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LAN AND LAMN FOR SYSTEMS OF INTERACTING DIFFUSIONS WITH BRANCHING AND IMMIGRATION

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ABSTRACT. – We consider parametric models for finite systems of branching diffusions with interactions and immigration of particles. Under conditions which link together the asymptotic behaviour of the process of particle configurations with smoothness of the parametrisation, we prove local asymptotic normality or local asymptotic mixed normality as the observation time tends to infinity. The limit theorems which are used follow from dividing the trajectory of the process of particle configurations into independent life-cycles between successive visits of the void configuration. © 2002 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Ce travail traite des modèles paramétriques pour des systèmes finis de diffusions avec interaction, branchement et immigration. Sous des hypothèses qui combinent le comportement asymptotique du processus des configurations des particules et la régularité de la paramétrisation, on démontre la propriété LAN (normalité asymptotique locale) ou LAMN (normalité mixte asymptotique locale) lorsque le temps d'observation tend vers l'infini. Les théorèmes limites utilisés sont obtenus en divisant la trajectoire du processus de configuration dans des cycles de vie entre des visites successives de la configuration vide. © 2002 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

This paper deals with statistical models for spatially branching particle systems. Such particle systems are of interest in models related to questions from population biology. They have been widely developed from a probabilistic point of view, see for instance Etheridge [6], Gorostiza and Wakolbinger [7] and Wakolbinger [30]. In this paper, we are concerned with statistical models for branching particle systems and restrict our attention to processes with finite particle configurations where particles are moving in $\mathbb{R}^d$. So

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particle configurations will be points $x = (x^1, \dots, x^l)$ in $(\mathbb{R}^d)^l$, $l \geqslant 0$. More precisely, we consider Markovian systems of particles where the joint motion of l particles

$$X^l = \begin{pmatrix} X^{1,l} \\ \vdots \\ X^{l,l} \end{pmatrix},$$

called l-point motion, during its random lifetime is governed by a stochastic differential equation

$$dX_t^{i,l} = b\big(X_t^{i,l}, X_t^l\big)\,dt + \sigma\big(X_t^{i,l}, X_t^l\big)\,dW_t^i, \quad 1 \leqslant i \leqslant l,$$

with independent m-dimensional Brownian motions $W^1, \dots, W^l$ and with Lipschitz coefficients b and σ.

A particle located at position $x^i \in \mathbb{R}^d$ at time $t > 0$ which belongs to a configuration $x = (x^1, \dots, x^l)$ of l particles branches with probability

$$\kappa\big(x^i, x\big)h + \mathrm{o}(h) \quad \text{as } h \to 0$$

in the small time interval $(t, t+h]$; $\kappa(.,.)$ is a continuous nonnegative function called the branching rate. When the particle "branches", it dies and gives rise to a random number of offspring, independently of the past, governed by the reproduction law

$$F\big(x^i, x, dn\big),$$

a probability measure on $\mathbb{N}_0$. The newborn particles choose their positions in space randomly, independently of the past and independently of the other newborn particles. Additionally, there is immigration of new particles at a configuration dependent rate $c(x)$. At each immigration time, exactly one particle immigrates, at a location which is chosen randomly in space. The resulting process of particle configurations $\varphi = (\varphi_t)_{t\geqslant 0}$ is a càdlàg process with values in the space S of all ordered finite configurations $x = (x^1, \dots, x^l)$ of arbitrary length $l \in \mathbb{N}_0$, with $x^j \in \mathbb{R}^d$.

We are interested in parametric statistical models for branching particle systems where the underlying drift function b, the branching rate κ, the reproduction law F, the distribution in space of the newborn particles, the immigration rate c and the distribution in space of the immigrating particle depend on some unknown k-dimensional parameter ϑ and where the resulting process φ of particle configurations can be observed continuously in time. Necessary and sufficient conditions for local absolute continuity of laws for such processes on a canonical path space as well as an explicit version of the corresponding likelihood ratio process have been obtained in Löcherbach [21]. In the present paper, we give conditions for local asymptotic normality (LAN) or local asymptotic mixed normality (LAMN). Once LAN or LAMN holds, it is possible to characterise asymptotically efficient estimators for the unknown parameter in ϑ and to determine asymptotically optimal estimation procedures. For the general statistical background, we refer the reader to Davies [3], Ibragimov and Khas'minskii [14], Le Cam and Yang [19] and Strasser [27]. Sharp developments of the log-likelihoods can also be useful in non-parametric situations in order to derive lower bounds for the rate of convergence of estimators – this has been used in Höpfner, Hoffmann and Löcherbach [11] in a context of non-parametric estimation of the branching rate.

We state the basic notions and assumptions in Section 2 and recall the formulas for the likelihood ratio process. Section 3 gives the conditions needed to prove LAN or LAMN. We suppose that φ is recurrent in the sense of Harris under ϑ and write m^{ϑ} for its invariant measure, $\bar{m}^{\vartheta}$ for an associated Campbell measure. A crucial condition is the logarithmic differentiability of $\xi \mapsto b^{\xi}(.,.), \kappa^{\xi}(.,.)$ etc. at $\xi = \vartheta$ in an $L^2(\bar{m}^{\vartheta})$-sense. Our main result is Theorem 3.7. It states the LAN or LAMN property at ϑ and the joint convergence of the score function martingales – corresponding to the motion part, the branching part and the immigration part of the experiment – together with their bracket processes to a limit process Y together with its bracket process. Y is either a Gaussian martingale with covariance being of diagonal type or a Gaussian martingale after independent time change. The limit theorems which are used in the null-recurrent case require a precise control of the tail of the life-cycle length of φ under ϑ (i.e. the time between successive visits to the void configuration) and follow by dividing the trajectory of φ into independent life-cycles and from known convergence results to stable processes. Section 4 – not in a statistical context but of interest in its own right – is devoted to some considerations concerning the explosion properties of branching diffusions. We consider the non-interactive case where particles are evolving independently. Here, the explosion probability is related to solutions of a backward stochastic differential equation, and in cases where the backward SDE admits only one solution, the branching diffusion cannot explode in finite time.

2. Basic assumptions

Write $R := \mathbb{R}^d$. We consider a stochastic process $\varphi = (\varphi_t)_{t \geqslant 0}$ of finite particle configurations with particles moving in R: φ has càdlàg paths taking values in the space

$$S = \bigcup_{l \geqslant 0} R^l \tag{2.1}$$

of ordered configurations where R^0 is the space $\{\Delta\}$ containing the void configuration. We suppose that φ satisfies the following assumptions.

Assumption 2.1. – For all $l > 0$ we have drift and diffusion coefficients

$$b(.,.): R \times R^l \to R \quad \text{and} \quad \sigma(.,.): R \times R^l \to \mathbb{R}^{d \times m}$$

which are globally Lipschitz continuous. Then l-particle motions

$$X^l = \begin{pmatrix} X^{1,l} \\ \vdots \\ X^{l,l} \end{pmatrix}$$

are solutions of the stochastic differential equation

$$dX_t^{i,l} = b\left(X_t^{i,l}, X_t^l\right) dt + \sigma\left(X_t^{i,l}, X_t^l\right) dW_t^i, \quad 1 \leqslant i \leqslant l, \tag{2.2}$$

during their random lifetime, with independent m-dimensional Brownian motions $W^1, \ldots, W^l$ driving the motion of every particle, for arbitrary $l > 0$. Define

$$a(y,z) := \sigma(y,z)\sigma^{\mathrm{T}}(y,z), \quad y \in R,\ z \in R^l,$$

and suppose that $a(y,z)$ is invertible for all y, z.

Assumption 2.2. – For all $l > 0$ there is a branching rate function $\kappa : R \times R^l \to \mathbb{R}_+$, a jump kernel π from $(R \times R^l, \mathcal{B}(R \times R^l))$ to $(R, \mathcal{B}(R))$ and a kernel F from $(R \times R^l, \mathcal{B}(R \times R^l))$ to $\mathbb{N}_0$, called the reproduction law, such that $F(y,z,\{0\}) > 0$, $F(y,z,\{1\}) = 0$ for all $y \in R$, $z \in R^l$. A particle in position $x^i \in R$ at time $t > 0$ belonging to a configuration $x = (x^1, \ldots, x^l)$ of l particles dies with position and configuration dependent rate

$$\kappa\left(x^i, x\right).$$

At its death time it gives rise to a random number of offspring particles according to the reproduction law

$$F\left(x^i, x, dn\right),$$

again depending on position and configuration of coexisting particles. Every newborn particle is then distributed randomly in space, independently of the other newborn particles and independently of the past up to time t, according to the law

$$\pi\left(x^i, x, dy\right) \text{ on } R.$$

Assumption 2.3. – For all $l \geqslant 0$ we are given an immigration rate function $c : R^l \to \mathbb{R}_+$ and a kernel ν from $(R^l, \mathcal{B}(R^l))$ to $(R, \mathcal{B}(R))$. Assume $c(\Delta) > 0$. Immigration of new particles occurs at configuration dependent rate c: If at time t there are l particles in positions $x = (x^1, \ldots, x^l)$, then one new particle will immigrate in $(t, t+h]$ with probability

$$c(x)h + \mathrm{o}(h) \quad \text{as } h \to 0.$$

The immigrating particle is distributed randomly in space, independently of the past, according to

$$\nu(x, dy)$$

on R, depending on the configuration x of already existing particles.

Assumption 2.4. – At branching or immigration times, the particles in the new configuration are rearranged randomly such that every permutation of particles has the same probability.

Notation 2.5. – For a function g defined on $R \times R^l$ write $g^i(z^1, \ldots, z^l) := g(z^i, z)$ for $z = (z^1, \ldots, z^l) \in R^l$. We call g symmetric if $g^i(z^{\pi(1)}, \ldots, z^{\pi(l)}) = g^{\pi(i)}(z^1, \ldots, z^l)$ holds for all permutations $\pi : \{1, \ldots, l\} \to \{1, \ldots, l\}$.

Assumption 2.6. –
(a) All functions $R \times R^l \ni (y,z) \mapsto b(y,z)$, $\sigma(y,z)$, $\kappa(y,z)$ and $c(z)$ are symmetric and continuous in (y,z) or z respectively. The kernels $F(y,z,.)$, $\pi(y,z,.)$ and $\nu(z,.)$ for $y \in R$, $z \in R^l$ are symmetric in (y,z) and continuous in (y,z) with respect to the weak convergence of probability measures.
(b) We shall assume that for all $l > 0$ the functions

$$R^l \ni x = \left(x^1,\dots,x^l\right) \mapsto c(x) + \sum_{i=1}^{l} \kappa\left(x^i,x\right),$$

$$x \mapsto \frac{c(x)}{c(x)+\sum_{i=1}^{l}\kappa(x^i,x)} \quad \text{and} \quad x \mapsto \frac{\kappa(x^i,x)}{c(x)+\sum_{i=1}^{l}\kappa(x^i,x)}, \quad 1 \leqslant i \leqslant l,$$

are continuous.

We give an example for possible models of the joint motion of particles which helps to understand the previous assumption.

Example 2.7. – We consider l-particle systems with mean field interaction as investigated for example in Sznitman [28] and Méléard [23]: Take $\tilde{b}: R \times R \to R$, $\tilde{\sigma}: R \times R \to \mathbb{R}^{d\times m}$ Lipschitz and write

$$b\left(x^i,x\right) := \frac{1}{l}\sum_{l=1}^{l}\tilde{b}\left(x^i,x^j\right), \qquad \sigma\left(x^i,x\right) := \frac{1}{l}\sum_{l=1}^{l}\tilde{\sigma}\left(x^i,x^j\right).$$

Other examples and references concerning branching particle systems can be found in Löcherbach [21, Chapter 5].

Since we wish to be able to distinguish between branching and immigration events, we suppose the following.

Assumption 2.8. – Either:
(a) For all $l > 0$, for all $x = (x^1,\dots,x^l) \in R^l$ and all $1 \leqslant i \leqslant l$ $\pi(x^i,x,\{x^i\}) = 0$.
Or:
(b) For all $l > 0$, for all $x = (x^1,\dots,x^l) \in R^l$ and $1 \leqslant i \leqslant l$ we have $\pi(x^i,x,\{x^i\}) = 1$ and $\nu(x,\{x^i\}) = 0$.

From now on, all functions and kernels b, σ, κ, c and π, F, ν defined above on $R \times R^l$ or on R^l for some $l \geqslant 0$ will be considered as functions and kernels on $R \times S$ or S respectively. We define $b(y,\Delta) = \sigma(y,\Delta) = \kappa(y,\Delta) := 0$, $\pi(y,\Delta,.) = F(y,\Delta,.) := 0$ the zero-measure for $y \in R$. For $x = (x^1,\dots,x^l)$ define

$$b(x) := \begin{pmatrix} b(x^1,x) \\ \vdots \\ b(x^l,x) \end{pmatrix} \tag{2.3}$$

and

$$a(x) := \begin{pmatrix} a(x^1,x) & 0 & \cdots & 0 \\ 0 & a(x^2,x) & 0 & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & a(x^l,x) \end{pmatrix}. \tag{2.4}$$

Let $D^*(\mathbb{R}_+, S)$ be the Skorokhod space of all càdlàg functions taking values in S with lifetime (due to possible explosion of the process) (cf. Dellacherie and Meyer [4, XIV, 23–24]). We write Ω^* for the subspace of $D^*(\mathbb{R}_+, S)$ consisting of all functions ψ with the following properties (i) and (ii) below.

(i) There is an increasing sequence of jump times $t_0 = 0 < t_1 \leqslant t_2 \leqslant \cdots$ with $t_n < t_{n+1}$ if $t_n < \infty$, $t_n \uparrow t_\infty$, with t_∞ the lifetime of ψ, such that for all $n \geqslant 0$ the function $\psi_{|[t_n, t_{n+1}[}$ is continuous taking values in some fixed R^l for some $l \geqslant 0$ depending on n and on ψ.

(ii) We have $l(\psi(t_n)) \neq l(\psi(t_{n+1}))$ for all $n \geqslant 0$.

We write φ for the canonical process on Ω^*, $\mathcal{A} := \sigma(\varphi_t, t \geqslant 0)$ and $\mathbb{F} := (\mathcal{F}_t)_{t \geqslant 0}$ for the filtration generated by φ; $\mathcal{F}_t := \bigcap_{T>t} \mathcal{F}_T^0$ with $\mathcal{F}_T^0 := \sigma(\varphi_r \colon r \leqslant T)$. Then as a special case of the construction given in Löcherbach [21], there is a unique probability measure $Q_x^{b,\sigma,\kappa,c,F,\pi,\nu}$ on Ω^* such that φ under $Q_x^{b,\sigma,\kappa,c,F,\pi,\nu}$ is strongly Markov, satisfying the model Assumptions 2.1–2.4 above, with $\varphi_0 = x \in S$ (cf. Löcherbach [21, Theorem 3.2]). We write $(T_n)_n$ for the successive jump times of φ, $T_0 = 0$, and T_∞ for the lifetime of φ. In the following, we consider parametric statistical models for branching particle systems where an unknown parameter ϑ governs the drift function b^ϑ, the branching rate function κ^ϑ, the immigration rate function c^ϑ, the reproduction law F^ϑ, the jump kernel π^ϑ and the immigration law ν^ϑ. Here ϑ belongs to some parameter set Θ, where $\Theta \subset \mathbb{R}^k$ is open such that $(b^\vartheta, \sigma, \kappa^\vartheta, c^\vartheta, F^\vartheta, \pi^\vartheta, \nu^\vartheta)$ satisfy 2.1–2.3, 2.6, 2.8 and such that the following non-explosion assumption holds

$$Q_{x,\vartheta}(T_\infty = \infty) = 1 \quad \text{for all } \vartheta \in \Theta, \tag{2.5}$$

where we write

$$Q_{x,\vartheta} := Q_x^{b^\vartheta,\sigma,\kappa^\vartheta,c^\vartheta,F^\vartheta,\pi^\vartheta,\nu^\vartheta} \tag{2.6}$$

for the law of the particle system under ϑ. Note that the diffusion coefficient will be fixed for the rest of the paper: we observe the process φ continuously in time. We refer the reader to Section 4 for some conditions ensuring that (2.5) holds.

As a consequence of (2.5), in the following we will work on the space $\Omega := \Omega^* \cap \{T_\infty = \infty\}$, equipped with the canonical σ-field $\mathcal{A}$ and canonical filtration $\mathbb{F}$ (we use the same notation as before for Ω^*). We introduce the following notation.

Notation 2.9. – The number of particles in a configuration $x \in S$ is given by $l(x) := l$ if $x \in R^l$. We will sometimes identify a configuration $x = (x^1, \ldots, x^l) \in R^l$ with the associated finite point measure on R

$$\mu_x := \sum_{i=1}^{l} \varepsilon_{x^i} \tag{2.7}$$

and write $x(f) := \int_R f(y)\mu_x(dy) = \sum_{i=1}^{l(x)} f(x^i)$ for $f : R \to \mathbb{R}$ measurable. In the same spirit, we write for $x \in S$ and functions $g : R \times S \to \mathbb{R}$ measurable

$$x(g) := x\big(g(., x)\big) := \sum_{i=1}^{l(x)} g\big(x^i, x\big), \quad \text{if } l(x) > 0,\ \Delta(g) := 0. \tag{2.8}$$

μ_x is an element of $(M^{p,l}, \mathcal{M}^{p,l})$, the space of all finite point measures of total mass l on $(R, \mathcal{B}(R))$, equipped with the topology of weak convergence and the corresponding Borel σ-field. We write

$$M^p := \bigcup_{l=0}^{\infty} M^{p,l},$$

with $M^{p,0}$ the space containing only the zero measure.

2.1. Likelihood ratio processes

We suppose that for all ϑ' and $\vartheta \in \Theta$ either 2.8(a) or 2.8(b) holds and that the law $Q_{x,\vartheta'}$ is locally absolute continuous with respect to $Q_{x,\vartheta}$ relative to $\mathbb{F}$ – by Theorem 5.12 of Löcherbach [21], it suffices to suppose that the following conditions (i)–(v) are fulfilled for all ϑ and $\vartheta' \in \Theta$.

(i) For all $l > 0, x = (x^1, \ldots, x^l) \in R^l$ and $1 \leqslant i \leqslant l$ $\kappa^{\vartheta}(x^i, x) = 0$ implies $\kappa^{\vartheta'}(x^i, x) = 0$ and $c^{\vartheta}(x) = 0$ implies $c^{\vartheta'}(x) = 0$.

(ii) For all $l > 0$, $x = (x^1, \ldots, x^l) \in R^l$, for all $1 \leqslant i \leqslant l$ the measure $\pi^{\vartheta'}(x^i, x, .)$ is absolute continuous with respect to $\pi^{\vartheta}(x^i, x, .)$ with density

$$p^{\vartheta'/\vartheta}\left(x^i, x, y\right) = \frac{d\pi^{\vartheta'}(x^i, x, \,.)}{d\pi^{\vartheta}(x^i, x, \,.)}(y).$$

We shall write $p^{\vartheta'/\vartheta}(x^i, x, z) := \prod_{k=1}^{l(z)} p^{\vartheta'/\vartheta}(x^i, x, z^k)$ for $z \in S$ where $\prod_{k=1}^{0} := 1$.

(iii) For all $l > 0, x = (x^1, \ldots, x^l) \in R^l$ and $1 \leqslant i \leqslant l$ absolute continuity $F^{\vartheta'}(x^i, x, .) \ll F^{\vartheta}(x^i, x, .)$ holds.

(iv) For all $l \geqslant 0$ and $x \in R^l$ $\nu^{\vartheta'}(x, .) \ll \nu^{\vartheta}(x, .)$ with density

$$q^{\vartheta'/\vartheta}(x, y) := \frac{d\,\nu^{\vartheta'}(x, \,.)}{d\,\nu^{\vartheta}(x, \,.)}(y).$$

Precise necessary and sufficient conditions for local absolute continuity of $Q_{x,\vartheta'}$ with respect to $Q_{x,\vartheta}$ relative to $\mathbb{F}$ are given in Theorem 5.12 of Löcherbach [21]. Thanks to Condition 2.8 we are able to distinguish between branching and immigration events and to introduce the following notation.

Notation 2.10. –

(a) We write $(T_n^I)_n$ for the subsequence of $(T_n)_n$ consisting of all immigration events, $T_0^I := 0$. Analogously, we define a subsequence $(T_n^B)_n$ of $(T_n)_n$ corresponding to branching events. We denote the position in space of the immigrating particle at time T_n^I by ζ_n^I, the position in space of the branching (i.e. dying) particle at time T_n^B by ζ_n^D and the configuration (given by a finite point measure on R) of the offspring particles at time T_n^B by ζ_n^B. Note that ζ_n^B can be the zero measure in case of a real death event.

(b) We define point measures associated to branching and to immigration events. Write

$$\mu^B(dt, dx, dy, dp) = \sum_{n \geqslant 1, T_n^B < \infty} \varepsilon_{(T_n^B, \varphi_{T_n^B-}, \zeta_n^D, \zeta_n^B)}(dt, dx, dy, dp) \tag{2.9}$$

on $(0,\infty)\times S\times R\times M^p$ and

$$\mu^I(dt,dx,dy)=\sum_{n\geqslant 1,T_n^I<\infty}\varepsilon_{(T_n^I,\varphi_{T_n^I-},\zeta_n^I)}(dt,dx,dy). \tag{2.10}$$

μ^B has the $(Q_{x,\vartheta},\mathbb{F})$-compensator

$$\begin{aligned}\nu^{B,\vartheta}(dt,dx,dy,dp)=dt\Bigg(\sum_{i=1}^{l(\varphi_{t-})}\kappa^\vartheta\big(\varphi_{t-}^i,\varphi_{t-}\big)\Bigg(\sum_{n=2}^{\infty}F^\vartheta\big(\varphi_{t-}^i,\varphi_{t-},\{n\}\big)\\ \times\int_{R^n}\bigotimes_{k=1}^n\pi^\vartheta\big(\varphi_{t-}^i,\varphi_{t-},du^k\big)\varepsilon_{(\varphi_{t-},\varphi_{t-}^i,\mu_{(u^1,\ldots,u^n)})}(dx,dy,dp)\\ +F^\vartheta\big(\varphi_{t-}^i,\varphi_{t-},\{0\}\big)\varepsilon_{(\varphi_{t-}^i,\varphi_{t-},0)}(dx,dy,dp)\Bigg)\Bigg).\end{aligned} \tag{2.11}$$

μ^I possesses the $(Q_{x,\vartheta},\mathbb{F})$-compensator

$$\nu^{I,\vartheta}(dt,dx,dy)=c^\vartheta(\varphi_{t-})\,dt\int_R\nu^\vartheta(\varphi_{t-},dy')\varepsilon_{(\varphi_{t-},y')}(dx,dy). \tag{2.12}$$

Remark 2.11. – Note that the definitions given above of branching times, branching positions etc. can be made precise, see Löcherbach [21], definition 5.10. We skip the precise definition since the meaning is intuitively clear.

In the following, we will work with occupation time measures m_t on $(S,\mathcal{B}(S))$ and η_t on $(R\times S,\mathcal{B}(R\times S))$, defined for $B\in\mathcal{B}(R)$ and $C\in\mathcal{B}(S)$ via

$$m_t(C):=\int_0^t 1_C(\varphi_s)\,ds,\qquad \eta_t(B\times C):=\int_0^t\varphi_s(B)1_C(\varphi_s)\,ds. \tag{2.13}$$

We still have to introduce further notation before being able to define the likelihood ratio process:

Consider the $\mathcal{M}^{2,c}_{loc}(Q_{x,\vartheta},\mathbb{F})$-martingale given by

$$M_s^{n,\vartheta}:=\begin{cases}0, & s<T_n,\\ \varphi_s-\varphi_{T_n}-\int_{T_n}^s b^\vartheta(\varphi_r)\,dr, & T_n\leqslant s<T_{n+1},\\ \varphi_{T_{n+1}-}-\varphi_{T_n}-\int_{T_n}^{T_{n+1}} b^\vartheta(\varphi_r)\,dr, & s\geqslant T_{n+1}.\end{cases} \tag{2.14}$$

Write $(\Gamma^{\vartheta'/\vartheta})_l^i(x):=(a^{-1}(b^{\vartheta'}-b^\vartheta))(x^i,x)$ and $\Gamma^{\vartheta'/\vartheta}:=((\Gamma^{\vartheta'/\vartheta})_l^1,\ldots,(\Gamma^{\vartheta'/\vartheta})_l^l)$. Then by Theorem 5.12 of Löcherbach [21], the likelihood ratio process $L_t^{\vartheta'/\vartheta}$ of $Q_{x,\vartheta'}$ to $Q_{x,\vartheta}$ relative to $\mathbb{F}$ is given by $\Lambda_t^{\vartheta'/\vartheta}:=\log L_t^{\vartheta'/\vartheta}=\sum_{i=1}^6(\Lambda i)^{\vartheta'/\vartheta}(t)$, where $\log(0):=-\infty$ and where

$$(\Lambda 1)^{\vartheta'/\vartheta}(t) = \sum_{k\geqslant 0} \int_0^t \left[1_{]\!]T_k,T_{k+1}]\!]}(s)\Gamma^{\vartheta'/\vartheta}(\varphi_{s-})\right]^{\mathrm{T}} dM_s^{k,\vartheta} - \frac{1}{2}\int_0^t \left(\Gamma^{\vartheta'/\vartheta}\right)^{\mathrm{T}} a\left(\Gamma^{\vartheta'/\vartheta}\right)(\varphi_s)\, ds, \tag{2.15}$$

and

$$(\Lambda 2)^{\vartheta'/\vartheta}(t) = \int_0^t \int_{S\times R\times M^p} \left[\log \frac{\kappa^{\vartheta'}}{\kappa^{\vartheta}} - \frac{\kappa^{\vartheta'}}{\kappa^{\vartheta}} + 1\right](y,x)\mu^B(ds,dx,dy,dp) + \int_0^t \int_{S\times R\times M^p} \left[\frac{\kappa^{\vartheta'}}{\kappa^{\vartheta}} - 1\right](y,x)\left(\mu^B - \nu^{B,\vartheta}\right)(ds,dx,dy,dp), \tag{2.16}$$

$$(\Lambda 3)^{\vartheta'/\vartheta}(t) = \int_0^t \int_{S\times R\times M^p} \left[\log \frac{F^{\vartheta'}}{F^{\vartheta}} - \frac{F^{\vartheta'}}{F^{\vartheta}} + 1\right]\left(y,x,\{l(p)\}\right)\mu^B(ds,dx,dy,dp) + \int_0^t \int_{S\times R\times M^p} \left[\frac{F^{\vartheta'}}{F^{\vartheta}} - 1\right]\left(y,x,\{l(p)\}\right)\left(\mu^B - \nu^{B,\vartheta}\right)(ds,dx,dy,dp), \tag{2.17}$$

$$(\Lambda 4)^{\vartheta'/\vartheta}(t) = \int_0^t \int_{S\times R\times M^p} \left[\log p^{\vartheta'/\vartheta} - p^{\vartheta'/\vartheta} + 1\right]\left(y,x,(p^1,\ldots,p^{l(p)})\right)\mu^B(ds,dx,dy,dp) + \int_0^t \int_{S\times R\times M^p} \left[p^{\vartheta'/\vartheta} - 1\right]\left(y,x,(p^1,\ldots,p^{l(p)})\right)\left(\mu^B - \nu^{B,\vartheta}\right)(ds,dx,dy,dp), \tag{2.18}$$

where $l(p) = l$ if $p \in M^{p,l}$ and where $(p^1,\ldots,p^{l(p)})$ is an arbitrary arrangement of the atoms of p. For the terms corresponding to immigration events, we can write

$$(\Lambda 5)^{\vartheta'/\vartheta}(t) = \int_0^t \int_{S\times R} \left[\log \frac{c^{\vartheta'}}{c^{\vartheta}} - \frac{c^{\vartheta'}}{c^{\vartheta}} + 1\right](x)\mu^I(ds,dx,dy) + \int_0^t \int_{S\times R} \left[\frac{c^{\vartheta'}}{c^{\vartheta}} - 1\right](x)\left(\mu^I - \nu^{I,\vartheta}\right)(ds,dx,dy) \tag{2.19}$$

and

$$(\Lambda 6)^{\vartheta'/\vartheta}(t) = \int_0^t \int_{S\times R} \left[\log q^{\vartheta'/\vartheta} - q^{\vartheta'/\vartheta} + 1\right](x,y)\mu^I(ds,dx,dy) \tag{2.20}$$

$$+\int_0^t \int_{S\times R} \left[q^{\vartheta'/\vartheta}-1\right](x,y)\left(\mu^I-\nu^{I,\vartheta}\right)(ds,dx,dy).$$

3. Local asymptotic normality and local asymptotic mixed normality for branching particle systems

In this section, we are going to prove the convergence of the experiment locally around some fixed point ϑ to a Gaussian shift experiment or to a mixed normal experiment respectively when time of the observation tends to infinity. We start with regularity conditions on the model.

3.1. Regularity conditions

We call error bound function any function $f: E\times\mathbb{R}_+\to\mathbb{R}_+$, E some measurable space, such that $f(e,.):\mathbb{R}_+\to\mathbb{R}_+$ is non-decreasing for every $e\in E$, $\lim_{c\downarrow 0} f(e,c)=0$ for all $e\in E$ and such that $f(.,c):E\to\mathbb{R}_+$ is measurable.

For some fixed point $\vartheta\in\Theta$ we impose the following conditions.

Condition D1(ϑ). – For all $l>0$ there exists some measurable function $\dot\Gamma_l^\vartheta : R^l\to\mathbb{R}^{ld\times k}$ such that for all $\vartheta'\in\Theta$

$$\left(\Gamma^{\vartheta'/\vartheta}-\dot\Gamma_l^\vartheta\,(\vartheta'-\vartheta)\right)^{\mathrm{T}} a\left(\Gamma^{\vartheta'/\vartheta}-\dot\Gamma_l^\vartheta\,(\vartheta'-\vartheta)\right)(x)\leqslant f^1_{\vartheta,l}(x,|\vartheta'-\vartheta|)\,|\vartheta'-\vartheta|^2$$

for all $x\in R^l$, (recall $a(x)$ which has been defined in (2.4)), where $f^1_{\vartheta,l}$ is an error bound function.

Define $\dot\Gamma^\vartheta(x):=\dot\Gamma_l^\vartheta(x)$ for $x\in S\cap R^l$, $l>0$, $\dot\Gamma^\vartheta(\Delta):=0$, $f^1_\vartheta(x,c)$ for $x\in S$ analogously. Suppose that the following integrability conditions (i) and (ii) are fulfilled.

(i) $\int_0^{\cdot}[(\dot\Gamma^\vartheta)^{\mathrm{T}} a\dot\Gamma^\vartheta](\varphi_s)\,ds$ is locally integrable with respect to $Q_{x,\vartheta}$ for all $x\in S$.

(ii) $\int_0^{\cdot} f^1_\vartheta(\varphi_s,\delta(\vartheta))\,ds$ is locally integrable with respect to $Q_{x,\vartheta}$ for all $x\in S$, for some constant $\delta(\vartheta)>0$.

Condition D2(ϑ). – There exists some measurable function $\dot\kappa^\vartheta : R\times S\to\mathbb{R}^k$ satisfying the following conditions (i) and (ii) for all $\vartheta'\in\Theta$.

(i)

$$\left(\frac{\kappa^{\vartheta'}}{\kappa^\vartheta}(y,z)-1-(\vartheta'-\vartheta)^{\mathrm{T}}\dot\kappa^\vartheta(y,z)\right)^2\leqslant f^2_\vartheta\left(y,z,|\vartheta'-\vartheta|\right)|\vartheta'-\vartheta|^2$$

for an error bound function f^2_ϑ such that

$$\int_0^{\cdot}\varphi_s\left(\kappa^\vartheta f^2_\vartheta(.,.,\delta(\vartheta))\right)ds \text{ is locally integrable w.r.t. } Q_{x,\vartheta}$$

for all $x\in S$, for some $\delta(\vartheta)>0$ (see 2.9 for the notation).

(ii)

$$\int_0^{\cdot} \varphi_s\big(\kappa^{\vartheta}\big[(\dot{\kappa}^{\vartheta})^{\mathrm{T}}\dot{\kappa}^{\vartheta}\big](.,.)\big)\,ds$$

is locally integrable with respect to $Q_{x,\vartheta}$ for all $x \in S$.

Condition D3(ϑ). – There exists some measurable function $\dot{F}^{\vartheta}: R \times S \times \mathbb{N}_0 \to \mathbb{R}^k$ satisfying the following conditions (i) and (ii) for all $\vartheta' \in \Theta$.

(i)

$$\sum_{n\geqslant 0} F^{\vartheta}(y,z,\{n\})\left(\frac{F^{\vartheta'}}{F^{\vartheta}}(y,z,\{n\}) - 1 - (\vartheta'-\vartheta)^{\mathrm{T}}\dot{F}^{\vartheta}(y,z,n)\right)^2 \leqslant f_{\vartheta}^3\big(y,z,|\vartheta'-\vartheta|\big)|\vartheta'-\vartheta|^2$$

for an error bound function f_{ϑ}^3 such that

$$\int_0^{\cdot} \varphi_s\big(\kappa^{\vartheta} f_{\vartheta}^3(.,.,\delta(\vartheta))\big)\,ds \text{ is locally integrable w.r.t. } Q_{x,\vartheta}$$

for all $x \in S$, for some $\delta(\vartheta) > 0$.

(ii)

$$\int_0^{\cdot} \varphi_s\left(\kappa^{\vartheta}\left(\sum_{n\geqslant 0} F^{\vartheta}(.,.,\{n\})\big[\big(\dot{F}^{\vartheta}\big)^{\mathrm{T}}\dot{F}^{\vartheta}\big](.,.,n)\right)\right)ds$$

is locally integrable with respect to $Q_{x,\vartheta}$ for all $x \in S$.

Condition D4(ϑ). – There exists some measurable function $\dot{p}^{\vartheta}: R \times S \times M^p \to \mathbb{R}^k$ satisfying the following conditions (i) and (ii) for all $\vartheta' \in \Theta$.

(i)

$$\sum_{n\geqslant 0} F^{\vartheta}(y,z,\{n\}) \int_{R^n} \bigotimes_{k=1}^{n} \pi^{\vartheta}\big(y,z,dp^k\big)\big[p^{\vartheta'/\vartheta}\big(y,z,\big(p^1,\ldots,p^{l(p)}\big)\big) - 1 - (\vartheta'-\vartheta)^{\mathrm{T}}\dot{p}^{\vartheta}(y,z,p)\big]^2 \leqslant f_{\vartheta}^4\big(y,z,|\vartheta'-\vartheta|\big)|\vartheta'-\vartheta|^2$$

(with definition $\bigotimes_{k=1}^{0} \pi^{\vartheta}(y,z,dp^k) := \varepsilon_0(dp)$, 0 being the zero measure), for an error bound function f_{ϑ}^4 such that

$$\int_0^{\cdot} \varphi_s\big(\kappa^{\vartheta} f_{\vartheta}^4(.,.,\delta(\vartheta))\big)\,ds \text{ is locally integrable w.r.t. } Q_{x,\vartheta}$$

for all $x \in S$, for some $\delta(\vartheta) > 0$.

(ii)

$$\int_0^{\cdot} \varphi_s\left(\kappa^{\vartheta}\left(\sum_{n\geqslant 0} F^{\vartheta}(.,.,\{n\}) \int_{R^n} \bigotimes_{k=1}^{n} \pi^{\vartheta}\big(.,.,dp^k\big)\big[(\dot{p}^{\vartheta})^{\mathrm{T}}\dot{p}^{\vartheta}\big](.,.,p)\right)\right)ds$$

is locally integrable with respect to $Q_{x,\vartheta}$ for all $x \in S$.

In the same manner we impose for the immigration terms the following

Condition D5(ϑ). – There is some measurable function $\dot{c}^{\vartheta} : S \to \mathbb{R}^k$ such that the following conditions (i) and (ii) hold for all $\vartheta' \in \Theta$.

(i)

$$\left(\frac{c^{\vartheta'}}{c^{\vartheta}} - 1 - (\vartheta' - \vartheta)^{\mathrm{T}} \dot{c}^{\vartheta}\right)^2 (x) \leqslant f_{\vartheta}^5\big(x, |\vartheta' - \vartheta|\big) |\vartheta' - \vartheta|^2$$

for all $x \in S$, for $f_{\vartheta}^5 : S \times \mathbb{R}_+ \to \mathbb{R}_+$ an error bound function such that

$$\int_0^{\cdot} c^{\vartheta}(\varphi_s) f_{\vartheta}^5\big(\varphi_s, \delta(\vartheta)\big)\, ds \text{ is locally integrable w.r.t. } Q_{x,\vartheta} \text{ for all } x \in S$$

for some $\delta(\vartheta) > 0$.

(ii)

$$\int_0^{\cdot} \big[(\dot{c}^{\vartheta})^{\mathrm{T}} \dot{c}^{\vartheta}\big](\varphi_s) c^{\vartheta}(\varphi_s)\, ds$$

is locally integrable w.r.t. $Q_{x,\vartheta}$ for all $x \in S$.

Condition D6(ϑ). – There is some measurable function $\dot{q}^{\vartheta} : S \times R \to \mathbb{R}^k$ such that the following conditions (i) and (ii) hold for all $\vartheta' \in \Theta$.

(i)

$$\int_R \big(q^{\vartheta'/\vartheta} - 1 - (\vartheta' - \vartheta)^{\mathrm{T}} \dot{q}^{\vartheta}\big)^2 (x, y) \nu^{\vartheta}(x, dy) \leqslant f_{\vartheta}^6(x, |\vartheta' - \vartheta|) |\vartheta' - \vartheta|^2$$

(with $\nu^{\vartheta}(x, dy)$ the immigration measure of Assumption 2.3) for all $x \in S$, for $f_{\vartheta}^6 : S \times \mathbb{R}_+ \to \mathbb{R}_+$ an error bound function such that

$$\int_0^{\cdot} c^{\vartheta}(\varphi_s) f_{\vartheta}^6\big(\varphi_s, \delta(\vartheta)\big)\, ds \text{ is locally integrable w.r.t. } Q_{x,\vartheta} \text{ for all } x \in S$$

for some $\delta(\vartheta) > 0$.

(ii)

$$\int_0^{\cdot} \int_R \big[(\dot{q}^{\vartheta})^{\mathrm{T}} \dot{q}^{\vartheta}\big](\varphi_s, y) \nu^{\vartheta}(\varphi_s, dy) c^{\vartheta}(\varphi_s)\, ds$$

is locally integrable w.r.t. $Q_{x,\vartheta}$ for all $x \in S$.

Under D1–D6, we can define score function martingales and information processes: We write $(M1)^\vartheta$ for the locally square integrable martingale given by

$$(M1)_t^\vartheta := \sum_{n\geqslant 0} \int_0^t \left[1_{]\!]T_n,T_{n+1}]\!]}(s)\dot{\Gamma}^\vartheta(\varphi_{s-})\right]^{\mathrm{T}} dM_s^{n,\vartheta}, \quad t \geqslant 0, \tag{3.21}$$

with $M^{n,\vartheta}$ as in (2.14). $(M1)^\vartheta$ possesses the angle bracket process

$$\left\langle (M1)^\vartheta \right\rangle_t = \int_S (I1)^\vartheta(x) m_t(dx), \tag{3.22}$$

where

$$(I1)^\vartheta(x) := \left((\dot{\Gamma}^\vartheta)^{\mathrm{T}} a \dot{\Gamma}^\vartheta\right)(x). \tag{3.23}$$

The second score function martingale is given as

$$(M2)_t^\vartheta := \int_0^t \int_{S\times R\times M^p} \dot{\kappa}^\vartheta(y,x)\left(\mu^B - \nu^{B,\vartheta}\right)(ds,dx,dy,dp), \quad t \geqslant 0. \tag{3.24}$$

Then $(M2)^\vartheta$ is in $\mathcal{M}_{loc}^{2,d}(Q_{x,\vartheta}, \mathbb{F})$, the set of all locally square integrable purely discontinuous $(Q_{x,\vartheta}, \mathbb{F})$-martingales, with predictable quadratic covariation process

$$\left\langle (M2)^\vartheta \right\rangle_t = \int_{R\times S} (I2)^\vartheta(y,z)\eta_t(dy,dz), \tag{3.25}$$

where

$$(I2)^\vartheta(y,z) := \left(\dot{\kappa}^\vartheta (\dot{\kappa}^\vartheta)^{\mathrm{T}}\right)(y,z)\kappa^\vartheta(y,z).$$

In the same way,

$$(M3)_t^\vartheta := \int_0^t \int_{S\times R\times M^p} \dot{F}^\vartheta\left(x,y,l(p)\right)\left(\mu^B - \nu^{B,\vartheta}\right)(ds,dx,dy,dp), \quad t \geqslant 0, \tag{3.26}$$

$$\left\langle (M3)^\vartheta \right\rangle_t = \int_{R\times S} (I3)^\vartheta(y,z)\eta_t(dy,dz) \tag{3.27}$$

with

$$(I3)^\vartheta(y,z) := \left(\sum_{n\geqslant 0} F^\vartheta(y,z,\{n\})\left[\dot{F}^\vartheta(\dot{F}^\vartheta)^{\mathrm{T}}\right](y,z,n)\right)\kappa^\vartheta(y,z), \tag{3.28}$$

and

$$(M4)_t^\vartheta := \int_0^t \int_{S\times R\times M^p} \dot{p}^\vartheta(x,y,p)\big(\mu^B - \nu^{B,\vartheta}\big)(ds,dx,dy,dp), \quad t \geqslant 0, \tag{3.29}$$

with predictable quadratic covariation process

$$\big\langle (M4)^\vartheta \big\rangle_t = \int_{R\times S} (I4)^\vartheta(y,z)\eta_t(dy,dz) \tag{3.30}$$

where

$$(I4)^\vartheta(y,z) := \left(\sum_{n\geqslant 0} F^\vartheta(y,z,\{n\}) \int_{R^n} \bigotimes_{k=1}^n \pi^\vartheta(y,z,dp^k)\big[\dot{p}^\vartheta(\dot{p}^\vartheta)^{\mathrm{T}}\big](y,z,p)\right)\kappa^\vartheta(y,z). \tag{3.31}$$

In the same way we define

$$(M5)_t^\vartheta := \int_0^t \int_{S\times R} \dot{c}^\vartheta(x)\big(\mu^I - \nu^{I,\vartheta}\big)(ds,dx,dy), \quad t \geqslant 0, \tag{3.32}$$

with predictable quadratic covariation

$$\big\langle (M5)^\vartheta \big\rangle_t = \int_S (I5)^\vartheta(x)m_t(dx) \tag{3.33}$$

with

$$(I5)^\vartheta(x) := \big(\dot{c}^\vartheta(\dot{c}^\vartheta)^{\mathrm{T}}\big)(x)c^\vartheta(x) \tag{3.34}$$

and

$$(M6)_t^\vartheta := \int_0^t \int_{S\times R} \dot{q}^\vartheta(x,y)\big(\mu^I - \nu^{I,\vartheta}\big)(ds,dx,dy), \quad t \geqslant 0, \tag{3.35}$$

with predictable quadratic covariation

$$\big\langle (M6)^\vartheta \big\rangle_t = \int_S (I6)^\vartheta(x)m_t(dx) \tag{3.36}$$

with

$$(I6)^\vartheta(x) := \left(\int_R \nu^\vartheta(x,dy)\big[\dot{q}^\vartheta(\dot{q}^\vartheta)^{\mathrm{T}}\big](x,y)\right)c^\vartheta(x). \tag{3.37}$$

We will also use the following overall information

$$I^{\vartheta}(y,z) := (I1)^{\vartheta}(z) + \sum_{i=2}^{4}(Ii)^{\vartheta}(y,z) + \sum_{i=5}^{6}(Ii)^{\vartheta}(z). \tag{3.38}$$

3.2. Recurrence

We make the following assumption concerning the asymptotic behaviour of φ under $Q_{x,\vartheta}$.

Condition R(ϑ). –

(a) φ is recurrent under $Q_{x,\vartheta}$ in the sense of Harris, admitting the void configuration Δ as recurrent point, with invariant measure m^{ϑ} normed to be a probability measure in the case of positive recurrence.

(b) Write $R_1 := \inf\{T_n\colon n > 0, \varphi_{T_n} = \Delta\}$, $R_n := \inf\{T_n > R_{n-1}\colon \varphi_{T_n} = \Delta\}$. If $m^{\vartheta}(S) = \infty$ (null-recurrence), then either

$$Q_{x,\vartheta}(R_2 - R_1 > t) \sim \frac{l(t)}{t^{\alpha}} \quad \text{for } t \to \infty \tag{3.39}$$

for some $\alpha \in (0,1)$, l slowly varying at infinity (cf., e.g., Bingham, Goldie, Teugels [1]) or

$$\int_0^x Q_{x,\vartheta}(R_2 - R_1 > t)\,dt \sim l(x) \quad \text{for } x \to \infty, \tag{3.40}$$

l slowly varying at infinity.

Note that if our goal were only to derive a quadratic decomposition of the log-likelihood ratio process, then it would be sufficient to presume quite weak conditions concerning the asymptotic behaviour of φ: We would just presume the existence of a sequence $u_n(\vartheta)$ of norming constants for the score function martingales such that remainder terms of type $u_n^2(\vartheta)\int_0^{tn} f(\varphi_s, c\cdot u_n(\vartheta))\,ds$ vanish in $Q_{x,\vartheta}$-probability as $n\to\infty$, for error bound functions f (compare also with Höpfner [10], assumptions A1(ϑ)–A3(ϑ)). In the (here quite natural) situation where φ under $Q_{x,\vartheta}$ is recurrent, this is verified if we have weak convergence of additive functionals $A_t := \int_0^t k(\varphi_s)\,ds$ of φ under $Q_{x,\vartheta}$, for measurable k which are integrable with respect to the invariant measure m^{ϑ} (this is even more). Weak convergence of such additive functionals is immediate in the positive recurrent case – however in the null-recurrent case it holds place only under the restrictive assumptions stated above under b) on the tails of the life cycle lengths:

Remark 3.1. –

(1) Condition R(ϑ) is necessary (and sufficient) for weak convergence of rescaled additive functionals of φ in the following sense: In case $m^{\vartheta}(S) = \infty$, suppose there exists a set $A \in \mathcal{B}(S)$ meeting $0 < m^{\vartheta}(A) < \infty$ and a sequence $r_n(\vartheta) \downarrow 0$ as $n\to\infty$ as well as a limit process V having continuous non-decreasing paths with $V_0 = 0$, $V_t \uparrow \infty$

as $t \to \infty$ (V is a time change process) such that for $n \to \infty$

$$\left(r_n^2(\vartheta) \int_0^{tn} 1_A(\varphi_s)\, ds \right)_{t \geqslant 0} \xrightarrow{\mathcal{L}} m^\vartheta(A) \cdot V \tag{3.41}$$

(weak convergence in $D(\mathbb{R}_+, \mathbb{R})$ under $Q_{x,\vartheta}$ for some $x \in S$). Then there is some $\alpha \in (0, 1]$ and some regularly varying sequence $(u_n(\vartheta))_n \in RV_{-\alpha/2}$ such that (3.41) even holds with $V = W^\alpha$ and $r_n(\vartheta) = u_n(\vartheta)$. W^α for $0 < \alpha < 1$ is the Mittag–Leffler process of index α, i.e. W^α is the process inverse to the stable increasing process S^α with index $\alpha \in (0, 1)$. Moreover, $W^1 \equiv id$. Recall that $(u_n(\vartheta))_n \in RV_{-\alpha/2}$ means that there is some function L which is regularly varying at infinity with index $-\alpha/2$ such that $u_n = L(n)$. In this case, (3.39) or (3.40) respectively necessarily hold. If $m^\vartheta(S) < \infty$, then $u_n(\vartheta) = 1/\sqrt{n}$ up to multiplication by a constant.

(2) Up to multiplication by a constant, the invariant measure m^ϑ is given by

$$m^\vartheta(C) := c \cdot E_{x,\vartheta} \left(\int_{R_1}^{R_2} 1_C(\varphi_s)\, ds \right)$$

for $C \in \mathcal{B}(S)$.

Proof of Remark 3.1(1). – The proof of the necessary part is a generalisation of the classical Darling–Kac theorem (see Darling and Kac [2] and Bingham, Goldie and Teugels [1], Theorem 8.11.3) which has been obtained by Touati [29], Theorem 10 and Lemma 5. This generalisation relies heavily on the fact that special functions f with $\int_S f(x) m^\vartheta(dx) \in (0, \infty)$ exist for the Harris process φ. We refer the reader to Höpfner and Löcherbach [13], Chapter 5 for the details. For the “sufficient” part, we refer to Resnick and Greenwood [26], Touati [29], Höpfner [9] and to Chapter 4 of Höpfner and Löcherbach [13] which gives a nice summary of the whole argument. □

The following martingale convergence theorem for null-recurrent cases is crucial for the sequel and has been obtained by Touati [29].

THEOREM 3.2 (Touati [29]). – *Suppose that* (3.39) *or* (3.40) *hold and write* $u_n(\vartheta) := (\Gamma(1-\alpha) Q_{x,\vartheta}(R_2 - R_1 > n))^{1/2}$ *in case of* (3.39) *and* $u_n(\vartheta) := (\frac{1}{n} \int_0^n Q_{x,\vartheta}(R_2 - R_1 > t)\, dt)^{1/2}$ *in case of* (3.40). *Consider a locally square integrable martingale* $M \in \mathcal{M}^2_{loc}$ *whose angle bracket* $\langle M \rangle$ *is a* m^ϑ*-integrable additive functional of* φ, *i.e.* $E_{\Delta,\vartheta}(\langle M \rangle_{R_1}) < \infty$. *Then we have for the rescaled sequence*

$$M^n := \left(\frac{1}{\sqrt{u_n(\vartheta)}} M_{tn} \right)_{t \geqslant 0}$$

in case of (3.39)

$$M^n \xrightarrow{\mathcal{L}} J^{1/2} B \circ W^\alpha \quad \text{as } n \to \infty,$$

and in case of (3.40)

$$M^n \xrightarrow{\mathcal{L}} J^{1/2} B \quad \text{as } n \to \infty$$

(*weak convergence in* $D(\mathbb{R}_+, \mathbb{R})$). *Here*, B *is a standard Brownian motion*, W^α *is a Mittag–Leffler process independent of* B *and* $J = E_{\Delta,\vartheta}(\langle M\rangle_{R_1})$.

Proof. – The basic idea of this theorem is to split the trajectory of φ into i.i.d. sequences of life-cycles and to use results concerning convergence to stable processes. See also Resnick and Greenwood [26], in particular for the necessary independence of B and W^α. □

3.2.1. When does R(ϑ) hold?

Example 3.3. –

(1) Suppose that the family of space and configuration dependent reproduction laws $F^\vartheta(.,.,dn)$ admits some fixed space and configuration independent subcritical law

$$\bar{F}^\vartheta(dn) \quad \text{such that} \quad \sum_{k\neq 1} k\bar{F}^\vartheta(\{k\}) < 1$$

as upper bound in the sense of convolution of probability measures: for every $y \in R$, $z \in S$ there is some $\hat{F}^\vartheta(y,z,dn)$ such that $F^\vartheta(y,z,.) * \hat{F}^\vartheta(y,z,.) = \bar{F}^\vartheta(.)$. Suppose that κ^ϑ is bounded away from zero (which guarantees that there is always a minimal amount of "branching") and that the immigration rate c^ϑ is bounded. Write

$$\varrho(t) := \inf\left\{ s \geqslant 0\colon \int_0^s \frac{\varphi_u(\kappa^\vartheta)}{l(\varphi_u)}\, du > t \right\}.$$

Then the time-changed process $\varphi \circ \varrho$ can be constructed in a coupled way with another branching diffusion $\bar{\varphi}$ where in $\bar{\varphi}$ particles branch at rate 1, reproduce according to $\bar{F}^\vartheta$, with an input of immigrating particles at constant rate, such that $\varphi \circ \varrho(t)$ is a subprocess (in the sense of subpopulations) of $\bar{\varphi}(t)$ for all t. Since $\bar{\varphi}$ is positive recurrent, with recurrent point Δ (see for instance Zubkov [31] and Pakes [24]), the same necessarily holds for $\varphi \circ \varrho$, hence for φ itself.

(2) Suppose that the reproduction law F^ϑ is constant in space and configuration, critical, admitting second moments $\sum_{k\geqslant 2} kF^\vartheta(\{k\}) = 2\beta$. Suppose further that branching and immigration occur at constant rates $\kappa^\vartheta \equiv a^\vartheta \in (0,\infty)$, $c^\vartheta \equiv b^\vartheta \in (0,\infty)$ such that $b^\vartheta < a^\vartheta \cdot \beta$. Then Zubkov [31], Theorem 2 shows that $\mathcal{L}(R_2 - R_1 \mid Q_{x,\vartheta})$ belongs to the domain of attraction of a positive stable law with index $\alpha = 1 - \frac{b^\vartheta}{a^\vartheta \cdot \beta}$.

Example 3.4. – We present an example for null-recurrent situations satisfying condition R(ϑ) with a certain space dependency. Suppose that $F^\vartheta, \kappa^\vartheta$ and c^ϑ depend only on the size l of a configuration and are independent of the space, F^ϑ being binary. Take some $0 < \alpha < \frac{1}{2}$ fixed and some parametric family $\{\kappa^\xi(k)\colon \xi \in \Theta,\ k \geqslant 1\}$ such that $\frac{1}{2} \leqslant \kappa^\xi(k) \leqslant 1$. Suppose that there is some fixed threshold k_0 such that for all $k \geqslant k_0$

$$c^\xi(k) = k + \alpha - k\kappa^\xi(k)$$

and

$$F^\xi(k,\{0\}) = \frac{1}{2\kappa^\xi(k)}, \qquad F^\xi(k,\{2\}) = 1 - F^\xi(k,\{0\}).$$

Note that $c^{\xi}(k) > 0$ for all $k \geqslant k_0$ and that for $l_t = l(\varphi_t)$ the number of particles at time t

$$Q_{x,\xi}(l_{t+h} = k+1 \mid l_t = k) = \left(\frac{k}{2} + \alpha\right) \cdot h + \mathrm{o}(h) \quad \text{as } h \to 0$$

and

$$Q_{x,\xi}(l_{t+h} = k-1 \mid l_t = k) = \frac{k}{2} \cdot h + \mathrm{o}(h) \quad \text{as } h \to 0,$$

$k \geqslant k_0$. Hence under every ξ, $(l_t)_{t\geqslant 0}$ restricted to $\{l_t \geqslant k_0\}$ can be considered as classical critical branching process with immigrations at constant rate α and branching at rate 1 for every particle. Define for every k a birth rate λ_k^{ξ} and a death rate μ_k^{ξ} by

$$\lambda_k^{\xi} := k\kappa^{\xi}(k)F^{\xi}(k, \{2\}) + c^{\xi}(k), \qquad \mu_k^{\xi} := k\kappa^{\xi}(k)F^{\xi}(k, \{0\})$$

and write

$$m_k^{\xi} := \frac{\lambda_0^{\xi} \cdots \lambda_{k-1}^{\xi}}{\mu_1^{\xi} \cdots \mu_k^{\xi}}.$$

By Karlin and McGregor [18] we know that only the asymptotic behaviour $k \to \infty$ of λ_k^{ξ} and μ_k^{ξ} determines the asymptotic behaviour of l_t. But asymptotically, m_k^{ξ} behaves like

$$m_k = \frac{\lambda_0 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k}$$

with $\lambda_k = \frac{1}{2}k + \alpha$ and $\mu_k = \frac{1}{2}k$, which is the invariant measure for a classical birth and death process. As a consequence we know that $Q_{x,\xi}(R_2 - R_1 > t) \sim C/t^{1-2\alpha}$, for some constant C, independently of ξ.

3.3. LAN and LAMN at ϑ

We now dispose of all the tools allowing to prove local asymptotic normality or local asymptotic mixed normality of the model. Write $\mathbb{F}^n := (\mathcal{F}_{nt})_{t\geqslant 0}$. For $\vartheta \in \Theta$ such that $\mathrm{R}(\vartheta)$ holds, let

$$(Mi)^{n,\vartheta}(t) := u_n(\vartheta)(Mi)^{\vartheta}_{tn}, \quad t \geqslant 0,\ 1 \leqslant i \leqslant 6, \tag{3.42}$$

be the rescaled score function martingales where

$$u_n(\vartheta) \begin{cases} = 1/\sqrt{n} & \text{if } m^{\vartheta}(S) = 1\,, \\ \sim (\Gamma(1-\alpha)Q_{x,\vartheta}(R_2 - R_1 > n))^{1/2} & \text{if } Q_{x,\vartheta}(R_2 - R_1 > \cdot) \in \\ & \quad RV_{-\alpha},\ 0 < \alpha < 1, \\ \sim (\frac{1}{n}\int_0^n Q_{x,\vartheta}(R_2 - R_1 > t)\,dt)^{1/2} & \text{if } \mathcal{L}(R_2 - R_1 \mid Q_{x,\vartheta}) \\ & \quad \text{is relatively stable,} \end{cases} \tag{3.43}$$

with RV_{α} the set of all regularly varying functions at infinity, with index α.

DEFINITION 3.5. – *We define a Campbell measure* $\bar{m}^\vartheta$ *on* $(R \times S, \mathcal{B}(R \times S))$ *associated to the invariant measure* m^ϑ *via*

$$\bar{m}^\vartheta(B \times C) := \int_S x(B) 1_C(x) m^\vartheta(dx)$$

for $B \in \mathcal{B}(R), C \in \mathcal{B}(S)$.

Remark 3.6. – Note that in cases where $b^\vartheta, \sigma,\ \kappa^\vartheta, F^\vartheta, \pi^\vartheta$ are purely position dependent and where $c^\vartheta(x)\nu^\vartheta(x, dy) \equiv \mu^\vartheta(dy)$ independent of $x \in S$, with notation

$$F^\vartheta \pi^\vartheta(y, A) := \sum_{k \neq 0} F^\vartheta(y, \{k\}) \sum_{l=1}^{k} \int_R \pi^\vartheta\big(y, du^l\big) 1_A\big(u^l\big),$$

$y \in R$, $A \in \mathcal{B}(R)$, we have a representation

$$\bar{m}^\vartheta(\cdot \times S) = \mu^\vartheta \left(\sum_{n=0}^{\infty} \big(U^\vartheta \kappa^\vartheta F^\vartheta \pi^\vartheta\big)^n \right) U^\vartheta$$

in cases where $\bar{m}^\vartheta(R \times S) < \infty$, with notation

$$U^\vartheta(y, A) := E_y \left(\int_0^\infty dt\, 1_A(X_t) \mathrm{e}^{-\int_0^t \kappa^\vartheta(X_r)\,dr} \right),$$

$y \in R$, $A \in \mathcal{B}(R)$, for a diffusion

$$dX_t = b^\vartheta(X_t)\,dt + \sigma(X_t)\,dW_t,$$

$(U^\vartheta \kappa^\vartheta)(x, dy) := U^\vartheta(x, dy)\kappa^\vartheta(y)$ (see Höpfner and Löcherbach [12]). This follows from conditioning on first branching events during one life cycle of φ. The same argument shows that in purely position dependent situations, in null-recurrent cases, $\bar{m}^\vartheta(A \times S) \equiv \infty$ for all $A \in \mathcal{B}(R)$, hence the projection of $\bar{m}^\vartheta$ onto R does no longer make sense in this case (compare to Höpfner and Löcherbach [12], Remark 2.4).

We impose the following integrability conditions.

Condition I(ϑ*).* –
(a) $f^1_\vartheta(\cdot, \delta(\vartheta)) \in L^1(m^\vartheta)$, $f^i_\vartheta(\cdot, \delta(\vartheta)) \in L^1(\kappa^\vartheta \bar{m}^\vartheta)$ for $i = 2, 3, 4$ and $f^i_\vartheta(\cdot, \delta(\vartheta)) \in L^1(c^\vartheta m^\vartheta)$ for $i = 5, 6$, where $\kappa^\vartheta \bar{m}^\vartheta(dy, dz) = \kappa^\vartheta(y, z)\bar{m}^\vartheta(dy, dz)$, $c^\vartheta m^\vartheta(dx) = c^\vartheta(x) m^\vartheta(dx)$.
(b) *Integrability of the information processes.*
Assume that $(Ii)^\vartheta \in L^1(m^\vartheta)$ component-wise for $i = 1, 5, 6$ and $(Ii)^\vartheta \in L^1(\bar{m}^\vartheta)$ component-wise for $i = 2, 3, 4$.
(c) Define

$$(Ji)^\vartheta := \int_S (Ii)^\vartheta(x) m^\vartheta(dx), \quad i = 1, 5, 6, \tag{3.44}$$

$$(Ji)^{\vartheta} := \int\limits_{R\times S} (Ii)^{\vartheta}(y,z)\bar{m}^{\vartheta}(dy,dz), \quad i = 2,3,4, \tag{3.45}$$

$$\Lambda = \Lambda(\vartheta) = \begin{pmatrix} (J1)^{\vartheta} & 0 & 0 & 0 & 0 & 0 \\ 0 & (J2)^{\vartheta} & 0 & 0 & 0 & 0 \\ 0 & 0 & (J3)^{\vartheta} & 0 & 0 & 0 \\ 0 & 0 & 0 & (J4)^{\vartheta} & 0 & 0 \\ 0 & 0 & 0 & 0 & (J5)^{\vartheta} & 0 \\ 0 & 0 & 0 & 0 & 0 & (J6)^{\vartheta} \end{pmatrix}, \tag{3.46}$$

$$J = J(\vartheta) := \sum_{i=1}^{6} (Ji)^{\vartheta} = \int\limits_{R\times S} I^{\vartheta}(y,z)\bar{m}^{\vartheta}(dy,dz) \tag{3.47}$$

with I^{ϑ} as in (3.38), and assume that J is invertible.

Fix some direction $h \in \mathbb{R}^k$ and some point $\vartheta \in \Theta$ with R(ϑ). Write $\vartheta_n := \vartheta + u_n(\vartheta)h$ for a sequence $u_n(\vartheta)$ of norming constants as in (3.43). We consider the sequence of filtered local models at ϑ

$$\big(\Omega_x, \mathcal{A}, \mathbb{F}^n, \{Q_{x,\vartheta+u_n(\vartheta)h}\colon h \in \Theta_{\vartheta,n}\}\big), \tag{3.48}$$

where $\Omega_x := \{\psi \in \Omega\colon \psi_0 = x\}$, $\Theta_{\vartheta,n} := \{h \in \mathbb{R}^k\colon \vartheta + u_n(\vartheta)h \in \Theta\}$, and prove LAN and LAMN respectively for the filtered local model (3.48). Denote by $L^{h/0}_{n,\vartheta}(t) := L^{\vartheta_n/\vartheta}_{t\cdot n}$ the likelihood ratio process of Q_{x,ϑ_n} to $Q_{x,\vartheta}$ relative to $\mathbb{F}^n$ and $(\Lambda i)^{h/0}_{n,\vartheta}(t) := (\Lambda i)^{\vartheta_n/\vartheta}(tn)$ for $i = 1,\ldots,6$, $\Lambda^{h/0}_{n,\vartheta}(t) := \Lambda^{\vartheta_n/\vartheta}(tn)$.

THEOREM 3.7. – *Consider some point* $\vartheta \in \Theta$ *fixed such that* D1(ϑ)–D6(ϑ), R(ϑ) *and* I(ϑ) *hold, some direction* $h \in \mathbb{R}^k$ *fixed and norming constants* $u_n(\vartheta)$ *as above in* (3.43).

(a) *The log-likelihood ratio process of* Q_{x,ϑ_n} *to* $Q_{x,\vartheta}$ *relative to* $\mathbb{F}^n$ *admits the decomposition*

$$(\Lambda i)^{h/0}_{n,\vartheta} = h^{\mathrm{T}}(Mi)^{n,\vartheta} - \frac{1}{2}h^{\mathrm{T}}\big\langle (Mi)^{n,\vartheta}\big\rangle h + Rem_{n,\vartheta,h}, \quad 1 \leqslant i \leqslant 6,$$

and

$$\Lambda^{h/0}_{n,\vartheta} = h^{\mathrm{T}}\bar{M}^{n,\vartheta} - \frac{1}{2}h^{\mathrm{T}}\big\langle \bar{M}^{n,\vartheta}\big\rangle h + Rem_{n,\vartheta,h},$$

where $\bar{M}^{n,\vartheta} := \sum_{i=1}^{6}(Mi)^{n,\vartheta}$, *with a process of remainder terms such that* $\sup_{s\leqslant t}|Rem_{n,\vartheta,h}(s)| \to 0$ *in* $Q_{x,\vartheta}$*-probability as* $n \to \infty$ *for all* $t \geqslant 0$.

(b) *Let* $M^{n,\vartheta} := ((M1)^{n,\vartheta},\ldots,(M6)^{n,\vartheta})^{\mathrm{T}}$ *be the array of rescaled score function martingales. Then we have weak convergence*

$$\big(M^{n,\vartheta}, \big\langle M^{n,\vartheta}\big\rangle\big) \to (Y, \langle Y\rangle) \quad \text{as } n \uparrow \infty,$$

(*weak convergence in* $D(\mathbb{R}_+, \mathbb{R}^{6k} \times \mathbb{R}^{6k\times 6k})$ *under* $Q_{x,\vartheta}$) *and*

$$\big(\bar{M}^{n,\vartheta}, \langle \bar{M}^{n,\vartheta}\rangle\big) \to \big(\bar{Y}, \langle \bar{Y}\rangle\big) \quad \textit{as } n \uparrow \infty,$$

(*weak convergence in* $D(\mathbb{R}_+, \mathbb{R}^{k} \times \mathbb{R}^{k\times k})$ *under* $Q_{x,\vartheta}$) *where* $Y = Y(\vartheta) = (Y^1, \ldots, Y^6)$ *is a continuous* $6k$*-dimensional limit process which is specified in* (3.49) *and* (3.50) *below.* $\bar{Y}$ *is given by* $\bar{Y} := \sum_{i=1}^{6} Y^i$. *Let* B *be a continuous* $6k$*-dimensional Gaussian martingale on some arbitrary probability space with covariance matrix* $\Lambda = \Lambda(\vartheta)$ *as in* (3.46) *above. Then*

$$Y = B, \quad \langle Y\rangle = \Lambda \quad \textit{if } m^{\vartheta}(S) = 1 \textit{ or if } (3.40) \textit{ holds}, \tag{3.49}$$

$$Y = B \circ W^{\alpha}, \quad \langle Y\rangle = \Lambda \cdot W^{\alpha} \quad \textit{if } Q_{x,\vartheta}(R_2 - R_1 > \cdot) \in RV_{-\alpha}, \quad 0 < \alpha < 1, \tag{3.50}$$

with B *independent of the Mittag–Leffler process* W^{α} *of index* α.

Proof. – (1) The proof of the decomposition of the log-likelihood ratio processes imitates a well-known scheme, see for instance the proof of Theorem 1 in Luschgy [22]. We just give the basic ideas, for the decomposition of $(\Lambda 2)^{h/0}_{n,\vartheta}$ – the other terms are treated analogously.

In a first step, we consider the martingale part $[\frac{\kappa^{\vartheta_n}}{\kappa^{\vartheta}} - 1] * (\mu^B - \nu^{B,\vartheta})$ of $(\Lambda 2)^{h/0}_{n,\vartheta}$. Define

$$\begin{aligned} Y^n_t &:= \int_0^{tn} \int_{S\times R\times M^p} \left[\frac{\kappa^{\vartheta_n}}{\kappa^{\vartheta}} - 1 - (\vartheta_n - \vartheta)^{\mathrm{T}} \dot{\kappa}^{\vartheta}\right](y,x)\big(\mu^B - \nu^{B,\vartheta}\big)(ds, dx, dy, dp) \\ &= \left[\frac{\kappa^{\vartheta_n}}{\kappa^{\vartheta}} - 1\right] * \big(\mu^B - \nu^{B,\vartheta}\big)_{t\cdot n} - h^{\mathrm{T}}(M2)^{n,\vartheta}(t). \end{aligned}$$

Then by assumption D2(ϑ), Y^n belongs to $\mathcal{M}^2_{loc}(Q_{x,\vartheta}, \mathbb{F}^n)$. Lenglart's domination theorem (cf. Jacod and Shiryaev [16, I.3.30]) yields

$$Q_{x,\vartheta}\big(\sup_{s\leqslant t} |Y^n(s)| \geqslant \varepsilon\big) \leqslant \frac{\eta}{\varepsilon} + Q_{x,\vartheta}\big(\langle Y^n\rangle_t \geqslant \eta\big) \tag{3.51}$$

for all $t \geqslant 0$, for all $\varepsilon > 0$ and $\eta > 0$. Using the form of the compensator $\nu^{B,\vartheta}$ of μ^B under $Q_{x,\vartheta}$ as in (2.11) and condition D2(ϑ) we achieve

$$\langle Y^n\rangle_t \leqslant |h|^2 u_n(\vartheta)^2 \int_0^{tn} \varphi_s\big(\kappa^{\vartheta} f^2_{\vartheta}(.,.,|\vartheta_n - \vartheta|)\big)\, ds. \tag{3.52}$$

Thanks to the assumptions on f^2_{ϑ} and due to condition R(ϑ), this last term tends to 0 in probability as $n \to \infty$. Using Lenglart's inequality, this leads to the required convergence

$$\sup_{s\leqslant t} |Y^n(s)| \to 0 \quad \text{in } Q_{x,\vartheta}\text{-probability as } n \to \infty.$$

In a second step it remains to consider terms

$$R^n(t) := \int\limits_0^{tn} \int\limits_{S\times R\times M^p} \left[\log \frac{\kappa^{\vartheta_n}}{\kappa^{\vartheta}} - \frac{\kappa^{\vartheta_n}}{\kappa^{\vartheta}} + 1 + \frac{1}{2}\big(u_n(\vartheta) h^{\mathrm{T}} \dot{\kappa}^{\vartheta}\big)^2\right](y,x)\mu^B(ds,dx,dy,dp).$$

These terms are treated exactly as in Luschgy [22], proof of Theorem 1, using Lenglart's inequality a couple of times.

The other terms corresponding to $i = 3,4,5,6$ are handled analogously. Thus the decomposition of $(\Lambda i)_{n,\vartheta}^{h/0}$ for $i = 2,\dots,6$ follows with $[(Mi)^{n,\vartheta}]$ instead of $\langle (Mi)^{n,\vartheta}\rangle$. We show that for $i = 2,\dots,6$,

$$\sup_{s\leqslant t} h^{\mathrm{T}}\big(\big[(Mi)^{n,\vartheta}\big] - \big\langle (Mi)^{n,\vartheta}\big\rangle\big)(s)h \to 0 \text{ in } Q_{x,\vartheta}\text{-probability} \tag{3.53}$$

which is seen as follows. For arbitrary $\varepsilon > 0$, for $i = 2$, we have

$$\begin{aligned} &h^{\mathrm{T}}\big(\big[(M2)^{n,\vartheta}\big] - \big\langle (M2)^{n,\vartheta}\big\rangle\big)(s)h \\ &\quad = u_n(\vartheta)^2 \int\limits_0^{sn} \int\limits_{S\times R\times M^p} \big(h^{\mathrm{T}}\dot{\kappa}^{\vartheta}\big)^2 1_{\{|u_n(\vartheta)h^{\mathrm{T}}\dot{\kappa}^{\vartheta}|>\varepsilon\}}\, d\mu^B \\ &\qquad - u_n(\vartheta)^2 \int\limits_0^{sn} \int\limits_{S\times R\times M^p} \big(h^{\mathrm{T}}\dot{\kappa}^{\vartheta}\big)^2 1_{\{|u_n(\vartheta)h^{\mathrm{T}}\dot{\kappa}^{\vartheta}|>\varepsilon\}}\, d\nu^{B,\vartheta} + R_{n,\vartheta,h,\varepsilon}(s), \end{aligned}$$

where $R_{n,\vartheta,h,\varepsilon} \in \mathcal{M}_{loc}^2(Q_{x,\vartheta}, \mathbb{F}^n)$ is given by

$$R_{n,\vartheta,h,\varepsilon}(s) = \int\limits_0^{sn} \int\limits_{S\times R\times M^p} \big(u_n(\vartheta)h^{\mathrm{T}}\dot{\kappa}^{\vartheta}\big)^2 1_{\{|u_n(\vartheta)h^{\mathrm{T}}\dot{\kappa}^{\vartheta}|\leqslant\varepsilon\}}\, d\big(\mu^B - \nu^{B,\vartheta}\big).$$

By assumption I(ϑ)

$$u_n(\vartheta)^2 \int\limits_0^{tn} \int\limits_{S\times R\times M^p} \big(h^{\mathrm{T}}\dot{\kappa}^{\vartheta}\big)^2 1_{\{|u_n(\vartheta)h^{\mathrm{T}}\dot{\kappa}^{\vartheta}|>\varepsilon\}}\, d\nu^{B,\vartheta}$$

and as a consequence also

$$u_n(\vartheta)^2 \int\limits_0^{tn} \int\limits_{S\times R\times M^p} \big(h^{\mathrm{T}}\dot{\kappa}^{\vartheta}\big)^2 1_{\{|u_n(\vartheta)h^{\mathrm{T}}\dot{\kappa}^{\vartheta}|>\varepsilon\}}\, d\mu^B$$

tend to 0 in $Q_{x,\vartheta}$-probability as $n \to \infty$. Moreover,

$$\langle R_{n,\vartheta,h,\varepsilon}\rangle(t) \leqslant \varepsilon^2 \int\limits_0^{tn} \int\limits_{S\times R\times M^p} \big(u_n(\vartheta)h^{\mathrm{T}}\dot{\kappa}^{\vartheta}\big)^2 d\nu^{B,\vartheta}$$

and

$$\int_0^{tn}\int_{S\times R\times M^p}\left(u_n(\vartheta)h^{\mathrm{T}}\dot{\kappa}^\vartheta\right)^2 d\nu^{B,\vartheta}=h^{\mathrm{T}}\left(u_n(\vartheta)^2\int_0^{tn}\varphi_s\left((I2)^\vartheta\right)ds\right)h$$

which in case of $m^\vartheta(S)=\infty$ tends to $[\int_{R\times S}(I2)^\vartheta(y,z)\bar{m}^\vartheta(dy,dz)]\cdot W_t^\alpha$ and in case $m^\vartheta(S)=1$ to $[\int_{R\times S}(I2)^\vartheta(y,z)\bar{m}^\vartheta(dy,dz)]\cdot t$. Hence $\langle R_{n,\vartheta,h,\varepsilon}\rangle(t)$ is bounded in $Q_{x,\vartheta}$-probability, and since ε may be chosen arbitrarily small, this leads to (3.53).

The decomposition of $(\Lambda 1)_{n,\vartheta}^{h/0}$ is shown analogously.

(2) It remains to prove the joint weak convergence of the rescaled score function martingales together with their angle bracket processes. Since $(M1)^\vartheta$ is a continuous martingale and $(M2)^\vartheta$–$(M6)^\vartheta$ are purely discontinuous martingales,

$$\left\langle\left((M1)^{n,\vartheta}\right)^l,\left((Mj)^{n,\vartheta}\right)^m\right\rangle\equiv 0$$

for all $1\leqslant l,m\leqslant k$ and for all $j=2,\dots,6$. $(Mi)^{n,\vartheta}$ and $(Mj)^{n,\vartheta}$ for $i=2,3,4$ and $j=5,6$ do not have any common jumps. Hence,

$$\left[\left((Mi)^{n,\vartheta}\right)^l,\left((Mj)^{n,\vartheta}\right)^m\right]\equiv 0$$

for all $1\leqslant l,m\leqslant k$, for all $i=2,3,4$ and $j=5,6$, and therefore also

$$\left\langle\left((Mi)^{n,\vartheta}\right)^l,\left((Mj)^{n,\vartheta}\right)^m\right\rangle\equiv 0$$

for all $1\leqslant l,m\leqslant k$, for all $i=2,3,4$ and $j=5,6$. Furthermore, for $i=2,3,4$ and $j=2,3,4$, $i\neq j$,

$$\langle(Mi)^{n,\vartheta},(Mj)^{n,\vartheta}\rangle_t=u_n(\vartheta)^2\int_{R\times S}(Iij)^\vartheta(y,z)\eta_{nt}(dy,dz),$$

(see (2.13) for the notation) and for $i=5,6$, $j=5,6$ with $j\neq i$,

$$\left\langle(Mi)^{n,\vartheta},(Mj)^{n,\vartheta}\right\rangle_t=u_n(\vartheta)^2\int_S(I56)^\vartheta(x)m_{nt}(dx),$$

where

$$(I23)^\vartheta(y,z):=\left(\sum_{n\geqslant 0}F^\vartheta(y,z,\{n\})\left[\dot{F}^\vartheta(y,z,n)\left(\dot{\kappa}^\vartheta\right)^{\mathrm{T}}(y,z)\right]\right)\kappa^\vartheta(y,z),\tag{3.54}$$

$$\begin{aligned}(I24)^\vartheta(y,z):=&\left(\sum_{n\geqslant 0}F^\vartheta(y,z,\{n\})\right.\\&\left.\times\left[\int_{R^n}\bigotimes_{k=1}^n\pi^\vartheta\left(y,z,dp^k\right)\dot{p}^\vartheta(y,z,p)\right]\left(\dot{\kappa}^\vartheta\right)^{\mathrm{T}}(y,z)\right)\kappa^\vartheta(y,z),\end{aligned}\tag{3.55}$$

$$(I34)^{\vartheta}(y,z) := \Bigg(\sum_{n\geqslant 0} F^{\vartheta}(y,z,\{n\}) \\ \times \Bigg[\int_{R^n} \bigotimes_{k=1}^{n} \pi^{\vartheta}(y,z,dp^k)\,\dot{p}^{\vartheta}(y,z,p)\Bigg]\big(\dot{F}^{\vartheta}\big)^{\mathrm{T}}(y,z,n)\Bigg)\kappa^{\vartheta}(y,z) \tag{3.56}$$

and

$$(I56)^{\vartheta}(x) := \Bigg(\int_R \nu^{\vartheta}(x,dy)\dot{q}^{\vartheta}(x,y)\Bigg)\big(\dot{c}^{\vartheta}\big)^{\mathrm{T}}(x)c^{\vartheta}(x). \tag{3.57}$$

Due to our differentiability assumptions Di(ϑ), all these terms (3.54) to (3.57) vanish. Consider for example $(I23)^{\vartheta}$ and let $\xi_n \to \vartheta$ such that $\xi_n - \vartheta = \delta_n e,\ \delta_n \downarrow 0$, for some arbitrary $e \in \mathbb{R}^k$ with $|e| = 1$. Then we get for $\tilde{e} \in \mathbb{R}^k$ arbitrary,

$$e^{\mathrm{T}}(I23)^{\vartheta}(y,z)\tilde{e} = \frac{1}{\delta_n}\Bigg(\sum_{n\geqslant 0} F^{\vartheta}(y,z,\{n\})\Bigg[\delta_n e^{\mathrm{T}}\dot{F}^{\vartheta}(y,z,n) - \frac{F^{\xi_n}}{F^{\vartheta}}(y,z,\{n\}) + 1\Bigg]\Bigg) \\ \times \big(\dot{\kappa}^{\vartheta}\big)^{\mathrm{T}}(y,z)\kappa^{\vartheta}(y,z)\tilde{e}$$

for all $y \in R$ and $z \in S$. By Jensen's inequality and assumption D3(ϑ),

$$\sum_{n\geqslant 0} F^{\vartheta}(y,z,\{n\})\Bigg[\delta_n e^{\mathrm{T}}\dot{F}^{\vartheta}(y,z,n) - \frac{F^{\xi_n}}{F^{\vartheta}}(y,z,\{n\}) + 1\Bigg] \\ \leqslant \Bigg(\sum_{n\geqslant 0} F^{\vartheta}(y,z,\{n\})\Bigg[\frac{F^{\xi_n}}{F^{\vartheta}}(y,z,\{n\}) - 1 - \delta_n e^{\mathrm{T}}\dot{F}^{\vartheta}(y,z,n)\Bigg]^2\Bigg)^{1/2} \\ \leqslant \sqrt{f_{\vartheta}^3(y,z,\delta_n)}\cdot\delta_n.$$

Here, $\sqrt{f_{\vartheta}^3(y,z,\delta_n)} \to 0$ as $n \to \infty$. Hence we have $e^{\mathrm{T}}(I23)^{\vartheta}(y,z)\tilde{e} \equiv 0$ for all e, $\tilde{e} \in \mathbb{R}^k$ which means $(I23)^{\vartheta}(y,z) \equiv 0$. The arguments for the other covariation terms are analogous.

Then the joint weak convergence of the rescaled score function martingales together with their angle brackets follows from Theorem 3.2 above of Touati: Note that by conditions I(ϑ) and R(ϑ), all martingales $(Mi)^{n,\vartheta}$ satisfy a Lindeberg condition

$$\frac{1}{u_n(\vartheta)}\int_0^{tn}\int |x|^2 1_{\{|x|>\varepsilon\sqrt{u_n(\vartheta)}\}}\nu^{(i)}(ds,dx) \to 0 \quad \text{in probability}$$

as $n \to \infty$, for all t and $\varepsilon > 0$, where $\nu^{(i)}(ds,dx)$ denotes the compensator of jumps of $(Mi)^{\vartheta}$. Then by Jacod and Shiryaev [16], VI.3.26, IX.1.19 and VI.6.1, convergence of $(Mi)^{n,\vartheta}$ to the limit process Y^i implies convergence of the pair $((Mi)^{n,\vartheta}, \langle (Mi)^{n,\vartheta}\rangle)$ to $(Y^i, \langle Y^i\rangle)$ and thus the assertion.

(3) The decomposition of $\Lambda_{n,\vartheta}^{h/0}$ follows from the decomposition of the $(\Lambda i)_{n,\vartheta}^{h/0}$ for $1 \leqslant i \leqslant 6$ since $\langle \bar{M}^{n,\vartheta}\rangle = \sum_{i=1}^{6}\langle (Mi)^{n,\vartheta}\rangle$, as $\langle (Mi)^{n,\vartheta}, (Mj)^{n,\vartheta}\rangle \equiv 0$ for $i \neq j$ which was shown above. This completes the proof of the theorem. □

Example 3.8. – We continue our Example 3.4. Suppose for simplicity, that the fixed threshold k_0 of 3.4. equals 0. We present a location-scale model for the family $\{\kappa^\xi\}$. Choose some fixed function $\kappa_0:\mathbb{R}_+\to[\frac{1}{2},1]$ which is continuously derivable, such that $\dot\kappa_0$ has compact support, $\dot\kappa_0$ being the derivative of κ_0. We define for $\xi=(\xi^1,\xi^2)\in(0,\infty)\times(0,\infty)$

$$\kappa^\xi(k):=\kappa_0\big(\xi^1+k\xi^2\big).$$

Then the integrability assumption I(ϑ) is clearly satisfied, and Theorem 3.7 is applicable since by 3.4 R(ϑ) also holds. Roughly speaking, null-recurrent situations are treatable in cases where the dependency on the parameter is restricted to small sets such that the integrability condition I(ϑ) holds and such that we dispose nevertheless of the limit theorems which we need.

Remark 3.9. – (1) In case of ergodicity (see for instance 3.3(1)), condition I(ϑ) is not restrictive since in this case m^ϑ is finite.

(2) In situations as in 3.3(2), condition I(ϑ) will *never* be fulfilled, since in this case all error-bound terms and all information terms depend only on the positions of particles, not on the whole configuration. As a consequence, the measure which is involved in condition I(ϑ) is the measure $\bar m^\vartheta(\cdot\times S)$ which – as pointed out in 3.5 – is identically equal to infinity. Hence it does not make sense in this situation to try to obtain locally quadratic approximations of likelihoods with renormalisation factors coming from a life-cycle decomposition of φ.

Remark 3.10. – Suppose we have LAN(ϑ) or LAMN(ϑ). As a well-known consequence of LAN(ϑ) or LAMN(ϑ), we are able to characterise efficiency of estimators for the unknown parameter ϑ. Consider estimators $(g_n)_{n\geqslant1}$ which are regular in the sense of Hájek (see Hájek [8]), i.e. satisfying

$$\forall h\colon\ \mathcal{L}\big(u_n(\vartheta)^{-1}\big(g_n-(\vartheta+u_n(\vartheta)h)\big)\mid Q_{x,\vartheta+u_n(\vartheta)h}\big)\xrightarrow{w}:F$$

where F is independent of h. (These estimators are approximately equivariant in small neighbourhoods of ϑ.)

Hájek [8] shows that under LAN(ϑ) necessarily $F=N(0,J^{-1})*Q$ for some probability measure Q, with J as in (3.47). Under LAMN(ϑ), Jeganathan [17] proves for sub-convex loss functions $l(.)$ that

$$\int l(z)F(dz)\geqslant\int\int P^{\langle\bar Y\rangle_1}(di)N\big(0,i^{-1}\big)(dv)l(v),$$

where $P^{\langle\bar Y\rangle_1}$ is the distribution of $\langle\bar Y\rangle_1$, with $\bar Y$ as in Theorem 3.7. This justifies calling an estimator g_n *regular* and *efficient at* ϑ, if $F=\mathcal{L}(Z\mid P_0)$ where $Z=\langle\bar Y\rangle_1^{-1}\bar Y_1$ is the maximum likelihood estimator in the limit model.

4. Some remarks on explosion for branching diffusions

4.1. The case without interactions

In a first step, we consider branching diffusions without interactions and immigrations: In case $d = 1$, suppose that all particles move and evolve independently, that the immigration rate c fulfils $c(.) \equiv 0$ (no immigrations) and that Assumption 2.8(b) holds. In the sequel, x will always denote a point in $\mathbb{R}$. Consider a filtered probability space $(\widetilde{\Omega}, (\mathcal{G}_t)_t, \mathcal{F}, P_x)$ satisfying the usual conditions and a $(P_x, (\mathcal{G}_t)_t)$-Brownian motion W, one-dimensional. Let $(\mathcal{G}_t)_t$ be the filtration generated by W and consider the diffusion process

$$X_t = x + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$$

of (2.2) driving one-particle motions of the branching diffusion process. Then, conditioning on the first branching event in the particle process, with

$$u(t,x) := Q_x^{b,\sigma,\kappa,0,F,\pi,\nu}(T_\infty \leqslant t),$$

we arrive at the expansion

$$u(t,x) = \int_0^t E_x\Big(\mathrm{e}^{-\int_0^s \kappa(X_u)\,du}\kappa(X_s)g\big(X_s, u(t-s, X_s)\big)\Big)\,ds, \tag{4.58}$$

where

$$g(x,s) := 1 - \sum_{k=0}^{\infty} F(x,\{k\})(1-s)^k \quad \text{for } |s| \leqslant 1.$$

In the following, we use techniques from the theory of backward stochastic differential equations (see for example El Karoui, Peng and Quenez [5] or Pardoux and Peng [25]) in order to be able to handle the Feynman–Kac type formula (4.58). We fix some $T > 0$ and put

$$Y_t := u(T-t, X_t) \quad \text{for } 0 \leqslant t \leqslant T,$$
$$e_t := \mathrm{e}^{-\int_0^t \kappa(X_u)\,du}, \qquad A_t := \kappa(X_t)g(X_t, Y_t).$$

Then the Markov property gives

$$\bar{Y}_t := e_t Y_t = \int_t^T E_x(e_s A_s \mid \mathcal{G}_t)\,ds.$$

Let $B_t := e_t A_t$. In a first step, suppose that $\kappa(.)$ is bounded. Then $\xi := \int_0^T B_s\,ds \in L^2(P_x, \mathcal{G}_T)$, hence by the representation theorem, $\xi = E_x(\xi) + \int_0^T Z_s\,dW_s$. Taking

$V_t := \xi - \int_0^t B_s\,ds$, we have

$$\int_t^T B_s\,ds = V_t = E_x(\xi) + \int_0^T Z_s\,dW_s - \int_0^t B_s\,ds$$

and

$$0 = V_T = E_x(\xi) + \int_0^T Z_s\,dW_s - \int_0^T B_s\,ds.$$

Thus we get for $\bar{Y}_t = E_x(V_t \mid \mathcal{G}_t)$

$$\bar{Y}_t = E_x(V_t \mid \mathcal{G}_t) - V_T = -\int_t^T Z_s\,dW_s + \int_t^T B_s\,ds.$$

Since $Y_t = (e_t)^{-1}\bar{Y}_t$ and $d(e_t)^{-1} = \kappa(X_t)(e_t)^{-1}\,dt$,

$$\begin{aligned} 0 = Y_T &= Y_t + \int_t^T \bar{Y}_s\,d(e_s)^{-1} + \int_t^T (e_s)^{-1}\,d\bar{Y}_s \\ &= Y_t + \int_t^T Y_s\kappa(X_s)\,ds - \int_t^T A_s\,ds + M_t, \end{aligned}$$

where M is a martingale. Taking conditional expectations, we thus have

$$Y_t = E_x\left(\left\{\int_t^T \kappa(X_s)\big(g(X_s, Y_s) - Y_s\big)\,ds\right\} \mid \mathcal{G}_t\right). \tag{4.59}$$

If $\kappa(.)$ is unbounded, take R such that $|x| \leqslant R/2$, write $\tau_R := \inf\{t\colon |X_t| \geqslant R\}$ and use a representation theorem for $\xi^R := \int_0^T B_{s\wedge\tau_R}\,ds \in L^2(P_x, \mathcal{G}_T)$. Then following the same arguments as above, one arrives at a backward representation for $Y_t^R := (e_{t\wedge\tau_R})^{-1}\,\bar{Y}_t^R$, with $\bar{Y}_t^R := \int_t^T E_x(B_{s\wedge\tau_R}\,ds \mid \mathcal{G}_t)$, and $R \to \infty$ gives the assertion (4.59).

THEOREM 4.1. – *Suppose that*
- $|\kappa(x)| \leqslant C(1 + |x|^p)$ *for* $p \leqslant 2$.
- $|g(x,u) - g(x,u')| \leqslant L|u - u'|$ *for all* $x \in \mathbb{R}$.
- $\|b\|_\infty < \infty$, $\|\sigma\|_\infty < \infty$.

We consider only solutions Y *of* (4.59) *with* $E_x(|Y_\tau|^2) < \infty$ *for all stopping times* $\tau \leqslant T$. *Then we have unicity for* (4.59) *in case* $p < 2$. *In case* $p = 2$, *unicity holds if additionally* $T < (\sqrt{8C(L+1)}\|\sigma\|_\infty)^{-1}$.

Proof. –
(1) In a first step we show that any solution Y of (4.59) satisfies

$$E_x\big(|Y_{T\wedge\tau_R}|^2\big) \leqslant 3, \tag{4.60}$$

if R is sufficiently large.

Write for short $f(s,x,u) := \kappa(x)(g(x,u)-u)$. Then (4.59) implies

$$Y_{t\wedge\tau_R} = E_x\left(Y_{T\wedge\tau_R} + \int_{t\wedge\tau_R}^{T\wedge\tau_R} f(s,X_s,Y_s)\,ds \mid \mathcal{G}_{t\wedge\tau_R}\right).$$

Thus with $A := \{|X_0| \leqslant R/2\}$,

$$1_A|Y_{t\wedge\tau_R}| \leqslant E_x\left(\left\{1_A|Y_{T\wedge\tau_R}| + \int_t^T |f(s\wedge\tau_R, X_{s\wedge\tau_R}, Y_{s\wedge\tau_R})|1_A\,ds\right\} \mid \mathcal{G}_{t\wedge\tau_R}\right). \quad (4.61)$$

Now

$$|f(s\wedge\tau_R, X_{s\wedge\tau_R}, Y_{s\wedge\tau_R})|1_A \leqslant C\big(1+R^p\big)(1+|Y_{s\wedge\tau_R}|1_A)$$

since $g(.,.)$ is bounded by 1. Moreover, $|Y_{T\wedge\tau_R}|1_A = (|Y_{T\wedge\tau_R}|1_A)(1_A\cdot 1_{\{\tau_R\leqslant T\}})$. Hence

$$E_x\big(|Y_{T\wedge\tau_R}|\,1_A \mid \mathcal{G}_{t\wedge\tau_R}\big) \leqslant E_x\big(|Y_{T\wedge\tau_R}|^2 1_A \mid \mathcal{G}_{t\wedge\tau_R}\big)^{1/2}\, E_x\big(1_A 1_{\{\tau_R\leqslant T\}} \mid \mathcal{G}_{t\wedge\tau_R}\big)^{1/2}$$

and thus for R sufficiently large

$$\begin{aligned} E_x\big(E_x\big(|Y_{T\wedge\tau_R}|1_A \mid \mathcal{G}_{t\wedge\tau_R}\big)^2\big) &\leqslant E_x\big(|Y_{T\wedge\tau_R}|^2\big)^{1/2} P_x(\tau_R\leqslant T)^{1/2} \\ &\leqslant \sqrt{2}\mathrm{e}^{-C_T R^2}\cdot\big(1+E_x\big(|Y_{T\wedge\tau_R}|^2\big)\big) \end{aligned}$$

since $\sqrt{x}\leqslant 1+x$. To obtain $P_x(\tau_R\leqslant T)\leqslant 2\mathrm{e}^{-2C_T R^2}$, note that for every R such that $R/2 \geqslant ||b||_\infty T + x$ the following holds:

$$P_x(\tau_R\leqslant T) = P_x\big(\sup_{t\leqslant T}|X_t|\geqslant R\big) \leqslant P_x\left(\sup_{t\leqslant T}\left|\int_0^t \sigma(X_s)\,dW_s\right| \geqslant \frac{R}{2}\right).$$

Write $M_t := \int_0^t \sigma(X_s)\,dW_s$, then $\langle M\rangle_t \leqslant \|\sigma\|_\infty^2\cdot t$, hence by an exponential inequality

$$P_x\left(\sup_{t\leqslant T}|M_t| \geqslant \frac{R}{2}\right) \leqslant 2\exp\left(-\frac{R^2}{4T\|\sigma\|_\infty^2}\right)$$

and thus the assertion with $C_T := 1/(8T\|\sigma\|_\infty^2)$.

As a consequence of the previous considerations, coming back to Eq. (4.61), we arrive at

$$E_x\big(|Y_{t\wedge\tau_R}|^2\big) \leqslant \sqrt{2}\mathrm{e}^{-C_T R^2}\cdot\big(1+E_x\big(|Y_{T\wedge\tau_R}|^2\big)\big) + C^2\big(1+R^p\big)^2\int_t^T \big(1+E_x(|Y_{s\wedge\tau_R}|^2)\big)\,ds$$

and hence by Gronwall's inequality, with $f(t) := E_x(|Y_{t\wedge\tau_R}|^2)$,

$$f(t) \leqslant \big(\big[\sqrt{2}\mathrm{e}^{-C_T R^2}\big(1+f(T)\big)\big]+1\big)\mathrm{e}^{C^2(1+R^p)^2(T-t)}. \quad (4.62)$$

Now for $t \to T$ we obtain

$$f(T) \leqslant \sqrt{2}\mathrm{e}^{-C_T R^2} + \sqrt{2}\mathrm{e}^{-C_T R^2} f(T) + 1.$$

If R is sufficiently large, $\sqrt{2}\mathrm{e}^{-C_T R^2} \leqslant 1/2$, and hence

$$E_x\left(|Y_{T\wedge\tau_R}|^2\right) \leqslant 3,$$

which is (4.60).

(2) Now let $(Y_t)_{t\leqslant T}$ and $(\bar{Y}_t)_{t\leqslant T}$ be two solutions of (4.59), $\Delta_t := Y_t - \bar{Y}_t$ and $\Delta f_s := f(s, X_s, Y_s) - f(s, X_s, \bar{Y}_s)$. Then

$$\Delta_{t\wedge\tau_R} = E_x\left(\Delta_{T\wedge\tau_R} + \int_{t\wedge\tau_R}^{T\wedge\tau_R} \Delta f_{s\wedge\tau_R}\, ds \mid \mathcal{G}_{t\wedge\tau_R}\right).$$

Since $|\Delta f_{s\wedge\tau_R}| \leqslant C(1+R^p)\cdot(L+1)\cdot|\Delta_{s\wedge\tau_R}|$, we have with $A := \{|X_0| \leqslant R/2\}$

$$1_A|\Delta_{t\wedge\tau_R}| \leqslant E_x\left(1_A|\Delta_{T\wedge\tau_R}| \mid \mathcal{G}_{t\wedge\tau_R}\right) + C(L+1)\left(1+R^p\right)\int_t^T E_x\left(1_A|\Delta_{s\wedge\tau_R}| \mid \mathcal{G}_{t\wedge\tau_R}\right) ds.$$

Thus Gronwall's inequality yields

$$E_x(|\Delta_{t\wedge\tau_R}|) \leqslant E_x(|\Delta_{T\wedge\tau_R}|)\cdot \mathrm{e}^{\alpha(1+R^p)(T-t)} \tag{4.63}$$

with $\alpha := C(L+1)$. Note that $\Delta_{T\wedge\tau_R} = 0$ if $\tau_R \geqslant T$. As a consequence,

$$\begin{aligned} E_x(|\Delta_{T\wedge\tau_R}|) &= E_x\left(|\Delta_{T\wedge\tau_R}| 1_{\{\tau_R\leqslant T\}}\right) \leqslant E_x\left(|Y_{T\wedge\tau_R}|^2 + |\bar{Y}_{T\wedge\tau_R}|^2\right)^{1/2}\cdot P_x(\tau_R \leqslant T)^{1/2} \\ &\leqslant \sqrt{6}\cdot\sqrt{2}\mathrm{e}^{-C_T R^2} \end{aligned}$$

by (4.60) and (1). This yields

$$E_x(|\Delta_{t\wedge\tau_R}|) \leqslant c_T\cdot \mathrm{e}^{-C_T R^2 + C(L+1)(T-t)R^p}$$

for some constant c_T depending on T. If $p < 2$, then $e^{-C_T R^2 + C(L+1)(T-t)R^p}$ clearly tends to 0 as $R \to \infty$. If $p = 2$, then $\mathrm{e}^{-C_T R^2 + C(L+1)(T-t)R^2}$ tends to 0 if $C_T - C(L+1)\times(T-t) > 0$ for all $t \leqslant T$ which is true if $T < 1/(\sqrt{8C(L+1)}\|\sigma\|_\infty)$. This finishes the proof. □

COROLLARY 4.2. – *Consider a branching diffusion without interactions of particles, with immigrations at bounded rate c, fulfilling Assumption* 2.8(b) *and all conditions of Theorem* 4.1. *Then the branching diffusion does not explode*: $Q_x^{b,\sigma,\kappa,c,F,\pi,\nu}(T_\infty < \infty) = 0$ *for all initial configurations $x \in S$.*

Proof. – (1) Consider first a branching diffusion with $c(.) \equiv 0$. Then by Theorem 4.1, for T sufficiently small, for all $x \in S \cap \mathbb{R}$, $Q_x^{b,\sigma,\kappa,0,F,\pi,\nu}(T_\infty \leqslant T) = u(T,x) = 0$ (note that $Y_t \equiv 0,\ 0 \leqslant t \leqslant T$, is always a solution of (4.59)). Using the Markov property, we thus arrive at $Q_x^{b,\sigma,\kappa,0,F,\pi,\nu}(T_\infty \leqslant t) = 0$ for all t, hence the assertion for every $x \in \mathbb{R}$. Now for $x = (x^1, \ldots, x^l) \in S$, by the independence assumptions,

$$Q_x^{b,\sigma,\kappa,0,F,\pi,\nu}(T_\infty < \infty) = 1 - \prod_{k=1}^{l} Q_{x^i}^{b,\sigma,\kappa,0,F,\pi,\nu}(T_\infty = \infty) = 0.$$

(2) A branching diffusion φ with bounded immigration rate $c \leqslant \|c\|_\infty$ can be constructed in a coupled way with a branching diffusion $\bar{\varphi}$ with immigrations at constant rate $\|c\|_\infty$ such that φ is a subprocess of $\bar{\varphi}$. It is clear, that $\bar{\varphi}$ cannot explode in finite time by (1), hence the same holds for φ. $\square$

Remark 4.3. – The relation of explosion problems for branching processes and non-linear partial differential equations is well-known, see for instance Ikeda and Watanabe [15] for treatment of explosion problems for branching Brownian motion. In a sense, our Theorem 4.1 is not really far from explosion problems for branching Brownian motion since drift and diffusion coefficients are supposed to be bounded. However, we give the proof of Theorem 4.1 in detail since the theory of partial differential equation is not involved at all – the arguments we are using are purely probabilistic ones.

4.2. The case with interactions

In situations with interactions, methods as in Section 4.1 (conditioning on first branching events etc.) are no more helpful, and the only methods we can use are coupling methods. We recall a result of Löcherbach [21], Proposition 5.13.

PROPOSITION 4.4. – *Suppose that* $0 < a \leqslant \kappa(.,.) \leqslant b < \infty$, *that* $c(.) \leqslant d < \infty$ *for some constants* a, b, d, *and that the family of reproduction laws* $F(x^i, x, .)$ *admits as upper bound* (*in the sense of convolution of probability measures*) *some law* $\bar{F}$ *with finite mean offspring number*:

$$\bar{F} \text{ is a probability measure on } \mathbb{N}_0 \setminus \{1\}, \quad \sum_{k \neq 1} k\,\bar{F}(\{k\}) < \infty,$$

and $F(x^i, x, .) * G(x^i, x, .) = \bar{F}(.)$ *for all* $(x^i, x) \in R \times S$ *for some kernel* G *from* $(R \times S, \mathcal{B}(R \times S))$ *to* $\mathbb{N}_0$. *Then* $Q_x^{b,\sigma,\kappa,c,F,\pi,\nu}(T_\infty < \infty) = 0$.

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