frac{d}{dt}\Phi_1 = (N - 1)P(q, t)$$

and (5.6) respectively where P is defined by (4.1). Since the right hand side of (5.6) is C^1 on $(0, \infty)^2$ and (A1), (A2) hold, the Picard theorem ensures the existence and uniqueness of the solution to the initial value problems

(5.6), (5.7) and (5.6), (5.8). The assertions concerning $q(\lambda, \alpha)$ all follow from the uniqueness and continuous dependence of solutions of (5.6), (5.7) on the data (λ, α).

To complete the proof of the lemma, we need only show that $Q(\lambda)$ is defined on all of $[\lambda, \infty)$ and that $\lim_{t\to\infty} Q(\lambda)(t) = 0$.

Clearly $Q(\lambda)(t) > 0$ and $Q(\lambda)'(t) < 0$ for all t in the domain of $Q(\lambda)$. In particular $Q(\lambda)(t) < t$ for $t > \lambda$.

Setting $S(t) = t^{1-N}\Phi_1(Q(\lambda)(t), t, \ldots, t)$ it follows from (5.5) that S satisfies

$$\frac{dS}{dt} = -(N-1)t^{-N}R(Q(\lambda)(t), t)$$

for all t in the domain of $Q(\lambda)$. Then arguing as in the proof of Lemma 2(*a*), we find that the domain of $Q(\lambda)$ contains $[\lambda, \infty)$, and that $\frac{dS}{dt}(t) < 0\ \forall t \in [\lambda, \infty)$. Thus $t^{1-N}\Phi_1(Q(\lambda)(t), t, \ldots, t) < \lambda^{1-N}\Phi_1(\lambda, \ldots, \lambda)$ for $t > \lambda$ and so if $\lim_{t\to\infty} Q(\lambda)(t) = b > 0$ we obtain a contradiction to the assumption (A3). Hence we have $\lim_{t\to\infty} Q(\lambda)(t) = 0$.

COROLLARY 5. — For $\lambda > 0$ set

$$g(\lambda) = \lambda^{1-N}\Phi_1(\lambda, \lambda, \ldots, \lambda) - (N-1)\int_\lambda^\infty t^{-N}R(Q(\lambda)(t), t)dt$$

where $Q(\lambda)$ is defined in Lemma 4. Then

a) $\quad -\infty < g(\lambda) < \lambda^{1-N}\Phi_1(\lambda, \lambda, \ldots, \lambda)$ for all $\lambda > 0$,

b) $\quad g : (0, \infty) \to \mathbb{R}$ is continuous,

c) $\quad \lim_{\lambda\to 0+} g(\lambda) = -\infty$ and $\lim_{\lambda\to\infty} g(\lambda) = +\infty$.

Proof. — *a*) By (A6), $R(Q(\lambda)(t), t) > 0$ for $t \in [\lambda, \infty)$ and so

$$g(\lambda) < \lambda^{1-N}\Phi_1(\lambda, \lambda, \ldots, \lambda).$$

Since $0 < Q(\lambda)(t) < t$, it follows from (A6) that

$$g(\lambda) \geqslant \lambda^{1-N}\Phi_1(\lambda, \lambda, \ldots, \lambda) - (N-1)\int_\lambda^\infty t^{-N}\{A + Bt^\beta\}\,dt \text{ with } \beta < N-1.$$

It follows that $g(\lambda) > -\infty$ and $\exists \lambda_0 > 0$ such that

$$g(\lambda) \geqslant \lambda^{1-N}\Phi_1(\lambda, \lambda, \ldots, \lambda) - 1 \qquad \forall \lambda > \lambda_0.$$

The limits in (*c*) now follow from (A4).

b) As above

$$|t^{-N}R(Q(\lambda)(t), t)| \leqslant t^{-N}\{A + Bt^\beta\} \quad \forall t \geqslant 1 \quad \text{and} \quad Q(\lambda)(t) \to Q(\lambda_0)(t)$$

pointwise on (λ_0, ∞) as $\lambda \to \lambda_0$. By the dominated convergence theorem (since $\beta < N - 1$), it follows that $g(\lambda) \to g(\lambda_0)$ as $\lambda \to \lambda_0$ and so g is continuous on $(0, \infty)$.

For $0 < \alpha \leqslant \lambda$, we set

$$\tau(\lambda, \alpha) = \lim_{r \to 0+} \tilde{T}(\lambda, \alpha)(r).$$

Recall from Lemma 1 (*a*) that for $0 < \alpha < \lambda$, $\tilde{T}(\lambda, \alpha)$ is strictly increasing on $(0,1]$ and that $\tilde{T}(\lambda, \lambda)(r) = \lambda^{1-N}\Phi_1(\lambda, \lambda, \ldots, \lambda)$ for $\lambda > 0$. Thus for $(\lambda, \alpha) \in \Delta$, we have

$$\tau(\lambda, \alpha) < \tilde{T}(\lambda, \alpha)(1) = \lambda^{1-N}\Phi_1(\alpha, \lambda, \ldots, \lambda)$$

and $\tau(\lambda, \lambda) = \lambda^{1-N}\Phi_1(\lambda, \lambda, \ldots, \lambda)$. From (A2) it follows that

$$\tau(\lambda, \alpha) < \tau(\lambda, \lambda) \qquad \forall(\lambda, \alpha) \in \Delta.$$

LEMMA 6. — *a*) $\forall(\lambda, \alpha) \in \Delta$, $\tau(\lambda, \alpha) > -\infty$.

b) $\tau : \Delta \to \mathbb{R}$ is continuous.

c) $\forall \lambda > 0$, $\tau(\lambda, .) : (0, \lambda) \to \mathbb{R}$ is strictly increasing, $\lim_{\alpha \to 0+} \tau(\lambda, \alpha) = -\infty$ and $\lim_{\alpha \to \lambda-} \tau(\lambda, \alpha) = g(\lambda)$, where g is the function defined in Corollary 5.

d) $\forall \alpha > 0$, $\tau(., \alpha) : (\alpha, \infty) \to \mathbb{R}$ is strictly increasing.

e) $g : (0, \infty) \to \mathbb{R}$ is strictly increasing.

Proof. — *a, b*) For $0 < p < 1$, it follows from (5.5) that,

$$\tilde{T}(p) = \tilde{T}(1) + (N-1) \int_p^1 \left[\frac{U(r)}{r}\right]^{-N} R\left(U'(r), \frac{U(r)}{r}\right)\left(\frac{U(r)}{r}\right)' dr$$

$$= \lambda^{1-N}\Phi_1(\alpha, \lambda, \ldots, \lambda) - (N-1) \int_\lambda^{\frac{U(p)}{p}} t^{-N} R(q(t), t) dt$$

where $\tilde{T} = \tilde{T}(\lambda, \alpha)$, $U = U(\lambda, \alpha)$ and $t, q = q(\lambda, \alpha)$ are as defined in Lemma 4. Since $0 < q(t) \leqslant t$, it follows from (A6) that $0 < R(q(t), t) \leqslant A + Bt^\beta$ for $t \geqslant \lambda$ where $\beta < N - 1$.

Since $\dfrac{U(p)}{p} > \lambda$ for $0 < p < 1$ and $\lim_{p \to 0} \dfrac{U(p)}{p} = +\infty$, we have that,

$$\lambda^{1-N}\Phi_1(\alpha, \lambda, \ldots, \lambda) - (N-1) \int_\lambda^\infty t^{-N}\{A + Bt^\beta\}\, dt < \tilde{T}(p) < \lambda^{1-N}\Phi_1(\alpha, \lambda, \ldots, \lambda).$$

In particular, $\tau(\lambda, \alpha) > -\infty$ since $\beta < N-1$, and $\tau(\lambda, \alpha) < \lambda^{1-N}\Phi_1(\alpha, \lambda, \ldots, \lambda)$. From (A3), it follows that $\lim_{\alpha \to 0} \tau(\lambda, \alpha) = -\infty$.

Furthermore by Lemma 4 and the dominated convergence theorem we see that

$$\tau(\lambda, \alpha) = \lambda^{1-N}\Phi_1(\alpha, \lambda, \ldots, \lambda) - (N-1) \int_\lambda^\infty t^{-N} R(q(\lambda, \alpha)(t), t) dt$$

converges to $\tau(\lambda_0, \alpha_0)$ as $(\lambda, \alpha) \to (\lambda_0, \alpha_0)$ in Δ. Thus τ is continuous on $(0, \infty)$. Similarly we have that $\lim_{\alpha \to \lambda -} \tau(\lambda, \alpha) = g(\lambda)$.

c) For

$$(\lambda, \alpha) \in \Delta, \qquad \tau(\lambda, \alpha) = \lim_{r \to 0+} \tilde{T}(\lambda, \alpha)(r)$$

$$= \lim_{r \to 0+} \left[\frac{U(r)}{r}\right]^{1-N} \Phi_1\left(U'(r), \frac{U(r)}{r}, \ldots, \frac{U(r)}{r}\right) \quad \text{where} \quad U = U(\lambda, \alpha)$$

$$= \lim_{t \to \infty} t^{1-N} \Phi_1(q(\lambda, \alpha)(t), t, \ldots, t) \quad \text{in the notation of Lemma 4}.$$

But, according to Lemma 4, if $0 < \alpha < \alpha_1 < \lambda$,

we have $$q(\lambda, \alpha)(t) < q(\lambda, \alpha_1)(t) \qquad \forall t \geqslant \lambda$$

and so from (A2) it follows that

$$\Phi_1(q(\lambda, \alpha)(t), t, \ldots, t) < \Phi_1(q(\lambda, \alpha_1)(t), t, \ldots, t) \qquad \forall t \geqslant \lambda.$$

Hence $\tau(\lambda, \alpha) \leqslant \tau(\lambda, \alpha_1)$.

To prove that in fact there is strict inequality, we set

$$S(t) = t^{1-N} \Phi_1(q(t), t, \ldots, t).$$

Then by (5.5) we have that $S = S(\lambda, \alpha)$ satisfies

$$\frac{d}{dt} S(t) = -(N-1) t^{-N} R(q(t), t) \qquad \text{for} \quad t > \lambda.$$

Suppose that $0 < \alpha < \alpha_1 < \lambda$ and that $\tau(\lambda, \alpha) = \tau(\lambda, \alpha_1)$.

Set
$$q(t) = q(\lambda, \alpha)(t), \qquad q_1(t) = q(\lambda, \alpha_1)(t),$$
$$S(t) = S(\lambda, \alpha)(t), \qquad S_1(t) = S(\lambda, \alpha)(t),$$

Then $\lim_{t \to \infty} S(t) = \lim_{t \to \infty} S_1(t)$ and so, for $\sigma > \lambda$

$$S_1(\sigma) - S(\sigma) = (N-1) \int_\sigma^\infty t^{-N} \{ R(q_1(t), t) - R(q(t), t) \} \, dt.$$

Thus $$0 \leqslant S_1(\sigma) - S(\sigma) \leqslant (N-1) \int_\sigma^\infty t^{-N} K t^\gamma \{ q_1(t) - q(t) \} \, dt \quad \text{for} \quad \sigma > \sigma_0$$

where $\gamma < 2(N-1)$, by (A7) since $\lim_{t \to \infty} q_1(t) = \lim_{t \to \infty} q(t) = 0$.

But

$$S_1(t) - S(t) = t^{1-N} \{ \Phi_1(q_1(t), t, \ldots, t) - \Phi_1(q(t), t, \ldots, t) \}$$

$$= t^{1-N} \{ q_1(t) - q(t) \} \int_0^1 \Phi_{11}(y q_1(t) + (1-y) q(t), t, \ldots, t) dy$$

$$\geqslant t^{1-N} \{ q_1(t) - q(t) \} C t^{2(N-1)} \qquad \text{where} \qquad C > 0$$

by (A2) provided that $t \geqslant t_0$.

Hence

$$0 \leqslant S_1(\sigma) - S(\sigma) \leqslant (N-1)C^{-1}K \int_\sigma^\infty t^{-2N+1+\gamma}\{ S_1(t) - S(t) \} \, dt \quad \text{for} \quad \sigma \geqslant \sigma_0.$$

Since $-2N+1+\gamma < -1$ it follows by a standard argument that $\exists \sigma_1 > \sigma_0$ such that $S_1(\sigma) \equiv S(\sigma)$ for all $\sigma > \sigma_1$ and consequently that $q_1(t) \equiv q(t)$ for all $t \geqslant t_1$. Thus $q_1(t) = q(t)$ for all $t \geqslant \lambda$ and so $\alpha = \alpha_1$.

This proves that for $0 < \alpha < \alpha_1 < \lambda$ we must have

$$\tau(\lambda, \alpha) < \tau(\lambda, \alpha_1).$$

d) Let $U = U(\lambda, \alpha)$ and set $w(r) = \dfrac{U(cr)}{c}$ for $r \in (0, 1]$ and $c \in (0, 1]$.

It is easy to check that w satisfies (5.1) and that

$$w(1) = \frac{U(c)}{c}, \qquad w'(1) = U'(c).$$

For $\lambda_1 > \lambda$, it follows from Lemma 1 (*a*) that $\exists$ a unique value of $c \in (0, 1)$ such that $\dfrac{U(c)}{c} = \lambda_1$. Furthermore, $0 < U'(c) < U'(1) = \alpha$. Setting $\alpha_1 = U'(c)$, where $\dfrac{U(c)}{c} = \lambda_1$, we thus have that $0 < \alpha_1 < \alpha < \lambda < \lambda_1$ and that $w = U(\lambda_1, \alpha_1)$.

But

$$\begin{aligned} \tilde{T}(\lambda_1, \alpha_1)(r) &= \left[\frac{w(r)}{r}\right]^{1-N} \Phi_1\left(w'(r), \frac{w(r)}{r}, \ldots, \frac{w(r)}{r}\right) \\ &= \left[\frac{U(cr)}{cr}\right]^{1-N} \Phi_1\left(U'(cr), \frac{U(cr)}{r}, \ldots, \frac{U(cr)}{r}\right) \\ &= \tilde{T}(\lambda, \alpha)(cr). \end{aligned}$$

Hence

$$\begin{aligned} \tau(\lambda_1, \alpha_1) &= \lim_{r \to 0+} \tilde{T}(\lambda_1, \alpha_1)(r) = \lim_{r \to 0+} \tilde{T}(\lambda, \alpha)(cr) \\ &= \lim_{r \to 0+} \tilde{T}(\lambda, \alpha)(r) \quad \text{since} \quad c \in (0, 1) \\ &= \tau(\lambda, \alpha). \end{aligned}$$

Thus $\tau(\lambda, \alpha) = \tau(\lambda_1, \alpha_1) < \tau(\lambda_1, \alpha)$ by part (*c*). This proves that $\tau(\cdot, \alpha)$ is strictly increasing on (α, ∞).

From parts (*c*) and (*d*), it follows easily that g is strictly increasing on $(0, \infty)$.

THEOREM 7. — Let $g : (0, \infty) \to \mathbb{R}$ be the function defined in Corollary 5.

a) There exists a unique value $\lambda^* \in (0, \infty)$ such that $g(\lambda^*) = 0$. There exists a continuously differentiable strictly decreasing function w on (λ^*, ∞) such that $0 < w(\lambda) < \lambda$ and

$$\{(\lambda, \alpha) \in \Delta : \tau(\lambda, \alpha) = 0\} = \{(\lambda, w(\lambda)) : \lambda \in (\lambda^*, \infty)\}.$$

Furthermore $\lim_{\lambda \to \lambda^*+} w(\lambda) = \lambda^*$ and $\lim_{\lambda \to \infty} w(\lambda) = 0$.

b) For $\lambda > \lambda^*$, let $U = U(\lambda, w(\lambda))$. For $\mu > \lambda$, there exists a unique value $c = c(\mu) \in (0, 1)$ such that $\frac{U(c)}{c} = \mu$. Then $U(\mu, w(\mu))(r) = \frac{U(cr)}{c}$ for $0 < r < 1$ and $w(\mu) = U'(c)$. The function $c : (\lambda, \infty) \to (0, 1)$ is continuously differentiable and strictly decreasing with $\lim_{\mu \to \infty} c(\mu) = 0$.

c) For $\lambda > 0$, let $U_t(\lambda) = U(\lambda, \lambda)$.
For $\lambda > \lambda^*$, let $U_c(\lambda) = U(\lambda, w(\lambda))$.

Then for $0 < \lambda \leqslant \lambda^*$, $U_t(\lambda)$ is the only solution of (2.22) to (2.26) and $U_t(\lambda)(0) = 0$.

For $\lambda > \lambda^*$, $U_t(\lambda)$ and $U_c(\lambda)$ are the only solutions of (2.22) and (2.26) and $U_t(\lambda)(0) = 0 < U_c(\lambda)(0)$.

As $\lambda \to \lambda^* +$, $U_c(\lambda)$ converges to $U_t(\lambda^*)$ uniformly on compact subsets of $(0, 1]$.

d) For $\lambda > \lambda^*$, let $R_c(\lambda) = U_c(\lambda)(0)$ be the radius of the cavity. Then R_c is a strictly increasing, concave continuous function on (λ^*, ∞) with

$$\lim_{\lambda \to \lambda^*+} R_c(\lambda) = 0 \quad \text{and} \quad \lim_{\lambda \to \infty} R_c(\lambda) = \infty.$$

Furthermore R_c is of class C^2 on (λ^*, ∞) and $\lim_{\lambda \to \infty} \frac{dR_c(\lambda)}{d\lambda} = 1$.

Proof. — By Corollary 5 (*b*) and (*c*) and Lemma 6 (*e*), there exists a unique value $\lambda^* \in (0, \infty)$ such that $g(\lambda^*) = 0$. Furthermore for $\lambda < \lambda^*$, $\tau(\lambda, \alpha) < 0$ for $0 < \alpha < \lambda$.

For $\lambda > \lambda^*$, there is a unique value $w(\lambda) \in (0, \lambda)$ such that $\tau(\lambda, w(\lambda)) = 0$ and $\lim_{\lambda \to \lambda^*} w(\lambda) = \lambda^*$.

Now fixing $\lambda > \lambda^*$ and setting $U = U(\lambda, w(\lambda))$ we have as in part (*d*) Lemma 6 that, for $\mu > \lambda$, $w(\mu) = U'(c(\mu))$ where $c(\mu)$ is the unique value of c in $(0, 1)$ such that $\frac{U(c)}{c} = \mu$. Since $\left(\frac{U(r)}{r}\right)' < 0$ for $0 < r < 1$, it follows that c is a continuously differentiable function of μ on (λ, ∞) and that

$$\frac{dc(\mu)}{d\mu} = \left\{\left[\frac{U(r)}{r}\right]'\right\}^{-1}\Bigg|_{r = c(\mu)} = \frac{r^2}{rU'(r) - U(r)}\Bigg|_{r = c(\mu)} < 0.$$

Hence $$\frac{dw(\mu)}{d\mu} = \mathrm{U}''(c(\mu))\frac{dc(\mu)}{d\mu} = \left.\frac{r^2\mathrm{U}''(r)}{r\mathrm{U}'(r)-\mathrm{U}(r)}\right|_{r=c(\mu)} < 0 \tag{5.9}$$

by Lemma 1(*a*). Also

$$\lim_{\mu\to\infty} w(\mu) = \lim_{\mu\to\infty} \mathrm{U}'(c(\mu)) = \lim_{r\to 0} \mathrm{U}'(r) = 0\,.$$

Thus we see that parts (*a*), (*b*) and (*c*) of the theorem are easy consequences of the preceding lemmas.

d) Let $\mu > \lambda > \lambda^*$ and set $\mathrm{U} = \mathrm{U}_c(\lambda)$. Then

$$\mathrm{R}_c(\mu) = \mathrm{U}(\mu, w(\mu))(0) = \frac{\mathrm{U}(0)}{c(\mu)} \tag{5.10}$$

and $$\frac{d\mathrm{R}_c(\mu)}{d\mu} = -\frac{\mathrm{U}(0)}{c(\mu)^2}\frac{dc(\mu)}{d\mu} = -\left.\frac{\mathrm{U}(0)}{r\mathrm{U}'(r) - \mathrm{U}(r)}\right|_{r=c(\mu)} > 0\,. \tag{5.11}$$

It now follows that R_c is of class C^2 on (λ, ∞) and that

$$\begin{aligned}\frac{d^2\mathrm{R}_c(\mu)}{d\mu^2} &= \left.\frac{\mathrm{U}(0)r\mathrm{U}''(r)}{[r\mathrm{U}'(r) - \mathrm{U}(r)]^2}\right|_{r=c(\mu)}\frac{dc(\mu)}{d\mu} \\ &= \left.\frac{\mathrm{U}(0)r^3\mathrm{U}''(r)}{[r\mathrm{U}'(r) - \mathrm{U}(r)]^3}\right|_{r=c(\mu)} < 0,\end{aligned}$$

by Lemma 1(*a*).

Since $\lim_{\mu\to\infty} c(\mu) = 0$, it follows from (5.10) that $\lim_{\mu\to\infty} \mathrm{R}_c(\mu) = \infty$, and from (5.11) that $\lim_{\mu\to\infty} \frac{d\mathrm{R}_c}{d\mu} = 1$.

It follows from (5.11) that R_c is strictly increasing on (λ^*, ∞) and so $\lim_{\lambda\to\lambda^*+} \mathrm{R}_c(\lambda) < \mathrm{R}_c(\lambda)$ for all $\lambda > \lambda^*$.

By Lemma 1(*a*), $\mathrm{R}_c(\lambda) < \mathrm{U}_c(\lambda)(r)$ for all $r \in (0, 1)$.

Thus $0 \leqslant \lim_{\lambda\to\lambda^*+} \mathrm{R}_c(\lambda) < \mathrm{U}_c(\lambda)(r)$ for all $\lambda > \lambda^*$ and all $r \in (0, 1)$.

But $\mathrm{U}_c(\lambda)(r)$ converges to $\lambda^* r$ as $\lambda \to \lambda^* +$ for each $r \in (0, 1)$. Hence we see that $\lim_{\lambda\to\lambda^*} \mathrm{R}_c(\lambda) = 0$.

Remarks. — 1. In the notation of Theorem 7, we find that for $\mu > \lambda > \lambda^*$,

$$\Phi_1(w(\mu), \mu, \ldots, \mu) = \Phi_1\left(\mathrm{U}'(r), \frac{\mathrm{U}(r)}{r}, \ldots, \frac{\mathrm{U}(r)}{r}\right)\Bigg|_{r=c(\mu)} \tag{5.12}$$

where $\mathrm{U} = \mathrm{U}(\lambda, w(\lambda))$, and

$$\frac{d\Phi_1}{d\mu} = \Phi_{11}w'(\mu) + (\mathrm{N} - 1)\Phi_{12} = (\mathrm{N} - 1)\mathrm{P}(w(\mu), \mu) \tag{5.13}$$

where the partial derivatives of Φ are evaluated at $(w(\mu), \mu, \ldots, \mu)$ and P is the function defined by (4.1). To verify (5.13) we note that by (5.9) and (5.12),

$$\begin{aligned}
&\Phi_{11}w'(\mu) + (\mathrm{N}-1)\Phi_{12} = \\
&=\Phi_{11}\left(\mathrm{U}'(r), \frac{\mathrm{U}(r)}{r}, \ldots, \frac{\mathrm{U}(r)}{r}\right)\frac{r^2\mathrm{U}''(r)}{[r\mathrm{U}'(r)-\mathrm{U}(r)]} \\
&\qquad\qquad + (\mathrm{N}-1)\Phi_{12}\left(\mathrm{U}'(r), \frac{\mathrm{U}(r)}{r}, \ldots, \frac{\mathrm{U}(r)}{r}\right)\Bigg|_{r=c(\mu)} \\
&= (\mathrm{N}-1)\mathrm{P}\left(\mathrm{U}'(r), \frac{\mathrm{U}(r)}{r}\right)\Bigg|_{r=c(\mu)} \quad \text{by (5.4)}.
\end{aligned}$$

2. These formulae give some information about the function V defined by (3.3) which relates the Cauchy traction problem to the displacement problem. In fact, for $\mu > \lambda > \lambda^*$, we have

$$\begin{aligned}
\mathrm{V}(\mu) &= \mu^{1-\mathrm{N}}\Phi_1(w(\mu), \mu, \ldots, \mu) \\
&= \left[\frac{\mathrm{U}(r)}{r}\right]^{1-\mathrm{N}}\Phi_1\left(\mathrm{U}'(r), \frac{\mathrm{U}(r)}{r}, \ldots, \frac{\mathrm{U}(r)}{r}\right)\Bigg|_{r=c(\mu)} \\
&= \tilde{\mathrm{T}}(\lambda, w(\lambda))(r)\,|_{r=c(\mu)} \quad \text{where} \quad \mathrm{U} = \mathrm{U}(\lambda, w(\lambda)). \qquad (5.14)
\end{aligned}$$

Hence $\lim_{\mu\to\infty} \mathrm{V}(\mu) = \lim_{r\to 0} \tilde{\mathrm{T}}(\lambda, w(\lambda))(r) = \tau(\lambda, w(\lambda)) = 0$ since $\lim_{\mu\to\infty} c(\mu) = 0$. Furthermore V is continuously differentiable and

$$\begin{aligned}
\frac{d\mathrm{V}(\mu)}{d\mu} &= \frac{d}{d\mu}\{\mu^{1-\mathrm{N}}\Phi_1(w(\mu), \mu, \ldots, \mu)\} \\
&= -(\mathrm{N}-1)\mu^{-\mathrm{N}}\Phi_1 + \mu^{1-\mathrm{N}}\frac{d}{d\mu}\Phi_1 \\
&= -(\mathrm{N}-1)\mu^{-\mathrm{N}}\mathrm{R}(w(\mu), \mu) < 0 \quad \text{by (5.12) and (A6)}.
\end{aligned}$$

Thus V is strictly decreasing.

6. COMPARISON OF THE ENERGIES OF SOLUTIONS

We consider the energies of the solutions of the problem (2.22) to (2.26). Recall that in the notation of section 5, $\mathrm{U}_t(\lambda)=\mathrm{U}(\lambda, \lambda)$ denotes the homogeneous deformation $r \to \lambda r$ which has no cavity. For $\lambda > \lambda^*$, $\mathrm{U}_c(\lambda)$ denotes the solution $\mathrm{U}(\lambda, w(\lambda))$ which has a cavity of radius $\mathrm{R}_c(\lambda) > 0$. From (2.10), it follows that,

$$\begin{aligned}
\mathrm{ED}_t(\lambda) \equiv \mathrm{E}(\mathrm{U}_t(\lambda)) &= \omega_\mathrm{N}\int_0^1 \Phi(\lambda, \lambda, \ldots, \lambda)r^{\mathrm{N}-1}dr \\
&= \frac{\omega_\mathrm{N}}{\mathrm{N}}\Phi(\lambda, \lambda, \ldots, \lambda) \qquad \text{for all} \quad \lambda > 0
\end{aligned} \qquad (6.1)$$

and

$$\mathrm{ED}_c(\lambda) \equiv \mathrm{E}(\mathrm{U}_c(\lambda)) = \omega_\mathrm{N} \int_0^1 \Phi\left(\mathrm{U}'(r), \frac{\mathrm{U}(r)}{r}, \ldots, \frac{\mathrm{U}(r)}{r}\right) r^{\mathrm{N}-1} dr \quad (6.2)$$

where $\mathrm{U} = \mathrm{U}_c(\lambda)$ for $\lambda > \lambda^*$.

LEMMA 8. — For $\lambda > \lambda^*$, let $\mathrm{U} = \mathrm{U}_c(\lambda) = \mathrm{U}(\lambda, w(\lambda))$ and $\tilde{\mathrm{T}} = \tilde{\mathrm{T}}(\lambda, w(\lambda))$. Then there exist positive constants $\mathrm{K}_1, \mathrm{K}_2$ and t_0 (depending on λ) such that

$$0 < \tilde{\mathrm{T}}(r) < \mathrm{K}_1 r^{\mathrm{N}-1-\beta} \qquad \text{for} \qquad 0 < r < r_0$$

and $$0 < \Phi_1\left(\mathrm{U}'(r), \frac{\mathrm{U}(r)}{r}, \ldots, \frac{\mathrm{U}(r)}{r}\right) < \mathrm{K}_2 r^{-\beta} \qquad \text{for} \qquad 0 < r < r_0$$

where β is the constant in (A6).

Proof. — We use the variables: $t = \dfrac{\mathrm{U}(r)}{r}$, $q(t) = \mathrm{U}'(r)$ introduced in Lemma 4.

Setting $\mathrm{S}(t) = t^{1-\mathrm{N}} \Phi_1(q(t), t, \ldots, t)$, we have that

$$\frac{d\mathrm{S}(t)}{dt} = -(\mathrm{N}-1)t^{-\mathrm{N}} \mathrm{R}(q(t), t) \qquad \text{for} \quad t > \lambda .$$

Since $$\lim_{t \to \infty} \mathrm{S}(t) = \lim_{r \to 0} \tilde{\mathrm{T}}(r) = \tau(\lambda, w(\lambda)) = 0,$$

$$\mathrm{S}(t) = (\mathrm{N}-1) \int_t^\infty s^{-\mathrm{N}} \mathrm{R}(q(s), s) ds$$

and so by (A6),

$$0 < \mathrm{S}(t) < (\mathrm{N}-1) \int_t^\infty s^{-\mathrm{N}} \{ \mathrm{A} + \mathrm{B} s^\beta \} ds \qquad \text{for} \qquad t \geqslant t_0$$

$$\leqslant \mathrm{K} t^{-\mathrm{N}+\beta+1} \qquad \text{for} \qquad t \geqslant t_0 .$$

Thus, $$0 < \tilde{\mathrm{T}}(r) < \mathrm{K}\left[\frac{\mathrm{U}(r)}{r}\right]^{-\mathrm{N}+\beta+1} \leqslant \frac{\mathrm{K} r^{\mathrm{N}-1-\beta}}{\mathrm{U}(0)^{\mathrm{N}-1-\beta}}$$

$$= \mathrm{K}_1 r^{\mathrm{N}-1-\beta} \qquad \text{for} \qquad 0 < r < r_0$$

and $$0 < \Phi_1\left(\mathrm{U}'(r), \frac{\mathrm{U}(r)}{r}, \ldots, \frac{\mathrm{U}(r)}{r}\right) = \left[\frac{\mathrm{U}(r)}{r}\right]^{\mathrm{N}-1} \tilde{\mathrm{T}}(r)$$

$$< \mathrm{K}\left[\frac{\mathrm{U}(r)}{r}\right]^\beta \leqslant \mathrm{K}\lambda^\beta r^{-\beta} = \mathrm{K}_2 r^{-\beta} \qquad \text{for} \quad 0 < r < r_0 .$$

Remark. — In the notation of Lemma 8, U satisfies

$$\frac{d}{dr}\{ r^{\mathrm{N}-1} \Phi_1 \} = (\mathrm{N}-1) r^{\mathrm{N}-2} \Phi_2 \qquad \text{for} \qquad 0 < r < 1,$$

where Φ_1 and Φ_2 are evaluated at $\left(\mathrm{U}'(r), \dfrac{\mathrm{U}(r)}{r}, \ldots, \dfrac{\mathrm{U}(r)}{r}\right)$. Furthermore,

$$r^{\mathrm{N}-1} \Phi_1 = \mathrm{U}(r)^{\mathrm{N}-1} \tilde{\mathrm{T}}(r)$$

and so by Lemma 1(*a*), $\frac{d}{dr}\{ r^{N-1}\Phi_1 \} > 0$ on $(0,1)$. According to Theorem 4.2 of [1], the function u: $\{ x \in \mathbb{R}^N : |x| \leqslant 1 \} \to \mathbb{R}^N$ defined by

$$u(x) = \frac{U(r)x}{r} \quad \text{for} \quad 0 < |x| \leqslant 1$$

is u a weak solution of (2.9) provided that

$$r^{N-1}\Phi_1 \quad \text{and} \quad r^{N-1}\Phi_2 \quad \text{belong to} \quad L^1(0,1).$$

By Lemma 8,

$$\int_0^1 |r^{N-1}\Phi_1|\, dr = \int_0^1 r^{N-1}\Phi_1 dr \leqslant \int_0^1 K_2 r^{N-1-\beta} dr < \infty$$

since $0 < \beta < N-1$. Also, for $0 < \varepsilon < 1$,

$$\begin{aligned}\int_\varepsilon^1 |r^{N-1}\Phi_2|\, dr &= \frac{1}{(N-1)} \int_\varepsilon^1 r \left| \frac{d}{dr}\{ r^{N-1}\Phi_1 \} \right| dr \\ &= \frac{1}{(N-1)} \int_\varepsilon^1 r \frac{d}{dr}\{ r^{N-1}\Phi_1 \}\, dr \\ &= \frac{1}{(N-1)} \left\{ r^N \Phi_1 \Big|_{r=\varepsilon}^{r=1} - \int_\varepsilon^1 r^{N-1}\Phi_1 dr \right\}.\end{aligned}$$

By Lemma 8, $$\int_0^1 |r^{N-1}\Phi_2|\, dr < \infty .$$

LEMMA 9. — For $\lambda > \lambda^*$, let $U = U_c(\lambda) = U(\lambda, w(\lambda))$ where w : $(\lambda^*, \infty) \to \mathbb{R}$ is the function defined in Theorem 7. Then,

$$\int_0^1 r^{N-1} \left| \Phi\left(U'(r), \frac{U(r)}{r}, \ldots, \frac{U(r)}{r} \right) \right| dr < \infty ,$$

$$\lim_{r \to 0} r^N \Phi\left(U'(r), \frac{U(r)}{r}, \ldots, \frac{U(r)}{r} \right) = 0$$

and $$ED_c(\lambda) = \frac{\omega_N}{N}\{ \Phi(w(\lambda), \lambda, \ldots, \lambda) + [\lambda - w(\lambda)]\Phi_1(w(\lambda), \lambda, \ldots, \lambda) \}. \quad (6.3)$$

Proof. — Since U satisfies (5.1), it is easily checked that U also satisfies the following « conservation law » for $0 < r < 1$,

$$\frac{d}{dr}\{ r^N \Phi - r^N U'(r)\Phi_1 + r^{N-1}U(r)\Phi_1 \} = N r^{N-1}\Phi ,$$

where Φ and Φ_1 are evaluated at $\left(U'(r), \frac{U(r)}{r}, \ldots, \frac{U(r)}{r}\right)$. Thus for $0 < s < 1$,

$$N \int_s^1 r^{N-1}\Phi dr = \{ r^N\Phi - r^N U'(r)\Phi_1 + r^{N-1}U(r)\Phi_1 \} \big|_{r=s}^{r=1} .$$

Since $U(1) = \lambda$, $U'(1) = w(\lambda)$ and

$$\lim_{r\to 0} r^{N-1}\Phi_1 = \lim_{r\to 0} U(r)^{N-1}\tilde{T}(\lambda, w(\lambda))(r) = U(0)^{N-1}\tau(\lambda, w(\lambda)) = 0 .$$

it follows that

$$\lim_{s\to 0}\left\{ s^N\Phi\left(U'(s), \frac{U(s)}{s}, \ldots, \frac{U(s)}{s}\right) + N\int_s^1 r^{N-1}\Phi dr \right\}$$
$$= \Phi(w(\lambda), \lambda, \ldots, \lambda) + [\lambda - w(\lambda)]\Phi_1(w(\lambda), \lambda, \ldots, \lambda) .$$

By (A5) we can assume that $\inf \Phi = 0$, and so it follows that

$$0 \leqslant \int_0^1 r^{N-1}\Phi dr = \int_0^1 r^{N-1} |\Phi| dr < \infty .$$

Hence $\lim_{s\to 0} s^N\Phi\left(U'(s), \frac{U(s)}{s}, \ldots, \frac{U(s)}{s}\right)$ exists and since $\int_0^1 r^{N-1} |\Phi| dr < \infty$, we must have

$$\lim_{s\to 0} s^N\Phi\left(U'(s), \frac{U(s)}{s}, \ldots, \frac{U(s)}{s}\right) = 0 .$$

Recalling that $ED_c(\lambda) = \omega_N \int_0^1 r^{N-1}\Phi dr$, the proof is complete.

THEOREM 10. — Let $ED_t(\lambda)$ and $ED_c(\lambda)$ be the energies defined by (6.1) and (6.2). Then,

$$ED_c(\lambda) < ED_t(\lambda) \qquad \text{for all} \quad \lambda > \lambda^* .$$

Furthermore, $ED_c(\lambda)$ is a continuously differentiable, strictly increasing function of λ on (λ^*, ∞) with

$$\frac{d}{d\lambda}\{ ED_c(\lambda) \} = \omega_N\Phi_1(w(\lambda), \lambda, \ldots, \lambda) > 0 \qquad \text{for} \quad \lambda > \lambda^* ,$$

$$\lim_{\lambda\to\lambda^*} ED_c(\lambda) = ED_t(\lambda^*) \qquad \text{and} \qquad \liminf_{\lambda\to\infty} \lambda^{-2N}\{ ED_t(\lambda) - ED_c(\lambda) \} > 0 .$$

Proof. — For $0 < \alpha \leqslant \lambda$, let $f(\lambda, \alpha) = \Phi(\alpha, \lambda, \ldots, \lambda) + [\lambda - \alpha]\Phi_1(\alpha, \lambda, \ldots, \lambda)$. Then for $\lambda > \lambda^*$,

$$\frac{N}{\omega_N}\{ ED_t(\lambda) - ED_c(\lambda) \} = f(\lambda, \lambda) - f(\lambda, w(\lambda))$$
$$= \frac{1}{2}[\lambda - w(\lambda)]^2\Phi_{11}(\gamma(\lambda), \lambda, \ldots, \lambda) > 0$$

by (A2), where $w(\lambda) < \gamma(\lambda) < \lambda$.

Since $\lim_{\lambda\to\infty} w(\lambda)=0$, there exists $\lambda_1 > 0$ such that

$$[\lambda - w(\lambda)]^2\Phi_{11}(\gamma(\lambda)\lambda, \ldots, \lambda) \geqslant C\lambda^{2N} \qquad \text{for} \quad \lambda \geqslant \lambda_1 .$$

Furthermore

$$\frac{d}{d\lambda}\,\mathrm{ED}_c(\lambda) = \frac{\omega_N}{N}\frac{d}{d\lambda}f(\lambda, w(\lambda)) = \frac{\omega_N}{N}\left\{\frac{\partial f}{\partial\lambda}(\lambda, w(\lambda)) + \frac{\partial f}{\partial\alpha}(\lambda, w(\lambda))\frac{dw}{d\lambda}(\lambda)\right\}$$
$$= \frac{\omega_N}{N}\{(N-1)\Phi_2 + \Phi_1 + [\lambda - w(\lambda)](N-1)\Phi_{12} + [\lambda - w(\lambda)]\Phi_{11}w'(\lambda)\} = \omega_N\Phi_1$$

by (5.12) where the partial derivatives of Φ are evaluated at $(w(\lambda), \lambda, \ldots, \lambda)$. Now,

$$\Phi_1(w(\lambda), \lambda, \ldots, \lambda) = \lambda^{N-1}\tilde{T}(\lambda, w(\lambda))(1)$$
$$> \lambda^{N-1}\tilde{T}(\lambda, w(\lambda))(0) = 0 ,$$

by Lemma 1(*a*). Hence $\frac{d}{d\lambda}\,\mathrm{ED}_c(\lambda) > 0$ for $\lambda > \lambda^*$.

Remarks. — 1. We can obtain some information about the energies of the solutions with cavities for the Cauchy traction problem. From (3.4), let

$$\mathrm{EC}_c(\lambda) = \mathrm{ED}_c(\lambda) - \frac{\omega_N\lambda}{N}\Phi_1(w(\lambda), \lambda, \ldots, \lambda)$$
$$= \frac{\omega_N}{N}\{\Phi(w(\lambda), \lambda, \ldots, \lambda) - w(\lambda)\Phi_1(w(\lambda), \lambda, \ldots, \lambda)\} ,$$

for $\lambda > \lambda^*$, by Lemma 10. Hence,

$$\frac{d}{d\lambda}\,\mathrm{EC}_c(\lambda) = \frac{\omega_N}{N}\left\{\Phi_1 w'(\lambda) + (N-1)\Phi_2 - w'(\lambda)\Phi_1 + w(\lambda)\frac{d\Phi_1}{d\lambda}\right\}$$
$$= \omega_N\frac{(N-1)}{N}R(w(\lambda), \lambda) > 0 \qquad \text{for} \quad \lambda > \lambda^* ,$$

by (A6). Thus the function $\mathrm{EC}_c(P)$ defined by (3.4) is strictly decreasing on $(0, P^*)$.

2. To obtain some information about the dead-load traction problem and to ensure that

$$\lim_{\lambda\to\infty}\mathrm{ED}_c(\lambda) = \lim_{P\to 0+}\mathrm{EC}_c(P) = \infty$$

we seem to need an extra hypothesis about Φ.

THEOREM 11. — In addition to the assumptions (A1) to (A7), suppose that

(A8) $$\lim_{(q,t)\to(0,\infty)} R(q, t) = \infty .$$

Then $\lim_{\lambda\to\infty} \Phi_1(w(\lambda), \lambda, \ldots, \lambda) = \infty$ and so $\lim_{\lambda\to\infty} \frac{d}{d\lambda} \mathrm{ED}_c(\lambda) = \lim_{\lambda\to\infty} \mathrm{ED}_c(\lambda) = \infty$.

Proof. — For $\lambda > \lambda^*$, let $U = U(\lambda, w(\lambda))$. In the notation of Lemma 4, we have

$$\frac{d}{dt} S(t) = -(N-1)t^{-N}R(q,t),t) \qquad \text{for} \quad t > \lambda$$

where $S(t) = t^{1-N}\Phi_1(q(t), t, \ldots, t)$. Since $\lim_{t\to\infty} S(t) = \tau(\lambda, w(\lambda)) = 0$, it follows that

$$S(t) = (N-1)\int_t^\infty s^{-N}R(q(s), s)ds\,.$$

Now given any $M > 0$, $\exists\varepsilon > 0$ and $t_0 > 0$ such that

$$R(q,t) > M \quad \text{if} \quad 0 < q < \varepsilon \quad \text{and} \quad t > t_0\,.$$

Since $q(t) \to 0$ as $t \to \infty$, it follows that $\exists t_0 > 0$ such that

$$R(q(t), t) > M \quad \text{for all} \quad t > t_0\,.$$

Hence
$$S(t) \geqslant Mt^{1-N} \qquad \text{for all} \quad t > t_0\,.$$

Thus $\Phi_1(q(t), t, \ldots, t) \geqslant M$ for all $t > t_0$ and so $\lim_{t\to\infty} \Phi_1(q(t), \ldots, t) = \infty$.

By (5.12),
$$\lim_{\mu\to\infty} \Phi_1(w(\mu), \mu, \ldots, \mu) = \lim_{r\to 0} \Phi_1\left(U'(r), \frac{U(r)}{r}, \ldots, \frac{U(r)}{r}\right) = \lim_{t\to\infty} \Phi_1(q(t), t, \ldots, t) = \infty\,.$$

The limits for $\mathrm{ED}_c(\lambda)$ and $\frac{d}{d\lambda}\mathrm{ED}_c(\lambda)$ follow from Theorem 10.

Remarks. 1. By (A8), we also have that $\lim_{\lambda\to\infty} \frac{d}{d\lambda}\mathrm{EC}_c(\lambda) = +\infty$ by (6.4).

Thus we conclude that $\lim_{P\to 0+} \mathrm{EC}_c(P) = \infty$. Furthermore Theorem 11 implies that the function D given by (3.5) for the dead-load traction problem has the property that $\lim_{\lambda\to\infty} D(\lambda) = \infty$.

2. When the function Φ has the special form (4.4),

$$R(q,t) = \frac{t\phi'(t) - q\phi'(q)}{t-q} \qquad \text{for} \qquad t \neq q\,.$$

Hence we see that the conditions (A1) to (A8) are all satisfied provided that ϕ and h satisfy the conditions (B1) to (B5) of section 4 and, in addition,

$$\lim_{t\to\infty} \phi'(t) = \infty\,.$$

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