ANNALES DE L'I. H. P., SECTION C

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Annales de l'I. H. P., section C, tome 1, nº 2 (1984), p. 109-145

<http://www.numdam.org/item?id=AIHPC_1984__1_2_109_0>

Ann. Inst. Henri Poincaré,
Vol. 1, n° 2, 1984, p. 109-145. *Analyse non linéaire*

The concentration-compactness principle in the Calculus of Variations. The locally compact case, part 1.

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ABSTRACT. — We present here a new method for solving minimization problems in unbounded domains. We first derive a general principle showing the equivalence between the compactness of all minimizing sequences and some strict sub-additivity conditions. The proof is based upon a compactness lemma obtained with the help of the notion of concentration function of a measure. We give various applications to problems arising in Mathematical Physics.

RÉSUMÉ. — Nous présentons ici une méthode nouvelle de résolution des problèmes de minimisation dans des domaines non bornés. Nous commençons par établir un principe général montrant l'équivalence entre la compacité de toutes les suites minimisantes et certaines conditions de sous-additivité stricte. La démonstration s'appuie sur un lemme de compacité obtenu à l'aide de la notion de fonction de concentration d'une mesure. Nous donnons diverses applications à des problèmes de physique mathématique.

Key-words: Concentration-compactness principle, minimization problem, unbounded domains, concentration function, rotating stars, Choquard-Pekar equations.
Subjet AMS Classification : 49 A 22, 47 H 15, 49 H 05, 58 E 30, 35 J 65, 81 C 05.

Annales de l'Institut Henri Poincaré - Analyse non linéaire - 0294-1449/84/02/109/37/$ 5,70

INTRODUCTION

In the Calculus of Variations or in Mathematical Physics, many *minimization problems* are given on *unbounded domains* like $\mathbb{R}^N$ for example. In general, the *invariance* of $\mathbb{R}^N$ by the *non-compact groups* of translations and dilations creates possible loss of compactness: as an illustration of these difficulties, recall that Rellich-Kondrakov theorem is no more valid in $\mathbb{R}^N$. The consequence of this fact is that, except for the special case of convex functionals, the standard convexity-compactness methods used in problems set in bounded domains fail to treat problems in unbounded domains.

In this series of papers, we present a general method—called *concentration, compactness method*—, which enables us to solve such problems. Roughly speaking in this paper and in the following one (Part 2), we show how this method enables us to solve problems with some form of « local compactness » or in other words problems which, if they were set in a bounded domain, would be solved by classical convexity-compactness methods. Subsequently we will study limiting cases when even in « local versions of the problem » loss of compactness may occur be the action of the group of dilations.

We first explain below that for general minimization problems, some sub-additivity inequalities hold. In the setting we take in this part (more general ones are given in Part 2), we consider minimization problems with constraints and the sub-additivity inequalities we obtain are for the infimum of the problem considered as a function of the value of the constraint. For a more precise statement we refer the reader to section 1. These inequalities are obtained by looking at special trial functions essentially consisting of two functions, one of which being sent to infinity by the use of translations.

We then show a general principle (concentration-compactness principle) which states that *all minimizing sequences are relatively compact if and only if the sub-additivity inequalities are strict*. The proof is based upon a lemma which, intuitively, indicates that the only possible loss of compactness for minimizing sequences occurs from the splitting of the functions at least in two parts which are going infinitely away from each other. And since this phenomenon is easily ruled out by the strict sub-additivity inequalities, we obtain some form of compactness. This crucial lemma is proved with the help of the notion of the concentration function of a measure —introduced by P. Lévy [*14*]. At this point, we want to emphasize that the arguments given in section I are only heuristic and that in all the examples we treat a rigorous proof has to be worked out, but always following the same general lines we give in section I.

We next apply this principle and its method of proof to various problems and examples. In this paper we consider the so-called *rotating stars problem* (see A. G. Auchmuty and R. Beals [*1*], [*2*], P. L. Lions [*16*], A. Friedman [*13*]):

$$(1)\quad \operatorname{Inf}\left\{ \int_{\mathbb{R}^3} j(\rho) + K(x)\rho dx - \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \rho(x)\rho(y)f(x-y)dxdy / \rho \geqslant 0,\ \rho \in L^1(\mathbb{R}^3),\ \int_{\mathbb{R}^3} \rho dx = \lambda \right\}$$

where K, f are given and j is a given convex function and where λ is a prescribed positive constant—representating the mass of the star-like fluid which density is given by ρ—. We will give below (section II) a complete solution of this problem by a direct application of our method.

We next treat the so-called Choquard-Pekar problem (see for example E. H. Lieb [*15*]):

$$(2)\quad \operatorname{Inf}\left\{ \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 + \frac{1}{2}V(x)u^2 dx - \frac{1}{4}\iint_{\mathbb{R}^N\times\mathbb{R}^N} u^2(x)u^2(y)\frac{1}{|x-y|}dxdy / u \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} u^2 dx = 1 \right\}.$$

We give below a necessary and sufficient condition (on V) for the solvability of this problem.

In part 2, we apply our methods to various variational problems associated with *nonlinear fields equations* such as for example:

$$(3)\qquad -\Delta u = f(x,u) \quad \text{in } \mathbb{R}^N, \qquad u \not\equiv 0, \quad u \underset{|x|\to\infty}{\longrightarrow} 0$$

where $f(x, t)$ is a given nonlinearity satisfying (for example): $f(x, 0) = 0$.

These equations also arise in the study of solitary waves in nonlinear Schrödinger equations (study of laser beams, see Suydam [*26*]) or in nonlinear Klein-Gordon equations (see W. Strauss [*25*], H. Berestycki and P. L. Lions [*3*] [*4*]).

We give in Part 2 sharp conditions ensuring the solvability of (3): the results we obtain contain the particular case when f is independent of x; this special case was studied by various authors (Z. Nehari [*22*]; G. H. Ryder [*24*]; M. Berger [*6*]; C. V. Coffmann [*8*], W. A. Strauss [*25*]; Coleman, Glazer and Martin [*9*]) and was settled in H. Berestycki and P. L. Lions [*3*] [*4*] – but all the results and methods in these references used heavily the spherical symmetry of this special case and thus could not be extended to the general case (3).

In Part 2, we also explain how our general method may apply to *unconstrained problems*, (ex. : Hartree-Fock problems), *problems with several constraints* (systems), problems in unbounded domains other than $\mathbb{R}^N$ (half-space, exterior domains...), problems invariant by translations only in some particular direction(s) (ex. : the vortex rings problem, see Fraenkel and Berger [*12*], H. Berestycki and P. L. Lions [*5*])...

Let us also mention that the method presented here also enables us to show the *orbital stability* of some standing waves in nonlinear Schrödinger equations (see T. Cazenave and P. L. Lions [*7*]). As a very particular consequence of our method, we present conditions for the solvability of problems in $\mathbb{R}^N$ which, if they were set in bounded domains, would always be solvable. The fact that conditions are needed was first noticed in M. J. Esteban and P. L. Lions [*11*].

The author would like to acknowledge the determinant contribution of S. R. S. Varadhan without whom this work would not have been possible.

The results and methods given in this paper and in the following one (Part 2) were announced in [*19*] [*20*].

I. THE HEURISTIC PRINCIPLE

I.1. General framework.

Let us first indicate the typical problem we want to look at: let H be a function space on $\mathbb{R}^N$ and let J, $\mathscr{E}$ be functionals defined on H (or on a subdomain of H) of the type:

$$(1) \qquad \mathscr{E}(u) = \int_{\mathbb{R}^N} e(x, \mathrm{A}u(x))dx\,; \qquad \mathrm{J}(u) = \int_{\mathbb{R}^N} j(x, \mathrm{B}u(x))dx$$

where $e(x, p), j(x, q)$ are real-valued functions defined on $\mathbb{R}^N \times \mathbb{R}^m$, $\mathbb{R}^N \times \mathbb{R}^n$ and j is nonnegative (for example); A, B are operators (possibly nonlinear) from H into E, F (functions spaces defined on $\mathbb{R}^N$ with values in $\mathbb{R}^m$, $\mathbb{R}^n$) which commute with translations of $\mathbb{R}^N$. To simplify, we assume: $\mathrm{J}(0) = 0$.

We consider the following minimization problem

$$(\mathrm{M}) \qquad \inf \{ \mathscr{E}(u)/u \in \mathrm{H},\ \mathrm{J}(u) = 1 \}.$$

We first imbed the problem (M) into a one parameter family of problems (M_λ)

$$(\mathrm{M}_\lambda) \qquad \mathrm{I}_\lambda = \inf \{ \mathscr{E}(u)/u \in \mathrm{H},\ \mathrm{J}(u) = \lambda \}$$

where $\lambda > 0$.

Our main assumption lies in the fact that we assume that it is possible to define « a problem at infinity »: for example we assume

$$(2) \qquad j(x, q) \to j^\infty(q), \qquad e(x, p) \to e^\infty(p) \quad \text{as} \quad |x| \to \infty$$

for all $p, q \in \mathbb{R}^m$, $\mathbb{R}^n$. The precise meaning of (2) has to be worked out in each problem. We then consider the problems at infinity

$$(\mathrm{M}_\lambda^\infty) \qquad \mathrm{I}_\lambda^\infty = \inf \{ \mathscr{E}^\infty(u)/u \in \mathrm{H},\ \mathrm{J}^\infty(u) = \lambda \}$$

where

$$\mathscr{E}^{\infty}(u) = \int_{\mathbb{R}^N} e^{\infty}(\mathrm{A}u(x))dx, \ \mathrm{J}^{\infty}(u) = \int_{\mathbb{R}^N} j^{\infty}(\mathrm{B}u(x))dx .$$

We will finally assume that we have some type of « *a priori* estimates » for problems (M_λ), $(\mathrm{M}_\lambda^\infty)$ insuring in particular that

$$\mathrm{K}_\lambda = \{ u \in \mathrm{H}, \mathrm{J}(u) = \lambda \} \neq \emptyset, \mathrm{I}_\lambda > -\infty$$

for all $\lambda > 0$ or for all $\lambda \in (0, 1]$, and that minimizing sequences for $(\mathrm{M}_\lambda) - (\mathrm{M}_\lambda^\infty)$ are bounded in H. Again, these *a priori* estimates are adapted to each particular problem.

REMARK I.1. — In what follows, we will also treat problems where:
— j is not nonnegative
— the constraint: $\mathrm{J}(u) = 1$ is replaced by: $\mathrm{J}(u) = 0$
— functions u in H take values in $\mathbb{R}^p$ instead of $\mathbb{R}$ (or $\mathbb{C}$)
— one has several constraints
— functions u in H are defined on unbounded domains different from $\mathbb{R}^N$.

For all those variants, the idea that we describe in the next section is adapted by straightforward arguments. ■

I.2. The sub-additivity conditions.

We first remark that we always have:

$$\mathrm{I}_\lambda \leqslant \mathrm{I}_\alpha + \mathrm{I}_{\lambda-\alpha}^{\infty}, \qquad \forall \alpha \in [0, \lambda[\tag{3}$$

where we agree: $\mathrm{I}_0 = 0$. Let us explain *heuristically* why (3) holds: indeed let $\varepsilon > 0$ and u_ε, v_ε be satisfy:

$$\begin{cases} \mathrm{I}_\alpha \leqslant \mathscr{E}(u_\varepsilon) \leqslant \mathrm{I}_\alpha + \varepsilon, & \mathrm{J}(u_\varepsilon) = \alpha \\ \mathrm{I}_{\lambda-\alpha}^{\infty} \leqslant \mathscr{E}^{\infty}(v_\varepsilon) \leqslant \mathrm{I}_{\lambda-\alpha}^{\infty} + \varepsilon, & \mathrm{J}^{\infty}(v_\varepsilon) = \lambda - \alpha ; \end{cases} \tag{4}$$

by a density argument we may assume that u_ε and v_ε have compact support and we denote by $v_\varepsilon^n = v_\varepsilon(. + n\chi)$ where χ is some given unit vector in $\mathbb{R}^N$. Since for n large enough, the distance between the supports of u_ε and v_ε^n is strictly positive and goes to $+\infty$ as n goes to $+\infty$, we deduce:

$$\begin{cases} \mathscr{E}(u_\varepsilon + v_\varepsilon^n) - \{ \mathscr{E}(u_\varepsilon) + \mathscr{E}^{\infty}(v_\varepsilon^n) \} \underset{n}{\to} 0 \\ \mathrm{J}(u_\varepsilon + v_\varepsilon^n) - \{ \mathrm{J}(u_\varepsilon) + \mathrm{J}^{\infty}(v_\varepsilon^n) \} \underset{n}{\to} 0 \end{cases} \tag{5}$$

and since $\mathscr{E}^{\infty}$, J^{∞} are translation-invariant, we finally obtain:

$$\begin{cases} \mathrm{I}_\alpha + \mathrm{I}_{\lambda-\alpha}^{\infty} \leqslant \lim\limits_n \mathscr{E}(u_\varepsilon + v_\varepsilon^n) = \mathscr{E}(u_\varepsilon) + \mathscr{E}(v_\varepsilon) \leqslant \mathrm{I}_\alpha + \mathrm{I}_{\lambda-\alpha}^{\infty} + 2\varepsilon \\ \lim\limits_n \{ \mathrm{J}(u_\varepsilon) + \mathrm{J}^{\infty}(v_\varepsilon^n) \} = \lambda \end{cases}$$

and by definition of I_λ we conclude:

$$I_\lambda \leqslant I_\alpha + I_{\lambda-\alpha} + 2\varepsilon$$

Let us now explain the typical results we may obtain using what we call the *concentration-compactness principle:* we first consider *the case when e* and *j* do depend on *x*, in this case we show that, for each fixed $\lambda > 0$, *all minimizing sequences of the problem* (M_λ) *are relatively compact if and only if the following strict subadditivity condition is satisfied:*

$$\text{(S.1)} \qquad I_\lambda < I_\alpha + I^\infty_{\lambda-\alpha}, \qquad \forall \alpha \in [0, \lambda[\,.$$

In the case when e and *j* do not depend on *x*, and thus (M_λ) is equivalent to (M^∞_λ) and is invariant by translations, then, for each fixed $\lambda > 0$, *all minimizing sequences of the problem* (M_λ) *are relatively compact up to a translation if and only if the following strict subadditivity condition is satisfied:*

$$\text{(S.2)} \qquad I_\lambda < I_\alpha + I_{\lambda-\alpha}, \qquad \forall \alpha \in]0, \lambda[\,;$$

recall that $I_\alpha = I^\infty_\alpha$, $\forall \alpha \in]0, \lambda]$.

The fact that (S.1) (resp. (S.2)) are necessary conditions for the compactness of all minimizing sequences is a consequence of the argument we gave to prove (3): indeed if for example

$$I_\lambda = I_\alpha + I^\infty_{\lambda-\alpha}$$

with $\alpha \in]0, \lambda[$ and if u_n, v_n denote minimizing sequences with compact supports of problems (M_α), $(M^\infty_{\lambda-\alpha})$. Obviously:

$$J^\alpha(\tilde{v}_n) = \lambda - \alpha, \qquad \mathscr{E}^\alpha(\tilde{v}_n) = \mathscr{E}(v_n) \underset{n}{\to} I^\infty_{\lambda-\alpha}$$

where $\tilde{v}_n = v_n(\cdot + \xi_n)$ and $\xi_n \in \mathbb{R}^N$. Choosing $|\xi_n|$ large enough, we may assume that: dist (Supp u_n, Supp $\tilde{v}_n$) $\underset{n}{\to} \infty$.

Therefore if we consider: $w_n = u_n + \tilde{v}_n$, w_n cannot be relatively compact since we can always find $\chi_n \in \mathscr{D}(\mathbb{R}^N)$ with *given norms* in H such that: $\int_{\mathbb{R}^N} w_n \chi_n dx = 0$. On the other hand, we find:

$$J(w_n) \underset{n}{\to} \lambda, \quad \mathscr{E}(w_n) = \lim_n \{ \mathscr{E}(u_n) + \mathscr{E}(v_n) \} = I_\alpha + I^\infty_{\lambda-\alpha} = I_\lambda$$

and we conclude.

In conclusion, we see that conditions (S.1)-(S.2) insure the *compactness* of all minimizing sequences and we will see in the following section—where we prove that (S.1)-(S.2) are sufficient conditions for the compactness of minimizing sequences—that (S.1)-(S.2) « prevent the possible splitting of minimizing sequences u_n by keeping u_n *concentrated* ».

I.3. The concentration-compactness lemma.

In this section, we show heuristically the fact that (S.1)-(S.2) insure the compactness of minimizing sequences. As we just said the argument we give below is heuristic but nevertheless, conveniently adapted and justified in all examples in sections below, will be the key argument that we will always use in the following sections.

The argument is based upon the following lemma, which admits many variants all obtained via similar proofs:

LEMMA I.1. — *Let $(\rho_n)_{n\geq 1}$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying:*

$$\rho_n \geqslant 0 \text{ in } \mathbb{R}^N, \qquad \int_{\mathbb{R}^N} \rho_n dx = \lambda \tag{6}$$

where $\lambda > 0$ is fixed. Then there exists a subsequence $(\rho_{n_k})_{k\geq 1}$ satisfying one the three following possibilities:

i) (compactness) there exists $y_k \in \mathbb{R}^N$ such that $\rho_{n_k}(\cdot + y_k)$ is tight i. e.*:*

$$\forall \varepsilon > 0, \exists R < \infty, \quad \int_{y_k + B_R} \rho_{n_k}(x)dx \geqslant \lambda - \varepsilon; \tag{7}$$

ii) (vanishing) $\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{y+B_R} \rho_{n_k}(x)dx = 0,$ *for all* $R < \infty$;

iii) (dichotomy) there exists $\alpha \in]0, \lambda[$ such that for all $\varepsilon > 0$, there exist $k_0 \geqslant 1$ and $\rho_k^1, \rho_k^2 \in L^1_+(\mathbb{R}^N)$ satisfying for $k \geqslant k_0$:

$$\left\{\begin{array}{l} \| \rho_{n_k} - (\rho_k^1 + \rho_k^2) \|_{L^1} \leqslant \varepsilon, \left| \int_{\mathbb{R}^N} \rho_k^1 dx - \alpha \right| \leqslant \varepsilon, \left| \int_{\mathbb{R}^N} \rho_k^2 dx - (\lambda - \alpha) \right| \leqslant \varepsilon \\ \text{dist}(\text{Supp } \rho_k^1, \text{Supp } \rho_k^2) \underset{k}{\rightarrow} +\infty . \end{array}\right. \tag{8}$$

Let us first explain how we use Lemma I.1 and we will then prove Lemma I.1. We consider first the case when e, j depend on x and we assume that (S.1) holds, and we take a minimizing sequences $(u_n)_{n\geq 1}$ of (M_λ):

$$\mathscr{E}(u_n) \underset{n}{\rightarrow} I_\lambda, \qquad J(u_n) = \lambda .$$

We apply Lemma I.1 with $\rho_n = j(x, Bu_n(x))$: we find a subsequence $(n_k)_{k\geq 1}$ such that (*i*), (*ii*) or (*iii*) holds for all $k \geqslant 1$. It is easy to see that (*ii*) cannot occur since we have in view of (S.1): $I_\lambda < I_\lambda^\infty$ and $J(u_n) = \lambda$. Next, if (*iii*) occurs we split u_n exactly as we split ρ_{n_k} (see the proof of Lemma I.1) and find, for all $\varepsilon > 0$, u_k^1, u_k^2 in H satisfying for k large enough: $u_{n_k} = u_k^1 + u_k^2 + v_k$

$$\left\{\begin{array}{ll} | J(u_k^1) - \alpha | \leqslant \varepsilon, & | J(u_k^2) - (\lambda - \alpha) | \leqslant \varepsilon ; \\ \text{dist}(\text{Supp } u_k^1, \text{Supp } u_k^2) \underset{k}{\rightarrow} \infty, & \| v_k \| \leqslant \varepsilon . \end{array}\right.$$

Replacing possibly α by $\lambda - \alpha$, we may assume without loss of generality that we have:

$$J(u_k^2) - J^\infty(u_k^2) \underset{k}{\to} 0, \qquad \varliminf_k \mathscr{E}(u_k^2) - \mathscr{E}^\infty(u_k^2) \geqslant 0$$

– see also the construction of ρ_k^1, ρ_k^2. Finally we obtain:

$$\begin{cases} I_\lambda = \lim_k \mathscr{E}(u_{n_k}) \geqslant \varliminf_k \{ \mathscr{E}(u_k^1) + \mathscr{E}^\infty(u_k^2) \} - \delta_\varepsilon \\ \qquad\qquad\qquad\quad \geqslant I_{\alpha-\varepsilon} + I^\infty_{\lambda-\alpha-\varepsilon} - \delta(\varepsilon); \end{cases}$$

and sending ε to 0 we find:

$$I_\lambda \geqslant I_\alpha + I^\infty_{\lambda-\alpha}$$

and this contradicts (S.1). The contradiction shows that (*ii*) cannot occur: therefore (*i*) occurs and we conclude easily if e, j do not depend on x. If e, j depend on x, we still need to show that y_k given in (*i*) is bounded. If it were not the case, we would deduce (taking a subsequence if necessary)

$$I_\lambda = \lim_k \mathscr{E}(u_{n_k}) \geqslant \varliminf_k \mathscr{E}^\infty(u_{n_k}), \qquad |y_k| \underset{k}{\to} \infty$$

$$\lambda = J(u_{n_k}) = \lim_k J^\infty(u_{n_k})$$

and thus: $I_\lambda \geqslant I_\lambda^\infty$, which again contradicts (S.1). ■

In conclusion, let us insist on the fact that the above argument is not rigorous but will be justified on all examples below. Let us also point out that, as we indicated, assumptions on H, e, j are needed insuring *a priori* estimates: more precisely we will assume that H, e, j are such that if (M_λ) was posed in a bounded domain instead of $\mathbb{R}^N$, then the solvability of (M_λ) would be insured by « usual » arguments using convexity-compactness methods. The role of (S.1) (or (S.2)) is to prevent (*ii*), (*iii*) from occuring and thus because of (*i*) to essentially reduce the problem to the case of a bounded domain.

We now prove Lemma I.1 using the notion of the concentration function of a measure (notion introduced by Lévy [*14*]) i. e. we consider the function: $Q_n(t) = \sup_{y\in\mathbb{R}^N} \int_{y+B_t} \rho_n(\xi)d\xi$. $(Q_n)_n$ is a sequence of nondecreasing, nonnegative, uniformly bounded functions on $\mathbb{R}_+$ and: $\lim_{t\to+\infty} Q_n(t) = \lambda$. By a classical lemma, there exist a subsequence $(n_k)_{k\geqslant 1}$ and a nondecreasing nonnegative function Q such that $Q_{n_k}(t) \underset{k}{\to} Q(t)$ for all $t \geqslant 0$—observe for example that Q_n is bounded in BV(0, T) for all $T < \infty$—. Obviously

$$\alpha = \lim_{t\to+\infty} Q(t) \in [0, \lambda].$$

If $\alpha = 0$, then (*ii*) occurs. If $\alpha = \lambda$, then (*i*) occurs: this is a classical consequence of the notion of concentration functions, see for example K. R. Parthasarathy [23]. We recall briefly the proof of this claim: indeed take $\mu > \frac{\lambda}{2}$ and observe that there exists $R = R(\mu)$ such that for all $k \geqslant 1$ we have

$$Q_{n_k}(R) = \sup_{y \in \mathbb{R}^N} \int_{y+B_R} \rho_{n_k}(\xi)d\xi > \mu$$

and thus there exists $y_k = y_k(\mu)$ satisfying:

$$\int_{y_k+B_R} \rho_{n_k}(\xi)d\xi > \mu .$$

We then set $y_k = y_k\left(\frac{\lambda}{2}\right)$ and we remark that we have necessarily:

$$| y_k(\mu) - y_k | \leqslant R\left(\frac{\lambda}{2}\right) + R(\mu) \qquad \text{for all} \quad \mu \geqslant \frac{\lambda}{2} .$$

Therefore setting $R'(\mu) = R\left(\frac{\lambda}{2}\right) + 2R(\mu)$, we find that there exists a sequence y_k in $\mathbb{R}^N$ such that for all $k \geqslant 1$ and for all $\mu \geqslant \frac{\lambda}{2}$ we have:

$$\int_{y_k+B_{R'}} \rho_{n_k}(\xi)d\xi > \mu ;$$

and (*ii*) is proved.

Finally if $\alpha \in (0, \lambda)$, we have to show that (*iii*) holds. Let $\varepsilon > 0$, choose R such that: $Q(R) > \alpha - \varepsilon$. Then for k large enough ($k \geqslant k_0$) we have: $\alpha - \varepsilon < Q_{n_k}(R) < \alpha + \varepsilon$. Furthermore we can find $R_k \underset{k}{\to} +\infty$ such that: $Q_{n_k}(R_k) \leqslant \alpha + \varepsilon$. Finally there exists $y_k \in \mathbb{R}^N$ such that:

$$\int_{y_k+B_R} \rho_{n_k}(\xi)d\xi \in]\alpha - \varepsilon, \alpha + \varepsilon[.$$

We then set: $\rho_k^1 = \rho_{n_k} 1_{y_k+B_R}$, $\rho_k^2 = \rho_{n_k} 1_{\mathbb{R}^N-(y_k+B_{R_k})}$. It is now clear that (8) holds since:

$$\int_{\mathbb{R}^N} \{ \rho_{n_k} - \rho_k^1 - \rho_k^2 \} \, dx = \int_{R \leqslant |x-y_k| \leqslant R_k} \rho_{n_k}(x)dx$$
$$\leqslant Q_{n_k}(R_k) - Q_{n_k}(R) + 2\varepsilon$$
$$\leqslant (\alpha + \varepsilon) - (\alpha - \varepsilon) + 2\varepsilon = 4\varepsilon .$$

■

REMARK I.2. — In the case when the elements in H have necessarily some smoothness, one replaces of course the characteristic functions $1_{y_k+B_R}$ by smooth cut-off functions: this argument will be detailed later on. ■

II. MINIMIZATION PROBLEMS IN L^1

II.1. Setting of the problem and main result.

We consider the following problem: find u minimizing:

$$(9)\quad I_\lambda = \inf\left\{ \int_{\mathbb{R}^N} j(u)dx - \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} u(x)u(y)f(x-y)dxdy / u \in K_\lambda \right\}$$

where $K_\lambda = \left\{ u \in L^q(\mathbb{R}^N)\cap L^1(\mathbb{R}^N),\ u \geqslant 0 \text{ a. e.}, \int_{\mathbb{R}^N} u dx = \lambda \right\}$ with $q = \frac{p+1}{p}$, $p \in]1, \infty[$ and where j, f satisfy:

$$(10)\qquad f \in M^p(\mathbb{R}^N)\ (^1), \qquad f \geqslant 0 \text{ a. e.};$$

$$(11)\qquad \begin{cases} j \text{ is a strictly convex, nonnegative function on } \mathbb{R}^+ \text{ satisfying:} \\ \lim_{t\to 0_+} j(t)t^{-1} = 0, \qquad \lim_{t\to\infty} j(t)t^{-q} = +\infty . \end{cases}$$

Such problems arise in Astrophysics and in Quantum Mechanics (Thomas Fermi theory): we refer the reader for more details to J. F. G. Auchmuty and R. Beals [*1*] [*2*], P. L. Lions [*16*], A. Friedman [*13*], a typical function f is:

$$(12)\qquad f(x) = \frac{1}{|x|} \quad \text{and} \quad N = 3;$$

of course in this case we have: $f \in M^3(\mathbb{R}^3)$ and $q = \frac{4}{3}$.

Let us immediately give our main result concerning this problem—we will see in sections II.3.4 various extensions of this result.

THEOREM II.1. — *Under assumptions* (10) *and* (11), *every minimizing sequence* $(u_n)_{n\geqslant 1}$ *of* (9) *is relatively compact in* $L^1(\mathbb{R}^N)$ *up to a translation if and only if the following condition holds:*

$$(S.2)\qquad I_\lambda < I_\alpha + I_{\lambda-\alpha}, \qquad \forall \alpha \in]0, \lambda[.$$

In addition if (S.2) *holds, for every minimizing sequence* $(u_n)_{n\geqslant 1}$ *there exists* $(y_n)_{n\geqslant 1}$ *in* $\mathbb{R}^N$ *such that* $u_n(\cdot + y_n)$ *is relatively compact in* $L^1 \cap L^q(\mathbb{R}^N)$ *and* $j(u_n(\cdot + y_n))$ *is relatively compact in* $L^1(\mathbb{R}^N)$. *In particular if* (S.2) *holds, there exists a minimum of problem* (9).

We give below examples where (S.2) holds and conditions insuring that (S.2) holds. The very formulation of the result above suggests that this result is a direct application of the concentration-compactness principle

(*) M^p denotes the Marcinkiewicz space or weak L^p space.

stated in the preceding section and indeed the proof given in section II.2 will follow the general lines we indicated.

REMARK II.1. — We will see extensions of assumption (10) in section II.3 below. Let us also mention that if j is a strictly convex function on $\mathbb{R}^+$ satisfying: $j(0) = 0$, $\lim_{t\to\infty} j(t)t^{-q} = +\infty$, $\underline{\lim}_{t\to 0_+} j(t)t^{-1} > -\infty$ we may consider $\tilde{j} = j - j'(0_+)t$ and this modification changes the value of I_λ by $(-j'(0_+)\lambda)$. We may then apply Theorem II.1. We will also discuss the cases when j is not strictly convex and when we only have: $\underline{\lim}_{t\to\infty} j(t)t^{-q} > C$, where $C = C(p, N)$ is a suitably choosen constant.

REMARK II.2. — In [1] was treated the case $N=3$, $f=\dfrac{1}{|x|}$ by a method using the symmetry of f (and additional assumptions on j) while in [16] general results were obtained but still using symmetry assumptions on f. In particular the infimum had to be restricted to the subset of K_λ consisting of functions with certain symmetries: the symmetry giving the necessary compactness as it is explained in P. L. Lions [17] [18]. Let us mention by the way the very interesting open problem: if (S.2) holds, does any minimum of (9) has (up to a translation) spherical symmetry? We know the answer of this question (and it is then positive) only if f is radial non-increasing.

REMARK II.3. — Let us recall the following well-known convolution inequalities:

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} u(x)u(y)f(x-y)dxdy \leqslant C_1 \| u \|^2_{L^r(\mathbb{R}^N)} \| f \|_{M^p(\mathbb{R}^N)}$$

where $r = (2p)/(2p-1)$, and thus using Hölder inequality we find:

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} u(x)u(y)f(x-y)dxdy \leqslant C_0 \| f \|_{M^p(\mathbb{R}^N)} \| u \|^q_{L^q(\mathbb{R}^N)} \| u \|^{2-q}_{L^1(\mathbb{R}^N)} \tag{13}$$

Therefore if j satisfies:

$$\underline{\lim}_{t\to\infty} j(t)t^{-q} > \frac{1}{2} C_0 \| f \|_{M^p(\mathbb{R}^N)} \lambda^{2-q}$$

we immediately obtain that $I_\mu > -\infty$ for all $\mu \in (0, \lambda]$, I_μ is continuous with respect to μ on $(0, \lambda]$ and that *any minimizing sequence* $(u_n)_{n\geqslant 1}$ *of* (9) *satisfies:*

$$\begin{cases} u_n \text{ is bounded in } L^1 \cap L^q \\ j(u_n) \text{ is bounded in } L^1 . \end{cases} \tag{14}$$

On the other hand if $f(x) = |x|^{-N/p}$ (thus $f \in M^p(\mathbb{R}^N)$) and if j satisfies:

$$\overline{\lim_{t\to\infty}}\, j(t)t^{-q} < \frac{1}{2} C_0 \| f \|_{M^p(\mathbb{R}^N)} \lambda^{2-q}$$

where C_0 is the best constant in inequality (13) (we will show in section VII that C_0 is achieved), then *it is possible to show that* $I_\lambda = -\infty$. Indeed, take $u \in \mathscr{D}_+(\mathbb{R}^N)$ such that: $\int_{\mathbb{R}^N} u(x)dx = \lambda$ and

$$\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} u(x)u(y)f(x-y)dxdy = K > k = \overline{\lim_{t\to\infty}}\, j(t)t^{-q}.$$

We then consider $u_\varepsilon(x) = \varepsilon^{-N}u\left(\frac{x}{\varepsilon}\right)$, clearly u_ε still lies in K_λ and we have:

$$\begin{aligned}
&\int_{\mathbb{R}^N} j(u_\varepsilon)dx - \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_\varepsilon(x)u_\varepsilon(y)f(x-y)dxdy \\
&\quad \leqslant \int_{\mathbb{R}^N} Cu_\varepsilon + \frac{k+K}{2} u_\varepsilon^q dx - \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_\varepsilon(x)u_\varepsilon(y)f(x-y)dxdy \\
&\quad \leqslant C\lambda + \frac{k+K}{2}\varepsilon^{-N/p} - K\varepsilon^{-N/p} = C\lambda - \left(\frac{K-k}{2}\right)\varepsilon^{-N/p} \underset{\varepsilon\to 0_+}{\longrightarrow} -\infty .
\end{aligned}$$

We next observe that if j is convex, $\lim_{t\to 0_+} j(t)t^{-1} = 0$ and if (10) holds then: $I_\lambda \leqslant 0$. Indeed take $u \in \mathscr{D}_+(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u(x)dx = \lambda$. We then consider $u_\varepsilon(x) = \varepsilon^N u(\varepsilon x)$, clearly u_ε still lies in K_λ and we find easily:

$$\int_{\mathbb{R}^N} j(u_\varepsilon(x))dx = \int_{\mathbb{R}^N} \frac{1}{\varepsilon^N} j(\varepsilon^N u(x))dx \underset{\varepsilon\to 0_+}{\longrightarrow} 0$$

and this shows: $I_\lambda \leqslant 0$. ■

Let us now give a few examples of situations where (S.2) holds: all examples are obtained using the following elementary Lemma:

LEMMA II.1. — *Let h be a real-valued function on* $[0, \lambda]$ *with* $\lambda > 0$ *satisfying:* $h(\theta\alpha) < \theta h(\alpha)$ *for all* $\alpha \in]0, \lambda[$, $\theta \in]1, \lambda/\alpha]$. *Then we have:* $h(\lambda) < h(\alpha) + h(\lambda - \alpha)$ *for all* $\alpha \in]0, \lambda[$.

Indeed if for example $\alpha \geqslant \lambda - \alpha$, we have:

$$h(\lambda) < \frac{\lambda}{\alpha} h(\alpha) = h(\alpha) + \frac{\lambda - \alpha}{\alpha} h(\alpha) \leqslant h(\alpha) + h(\lambda - \alpha).$$

Using this observation we deduce from Theorem II.1:

COROLLARY II.1. — *We assume* (10), (11) *and either:*

(15) $\forall t \geqslant 1$, a. e. $\xi \in \mathbb{R}^N$, $\qquad f(t\xi) \geqslant t^{-m} f(\xi)$, $\qquad$ *for some* $\quad m \in (0, N)$

or

(16) $\forall \theta \geqslant 1, \forall t \geqslant 0, \qquad j(\theta t) \leqslant \theta^2 j(t)$.

Then (S.2) *holds if and only if* $I_\lambda < 0$. *If this is the case the conclusions of Theorem* II.1 *hold.*

REMARK II.4. — Of course if $f \equiv 0$, (10), (15) hold but $I_\lambda = 0$ and (S.2) does not hold. Let us also mention that *if* (15) *holds,* $f \not\equiv 0$ *and if* $\lim_{t \to 0+} j(t) t^{-(N+m)/N} = 0$, *then* $I_\lambda < 0$ *for all* $\lambda > 0$: indeed take $u \in \mathscr{D}_+(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u(x)dx = \lambda$ and consider $u_\varepsilon(x) = \varepsilon^N u(\varepsilon x)$. This is the case for example if $f(x) = \dfrac{1}{|x|^{N/p}}$ with $m = N/p$ $\left(\text{one finds again the exponent } q = \dfrac{N+m}{N}\right)$.

In addition if either (15) holds and $f \not\equiv 0$ or if (16) holds with θ^2 replaced by θ^ν for any $\nu < 2$, then $I_\lambda < 0$ for λ large enough: more precisely there exists $\lambda_0 \geqslant 0$ such that $I_\lambda = 0$ for $\lambda \in (0, \lambda_0]$ and $I_\lambda < 0$ for $\lambda > \lambda_0$. This is proved by considering either $u(\varepsilon x)$ or $\dfrac{1}{\varepsilon} u(x)$. ■

Proof of Corollary II.1. — Let $\alpha \in (0, \lambda)$, we are going to prove that if $\theta \in]1, \lambda/\alpha]$ then we have: $I_{\theta\alpha} < \theta I_\alpha$, provided $I_\alpha < 0$. By a straightforward modification of Lemma II.1, we will conclude that (S.2) holds provided $I_\lambda < 0$. Now if (S.2) holds, necessarily $I_\lambda < 0$ since (*cf.* Remark II.3) $I_\mu \leqslant 0$ for all $\mu \geqslant 0$. Therefore we may assume that $I_\alpha < 0$. If (15) holds, we take any u in K_α and we set: $v = u\left(\dfrac{\cdot}{\theta^{1/N}}\right)$, clearly $v \in K_{\theta\alpha}$ and thus

$$\begin{aligned} I_{\theta\alpha} &\leqslant \inf_{u \in K_\alpha} \left\{ \theta \int_{\mathbb{R}^N} j(u)dx - \frac{\theta^2}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} u(x)u(y) f(\theta(x-y)) dxdy \right\} \\ &\leqslant \inf_{u \in K_\alpha} \left\{ \theta \int_{\mathbb{R}^N} j(u)dx - \frac{\theta^{2-m/N}}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} u(x)u(y) f(x-y) dxdy \right\} \\ &\leqslant \theta^{2-m/N} I_\alpha < \theta I_\alpha < 0 . \end{aligned}$$

On the other hand if (16) holds, we take any u in K_α and we set: $v = \theta u$, clearly $v \in K_{\theta\alpha}$ and thus

$$\begin{aligned} I_{\theta\alpha} &\leqslant \inf_{u \in K_\alpha} \left\{ \int_{\mathbb{R}^N} j(\theta u)dx - \frac{\theta^2}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} u(x)u(y) f(x-y) dxdy \right\} \\ &\leqslant \theta^2 I_\alpha < \theta I_\alpha < 0 . \end{aligned}$$

■

REMARK II.5. — Looking closely at the above proof, one sees it is possible to replace (15) by:

$$(15') \qquad \forall t \geqslant 1, \ \exists \gamma > 0, \ \text{a. e. } \xi \in \mathbb{R}^N, \qquad f(t\xi) \geqslant (t^{-N} + \gamma) f(\xi).$$

II.2. **Proof of Theorem** II.1.

We first recall that the argument indicated in section I.2 is easily justified here and thus we see that we always have:

$$I_\lambda \leqslant I_\alpha + I_{\lambda-\alpha} \qquad \text{for all} \quad \alpha \in (0, \lambda)$$

and if there exists $\alpha \in (0, \lambda)$ such that the equality holds, then we can build some minimizing sequence which will not be relatively compact. Therefore (S.2) is a necessary condition for the compactness of minimizing sequences. We now have to show the converse and we thus assume that (S.2) holds. Then take a minimizing sequence (u_n) for (9) and recall that, in view of Remark II.3, we have:

$$\begin{cases} u_n \text{ is bounded in } L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \\ j(u_n) \text{ is bounded in } L^1(\mathbb{R}^N). \end{cases}$$

We may apply Lemma I.1 (with $\rho_n = u_n$) and we find a sequence that we still denote by (u_n) satisfying (*i*) or (*ii*) or (*iii*) for all $n \geqslant 1$. Exactly as in section I.3, we will rule out the possibility that (u_n) satisfies (*ii*) or (*iii*) by the use of (S.2) and we will conclude using the compactness obtained in (*i*).

Step 1 *: Dichotomy does not occur.*

If (*iii*) (in Lemma I.1) occurs, there exists $\alpha > 0$ such that for any fixed $\varepsilon > 0$, we may find u_n^1, u_n^2 satisfying for large n the condition (8) and furthermore recall from the proof of Lemma I.1 that we may assume:

$$u_n = u_n^1 + u_n^2 + v_n, \quad 0 \leqslant u_n^1, u_n^2, v_n \leqslant u_n, \quad u_n^1 u_n^2 = u_n^1 v_n = u_n^2 v_n = 0 \text{ a. e.}.$$

Finally we denote by: $d_n = \text{dist}\,(\text{Supp}\, u_n^1, \text{Supp}\, u_n^2)$, $\alpha_n = \int_{\mathbb{R}^N} u_n^1 dx$, $\beta_n = \int_{\mathbb{R}^N} u_n^2 dx$. We may assume without loss of generality that: $d_n \underset{n}{\to} \infty$, $\alpha_n \underset{n}{\to} \alpha$, $\beta_n \underset{n}{\to} \bar{\beta}$ with $\bar{\alpha} \in (0, \lambda)$, $\bar{\beta} \in (0, \lambda - \bar{\alpha})$, $|\bar{\alpha} - \alpha| \leqslant \varepsilon$, $|\bar{\beta} - (\lambda - \alpha)| \leqslant \varepsilon$.

We first notice that we have:

$$\int_{\mathbb{R}^N} j(u_n) dx \geqslant \int_{\mathbb{R}^N} j(u_n^1) dx + \int_{\mathbb{R}^N} j(u_n^2) dx.$$

In addition, we have:

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n(x)u_n(y)f(x-y)dxdy = \iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n^1(x)u_n^1(y)f(x-y)dxdy$$
$$+\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n^2(x)u_n^2(y)f(x-y)dxdy + 2\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n^1(x)u_n^2(y)f(x-y)dxdy$$
$$+ 2\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n(x)v_n(y)f(x-y)dxdy - \iint_{\mathbb{R}^N\times\mathbb{R}^N} v_n(x)v_n(y)f(x-y)dxdy\,.$$

And we remark that we have:

$$\left|\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n(x)v_n(y)f(x-y)dxdy\right|$$
$$\leqslant \|f\|_{M^p}\|u_n\|_{L^{2p/(2p-1)}}\|v_n\|_{L^{2p/(2p-1)}} \leqslant C\|v_n\|_{L^1}^{1-q/2} \leqslant C\varepsilon^{1-q/2}\,;$$
$$\left|\iint_{\mathbb{R}^N\times\mathbb{R}^N} v_n(x)v_n(y)f(x-y)dxdy\right| \leqslant C\|v_n\|_{L^1}^{2-q} \leqslant C\varepsilon^{2-q}\,.$$

We next claim that we have:

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n^1(x)u_n^2(y)f(x-y)dxdy \underset{n}{\to} 0\,;$$

indeed remarking that for all $\delta > 0$, $f_\delta = f1_{(|f|\geqslant\delta)}$ lies in $L^q(\mathbb{R}^N)$ for all $1 \leqslant q < p$, we deduce:

$$\left|\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n^1(x)u_n^2(y)f(x-y)dxdy\right|$$
$$\leqslant \delta\lambda^2 + \left|\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n^1(x)u_n^2(y)f_\delta(x-y)dxdy\right|$$
$$\leqslant \delta\lambda^2 + \left|\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n^1(x)u_n^2(y)f_\delta(x-y)1_{(|x-y|\geqslant d_n)}dydy\right|$$

since $d_n = \text{dist}\,(\text{Supp}\; u_n^1, u_n^2)$. And the last integral may be bounded by:

$$\|u_n^1\|_{L^{2q/(2q-1)}}\|u_n^2\|_{L^{2q/(2q-1)}}\|f_\delta 1_{(|x|\geqslant d_n)}\|_{L^q(\mathbb{R}^N)}$$

and we conclude since $\dfrac{2q}{2q-1} \in [1, (p+1)/p]$ if q is choosen in $[1, p[$ near p and since $\|f_\delta 1_{(|x|\geqslant d_n)}\|_{L^q(\mathbb{R}^N)} \underset{n}{\to} 0$.

Combining these inequalities we find:

$$\begin{aligned} I_\lambda &= \lim_{n\to\infty} \left\{ \int_{\mathbb{R}^N} j(u_n)dx - \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n(x)u_n(y)f(x-y)dxdy \right\} \\ &\geqslant \liminf_{n\to\infty} \left\{ \int_{\mathbb{R}^N} j(u_n^1)dx - \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n^1(x)u_n^1(y)f(x-y)dxdy \right\} \\ &+ \liminf_{n\to\infty} \left\{ \int_{\mathbb{R}^N} j(u_n^2)dx - \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n^2(x)u_n^2(y)f(x-y)dxdy \right\} - \delta(\varepsilon) \\ &= I_{\bar\alpha} + I_{\bar\beta} - \delta(\varepsilon) \end{aligned}$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0_+$.

Sending ε to 0, we obtain a contradiction with (S.2) and the contradiction shows that *(iii)* cannot occur.

Step 2: Vanishing does not occur.

If *(ii)* occurs then arguing in a way similar to the preceding, we obtain:

$$\begin{aligned} &\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n(x)u_n(y)f(x-y)dxdy \\ &\leqslant \delta\lambda^2 + \iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n(x)u_n(y)f_\delta(x-y)dxdy \\ &\leqslant \delta\lambda^2 + \iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n(x)u_n(y)g_\delta^R(x-y)1_{(|x-y|\leqslant R)}dxdy + \|u_n\|^2_{L^{2q/(2q-1)}}\|f_\delta^R\|_{L^q} \end{aligned}$$

where q is choosen in $[1, p[$ such that $\dfrac{2q}{2q-1} \in \left[1, \dfrac{p+1}{p}\right]$ and where f_δ^R, g_δ^R are given by: $g_\delta^R = f_\delta \wedge R$, $f_\delta^R = (f_\delta - R)^+ 1_{(|x|\leqslant R)} + f_\delta 1_{(|x|>R)}$. To conclude we observe:

$$\|f_\delta^R\|_{L^q} \to 0 \quad \text{as} \quad R \to +\infty, \qquad \text{for each fixed} \quad \delta > 0;$$

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n(x)u_n(y)g_\delta^R(x-y)1_{(|x-y|\leqslant R)}dxdy \leqslant R\int_{\mathbb{R}^N} u_n(x)\int_{|x-y|\leqslant R} u_n(y)dydx \leqslant RQ_n(R)\lambda.$$

Therefore sending n to $+\infty$, then R to $+\infty$ and finally δ to 0, we find:

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} u_n(x)u_n(y)f(x-y)dxdy \underset{n}{\to} 0.$$

This would imply: $I_\lambda \geqslant 0$, and this again contradicts (S.2).

Step 3 : Conclusion.

We have proved the existence of $(y_n)_{n\geq 1}$ in $\mathbb{R}^N$ such that (7) holds. We then denote by $\bar{u}_n = u_n(\cdot + y_n)$. Taking a subsequence if necessary we may assume that $\bar{u}_n$ converges weakly in $L^\alpha(\mathbb{R}^N)$ to some u for $1 \leq \alpha \leq \dfrac{p+1}{p} = q$ and $u \in L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$. In addition the convex functional $\int_{\mathbb{R}^N} j(u)dx$ being strongly lower semicontinuous on L^q by Fatou's lemma is weakly lower semicontinuous and we have:

$$\liminf_n \int_{\mathbb{R}^N} j(\bar{u}_n)dx \geq \int_{\mathbb{R}^N} j(u)dx .$$

In addition, using (7), it is easy to deduce: $\int_{\mathbb{R}^N} udx = \lambda.$

We next show that u is a minimum i. e. that we have:

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \bar{u}_n(x)\bar{u}_n(y)f(x-y)dxdy \underset{n}{\to} \iint_{\mathbb{R}^N\times\mathbb{R}^N} u(x)u(y)f(x-y)dxdy .$$

Indeed we have:

$$\begin{aligned} &\left|\left\{\iint_{\mathbb{R}^N\times\mathbb{R}^N} \bar{u}_n(x)\bar{u}_n(y)f(x-y)dxdy\right\}^{1/2}\right. \\ &\qquad \left. -\left\{\iint_{\mathbb{R}^N\times\mathbb{R}^N} u(x)u(y)f(x-y)dxdy\right\}^{1/2}\right| \\ &\qquad \leq \left|\iint_{\mathbb{R}^N\times\mathbb{R}^N} (\bar{u}_n(x)-u(x))(\bar{u}_n(y)-u(y))f(x-y)dxdy\right|^{1/2} \end{aligned}$$

and

$$\begin{aligned} &\left|\iint_{\mathbb{R}^N\times\mathbb{R}^N} (\bar{u}_n(x)-u(x))(\bar{u}_n(y)-u(y))f(x-y)dxdy\right| \\ &\qquad \leq \delta\lambda^2 + \varepsilon_\delta(R) + \left|\iint_{\mathbb{R}^N\times\mathbb{R}^N} (\bar{u}_n(x)-u(x))(\bar{u}_n(y)-u(y))g_\delta^R(x-y)dxdy\right| . \end{aligned}$$

We now introduce:

$$v_n(x) = \int_{\mathbb{R}^N} \{\bar{u}_n(y) - u(y)\}\, g_\delta^R(x-y)dy ;$$

since $g_\delta^R(x - \cdot) \in L^1 \cap L^\infty$, we have for all $x \in \mathbb{R}^N$:

$$v_n(x) = \int_{\mathbb{R}^N} \bar{u}_n(y)g_\delta^R(x-y)dy - \int_{\mathbb{R}^N} u(y)g_\delta^R(x-y)dy \underset{n}{\to} 0 .$$

In addition:

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\bar{u}_n(y)g_\delta^R(x-y)dy\right)dx = \left(\int_{\mathbb{R}^N}g_\delta^R dx\right)\cdot\left(\int_{\mathbb{R}^N}\bar{u}_n dy\right)$$

$$= \lambda\int_{\mathbb{R}^N}g_\delta^R dx$$

$$= \left(\int_{\mathbb{R}^N}g_\delta^R dx\right)\cdot\left(\int_{\mathbb{R}^N}u dy\right)$$

$$= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}u(y)g_\delta^R(x-y)dy\right)dx$$

and this yields easily:

$$\int_{\mathbb{R}^N}\bar{u}_n(y)g_\delta^R(x-y)dy \underset{n}{\rightarrow} \int_{\mathbb{R}^N}u(y)g_\delta^R(x-y)dy \quad \text{in} \quad L^1(\mathbb{R}^N)$$

or $v_n(x) \underset{n}{\rightarrow} 0$ in $L^1(\mathbb{R}^N)$. Therefore: $v_n \underset{n}{\rightarrow} 0$ in $L^\alpha(\mathbb{R}^N)$ for $1 \leqslant \alpha < \infty$, and thus we find:

$$\int_{\mathbb{R}^N}\bar{u}_n v_n dx \underset{n}{\rightarrow} 0$$

i. e.:

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}(\bar{u}_n(x)-u(x))(\bar{u}_n(y)-u(y))f(x-y)dxdy \rightarrow 0.$$

This proves that u is a minimum of (9) and thus:

$$\int_{\mathbb{R}^N}j(\bar{u}_n)dx \underset{n}{\rightarrow} \int_{\mathbb{R}^N}j(u)dx.$$

Since j is strictly convex, we deduce now that actually $\bar{u}_n \underset{n}{\rightarrow} u$ in measure. Indeed observe that by similar arguments $v_n = \dfrac{\bar{u}_n + u}{2}$ satisfies also:

$$\int_{\mathbb{R}^N}j(v_n)dx \underset{n}{\rightarrow} \int_{\mathbb{R}^N}j(u)dx$$

therefore we have: $j(\bar{u}_n) + j(u) - 2j(v_n) \underset{n}{\rightarrow} 0$ in L^1.

Let $K < \infty$, and denote by:

$$0<\delta_K(\lambda)=\inf\left\{j(s)+j(t)-2j\left(\frac{s+t}{2}\right)/0\leqslant s\leqslant K,\, 0\leqslant t\leqslant K,\, |s-t|\geqslant\lambda\right\}$$

for all $\lambda > 0$. Remarking that

$$\text{meas}\,(\bar{u}_n(x)\geqslant K \quad \text{or} \quad u(x)\geqslant K)\leqslant\frac{2\lambda}{K}$$

we deduce for all $\gamma > 0$:

$$\begin{aligned}\text{meas}\,(|\bar{u}_n - u| \geqslant \gamma) &\leqslant \frac{2\lambda}{K} + \text{meas}\,(\delta_K(|\bar{u}_n - u|) \geqslant \delta_K(\gamma)) \\ &\leqslant \frac{2\lambda}{K} + (\delta_K(\gamma))^{-1}\int_{\mathbb{R}^N} \delta_K(|\bar{u}_n - u|)dx \\ &\leqslant \frac{2\lambda}{K} + (\delta_K(\gamma))^{-1}\int_{\mathbb{R}^N} \{ j(\bar{u}_n) + j(u) - 2j(v_n) \}\, dx\,;\end{aligned}$$

therefore $\bar{u}_n \underset{n}{\to} u$ in measure. Using (7) and the consequence of $\int_{\mathbb{R}^N} j(\bar{u}_n)dx$, this implies easily: $\bar{u}_n \underset{n}{\to} u$ in $L^1 \cap L^q$ and $j(\bar{u}_n) \underset{n}{\to} j(u)$ in L^1. ∎

REMARK II.6. — Let us point out that the strict convexity of j is assumed only to insure L^p convergences and that if we no longer assume that j is strictly convex but merely convex then (S.2) is equivalent to the fact that every minimizing sequence is weakly relatively compact in $L^1(\mathbb{R}^N)$, and if (S.2) holds then this compactness insures that all converging subsequences are, up to a translation, converging to a minimum. ∎

II.3. Extensions.

We now consider the case of problem (9) where f is taken to be more general than in (10): we will assume that f satisfies:

$$(10')\quad \begin{cases} f^+ \in M^{p_1}(\mathbb{R}^N) + M^{p_2}(\mathbb{R}^N) & \text{with} \quad 1 \leqslant p_1 \leqslant p_2 < \infty \\ f^- \in M^{\bar{p}_1}(\mathbb{R}^N) + M^{\bar{p}_2}(\mathbb{R}^N) & \text{with} \quad 1 \leqslant \bar{p}_1 \leqslant \bar{p}_2 < \infty \end{cases}$$

where we agree: $M^1(\mathbb{R}^N) = L^1(\mathbb{R}^N)$. We then denote by $q = \dfrac{p_1 + 1}{p_1}$
As in remark II.3, we then have for all $u \in L^1 \cap L^q$:

$$\left| \iint_{\mathbb{R}^N \times \mathbb{R}^N} u(x)u(y)f_1(x-y)dxdy \right| \leqslant C_0 \| f_1 \|_{M^{p_1}} \| u \|_{L^q}^q \| u \|_{L^1}^{2-q}$$

where $f^+ = f_1 + f_2$ and $f_1 \in M^{p_1}$, $f_2 \in M^{p_2}$ and where C_0 is the smallest positive constant such that the above inequality holds for all u.
We will then assume:

$$(11')\quad \begin{cases} j \text{ is strictly convex, nonnegative on } \mathbb{R}^+ \text{ and: } \lim\limits_{t \to 0_+} j(t)t^{-1} = 0 \\ \varliminf\limits_{t \to \infty} j(t)t^{-q} > \dfrac{1}{2} C_0 \| f_1 \|_{M^{p_1}} \lambda^{2-q}. \end{cases}$$

Then with a similar proof, we extend Theorem II.1 as follows:

THEOREM II.2. — *Under assumptions* (10'), (11'), *every minimizing sequence*

$(u_n)_{n\geq 1}$ *of* (9) *is relatively compact in* $L^1(\mathbb{R}^N)$ *up to a translation if and only if* (S.2) *holds. If* (S.2) *holds, for every minimizing sequence* $(u_n)_{n\geq 1}$, *there exists* $(y_n)_{n\geq 1}$ *in* $\mathbb{R}^N$ *such that* $u_n(\cdot + y_n)$ *is relatively compact in* $L^1(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$, $j(u_n\cdot + y_n))$ *and* $u_n(x + y_n)u_n(y + y_n)f^-(x - y)$ *is relatively compact in* $L^1(\mathbb{R}^N)$ *and in* $L^1(\mathbb{R}^N \times \mathbb{R}^N)$ *respectively. In particular there exists a minimum of* (9) *if* (S.2) *holds.*

Similar remarks to those made above can be formulated but we will skip them. Instead let us discuss one example arising in Thomas-Fermi theory (*cf.* [*16*]):

Example. — $N = 3$, $f(x) = A\dfrac{e^{-\mu|x|}}{|x|} - B\dfrac{e^{-\nu|x|}}{|x|}$, $j(t) = t^\alpha$ with $\alpha > 1$, where A, B, μ, ν are given positive constants satisfying:

$$\text{either} \quad \mu \geqslant \nu,\ A > B \qquad \text{or} \quad \mu < \nu$$

(remark that if this is not the case then $f \leqslant 0$ and (9) has no solution). Clearly (10′) holds with $p_1 = p_2 = \bar{p}_1 = \bar{p}_2 = 3$ if $A > B$ or $A = B$ (and $\mu < \nu$) or with $p_1 = p_2$ arbitrary in $]1, \infty[$ and $\bar{p}_1 = \bar{p}_2 = 3$ if $A < B$ (and thus $\mu < \nu$).

CASE 1. — $A > B$, μ, ν arbitrary or $A = B$ and $\mu < \nu$: then (11′) holds if $\alpha > \frac{4}{3}$ $\left(\text{or} \geqslant \frac{4}{3} \text{ if } \lambda \text{ is small enough}\right)$. Using the proof of Corollary II.1, we see that (S.2) holds if $\alpha \leqslant 2$ and if $I_\lambda < 0$. Therefore in this case and if $\alpha \in \left]\frac{4}{3}, 2\right]$, (S.2) holds if and only if $I_\lambda < 0$.

CASE 2. — $A < B$, $\mu < \nu$: then (11′) holds for all $\alpha > 1$. And if $\alpha \leqslant 2$, then (S.2) holds if and only if $I_\lambda < 0$. ■

II.4. **Translation-dependent problems.**

With the same motivations from Physics as before (*cf.* [*1*] [*2*] [*13*])) we consider now the problem:

$$(9') \qquad I_\lambda = \inf\{\, \mathscr{E}(u)/u \in K_\lambda \,\}$$

where K_λ is given as before and $\mathscr{E}$ is defined by:

$$\mathscr{E}(u) = \int_{\mathbb{R}^N} j(u)dx + \int_{\mathbb{R}^N} V(x)u(x)dx - \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} u(x)u(y)f(x - y)dxdy$$

and f, j still satisfy (10)-(11) (we could as well assume (10')-(11')). Finally we assume that the potential V satisfies:

(17) $$V \in C_b(\mathbb{R}^N), \qquad V(x) \to V_\infty \quad \text{as} \quad |x| \to \infty;$$

(we will see below extensions of this condition).

We then consider:

$$\mathcal{E}^\infty(u) = \int_{\mathbb{R}^N} j(u)dx + V_\infty \int_{\mathbb{R}^N} u(x)dx - \frac{1}{2}\iint_{\mathbb{R}^N \times \mathbb{R}^N} u(x)u(y)f(x-y)dxdy$$

$$I_\lambda^\infty = \inf\{\mathcal{E}^\infty(u)/u \in K_\lambda\};$$

and we set $I_0 = 0$. Then we have:

THEOREM II.3. — *We assume* (10)-(11) *and* (17). *Then every minimizing sequence* $(u_n)_{n\geq 1}$ *of* (9) *is relatively compact in* $L^1(\mathbb{R}^N)$ *if and only if the following condition holds:*

(S.1) $$I_\lambda < I_\alpha + I_{\lambda-\alpha}^\infty, \qquad \forall \alpha \in [0, \lambda[.$$

If this condition holds, then any minimizing sequence $(u_n)_{n\geq 1}$ *is relatively compact in* $L^1 \cap L^q$ *and* $j(u_n)$ *is relatively compact in* L^1 *; in particular there exists a minimum in* (9'). –

Proof. — We follow the proof of Theorem II.1. Concerning step 1, we just remark that in view of the construction of u_n^1, u_n^2, two cases are possible: we have $u_n^1 = u_n 1_{y_n + B_R}$, $u_n^2 = u_n 1_{\mathbb{R}^N - (y_n + B_{R_n})}$ and if y_n is bounded, then:

$$\int_{\mathbb{R}^N} Vu_n^2 dx = \int_{|x-y_n|\geq R_n} Vu_n^2 dx = V_\infty \int_{\mathbb{R}^N} u_n^2 dx + \int_{|x-y_n|\geq R_N} \{V - V_\infty\} u_n^2 dx$$

and:

$$\int_{|x-y_n|\geq R_n} |V - V_\infty| u_n^2 dx \underset{n}{\to} 0 \qquad \text{since} \qquad R_n \underset{n}{\to} \infty.$$

On the other hand if y_n is unbounded, then taking a subsequence if necessary we may assume $|y_n| \underset{n}{\to} \infty$ and thus:

$$\left| \int_{\mathbb{R}^N} Vu_n^1 dx - \int_{\mathbb{R}^N} V^\infty u_n^1 dx \right| \leq \int_{|x-y_n|\leq R} |V - V^\infty| u_n^1 dx \underset{n}{\to} 0.$$

Therefore, arguing as in Step 1 above, we find:

either $$I_\lambda \geq I_{\bar\alpha} + I_{\bar\beta}^\infty - \delta(\varepsilon)$$

or $$I_\lambda \geq I_{\bar\alpha}^\infty + I_{\bar\beta} - \delta(\varepsilon)$$

and sending ε to 0, this contradicts the case $\alpha \in]0, \lambda[$ in (S.1).

Concerning Step 2, we observe that if $Q_n(t) \underset{n}{\to} 0$ for all $t < \infty$, then:

$$\left| \int_{\mathbb{R}^N} Vu_n dx - \int_{\mathbb{R}^N} V^\infty u_n dx \right| \leqslant \| V - V^\infty \|_{L^-} \int_{|x| \leqslant R} u_n dx + \int_{|x| \geqslant R} | V - V^\infty | u_n dx$$
$$\leqslant \| V - V^\infty \|_{L^-} Q_n(R) + \lambda \sup_{|x| \geqslant R} | V(x) - V^\infty |$$

and we find $I_\lambda \geqslant I_\lambda^\infty$ sending n to $+\infty$ and then R to $+\infty$. This contradicts the case $\alpha = \lambda$ in (S.1).

Therefore (*i*) in Lemma I.1 holds and if y_n is unbounded, we may assume, taking a subsequence if necessary, that $|y_n| \underset{n}{\to} +\infty$. This implies in particular that for all $R < \infty$: $\int_{B_R} u_n dx \underset{n}{\to} 0$ and by the same argument as above we deduce:

$$\left| \int_{\mathbb{R}^N} Vu_n dx - \int_{\mathbb{R}^N} V^\infty u_n dx \right| \underset{n}{\to} 0$$

and again this contradicts the case $\alpha = \lambda$ in (S.1).

This shows that y_n is bounded or in other words that u_n is weakly relatively compact in $L^1(\mathbb{R}^N)$. And we conclude as in the proof of Theorem II.1.

REMARK II.7. — Looking carefully at the above proof, we see that *we may replace* (17) by:

$$(17') \begin{cases} V = V_1 + V_2 \quad \text{with} \quad V_2 \in C_b(\mathbb{R}^N), \quad V_1^+ \in L^{p_1} + L^{p_2}, \quad V_1^- \in L^{q_1} + L^{q_2} \\ \quad \text{where} \quad 1 \leqslant p_1 \leqslant p_2 < \infty, \quad q_1 \leqslant q_2 < \infty \end{cases}$$

and $q_1 = q/(q-1)$ (it is even possible to consider the case when $V_2 \in L^\infty(\mathbb{R}^N)$ but it would involve technicalities below). We then set $V^\infty = \liminf_{|x| \to \infty} V_2(x)$ and define $\mathscr{E}^\infty$, I_λ^∞ as before. Then in this situation (S.1) holds is a sufficient condition for the compactness of minimizing sequences and is a necessary condition if $V_2 \xrightarrow[|x| \infty \infty]{} V^\infty$.

Let us give one simple example where it is easy to check (S.1):

COROLLARY II.2. — *We assume* (10), (11), (16) *and* (17). *Then if* $V \geqslant 0$ *and if* $I_\lambda < 0$, (S.1) *holds if and only if* : $I_\lambda < I_\lambda^\infty$. *If this is the case, the conclusions of Theorem* II.3 *hold.*

REMARK II.8. — If $V \geqslant 0$ is given then in many cases it is possible to see that, for λ large enough, $I_\lambda < 0$. In addition if $V \geqslant 0$ and if $I_\lambda < 0$, the result above implies that theorem II.3 may be applied if we know: $I_\lambda < I_\lambda^\infty$. We now claim that this is the case if $V(x) < V^\infty$ $\forall x \in \mathbb{R}^N$: indeed if we had $I_\lambda = I_\lambda^\infty$, by Corollary II.1, there would exist a minimum of the problem at infinity I_λ^∞: let us denote by u_0 such a minimum, then

$$\int_{\mathbb{R}^N} V u_0 dx < V^\infty \int_{\mathbb{R}^N} u_0 dx$$

and thus $I_\lambda \leqslant \mathscr{E}(u_0) < I_\lambda^\infty$. The contradiction shows that indeed we have: $I_\lambda < I_\lambda^\infty$.

Let us also point out that if $V(x) \geqslant V^\infty$, $\forall x \in \mathbb{R}^N$, then we have $I_\lambda = I_\lambda^\infty$ and (S.1) does not hold.

Finally let us notice that, since we may add constants to V without changing the minimization problem, we may always « normalize » V in such a way that $V \geqslant 0$ and more precisely: $\inf_{\mathbb{R}^N} V = 0$. ■ ■

Proof of Corollary II.2. — We follow the proof of Corollary II.1, take $\alpha > 0$ such that $I_\alpha < 0$ and consider $I_{\alpha\theta}$ for some $\theta > 1$, then:

$$\begin{aligned} I_{\theta\alpha} &= \inf_{u \in K_\alpha} \mathscr{E}(\theta u) \\ &= \inf_{u \in K_\alpha} \left\{ \int_{\mathbb{R}^N} j(\theta u) dx + \theta \int_{\mathbb{R}^N} V u dx + -\theta^2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} u(x) u(y) f(x-y) dx dy \right\} \\ &\leqslant \inf_{u \in K_\alpha} \left\{ \theta^2 \int_{\mathbb{R}^N} j(u) dx + \theta \int_{\mathbb{R}^N} V u dx + -\theta^2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} u(x) u(y) f(x-y) dx dy \right\} \end{aligned}$$

and using the fact that V is nonnegative we find:

$$I_{\theta\alpha} \leqslant \theta^2 I_\alpha < \theta I_\alpha < 0 ,$$

This implies: $I_\lambda < I_\alpha + I_{\lambda-\alpha}$, since we assume $I_\lambda < 0$. Remarking finally that $I_\mu \leqslant I_\mu^\infty$, we conclude easily. ■

We conclude this section by a fundamental counter-example: the concentration-compactness principle states that the subadditivity conditions (S.1), (S.2) are necessary and sufficient conditions for the relative compactness of *all* minimizing sequences. But it may happen that (S.1), (S.2) do not hold but still there exists a minimum in the original minimization problem: in this case our principle only indicates that the problem is not well-posed in the sense that there exists some minimizing sequence which is not relatively compact (even up to a translation in the translation-invariant case). We give now one example of such a phenomenon:

Example. — Take $N = 3$, $f(x) = \frac{1}{|x|}$, $j(t) = \frac{1}{q} t^q$ where q is taken in $\left]\frac{4}{3}, 2\right[$. In view of Theorem II.1 and Corollary II.1, there exists a solution u_0 of the minimization problem:

$$I_\lambda^\infty = \mathscr{E}^\infty(u_0) = \inf \{ \mathscr{E}^\infty(u) / u \in K_\lambda \}$$

where

$$\mathcal{E}^{\infty}(u) = \int_{\mathbb{R}^N} \frac{1}{q} u^q dx - \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} u(x)u(y) \, | x - y |^{-1} dxdy$$

and λ is an arbitrary positive constant. Looking at the Euler equation, it is an easy exercise to check that u_0 has compact support (see [1] [2] [13]) say: Supp $u_0 \subset B_{R_0}$. Then if we consider: $V \in C^\infty(\mathbb{R}^N)$, $V \equiv 0$ if $|x| \leqslant R_0$, $V(x) > 0$ for $|x| > R_0$, $V(x) \to 0$ as $|x| \to \infty$; and if we denote by I_λ the infinum corresponding to (9') then we have:

$$I_\lambda = I_\lambda^\infty, \quad I_\lambda < I_\alpha + I_{\lambda-\alpha}^\infty \qquad \forall \alpha \in]0, \lambda[.$$

Therefore (S.1) does not hold, but still there exists a minimum of (9') namely u^0 since $\int_{\mathbb{R}^N} V u^0 dx = 0$!

III. CHOQUARD-PEKAR PROBLEMS

III.1. Main results.

Motivated by quantum mechanics (see E. H. Lieb [15]) and statistical physics (see Donsker and S. R. S. Varadhan [10]) we consider the following problem: find u minimizing

$$(18) \quad I_\lambda = \inf \left\{ \int_{\mathbb{R}^3} \frac{1}{2} |\nabla u|^2 + \frac{1}{2} V(x) u^2 dx + \right. \\ \left. - \frac{1}{4} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x - y|} dxdy / u \in K_\lambda \right\}$$

where

$$K_\lambda = \{ u \in H^1(\mathbb{R}^3), \, |u|_{L^2(\mathbb{R}^3)}^2 = \lambda \} .$$

We need the term $\int_{\mathbb{R}^3} V(x) u^2(x) dx$ to be meaningful on $H^1(\mathbb{R}^3)$, therefore we assume (more general cases will be discussed later on):

$$(19) \qquad V \in L^p(\mathbb{R}^3) + L^q(\mathbb{R}^3), \qquad \text{with} \quad \frac{3}{2} \leqslant p, q < \infty .$$

A typical example (relevant for Physics) is the Coulomb potential:

$$V(x) = - \sum_{i=1}^{m} \frac{z_i}{|x - x_i|}, \qquad \text{with} \quad z_i > 0, \, x_i \in \mathbb{R}^3 (1 \leqslant i \leqslant m) .$$

Since (19) implies that V, in some weak sense, vanishes at infinity; the « problem at infinity » is given by:

$$(20)\quad I_\lambda^\infty = \inf\left\{ \int_{\mathbb{R}^3} \frac{1}{2} |\nabla u|^2 dx - \frac{1}{4} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dxdy / u \in K_\lambda \right\}$$

Obviously, the considerations of section I are easily justified and we obtain in a straightforward way:

$$I_\lambda^\infty \leqslant I_\alpha + I_{\lambda-\alpha}^\infty, \qquad \forall \alpha \in [0, \lambda[\,.$$

Notice also that we have; choosing some u in K_λ and denoting by $u_\sigma = \left(\frac{\cdot}{\sigma}\right)$ for $\sigma > 0$.

$$I_\lambda^\infty \leqslant \int_{\mathbb{R}^3} \frac{1}{2} |\nabla(\sigma^{-3/2} u_\sigma)|^2 dx - \frac{1}{4} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{[\sigma^{-3/2}u_\sigma(x)]^2[\sigma^{-3/2}u_\sigma(y)]^2}{|x-y|} dxdy$$

or

$$I_\lambda^\infty \leqslant \frac{1}{\sigma^2} \int_{\mathbb{R}^3} \frac{1}{2} |\nabla u|^2 dx - \frac{1}{\sigma} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dxdy < 0$$

for $\sigma > 0$ large enough.

Our main result is the following:

THEOREM III.1. — 1) *If* $V \equiv 0$, *every minimizing sequence* $(u_n)_n$ *is relatively compact up to a translation in* $H^1(\mathbb{R}^3)$. *In particular if* $V \equiv 0$, $I_\lambda = I_\lambda^\infty$ *has a minimum.*

2) *If* $V \not\equiv 0$, *every minimizing sequence* $(u_n)_n$ *is relatively compact in* $H^1(\mathbb{R}^3)$ *if and only if the following condition holds:*

$$(21)\qquad I_\lambda < I_\lambda^\infty .$$

3) *This condition holds for all* $\lambda > 0$ *if* $V \leqslant 0$, $V \not\equiv 0$ *or for* λ *large enough if* V *is negative somewhere. Finally if* $V \geqslant 0$, $V \not\equiv 0$, *there is no minimum in the problem* (18).

REMARK III.1. — This result illustrates the striking differences between problems in unbounded domains and those in bounded regions. Indeed if we replace $\mathbb{R}^3$ by any bounded region, (18) has a minimum for all potentials V.

REMARK III.2. — In the case $V \equiv 0$, the existence of a minimum was proved by E. H. Lieb [*15*] by the use of symmetrization arguments (which cannot be used in the case of general potentials V); in addition it is proved in [*15*] that the minimum is unique up to a translation (and a change of sign). Later we proved (in P. L. Lions [*21*]) that *for all potentials* V *with spherical symmetry, there exists a minimum of* (18) *when we restrict the infimum to functions with spherical symmetry.*

In the case when $V \geqslant 0$, $V \not\equiv 0$, we prove here that there is no minimum and that for any minimizing sequence $(u_n)_n$ there exists $(y_n)_n$ in $\mathbb{R}^3$ such that $u_n(\cdot + y_n)$ converges in H^1 to be the minimum of the problem with $V \equiv 0$ and that $|y_n| \to \infty$. If we assume in addition that V has spherical symmetry, these considerations only show that

$$I_\lambda < \operatorname{Inf}\left\{ \int_{\mathbb{R}^3} \frac{1}{2} |\nabla u|^2 + \frac{1}{2} V(|x|) u^2 dx - \frac{1}{4} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} dx/dy / u \in K_\lambda^S \right\}$$

where $K_\lambda^S = \{ u \in K_\lambda / u(x) = u(|x|) \}$.

In the next section we prove Theorem III.1; the following sections being devoted to various variants and extensions of this problem and of Theorem III.1.

III.2. **Proof of Theorem** III.1.

We first make a few preliminary observations: let $(u_n)_n$ be a minimizing sequence of (18), then we have:

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u_n^2(x) u_n^2(x)}{|x-y|} dxdy \leqslant C |u_n|^4_{L^{12/5}(\mathbb{R}^3)} \quad \text{(convolution inequalities)}$$

$$\leqslant C |u_n|^3_{L^2(\mathbb{R}^3)} |u_n|_{L^6(\mathbb{R}^3)} \quad \text{(Hölder inequalities)}$$

$$\leqslant C |u_n|^3_{L^2(\mathbb{R}^3)} |\nabla u_n|_{L^2(\mathbb{R}^3)} \quad \text{(Sobolev inequalities)}$$

while if $V = V_1 + V_2$ with $V_1 \in L^p$, $V_2 \in L^q$ and $\frac{3}{2} \leqslant p, q < \infty$, we have:

$$\int_{\mathbb{R}^N} |V_1| u_n^2 dx \leqslant |V_1|_{L^p} |u_n^2|_{L^{p'}} \leqslant C |u_n|_{L^6}^{3/p} \leqslant C |\nabla u_n|_{L^2}^{3/p}$$

and similarly for V_2. If p or $q = \frac{3}{2}$, we argue as follows:

$$\int_{\mathbb{R}^3} |V_1| u_n^2 dx \leqslant \int_{\mathbb{R}^3} \operatorname{Min}\{|V_1|, M\} u_n^2 dx + M\lambda$$

$$\leqslant \delta(M) |\nabla u_n|^2_{L^2(\mathbb{R}^3)} + M(\lambda)$$

where $\delta(M) = |\operatorname{Min}(|V_1|, M)|_{L^p} \to 0$ as $M \to \infty$.

These inequalities show that u_n *is bounded in* $H^1(\mathbb{R}^3)$ and thus $I_\lambda > -\infty$.

We next claim that we have always:

$$(28) \qquad I_{\theta\lambda} < \theta I_\lambda, \qquad \forall \lambda > 0, \forall \theta \in]1, \infty[.$$

Recalling Lemma II.1, this yields:

$$I_\lambda < I_\alpha + I_{\lambda-\alpha} \leqslant I_\alpha + I_{\lambda-\alpha}^\infty, \qquad \forall \alpha \in]0, \lambda[.$$

In particular we see that *condition* (S.2) *holds*; while (S.1) *holds if and only if* (21) *holds* and this immediately explains Theorem III.1 in view of the heuristic principle and method given in section I. To prove (22), we observe that:

$$I_{\theta\lambda}=\text{Inf}\left\{ \theta\int_{\mathbb{R}^3}\frac{1}{2}|\nabla u|^2+\frac{1}{2}Vu^2dx-\frac{\theta^2}{4}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}dxdy/u\in K_\lambda\right\}$$

and (22) is proved as soon as we know that in (18), we may restrict the infimum to elements u of K_λ such that:

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} u^2(x)u^2(y)\,|x-y|^{-1}dxdy \geqslant \alpha > 0$$

for some $\alpha > 0$. If this were not the case, there would exist a minimizing sequence $(u_n)_n$—thus bounded in $H^1(\mathbb{R}^3)$—satisfying:

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} u^2(x)u^2(y)\,|x-y|^{-1}dxdy \underset{n}{\rightarrow} 0\,.$$

This would imply in particular that u_n converges weakly in H^1 to 0, therefore u_n^2 converges weakly in L^β to 0 for $1 < \beta \leqslant 3$ and thus

$$\int_{\mathbb{R}^3} Vu_n^2dx \underset{n}{\rightarrow} 0\,.$$

This would then imply that $I_\lambda \geqslant 0$, contradicting the scaling argument made before Theorem III.1.

We are going to prove that *every minimizing sequence* (u_n) *is relatively compact in* H^1 *up to a translation.* Of course we are going to involve the concentration-compactness lemma (I.1) with $\rho_n = u_n^2$. We first rule out the possibility of vanishing: indeed if we had

$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}^3}\int_{y+B_R} u_{n_k}^2(x)dx = 0, \qquad \text{for all} \quad R < \infty\,;$$

for a subsequence n_k; then by the same argument as in Step 2 of section II.2, we prove that this would yield:

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} u_n^2(x)u_n^2(y)\,|x-y|^{-1}dxdy \underset{n}{\rightarrow} 0$$

and we already saw that this is not possible.

To prove dichotomy does not occur, we have to modify the construction of ρ_k^1, ρ_k^2 in the proof of Lemma I.1 according to the:

LEMMA III.1. — *Under the assumptions and notations of Lemma* I.1, *if we assume in addition that* $\rho_n = u_n^2$ *with* u_n *bounded in* $H^1(\mathbb{R}^N)$; *there* *exists a subsequence n_k such that either compactness ((i)) occurs, either vanishing ((ii)) occurs, or dichotomy occurs as follows: there exists $\alpha \in\,]0, \lambda[$ such that for all $\varepsilon > 0$, there exist $k_0 \geqslant 1$, u_k^1, u_k^1 bounded in $H^1(\mathbb{R}^N)$ satisfying for $k \geqslant k_0$:*

$$\left\{\begin{array}{l} \| u_{n_k} - (u_k^1 + u_k^2) \|_{L^p} \leqslant \delta_p(\varepsilon) \underset{\varepsilon \to 0_+}{\longrightarrow} 0, \qquad \textit{for} \quad 2 \leqslant p < 6\,; \\ \left| \displaystyle\int_{\mathbb{R}^N} (u_k^1)^2 dx - \alpha \right| \leqslant \varepsilon, \left| \displaystyle\int_{\mathbb{R}^N} (u_k^2)^2 dx - (\lambda - \alpha) \right| \leqslant \varepsilon\,; \\ \text{dist}\,(\text{Supp}\, u_k^1, \text{Supp}\, u_k^2) \underset{n}{\rightarrow} +\infty\,; \\ \underline{\lim}_k \displaystyle\int_{\mathbb{R}^N} \{ | \nabla u_{n_k} |^2 - | \nabla u_k^1 |^2 - | \nabla u_k^2 |^2 \}\, dx \geqslant 0\,. \end{array}\right.$$

It is now easy to rule out dichotomy: indeed if it would occur, we would easily deduce as we did before:

$$I_\lambda \geqslant I_\alpha + I_{\lambda-\alpha}^\infty \qquad \text{or} \qquad I_\lambda \geqslant I_\alpha^\infty + I_{\lambda-\alpha}$$

contradicting the strict subadditivity inequalities we already obtained. Let us now prove Lemma III.1: we therefore assume that $Q_{n_k}(t)$ converges for all $t \geqslant 0$ to some nondecreasing function Q satisfying:

$$0 \leqslant Q(t) \nearrow \alpha \in\,]0, \lambda[\qquad \text{as} \qquad t \nearrow +\infty\,.$$

Let $\varepsilon > 0$, choose R_0 such that $Q(R) \geqslant \alpha - \varepsilon$ if $R \geqslant R_0$. Next let ξ, φ be cut-off functions: $0 \leqslant \xi \leqslant 1$, $0 \leqslant \varphi \leqslant 1$, $\xi \equiv 1$ if $| x | \leqslant 1$, $\xi \equiv 0$ if $| x | \geqslant 2$, $\varphi \equiv 0$ if $| x | \leqslant 1$, $\varphi \equiv 1$ if $| x | \geqslant 2$, $\xi, \varphi \in \mathcal{D}(\mathbb{R}^N)$ and let ξ_μ, φ_μ denote $\xi\left(\frac{\cdot}{\mu}\right)$, $\varphi\left(\frac{\cdot}{\mu}\right)$. We have for $R \geqslant 1$ and for v in $H^1(\mathbb{R}^3)$ with $\| v \|_{H^1} \leqslant M$, where $M \geqslant \sup_n \| u_n \|_{H^1}$:

$$\left| \int_{\mathbb{R}^N} | \nabla(\xi_R v) |^2 dx - \int_{\mathbb{R}^N} \xi_R^2 | \nabla v |^2 dx \right| \leqslant \frac{C}{R}$$

$$\left| \int_{\mathbb{R}^N} | \nabla(\varphi_R v) |^2 dx - \int_{\mathbb{R}^N} \varphi_R^2 | \nabla v |^2 dx \right| \leqslant \frac{C}{R}$$

for some constant $C = C(M) > 0$. We choose R_1 large enough: $\dfrac{C}{R_1} \leqslant \varepsilon$. Of course we may assume $R_1 \geqslant R_0$ and thus $Q(R_1) \geqslant \alpha - \varepsilon$. Then for k large enough: $Q_{n_k}(R_1) \in [\alpha - 2\varepsilon, \alpha + \varepsilon]$. Let $y_k \in \mathbb{R}^3$ be such that:

$$Q_{n_k}(R_1) = \int_{y_k + B_{R_1}} (u_{n_k})^2 dx\,.$$

We then set: $u_k^1 = \xi_{R_1}(\cdot - y_k)u_{n_k}$. Then for k large enough we find: $\left| \int_{\mathbb{R}^N} (u_k^1)^2 dx - \alpha \right| \leqslant 2\varepsilon$; and

$$\left| \int_{\mathbb{R}^N} |\nabla u_k^1|^2 dx - \int_{\mathbb{R}^N} \xi_{R_1}^2(x - y_k) |\nabla u_{n_k}|^2 dx \right| \leqslant \varepsilon .$$

We finally consider $\varphi_k = \varphi_{R_k}(\cdot - y_k)$ and $u_k^2 = \varphi_k u_{n_k}$; where $R_k \underset{k}{\to} +\infty$ is such that: $Q_{n_k}(2R_k) \leqslant \alpha + 2\varepsilon$.

To conclude, we first observe that:

$$\begin{aligned} \int_{\mathbb{R}^N} |u_{n_k} - (u_k^1 + u_k^2)|^2 dx &= \int_{\mathbb{R}^N} \{ 1 - \xi_{R_1}^2(x - y_k) - \varphi_k^2(x) \} u_{n_k}^2(x) dx \\ &\leqslant \int_{R_1 \leqslant |x - y_k| \leqslant 2R_k} u_{n_k}^2(x) dx \\ &\leqslant Q_{n_k}(2R_k) - Q_{n_k}(R_1) \leqslant 4\varepsilon ; \end{aligned}$$

therefore by Hölder and Sobolev inequalities we also have:

$$|u_{n_k} - (u_k^1 + u_k^2)|_{L^p} \leqslant C\varepsilon^\theta ; \qquad \text{with} \quad \theta = (6 - p)/2p .$$

And the proof of Lemma III.1 is completed.

REMARK III.3. — Of course in Lemma III.1, we may replace H^1 by $W^{m,p}$ for all $m > 0$, $p \geqslant 1$ and the relation $\rho_n = u_n^2$ by $\rho_n = |u_n|^\alpha$ for all $p \leqslant \alpha < \dfrac{Np}{N - mp}$. ■

At this point, we have proved that any minimizing sequence satisfies the following compactness criterion: $\exists (y_n)_n \in \mathbb{R}^3$ such that:

$$\forall \varepsilon > 0, \exists R < \infty, \qquad \int_{y_n + B_R} u_n^2 dx \geqslant \lambda - \varepsilon .$$

We then denote by $\tilde{u}_n = u_n(\cdot + y_n)$. The above property implies obviously that if $\tilde{u}_n$ (or a subsequence) converges weakly in H^1, a. e. on $\mathbb{R}^3$ and in L^p_{loc} (for $2 \leqslant p < 6$) to some u, we have:

$$\int_{B_R} \tilde{u}^2 dx \geqslant \lambda - \varepsilon$$

and thus $\int_{\mathbb{R}^3} \tilde{u}^2 dx = \lambda$, i. e. $\tilde{u}_n$ *converges strongly in* L^2 to $\tilde{u}$.

By Hölder inequalities, $\tilde{u}_n$ converges strongly to $\tilde{u}$ in L^p for $2 \leqslant p < 6$. Next, two cases are possible: either $I_\lambda < I_\lambda^\infty$ or $I_\lambda = I_\lambda^\infty$. If $I_\lambda < I_\lambda^\infty$, we claim that (y_n) remains bounded, if it were not the case we would deduce easily from the above informations that:

$$\int_{\mathbb{R}^3} \mathrm{V} u_{n_k}^2 dx = \int_{\mathbb{R}^3} \mathrm{V}(x + y_{n_k})(\tilde{u}_{n_k})^2 dx \underset{k}{\rightarrow} 0, \qquad \text{if} \quad |y_{n_k}| \rightarrow \infty;$$

hence $\mathrm{I}_\lambda \geqslant \mathrm{I}_\lambda^\infty$. Now if (y_n) remains bounded, we see that u_n converges strongly to some u in L^p for $2 \leqslant p < 6$; in that case it is easy to show that u is then a minimum and thus *a posteriori*:

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \underset{n}{\rightarrow} \int_{\mathbb{R}^3} |\nabla u|^2 dx$$

showing the compactness in H^1. In the second case, that is if $\mathrm{I}_\lambda = \mathrm{I}_\lambda^\infty$, we argue as before if (y_n) is bounded while if $|y_{n_k}| \underset{k_1}{\rightarrow} \infty$, then we check in a straightforward way that $\tilde{u}_{n_k}$ converges in H^1 to $\tilde{u}$ which is a minimum of the problem $\mathrm{I}_\lambda^\infty$.

There just remains to prove part 3) of Theorem III.1. To do so, we first remark that we just proved that in the case $\mathrm{V} \equiv 0$, there exists a minimum u_λ which by the uniqueness results of E. H. Lieb [*15*] satisfies: $u_\lambda(x) = \lambda^2 u_1(\lambda x)$. In addition we may assume that u_λ is spherically symmetric, positive in $\mathbb{R}^3$ and exponentially decreasing at infinity. Therefore if $\mathrm{V} \leqslant 0$, $\mathrm{V} \not\equiv 0$, we have:

$$\int_{\mathbb{R}^3} \mathrm{V}(x) u^2(x) dx < 0$$

and thus $\mathrm{I}_\lambda < \mathrm{I}_\lambda^\infty$. Now if V is only negative somewhere i. e. if $\mathrm{V}^- \not\equiv 0$ on a set of positive measure and denoting by y_0 a Lebesgue point of V and of this set we see that:

$$\int_{\mathbb{R}^3} \mathrm{V}(x) u_\lambda^2(x - y_0) dx = \int_{\mathbb{R}^3} \mathrm{V}(x) \lambda^3 u_1^2(\lambda(x - y_0)) dx \underset{\lambda \rightarrow +\infty}{\longrightarrow} \mathrm{V}(y_0) < 0,$$

therefore for λ large enough, we deduce: $\mathrm{I}_\lambda < \mathrm{I}_\lambda^\infty$.

Finally if $\mathrm{V} \geqslant 0$, $\mathrm{V} \not\equiv 0$, it is clear that we have: $\mathrm{I}_\lambda \geqslant \mathrm{I}_\lambda^\infty$. And since the reversed inequality always hold, we have $\mathrm{I}_\lambda = \mathrm{I}_\lambda^\infty$. If a minimum u of (18) would exist, this would imply:

$$\mathrm{I}_\lambda \geqslant \mathrm{I}_\lambda^\infty + \int_{\mathbb{R}^3} \mathrm{V} u^2 dx = \mathrm{I}_\lambda + \int_{\mathbb{R}^3} \mathrm{V} u^2 dx \geqslant \mathrm{I}_\lambda$$

and thus $u \equiv 0$ on a set of positive measure. But since u is a minimum of (13), u is an eigenfunction corresponding to the first eigenvalue of the Schrödinger operator $-\Delta + \mathrm{W}$ with $\mathrm{W} = \mathrm{V} - \frac{1}{2}\left(u^2 * \frac{1}{|x|}\right)$ and by well-known results u cannot vanish on a set of positive measure. ■

III.3. Variants and extensions.

It is possible to treat by analogous methods similar problems where $|Du|^2$ is replaced by arbitrary Sobolev norms and $\mathbb{R}^3$ by $\mathbb{R}^N$, where the potential V and the kernel $\dfrac{1}{|x-y|}$ are replaced by more general ones, where we add in the functional local terms like: $\int_{\mathbb{R}^3} f(x, u)dx$. Two relevant examples of this sort are:

EXAMPLE III.1. — $f(x, u) = f(u)$, where $f \in C^1(\mathbb{R})$, $f(0) = f'(0) = 0$. In that case we only need to assume:

$$\overline{\lim_{t \to 0}}\, f^-(t)t^{-2} < \infty, \qquad \overline{\lim_{|t| \to \infty}}\, f^-(t)\,|t|^{-10/3} = 0$$

and then the concentration-compactness principle holds.

EXAMPLE III.2. — $f(x, u) = f(x)u$, where, to simplify, $f \in L^2 + L^{6/5}$. Again the concentration-compactness principle holds.

We only mention one possible extension: we replace (18) by

$$(18') \quad I_\lambda = \operatorname{Inf}\left\{ \int_{\mathbb{R}^N} \frac{1}{2} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \frac{1}{2} V(x)u^2 dx + \frac{1}{4} \iint_{\mathbb{R}^N \times \mathbb{R}^N} u^2(x)u^2(y)W(x-y)dxdy / u \in K_\lambda \right\}$$

where $K_\lambda = \{ u \in H^1(\mathbb{R}^N) / |u|^2_{L^2} = \lambda \}$.

And where we assume (to simplify):

$$(19') \quad \begin{cases} V^+ \in L^1_{loc}\,; \forall \delta > 0,\ V^+ 1_{(V^+ \geq \delta)} \in L^p \text{ with } p \geq \frac{N}{2} \text{ if } N \geq 3,\ p > 1 \text{ if } N \leq 2; \\ V^- \in L^p + L^q \text{ with } \infty > p, q \geq \frac{N}{2} \text{ if } N \geq 3,\ p, q > 1 \text{ if } N \leq 2\,; \end{cases}$$

$$(23) \quad \begin{cases} a_{ij}(x) \in C_b(\mathbb{R}^N);\ \exists \nu > 0,\ \forall \xi \in \mathbb{R},\ a_{ij}(x)\xi_i\xi_j \geq \nu\,|\,\xi^2 \quad \text{a. e.} \quad x \in \mathbb{R}^N; \\ a_{ij}(x) \to \bar{a}_{ij} \quad \text{as} \quad |x| \to \infty; \end{cases}$$

$$(24) \quad W \in M^p(\mathbb{R}^N) + M^q(\mathbb{R}^N) \quad \text{with} \quad p, q > \frac{N}{2} \text{ if } N \geq 3, p, q > 1 \text{ if } N \leq 2.$$

Of course, we have now to replace the problem at infinity (20) by

$$(20') \quad I_\lambda^\infty = \operatorname{Inf}\left\{ \int_{\mathbb{R}^N} \frac{1}{2} \bar{a}_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx + - \frac{1}{4} \iint_{\mathbb{R}^N \times \mathbb{R}^N} u^2(x)u^2(y)W(x-y)dxdy / u \in K_\lambda \right\}.$$

We then have the:

THEOREM III.2. — *We assume* (19'), (23), (24). *Then* (21) *is a necessary and sufficient condition for the relative compactness in* $H^1(\mathbb{R}^N)$ *of all minimizing sequences. If* $V \equiv 0$, $a_{ij}(x) \equiv \bar{a}_{ij}$ *then* $I_\lambda < 0$ *is a necessary and sufficient condition for the relative compactness up to a translation in* $H^1(\mathbb{R}^N)$ *of all minimizing sequences.*

The proof of Theorem III.2 is very much the same than the proof of Theorem III.1, we only describe two technical points which are necessary in order to complete the proof. First if $u_n \underset{n}{\rightharpoonup} 0$ weakly in $H^1(\mathbb{R}^N)$ then we have for all $\delta > 0$:

$$\int_{\mathbb{R}^N} V^+ u_n^2 dx \leqslant \delta\lambda + \int_{\mathbb{R}^N} V^+ 1_{(V^+ \geqslant \delta)} u_n^2 dx$$

and this yields: $\displaystyle\int_{\mathbb{R}^N} V^+ u_n^2 dx \underset{n}{\rightarrow} 0.$

Next, if $\tilde{u}_n(\cdot) = u_n(\cdot + y_n)$ converges weakly in $H^1(\mathbb{R}^N)$ to some $\tilde{u}$ with $|y_n| \underset{n}{\rightarrow} \infty$, then we have:

$$\begin{aligned}\underline{\lim}_n \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} dx &\geqslant \underline{\lim}_n \int_{\mathbb{R}^N} a_{ij}(x + y_n) \frac{\partial \tilde{u}_n}{\partial x_i} \frac{\partial \tilde{u}_n}{\partial x_j} dx \\ &\geqslant \underline{\lim}_n \int_{B_R} \bar{a}_{ij}(x + y_n) \frac{\partial \tilde{u}_n}{\partial x_i} \frac{\partial \tilde{u}_n}{\partial x_j} dx \\ &\geqslant \int_{B_R} \bar{a}_{ij} \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial \tilde{u}}{\partial x_j}, \text{ for all } R < \infty;\end{aligned}$$

and we deduce:

$$\underline{\lim}_n \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} dx \geqslant \int_{\mathbb{R}^N} \bar{a}_{ij} \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial \tilde{u}}{\partial x_j} dx.$$

REMARK III.4. — It is possible to show as in the proof of Theorem III.1 that if $I_\lambda < 0$ then all minimizing sequences are relatively compact up to a translation. In addition if $(a_{ij}(x)) \geqslant (\bar{a}_{ij})$ for all $x \in \mathbb{R}^N$ and if $V \geqslant 0$, then $I_\lambda = I_\lambda^\infty$ and there is no minimum of (18').

REMARK III.5. — In the case when $(a_{ij}(x))$ does not converge as $|x| \to \infty$, it is possible to argue as follows: assume there exist $(\bar{a}_{ij})$, $(\underline{a}_{ij})$ satisfying:

$$\begin{aligned}\forall \xi \in \mathbb{R}^N,\ \bar{a}_{ij}\xi_i\xi_j &\geqslant \lim \sup_{|x| \to \infty} \text{ess } a_{ij}(x)\xi_i\xi_j \\ &\geqslant \lim \inf_{|x| \to \infty} \text{ess } a_{ij}(x)\xi_i\xi_j \geqslant \underline{a}_{ij}\xi_i\xi_j\end{aligned}$$

and let $\bar{I}_\lambda^\infty$, $\underline{I}_\lambda^\infty$ the infinum given by (20′) corresponding to $\bar{a}_{ij}$, $\underline{a}_{ij}$ respectively. We always have:

$$I_\lambda \leqslant I_\alpha + \bar{I}_{\lambda-\alpha}^\infty \qquad \forall \alpha \in [0, \lambda[$$

while if $I_\lambda < I_\alpha + \underline{I}_{\lambda-\alpha}^\infty$, $\forall \alpha \in [0, \lambda[$, then all minimizing sequences are relatively compact in $H^1(\mathbb{R}^N)$.

REMARK III.6. — Of course, if we assume that: $(a_{ij}(x)) \leqslant (\bar{a}_{ij})$ on $\mathbb{R}^N$ and if $V \not\equiv 0$, $V \leqslant 0$ then (21) holds if and only if $I_\lambda < 0$. Indeed either $I_\lambda^\infty = 0$ and then (21) holds, or $I_\lambda^\infty < 0$ and by Theorem III.2 there exists u minimum for the problem I_λ^∞: we then conclude remarking that we have:

$$\int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx \leqslant \int_{\mathbb{R}^N} \bar{a}_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx$$

$$\int_{\mathbb{R}^N} V u^2 dx < 0 .$$

III.4. **On the Euler equation.**

Clearly enough, if u is a minimum of (18′), then u solves the following Euler equation:

$$(25) \qquad -\frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + Vu + \theta u = (u^2 * W)u \quad \text{in} \quad \mathscr{D}'(\mathbb{R}^N)$$

where $\theta \in \mathbb{R}$ is a Lagrange multiplier. It is often of interest for Mathematical Physics to investigate directly the solutions of the equation (25)—where θ is now a fixed parameter. To find one solution, one may take advantage of the homogeneity of the nonlinearity by looking at the following minimization problem:

$$(26) \quad I_\lambda = \operatorname{Inf}\left\{ -\iint_{\mathbb{R}^N \times \mathbb{R}^N} u^2(x) u^2(y) W(x-y) dx dy / u \in H^1(\mathbb{R}^N), J(u) = \lambda \right\}$$

with:

$$J(u) = \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + Vu^2 + \theta u^2 dx .$$

Indeed if u is a minimum of (26), then there exists $\sigma > 0$ such that σu is a solution of (25) – by the way, let us mention that here and in all the minimization problems considered in section III, the solutions we find are positive on $\mathbb{R}^N$.

Next, if we want to solve (26) we have to consider the problem at infinity:

$$(27) \quad I_\lambda^\infty = \operatorname{Inf}\left\{ -\iint_{\mathbb{R}^N \times \mathbb{R}^N} u^2(x) u^2(y) W(x-y) dx dy / u \in H^1(\mathbb{R}^N), J^\infty(u) = \lambda \right\}$$

with:

$$J^{\infty}(u) = \int_{\mathbb{R}^N} \bar{a}_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \theta u^2 dx$$

and where we assume (23) and $\theta > 0$ (for example).

Before stating our main result, let us point out that one clearly has:

(28) $$I_\lambda = \lambda^2 I_1 \leqslant I_\lambda^\infty = \lambda^2 I_1^\infty \leqslant 0 .$$

THEOREM III.3. — *Assume* (19'), (23) *and*

(29) $$\begin{cases} W \not\equiv 0 ; W \in M^p + M^q \quad \textit{with} \quad \dfrac{N}{4} < p, q < \infty \quad \textit{if} \quad N \geqslant 5 ; \\ W \in L^1 + M^q \quad \textit{with} \quad 1 < q < \infty \quad \textit{if} \quad N \leqslant 4 ; \end{cases}$$

(30) $$\exists \nu > 0, \forall u \in H^1(\mathbb{R}^N), \quad J(u) \geqslant \nu \| u \|_{H^1}^2 .$$

Then (21) *is a necessary and sufficient condition for the relative compactness in* $H^1(\mathbb{R}^N)$ *of all minimizing sequences of* (26). *In the particular case when* $a_{ij}(x) \equiv \bar{a}_{ij}$, $V \equiv 0$, *every minimizing sequence of* (27) *is relatively compact in* $H^1(\mathbb{R}^N)$ *up to a translation if and only if*: $I_\lambda < 0$.

REMARK III.7. — The analogues of Remark III.4-6 hold in the above situation. In addition it is possible to treat cases with more general W if $N \leqslant 4$.

REMARK III.8. — The condition (30) is essentially necessary in order to obtain a positive solution of (25)—as we do in the preceding result. Indeed if $u_0 \geqslant 0$, $u_0 \not\equiv 0$ solves (25) then 0 is the first eigenvalue of the operator $\left(- \dfrac{\partial}{\partial x_i} \left(a_{ij} \dfrac{\partial}{\partial x_j} \right) + V + \theta - (u_0^2 * W) \right)$ in $H^1(\mathbb{R}^N)$ and this implies: $J(u) \geqslant \int_{\mathbb{R}^N} \{ u_0^2 * W \} u^2 dx \geqslant 0$, $\forall u \in H^1(\mathbb{R}^N)$ as soon as $W \geqslant 0$. If W admits both signs, it is possible to relax a little bit (30) but we will skip such extensions.

We present now a rough sketch of the proof of Theorem III.3: let u_n be a minimizing sequence of (2). Because of (30), u_n is clearly bounded in $H^1(\mathbb{R}^N)$. We then set:

$$\tilde{\rho}_n = | \nabla u_n |^2 + u_n^2$$

and we consider the concentration function Q_n of $\tilde{\rho}_n$. Let $\mu_n = \lim_{t \uparrow \infty} Q_n(t)$. Without loss of generality we may assume that $\mu_n \underset{n}{\rightarrow} \mu > 0$. We will apply the concentration-compactness lemma on $\rho_n = \dfrac{1}{\mu_n} \tilde{\rho}_n$. This will yield Theorem III.3 since, in view of (23), the strict subadditivity condition

(S.1) is equivalent to (21), while the condition (S.2) is equivalent to $I_\lambda < 0$.

Indeed, if we apply Lemma I.1, we first observe that vanishing ((*ii*)) cannot occur: indeed if it occurs, then by an argument similar to the one used in Step 2 of the proof of Theorem II.1, we obtain:

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} u_n^2(x)u_n^2(y)W(x-y)dxdy \underset{n}{\to} 0$$

and this contradicts the fact that $I_\lambda < 0$ since $I_\lambda < I_\lambda^\infty \leqslant 0$.

If dichotomy occurs, arguing as in the proof of Lemma III.1, we see that we can find u_n^1, u_n^2 satisfying: $\exists y_n \in \mathbb{R}^N$. $\exists R_0' < \infty$

$$\begin{cases} \| u_n - (u_n^1 + u_n^2) \|_{H^1} \leqslant \varepsilon \,; \; | \, \| u_n^1 \|_{H^1}^2 - \tilde{\alpha} \, | \leqslant \varepsilon \,; \; \mathrm{Supp}\, u_n^1 \subset (y_n + B_{R_0'}) \\ | \, \| u_n^2 \|_{H^1}^2 - (1-\tilde{\alpha}) \, | \leqslant \varepsilon \,; \; \mathrm{dist}\,(\mathrm{Supp}\, u_n^1, \mathrm{Supp}\, u_n^2) \underset{n}{\to} \infty \end{cases}$$

for some $\tilde{\alpha} \in (0, 1)$. This immediately yields:

$$I_\lambda \geqslant \underline{\lim}_n \left\{ - \iint_{\mathbb{R}^N \times \mathbb{R}^N} (u_n^1)^2(x)(u_n^1)^2(y)W(x-y)dxdy \right\} + \underline{\lim}_n \left\{ - \iint_{\mathbb{R}^N \times \mathbb{R}^N} (u_n^2)^2(x)(u_n^2)^2(y)W(x-y)dxdy \right\};$$

and without loss of generality we may assume that there exists $\alpha \in \,]0, \lambda[$ such that for some constant C:

$$| J(u_n^1) - \alpha | \leqslant C\varepsilon, \qquad | J(u_n^2) - (\lambda - \alpha) | \leqslant C\varepsilon .$$

Two cases are then possible: either $| y_n |$ (or a subsequence) $\underset{n}{\to} \infty$, then because of (23): $\lim_n | J(u_n^1) - J^\infty(u_n^1) | = 0$; or y_n remains bounded and then:

$$\lim_n | J(u_n^2) - J^\infty(u_n^2) | = 0 .$$

Combining these equalities-inequalities, we deduce letting $\varepsilon \to 0$:

$$I_\lambda \geqslant I_\alpha^\infty + I_{\lambda-\alpha} \qquad \text{or} \qquad I_\lambda \geqslant I_\alpha + I_{\lambda-\alpha}^\infty$$

and this is not possible since (21) holds and since (21) is equivalent to (S.1) as we remarked before.

Therefore we are always in the situation of « compactness » ((*i*)): there exist $(y_n)_n \subset \mathbb{R}^N$ such that:

$$\forall \varepsilon > 0, \exists R < \infty, \qquad \int_{|\bar{x} - y_n| \geqslant R} | \nabla u_n |^2 + u_n^2 dx \leqslant \varepsilon .$$

If (21) holds, it is easy to show as we did before that (y_n) must remain bounded. Therefore we may take as well: $y_n = 0$. Next, if u_n converges weakly and a. e. to some u in $H^1(\mathbb{R}^N)$, by Rellich Theorem u_n converges strongly to u in $L^2(B_R)$ and the above property shows that u_n converges strongly to u in $L^2(\mathbb{R}^N)$. Using then Sobolev embeddings and Hölder inequalities, we see that u_n converges strongly to u in $L^p(\mathbb{R}^N)$ for $2 \leqslant p < \frac{2N}{N-2}$. And this yields:

$$I_\lambda = - \iint_{\mathbb{R}^N \times \mathbb{R}^N} u^2(x)u^2(y)W(x-y)dxdy, \qquad J(u) \leqslant \lambda .$$

We then claim that $J(u) = \lambda$. Indeed if this were not the case, we would have: $I_\lambda \geqslant I_\alpha$ for some $\alpha \in]0, \lambda[$, and this would contradict (S1) since $I^\infty_{\lambda-\alpha} \leqslant 0$. Therefore u is a minimum and the convergence of u_n to u is strong in H^1. ■

REMARK III.9. — The above proof shows that in the concentration-compactness method, Lemma I.1 needs not to be used with quantities ρ_n directly related to the constraint but that there exists a certain flexibility in the choice of ρ_n which only has to be a nonnegative quantity giving some control on the constraint functional.

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(Manuscrit reçu en juin 1983)