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CLARK–OCONE FORMULAS AND POINCARÉ INEQUALITIES ON THE DISCRETE CUBE

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ABSTRACT. – We establish Poincaré inequalities for the continuous time random walk on the cube $\{-1,+1\}^d$. A first method is based on the study of cylindrical functionals. A Poincaré inequality is proved for these functionals and extended to arbitrary functionals. A second method is based on martingale representation formulas. A whole family of Clark–Ocone formulas is then available, which leads to the corresponding family of Poincaré inequalities. These various inequalities are compared through examples. We also show that the cylindrical method extends to some asymmetric continous time random walks on $\{-1,+1\}^d$. © 2001 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Nous établissons des inégalités de Poincaré pour la marche aléatoire à temps continu sur le cube $\{-1,+1\}^d$. Une première méthode consiste à étudier en premier lieu les fonctionnelles cylindriques. La seconde méthode exploite des formules de représentation par martingales, dites formules de Clark–Ocone. De telles formules sont mises en évidence et conduisent à toute une famille d'inégalités de Poincaré. Nous comparons ces inégalités par des exemples. Nous montrons par ailleurs que la méthode cylindrique s'applique à certaines marches aléatoires dissymétriques à temps continu sur le cube $\{-1,+1\}^d$. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

The main purpose of this work is to establish Clark–Ocone formulas and Poincaré inequalities for the continuous time random walk $(B_t)_{t\geqslant 0}$ on the discrete cube $\{-1,+1\}^d$. To explain in this introduction some of the motivations and aspects of this work, let us consider $d=1$ for simplicity and give a short description of the random walk. The law of $(B_t)_{t\geqslant 0}$ is given by its infinitesimal generator $\mathbf{L}$

$$\mathbf{L}(f) = (f\circ\tau - f),$$

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where $\tau(x) = -x$. The process starts at B_0, waits an exponential time T_1 of parameter 1, then jumps to its neighbour $-B_0 = B_{T_1}$, and so on, the waiting times $(T_n - T_{n-1})_{n\geqslant 1}$ being mutually independent. For a fixed time $t \geqslant 0$, the law of B_t is just an asymmetric Bernoulli measure with weights $(1 \pm \mathrm{e}^{-2t})/2$.

Our first motivation to find a Clark–Ocone formula was Poincaré inequalities on the path space. Given a cylindrical functional $F = f(B_t)$, it is easy to see that

$$\mathbb{E}\big(F^2\big) - \mathbb{E}\big(F\big)^2 \leqslant \frac{1}{4}\big(1 - \mathrm{e}^{-4t}\big)\mathbb{E}\big((DF)^2\big), \tag{1}$$

where the discrete derivative DF is the cylindrical functional $(f \circ \tau - f)(B_t)$. As was shown in [1], there are several manners to extend this Poincaré inequality to general cylindrical functionals. One of them is the following. If $F(B) = f(B_{t_1}, \ldots, B_{t_n})$ with $0 = t_0 < t_1 < \cdots < t_n$, then

$$\begin{aligned}\mathbb{E}\big(F^2\big) - \mathbb{E}(F)^2 &\leqslant \mathcal{E}_{\text{cyl}}(F) \\ &= \frac{1}{4}\sum_{i=1}^{n}\big(1 - \mathrm{e}^{-4(t_i - t_{i-1})}\big)\mathbb{E}\left(\left(\sum_{k=i}^{n} \mathrm{e}^{-2(t_k - t_i)}\,\widetilde{D}_k F \circ \tau_{i,k-1}\right)^2\right),\end{aligned} \tag{2}$$

where the discrete derivatives of F are defined by

$$\big(\widetilde{D}_k F \circ \tau_{i,k-1}\big)(B) = \big(\widetilde{D}_k f\big)(B_{t_1}, \ldots, B_{t_{i-1}}, \tau B_{t_i}, \ldots, \tau B_{t_{k-1}}, B_{t_k}, \ldots, B_{t_N}), \tag{3}$$

and

$$\begin{aligned}\big(\widetilde{D}_k f\big)(x_1, \ldots, x_N) &= x_k\big(f(x_1, \ldots, \tau x_k, \ldots, x_N) - f(x_1, \ldots, x_N)\big) \\ &= \big(f(x_1, \ldots, x_{k-1}, -1, x_{k+1}, \ldots, x_N) \\ &\quad - f(x_1, \ldots, x_{k-1}, +1, x_{k+1} \ldots, x_N)\big).\end{aligned}$$

At this stage, it is tempting to draw a parallel with the Ornstein–Uhlenbeck process $(X_t)_{t\geqslant 0}$ on $\mathbb{R}$, because in that case, for a cylindrical functional $F = f(X_{t_1}, \ldots, X_{t_n})$ with a smooth f on $\mathbb{R}^n$, we have that

$$\mathbb{E}\big(F^2\big) - \mathbb{E}(F)^2 \leqslant \sum_{i=1}^{n}\big(1 - \mathrm{e}^{-2(t_i - t_{i-1})}\big)\mathbb{E}\left(\left(\sum_{k=i}^{n} \mathrm{e}^{-(t_k - t_i)}\,\partial_k f(X_{t_1}, \ldots, X_{t_n})\right)^2\right). \tag{4}$$

Actually, this inequality is a particular case of the following theorem, which deals with functionals of the whole path.

THEOREM 1.1. – *Let $(\mathcal{F}_t)_{t\geqslant 0}$ be the filtration associated to the Ornstein–Uhlenbeck process $(X_t)_{t\geqslant 0}$. Let $T > 0$ and $F(X)$ be an $\mathcal{F}_T$-measurable and $\mathbf{L}^2$ functional in the domain of the Malliavin gradient operator, and let DF be its Malliavin derivative. Then*

$$\mathbb{E}\big(F\big)^2 - \mathbb{E}(F)^2 \leqslant \mathbb{E}\int_0^T\left(\big(\dot{\widehat{DF}}\big)_t - \int_t^T \mathrm{e}^{-(s-t)}\big(\dot{\widehat{DF}}\big)_s\,ds\right)^2 2\,dt. \tag{5}$$

A simple way to prove this theorem [3] is to use the Clark–Ocone formula

$$F - \mathbb{E}(F) = \int_0^T H_t \sqrt{2}\, d\omega_t,$$

where H is the predictable projection of $(\widehat{\dot{D}F})_t - \int_t^T \mathrm{e}^{-(s-t)} (\widehat{\dot{D}F})_s\, ds$. The coefficient $\mathrm{e}^{-(t-s)}$ reflects here a constant strictly positive curvature, as do the coefficients $\mathrm{e}^{-(t_k - t_i)}$ in (4). This representation formula is well known in general for diffusion processes on Riemannian manifolds, and may be obtained from the Bismut integration by part formula (see [6] for example).

If we come back to the case of the cube, we may wonder whether there exists or not a Poincaré inequality such as (5) on the cube, and for what class of functionals.

A first method is to start from inequality (2) with cylindrical functionals and to identify the energy $\mathcal{E}_{\mathrm{cyl}}(F)$ with an energy that may be defined for all functionals, under some integrability conditions. It is shown in Section 3 that

$$\mathcal{E}_{\mathrm{cyl}}(F) = \mathbb{E} \int_0^T \left(D_t F - \int_t^T 2\mathrm{e}^{-2(s-t)} (-1)^{N(]t,s[)} \big(D_s F \circ \varepsilon_t^+ \big) \widetilde{N}(ds) \right)^2 dt, \tag{6}$$

where $D_t F$ is the gradient in Poisson spaces, N the Poisson point process of jumps, and $\widetilde{N}$ the associated compound Poisson measure. Precise notations and definitions will be given in Section 2. It then follows from the Poincaré inequality (2) and equality (6) that

$$\mathbb{E}\big(F^2\big) - \mathbb{E}(F)^2 \leqslant \mathcal{E}_{\mathrm{cyl}}(F) \tag{7}$$

for all cylindrical functionals F, thus allowing extensions to a wide class of functionals.

According to the comments following Theorem 1.1, another method to investigate Poincaré inequalities on the cube is to look for a Clark–Ocone formula. The martingale representation formula

$$F - \mathbb{E}(F) = \int_0^T H_t \widetilde{N}(dt), \tag{8}$$

where H is the predictable projection of DF was proved by Picard in [9]. This identity is a consequence of an isometry formula which plays the role of the integration by parts formula for diffusions. However, the representation (8) seems to reflect a zero curvature, and therefore does not compare to the Ornstein–Uhlenbeck process, whose curvature is constant and equal to 1. In Section 4 we show that actually a whole family of formulas such as (8) is available. Namely, we show that for every $\alpha \in \mathbb{R}$ and every $\mathcal{F}_T$-measurable functional F satisfying some integrability conditions,

$$F - \mathbb{E}(F) = \int_0^T H_t^\alpha \widetilde{N}(dt), \tag{9}$$

were H^α is the predictable projection of $D_tF + \alpha\int_t^T 2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s[)}\cdot(D_sF\circ\varepsilon_t^+)\widetilde{N}(ds)$.

It turns out that the processes H^α are all equal. This behaviour differs from those of diffusions, where α stands for curvature. Our motivation for this work was actually an attempt to understand Bismut formulas and curvature in the discrete setting. As suggested by Remark 4.2 below, the Poincaré inequality (7) coming from the cylindrical method would make us think of a constant curvature equal to 4, whereas the Clark–Ocone formula of parameter α makes it confused.

Introducing the parameter α seems artificial, but its strength lies in the applications. Indeed, we deduce from (9) that under some integrability conditions

$$\mathbb{E}(F^2) - \mathbb{E}(F)^2 \leqslant \mathcal{E}_\alpha(F),$$

where the energy of parameter α is given by

$$\mathcal{E}_\alpha(F) = \mathbb{E}\int_0^T \left(D_tF + \alpha\int_t^T 2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s[)}\big(D_sF\circ\varepsilon_t^+\big)\widetilde{N}(ds)\right)^2 dt.$$

We recover here the Poincaré inequality (7) when $\alpha = -1$ since $\mathcal{E}_{-1}(F) = \mathcal{E}_{\text{cyl}}(F)$, and when $\alpha = 0$ the Poincaré inequality of [1].

In Section 5 we give examples and compare the energies $\mathcal{E}_\alpha(F)$ with fixed F and varying α's. It turns out that in some cases $\alpha = 0$ gives the lowest energy, and in other cases $\alpha = -1$ does. We also show that for any $\alpha_0 \in \mathbb{R}$ ($\alpha_0 \neq -1/2$ for technical reasons), there exists a functional F for wich $\mathcal{E}_{\alpha_0}(F)$ is minimum.

In the last sections some of the preceding results are extended to asymmetric random walks, that spend “more” time on one point than on the other. More precisely, in Section 6 we extend the Poincaré inequality (2) for cylindrical functionals. Unfortunately, the energy $\mathcal{E}_{\text{cyl}}(F)$ itself is no more equal to the counterpart of $\mathcal{E}_{-1}(F)$. However, when we let the mesh of the partition $(t_1,\ldots,t_n)$ tend to 0, the energy $\mathcal{E}_{\text{cyl}}^{(n)}(F)$ converges towards $\mathcal{E}_{-1}(F)$. We then deduce the Poincaré inequality (7) for a large class a functionals. Unlike the symmetric case, we miss the isometry formula giving the martingale representation (8), which is false in the asymmetric case. This lack of Clark–Ocone formula prevents us to get easily the corresponding Poincaré inequality as it is done in Section 4. It rehabilitates then the “cylindrical method” based on the study of cylindrical functionals.

Finally, in Section 7 we give examples and compare the variance and the energies for some functionals. In particular, we examine the number N_T of jumps of the random walk up to time T. We show that its variance exceeds its energy of parameter 0

$$\mathcal{E}_0(F) = \mathbb{E}\int_0^T (D_tF)^2\,dt$$

though N_T is a very “reasonable” functional. We show by this way that the Poincaré inequality of parameter 0 is no more true if the random walk is not symmetric.

2. Stochastic calculus on Poisson spaces

We recall in this section the tools that will be needed in the sequel. We start with the description of the random walk.

2.1. Description of the random walk and of the associated point process

Our continous time random walk $(B_t)_{t\geqslant 0}$ on the discrete cube $\{-1,+1\}^d$ is given by its infinitesimal generator $\mathbf{L}$

$$\mathbf{L}(f)=\sum_{j=1}^{d}\bigl(f\circ\tau^{(j)}-f\bigr),$$

where $\tau^{(j)}$ acts on the jth coordinate of the cube: If $x\in\{-1,+1\}^d$, then

$$\tau^{(j)}(x)=\bigl(x^{(1)},\ldots,-x^{(j)},\ldots,x^{(d)}\bigr).$$

In other words, the process starts at B_0, waits an exponential time T_1, then jumps to a neighbour B_{T_1} of B_0, and so on.

The path of the walk is characterized by the times T_n and the directions of jumps j_n. Therefore the walk B may be modelized by a Poisson point process N on $U=\mathbb{R}^+\times J=\mathbb{R}^+\times\{1,\ldots,d\}$, each point (t,j) corresponding to a jump of the random walk at time t and direction j.

More precisely, we call Ω the set of atomic measures ω on U, and endow U with the measure $\widehat{N}(dt\times\{j\})=dt$. Then, let N be the random measure on U defined by $N(\omega,A)=\omega(A)$ for $\omega\in\Omega$, and $\mathbb{P}$ be the probability measure on Ω under which N is a random Poisson measure of intensity (or compensator) $\widehat{N}$. We also let $\widetilde{N}=N-\widehat{N}$ be the compound Poisson measure.

If we order the atoms $(T_n,j_n)_{n\geqslant 1}$ of the measure N, then T_n are the jumping times and j_n the jumping directions of the random walk, that is: $B_{T_1}=\tau^{(j_1)}(B_0)$, and $B_{T_n}=\tau^{(j_n)}(B_{T_{n-1}})=\tau^{(j_n)}\circ\ldots\circ\tau^{(j_1)}(B_0)$.

2.2. Filtered space and predictable processes

Let $(\mathcal{F}_t)_{t\geqslant 0}$ be the filtration $\sigma(N(A);A\in\mathcal{B}([0,t]\times\{1,\ldots,d\}))$, $t\geqslant 0$, and let $\mathcal{F}=\mathcal{F}_{\infty^-}=\bigvee_{t\geqslant 0}\mathcal{F}_t$. An $\mathcal{F}$-measurable functional F is $\mathcal{F}_t$ (resp. $\mathcal{F}_{t^-}$)-measurable if and only if $F(\omega)=F(\omega_{|[0,t]\times J})$ (resp. $F(\omega)=F(\omega_{|[0,t[\times J})$). Indeed, every $\mathcal{F}_t$-measurable functional F meets the condition $F(\omega)=F(\omega_{|[0,t]\times J})$ since it holds for all $F=\omega(A)$, $A\in\mathcal{B}([0,t]\times J)$. To prove the converse, notice that for every $A\in\mathcal{B}(U)$ the functional $\omega_{|[0,t]\times J}(A)$ is $\mathcal{F}_t$-measurable, so that the map

$$\begin{aligned}(\Omega,\mathcal{F}_t)&\longrightarrow(\Omega,\mathcal{F})\\ \omega&\longmapsto\omega_{|[0,t]\times J}\end{aligned}$$

is measurable. Therefore F is $\mathcal{F}_t$-measurable if $F=F(\omega_{|[0,t]\times J})$.

Notice that the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ is right-continuous. Indeed, if F is $\mathcal{F}_{t^+}$-measurable, then $F(\omega)=F(\omega_{|[0,t+\varepsilon]\times J})$ for each ω and $\varepsilon>0$. As $\omega_{|[0,t+\varepsilon]\times J}=\omega_{|[0,t]\times J}$ for ε small enough (depending on ω), $F=F(\omega_{|[0,t]\times J})$, and F is $\mathcal{F}_t$-measurable. Thus $\mathcal{F}_{t^+}=\mathcal{F}_t$.

In what concerns predictable processes, we have a similar property. Recall first that the predictable σ-algebra $\mathcal{P}$ is generated by the adapted left-continuous processes, or equivalently by processes of the form $\mathbf{1}_{]u,v]}F(\omega)$ where F ranges over the bounded $\mathcal{F}_u$-measurable functionals, and $u < v \leqslant \infty$. In our case, a process Y is predictable if and only if $Y_t(\omega) = Y_t(\omega_{|[0,t[\times J})$. In order to prove it is a necessary condition, notice that it is satisfied by all processes $\mathbf{1}_{]u,v]}F(\omega)$ where F is a bounded $\mathcal{F}_u$-measurable functional. Then apply a monotone class argument. Conversely, for all $A \in \mathcal{B}(\mathbb{R} \times J)$ the process $(\omega_{|[0,t[\times J})(A)$ is adapted and left-continuous, hence predictable. It follows that the map

$$\begin{aligned} (\mathbb{R} \times \Omega, \mathcal{P}) &\longrightarrow \big(\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}\big) \\ (t, \omega) &\longmapsto (t, \omega_{|[0,t[\times J}) \end{aligned}$$

is measurable. Therefore if $Y_t = Y_t(\omega_{|[0,t[\times J})$, the process Y is predictable.

Now we introduce the predictable projection of a bounded or non-negative process W. It is the unique predictable process ${}^pW$ which satisfies

$${}^pW_\tau \overset{\text{a.s.}}{=} \mathbb{E}(W_\tau \mathcal{F}_{\tau^-}) \tag{10}$$

for any predictable time τ. (See [5, Chap. VI.2] for proofs and more details.) If W is neither bounded nor negative, then its predictable projection is defined if ${}^p|W|$ takes finite values only, and in that case

$${}^pW = \big({}^p(W_+) - {}^p(W_-)\big).$$

The property (10) is still satisfied, and still characterizes ${}^pW$. Finally, it will be useful to consider ${}^pW = ({}^p(W_+) - {}^p(W_-))$ on the set $\{{}^p|W| < \infty\}$ in the general case. We present here several properties that will be useful in the next sections.

LEMMA 2.1. – *Let W be a bounded or non-negative process. Then for all $j \in \{1, \ldots, d\}$*

$$\mathbb{E}\int W_t \widehat{N}(dt, j) = \mathbb{E}\int {}^pW_t \widehat{N}(dt, j).$$

Since $\widehat{N}(dt, j) = dt$ and $\mathbb{E}W_t = \mathbb{E}\,{}^pW_t$, Lemma 2.1 is a direct consequence of Fubini's theorem.

The following lemma gives a very useful expression of ${}^pW$.

LEMMA 2.2. – *The predictable projection ${}^pW$ of the process W is given by*

$${}^pW_t(\omega) = \int W_t(\omega_{|[0,t[\times J} + \bar{\omega}_{|[t,\infty[\times J})\, d\mathbb{P}^t(\bar{\omega}) = \big(\mathbb{E}^t W_t\big)(\omega_{|[0,t[}), \tag{11}$$

where $\mathbb{P}^t$ denotes the law of a Poisson process on $[t, \infty[\times J$ with intensity ds on each $[t, \infty[\times\{j\}$, and where $\mathbb{E}^t$ denotes the expectation with respect to $\mathbb{P}^t$.

Proof. – Assume first that (11) is satisfied for all processes of the form $W_t(\omega) = \mathbf{1}_{[0,u[}(t)F(\omega)$ where $u \geqslant 0$, $F = f(T_1, \ldots, T_m)$ and f is a smooth bounded function on $\mathbb{R}^m$. Then, by a monotone class argument, (11) extends to all bounded processes (see [5, p. 114] for the use of the monotone class theorem) and then to all non-negative processes (applying (11) to $W \wedge n$). Finally, we consider an arbitrary process W. Applying (11) to $|W|$ shows that ${}^p|W|_t(\omega)$ is finite if and only if $W_t(\omega_{|[0,t[\times J} + \bar{\omega}_{|[t,\infty[\times J})$ is integrable with respect to $\mathbb{P}^t(\bar{\omega})$. Therefore, using once more equality (11) with W_+ and W_- yields (11) for W on the set $\{{}^p|W| < \infty\}$.

Now let us consider the process $W_t(\omega) = \mathbf{1}_{[0,u[}(t)F(\omega)$ where $F = f(T_1,\ldots,T_m)$ and f is a smooth bounded function on $\mathbb{R}^m$. Then ${}^pW$ is given gy

$$
{}^pW_t(\omega) = \mathbf{1}_{[0,u[}(t)M_{t^-}(\omega)
$$

where M_t is the càd-làg version of the martingale $\mathbb{E}(F|\mathcal{F}_t)$. Therefore it is enough to show that M_{t^-} coincides with

$$
H_t(\omega) = \int F(\omega_{|[0,t[\times J} + \bar{\omega}_{|[t,\infty[\times J})\,d\mathbb{P}^t(\bar{\omega}),
$$

which is equivalent to saying that $H_t \overset{\text{a.s.}}{=} \mathbb{E}(F|\mathcal{F}_{t^-})$ and that $t \mapsto H_t(\omega)$ is left-continuous.

We first show that $H_t \overset{\text{a.s.}}{=} \mathbb{E}(F|\mathcal{F}_{t^-})$. To this end, let G be a bounded and $\mathcal{F}_{t^-}$-measurable functional. Then $G(\omega) = G(\omega_{|[0,t[\times J})$ and

$$
\begin{aligned}
\mathbb{E}(GH_t) &= \int G(\omega_{|[0,t[\times J})F(\omega_{|[0,t[\times J} + \bar{\omega}_{|[t,\infty[\times J})\,d\mathbb{P}(\omega)\,d\mathbb{P}^t(\bar{\omega}) \\
&= \int (GF)(\omega_{|[0,t[\times J} + \bar{\omega}_{|[t,\infty[\times J})\,d\mathbb{P}(\omega)\,d\mathbb{P}^t(\bar{\omega}).
\end{aligned}
$$

Since $(\omega_{|[0,t[\times J} + \bar{\omega}_{|[t,\infty[\times J})$ has exactly the law $\mathbb{P}$ under $\mathbb{P}(\omega) \otimes \mathbb{P}^t(\bar{\omega})$, we get that $\mathbb{E}(GH_t) = \mathbb{E}(GF)$, and thus $H_t = \mathbb{E}(F|\mathcal{F}_{t^-})$.

Then we show that H_t is left-continuous. Fix $t > 0$ and let $(t_n)_{n \geqslant 0}$ be an increasing sequence towards t. Let also $\delta_n = t - t_n$. For n large enough ($n \geqslant n_0(\omega)$), $\omega([t_n,t[) = 0$, and the number of jumps up to time t or t_n is $m_1 = N_t(\omega)$. Let us denote $T_1 = T_1(\omega),\ldots,T_{m_1} = T_{m_1}(\omega)$, and let $\bar{T}_k = T_k(\bar{\omega}_{|[t,\infty[\times J}) \in [t,\infty[$ be the kth jump of $\bar{\omega}$. Then, under $\mathbb{P}^{t_n}$, the sequence of jumps $(\bar{T}_1,\ldots,\bar{T}_k,\ldots)$ has the law of $(\bar{T}_1+\delta_n,\ldots,\bar{T}_k+\delta_n,\ldots)$ under $\mathbb{P}^t$. Therefore,

$$
\begin{aligned}
H_{t_n}(\omega) &= \int F(\omega_{|[0,t_n[\times J} + \bar{\omega}_{|[t_n,\infty[\times J})\,d\mathbb{P}^{t_n}(\bar{\omega}) \\
&= \int f(T_1,\ldots,T_{m_1},\bar{T}_1,\ldots,\bar{T}_{m-m_1})\,d\mathbb{P}^{t_n}(\bar{\omega}) \\
&= \int f(T_1,\ldots,T_{m_1},\bar{T}_1+\delta_n,\ldots,\bar{T}_{m-m_1}+\delta_n)\,d\mathbb{P}^t(\bar{\omega}) \\
&\xrightarrow{n\to\infty} \int f(T_1,\ldots,T_{m_1},\bar{T}_1,\ldots,\bar{T}_{m-m_1})\,d\mathbb{P}^t(\bar{\omega}) \\
&= H_t(\omega).
\end{aligned}
$$

Since f is assumed to be smooth and bounded, we used the dominated convergence theorem in the last-but-one line. Thus H_t is left-continuous, and the proof of Lemma 2.2 is complete. □

LEMMA 2.3 (Jensen's inequality). – *Let W be a process that admits a predictable projection. Then*

$$\left({}^p W\right)^2 \leqslant {}^p\left(W^2\right).$$

More generally, $\varphi({}^p W) \leqslant {}^p\varphi(W)$ for any positive convex function φ on $\mathbb{R}$. If ${}^p|W|$ is not necesseraly finite, we still have that

$$\left({}^p W\right)^2 \mathbf{1}_{\{{}^p|W|<\infty\}} \leqslant {}^p\left(W^2\right). \tag{12}$$

Proof. – Assume first that W admits a predictable projection. For every predictable time τ we have

$$\varphi\left({}^p W_\tau\right) = \varphi\left(\mathbb{E}(W_\tau|\mathcal{F}_{\tau^-})\right) \leqslant \mathbb{E}\left(\varphi(W_\tau)|\mathcal{F}_{\tau^-}\right) \leqslant {}^p\varphi(W)_\tau.$$

It follows that $\varphi({}^p W) \leqslant {}^p\varphi(W)$. If ${}^p|W|$ is not necessarily finite, the expression of ${}^p W$ is given by the previous lemma. If ${}^p|W|_t(\omega) = \infty$ then obiously $({}^p W_t(\omega))^2 \mathbf{1}_{\{{}^p|W|_t(\omega)<\infty\}} = 0 \leqslant {}^p(W^2)$, and on the set ${}^p|W|_t(\omega) < \infty$ we have

$$\begin{aligned}\left({}^p W_t(\omega)\right)^2 &= \left((\mathbb{E}^t W_t)(\omega_{|[0,t[\times J})\right)^2 \\ &\leqslant (\mathbb{E}^t (W_t)^2)(\omega_{|[0,t[\times J}) \\ &= {}^p\left(W^2\right)_t(\omega).\end{aligned}$$

The proof of Lemma 2.3 is thus complete. □

2.3. Derivative operator

In this part we introduce (following [8] and [9]) the derivative of a functional F defined on Ω, wich will play the role of the Malliavin derivative. Several operators have already been defined on Poisson spaces. In [2,4] or [10] an operator is defined by shifting the atoms T_n of the point process. In this work we need a different operator obtained by adding atoms, but which is no more a derivation. Let us now be more precise. The derivative of a functional F defined on Ω is the process $D_u F = F \circ \varepsilon_u^+ - F \circ \varepsilon_u^-$, where, if $u = (t, j)$, the transformation ε_u^+ (resp. ε_u^-) adds an atom at time t and direction j if there was none, and removes all other atom at time t (resp. removes the probably existing atom at time t), that is: $\varepsilon_u^+(\omega) = \omega_{U\setminus\{t\}\times J} + \delta_u$ and $\varepsilon_u^-(\omega) = \omega_{U\setminus\{t\}\times J}$. Notice that $\varepsilon_u^+(\omega) = \omega$ for $\mathbb{P} \times N$-almost every (u, ω) and that $\varepsilon_u^-(\omega) = \omega$ $\mathbb{P} \times \widehat{N}$-almost everywhere. Let us denote also by μ the measure $\mathbb{P} \times (N + \widehat{N})$ on $\Omega \times (\mathbb{R}^+ \times J)$.

The following essential proposition is proved in [9, Théorème 1].

PROPOSITION 2.4 (Isometry formula). – *Let P be a Poisson point process on a space V with intensity $\widehat{P}$. Let Z_v be a positive, $\mathbb{P} \times P$ or $\mathbb{P} \times \widehat{P}$-integrable process such that $D_v Z_v = 0$ for all $v \in V$. Then*

$$\mathbb{E}\left(\int_V Z_v \, dP(v)\right) = \mathbb{E}\left(\int_V Z_v \, d\widehat{P}(v)\right). \tag{13}$$

We first use this isometry formula to show basic properties of ε^+ and ε^-.

PROPOSITION 2.5. – *Let G be a functional on Ω. Recall that μ is the measure $\mathbb{P} \times (N + \widehat{N})$ on $\Omega \times U$.*

(i) *If $G = 0$ almost surely, then $G \circ \varepsilon_u^+ = 0$ and $G \circ \varepsilon_u^- = 0$ for μ-almost every (ω, u). It follows that the process DF is well-defined in the sense that if $F = H$ almost-surely, then DF and DH coincide μ-almost everywhere.*

(ii) *If $G = 0$ almost surely, then $G \circ \varepsilon_u^\pm \circ \varepsilon_v^\pm = 0$ for $\mu \times (N + \widehat{N})$-almost every (ω, u, v).*

Proof of (i). – Since $\varepsilon_u^+(\omega) = \omega$ $\mathbb{P} \times N$-almost everywhere, $G \circ \varepsilon_u^+ = 0$ $\mathbb{P} \times N$-almost everywhere. Applying Proposition 2.4 to the process $|G \circ \varepsilon_u^+|$ yields

$$\mathbb{E} \int |G \circ \varepsilon_u^+| \, d\widehat{N} = \mathbb{E} \int |G \circ \varepsilon_u^+| \, dN = \mathbb{E}\left(|G| \int dN \right) = 0.$$

Therefore $G \circ \varepsilon_u^+$ also vanishes $\mathbb{P} \times \widehat{N}$ almost everywhere. Similarly, applying Proposition 2.4 to $|G \circ \varepsilon_u^-|$ implies that $G \circ \varepsilon_u^- = 0$ μ-almost everywhere. The proof of (i) is thus complete. □

Although (i) shows that DF is defined μ-a.e. for an almost surely defined functional F, we must be very careful with the process DF, since $F = H$ almost surely does *not* imply that DF and DH are indistinguishable processes. Take for instance $F = \mathbf{1}_{\{T_1 = 1\}}$. Then $F \overset{\text{a.s.}}{=} 0$, but $\mathbb{P}\{\exists u;\ D_u F \neq 0\} = \mathbb{P}\{\exists u;\ F \circ \varepsilon_u^+ \neq 0\} = \mathbb{P}\{T_1 \geqslant 1\} > 0$.

In order to prove (ii) we need the following corollary which extends the isometry formula (13) to processes with several parameters. It is proved by iteration in [9].

COROLLARY 2.6. – *Let $Z_{\mathbf{u}}$, $\mathbf{u} = (u_1, \ldots, u_k) \in U^k$, be a non-negative k parameters measurable process such that $Z_{\mathbf{u}} = 0$ as soon as two parameters $u_l = (t_l, j_l)$ and $u_{l'} = (t_{l'}, j_{l'})$ have equal times $t_l = t_{l'}$. Assume that for all $(\alpha_1, \ldots, \alpha_k)$ with $\alpha_j \in \{+, -\}$ and all $\mathbf{u} = (u_1, \ldots, u_k)$*

$$Z_{\mathbf{u}} \circ \varepsilon_{u_1}^{\alpha_1} \circ \cdots \circ \varepsilon_{u_k}^{\alpha_k} = Z_{\mathbf{u}}.$$

Define also $N^+ = N$ and $N^- = \widehat{N}$. Then

$$\mathbb{E}\left(\int Z_{\mathbf{u}} \, dN^{\alpha_1}(u_1) \cdots dN^{\alpha_k}(u_k) \right)$$

does not depend on the sequence $(\alpha_1, \ldots, \alpha_k)$.

Proof of (ii). – Assume that $G \overset{\text{a.s.}}{=} 0$. If $u_i = (t_i, j_i)$, $\alpha_1, \alpha_2 \in \{+, -\}$ and if $t_1 = t_2$, we already know by (i) that $G \circ \varepsilon_{u_1}^{\alpha_1} \circ \varepsilon_{u_2}^{\alpha_2} = G \circ \varepsilon_{u_1}^{\alpha_1} = 0$ for μ-almost all (ω, u_1). In order to show that $\mathbf{1}_{\{t_1 \neq t_2\}} G \circ \varepsilon_{u_1}^\pm \circ \varepsilon_{u_2}^\pm = 0$ $\mu \times N^\pm$ almost everywhere, take $k = 2$ and $Z_{(u_1, u_2)} = \mathbf{1}_{\{t_1 \neq t_2\}} G \circ \varepsilon_{u_1}^{\alpha_1} \circ \varepsilon_{u_2}^{\alpha_2}$. By Corollary 2.6 we get that

$$\mathbb{E} \int_{t_1 \neq t_2} |G \circ \varepsilon_{u_1}^{\alpha_1} \circ \varepsilon_{u_2}^{\alpha_2}| \, dN^{\beta_1}(u_1) \, dN^{\beta_2}(u_2)$$

$$= \mathbb{E} \int_{t_1 \neq t_2} |G \circ \varepsilon_{u_1}^{\alpha_1} \circ \varepsilon_{u_2}^{\alpha_2}| \, dN^{\alpha_1}(u_1) \, dN^{\alpha_2}(u_2)$$

$$= \mathbb{E}\left(|G| \int_{t_1 \neq t_2} dN^{\alpha_1}(u_1) \, dN^{\alpha_2}(u_2) \right) = 0.$$

Thus $G \circ \varepsilon_{u_1}^{\pm} \circ \varepsilon_{u_2}^{\pm} = 0$ for $\mu \times (N + \widehat{N})$-almost every (ω, u_1, u_2), and the proof of part (ii) of Proposition 2.5 is complete. □

Before turning to the next paragraph, we give the expression of the derivative DF when F is a cylindrical functional. First take $F = f(B_t)$. If there is no jump at time s with $s \leqslant t$, then $D_{(s,j)}F$ is just $f(\tau^{(j)}B_t) - f(B_t) = D^{(j)}f(B_t)$ where $D^{(j)}f$ is the function $f \circ \tau^{(j)} - f$ on $\{-1,+1\}^d$. If the walk jumps in direction j at time $s \leqslant t$, then $D_{(s,j)}F$ is $f(B_t) - f(\tau^{(j)}B_t) = -D^{(j)}f(B_t)$. Therefore

$$D_{(s,j)}F = \mathbf{1}_{\{s \leqslant t\}} (-1)^{N(\{(s,j)\})} D^{(j)} f(B_t)$$

μ-almost everywhere. More generally, let $F = f(B_{t_1}, \ldots, B_{t_n})$. As before, adding an atom at time s moves up to time t_n, and

$$D_{(s,j)}F = (-1)^{N(\{(s,j)\})} \big(f(B_{t_1}, \ldots, B_{t_{i-1}}, \tau^{(j)}B_{t_i}, \ldots, \tau^{(j)}B_{t_n}) - f(B_{t_1}, \ldots, B_{t_n}) \big)$$

on $t_{i-1} < s \leqslant t_i$ and μ-almost everywhere.

2.4. Asymmetric random walks

One may skip this paragraph for the reading of the core of the paper (up to Section 5). We shall meet the processes described here in Sections 6 and 7 only. In these sections, we let the process spend more time in one direction than in the other. More precisely, we wish to consider the generator

$$\mathbf{L} f(x) = \sum_{j=1}^{d} \lambda_j(x^{(j)}) \big(f \circ \tau^{(j)} - f \big)(x),$$

where $\lambda_j \geqslant 0$ on $\{+1,-1\}$. If the random walk stands at x, then it will jump in direction j with rate $\lambda_j(x^{(j)})$. As it depends on $x^{(j)}$ only, the process $(B_t)_{t \geqslant 0}$ associated to $\mathbf{L}$ has independent coordinates and each $(B_t^{(j)})_{t \geqslant 0}$ is an asymmetric continous time random walk on $\{+1,-1\}$, jumping from -1 to $+1$ with rate $\lambda_j(-1)$ and from $+1$ to -1 with rate $\lambda_j(+1)$. Therefore, if we take for simplicity $d = 1$ and lay down the superscript $_{(j)}$, the waiting time T_1 has an exponential law of parameter $\lambda(B_0)$, and more generally $T_{n+1} - T_n$ is exponential of parameter $\lambda(B_{T_n})$.

The invariant measure associated to our process is the product measure $\mu_\lambda = \bigotimes_{j=1}^{d} \mu_{\lambda_j}$ with $\mu_{\lambda_j}(+1) = \lambda_j(-1)/\Lambda_j$ and $\mu_{\lambda_j}(-1) = \lambda_j(+1)/\Lambda_j$, where $\Lambda_j = \lambda_j(-1) + \lambda_j(+1)$.

In the previous paragraphs the case $\lambda_j \equiv 1$ was considered. The random point process associated to the jumps $N = \sum_{n \geqslant 1} \delta_{(T_n, j_n)}$ was a random Poisson measure. This is no more the case here, since the law of the waiting time $(T_{n+1} - T_n)$ depends on the past of the process through its position x at time T_n. Let $\mathbb{P}_\lambda$ be the law on Ω of this point process. Then $\mathbb{P}_\lambda$ is determined by its compensator $\widehat{N}_\lambda$ given by (see for example [7, Theorem 1.33, p. 136])

$$\widehat{N}_\lambda\big(dt, \{j\}\big) = \lambda_j\big(B_{t^-}^{(j)}\big)\,dt.$$

The filtration $(\mathcal{F}_t)_{t\geqslant 0}$ considered is the same as for the symmetric random walk. Therefore it still holds that a functional F is $\mathcal{F}_t$-measurable if and only if $F = F(\omega_{|[0,t]\times J})$. Similarly, $Y = (Y_t(\omega_{|[0,t[\times J}))_{t\geqslant 0}$ still characterizes the predictable processes.

The predictable projection of a bounded or non-negative process W is defined as in the symmetric case by

$$^pW_\tau \overset{\text{a.s.}}{=} \mathbb{E}_\lambda(W_\tau|\mathcal{F}_{\tau^-}) \tag{14}$$

for any predictable time τ. Then Lemma 2.1 still holds. Indeed, $\widehat{N}_\lambda(dt, j)$ is a predictable measure (see [5, Chap. VI.2]), so that for all $j \leqslant d$ and all bounded or non-negative processes W,

$$\mathbb{E}_\lambda \int W_t \widehat{N}_\lambda(dt, j) = \mathbb{E}_\lambda \int {}^pW_t \widehat{N}_\lambda(dt, j).$$

Lemma 2.2 giving the expression of ^pW also holds for $\mathbb{P}_\lambda$, replacing the law $\mathbb{P}^t$ of the translated point process by the law $\mathbb{P}_\lambda^{t,B_{t-}}$ of a point process on $[t,\infty[\times J$ starting at $x_{t^-} = B_{t^-}$ and with compensator $\lambda_j(x_{s^-}^{(j)})\,ds$ on each $[t,\infty[\times\{j\}$. It yields

$$\begin{aligned} {}^pW_t(\omega) &= \int W_t(\omega_{|[0,t[\times J} + \bar\omega_{|[t,\infty[\times J})\,d\mathbb{P}_\lambda^{t,B_{t-}}(\bar\omega) \\ &= \big(\mathbb{E}_\lambda^t W_t\big)(\omega_{|[0,t[}). \end{aligned} \tag{15}$$

We turn now to the derivative operator, which was already defined in the previous paragraph. The definition of the transformations $\varepsilon_u^\pm$ does not involve any measure, so that we may wonder if Proposition 2.5 is still true for $\mathbb{P}_\lambda$. The stake is to be able to define the process DF if the functional F is only almost surely defined. This is indeed the case. To prove it, we need to compare the asymmetric random walk and the symmetric one. It is shown in [7, Chap. IV.2 and IV.4] that $\mathbb{P}_\lambda$ is locally equivalent to $\mathbb{P}$, which means that for each $t \geqslant 0$, the restrictions of $\mathbb{P}_\lambda$ and of $\mathbb{P}$ to $\mathcal{F}_t$ meet $\mathbb{P}_{\lambda|\mathcal{F}_t} \sim \mathbb{P}_{|\mathcal{F}_t}$. More precisely, the density process Z such that $\mathbb{P}_{\lambda|\mathcal{F}_t}(d\omega) = Z_t(\omega)\mathbb{P}_{|\mathcal{F}_t}(d\omega)$ is given in [7, Chap. III.5] by

$$Z_t(\omega) = \exp\left(\int_0^t \big(1 - \lambda(x_{s^-})\big)\,ds\right) \prod_{n;\,T_n\leqslant t} \lambda(x_{T_n^-}) > 0.$$

Therefore, the measures $\mu_\lambda = \mathbb{P}_\lambda \times (\widehat{N}_\lambda + N)$ and μ are locally equivalent, and Proposition 2.5 is easily extended as follows.

PROPOSITION 2.7. – *Let $T > 0$ be finite and let G be an $\mathcal{F}_T$-measurable functional on Ω.*

(i) *If $G = 0$ $\mathbb{P}_\lambda$-almost surely, then $G \circ \varepsilon_u^+ = 0$ and $G \circ \varepsilon_u^- = 0$ for μ_λ-almost every (ω, u). It follows that the process DF is well-defined in the sense that if F and H coincide $\mathbb{P}_\lambda$-almost-surely and are $\mathcal{F}_T$-measurable, then DF and DH coincide μ_λ-almost everywhere.*

(ii) *If $G = 0$ $\mathbb{P}_\lambda$-almost surely, then and $G \circ \varepsilon_u^\pm \circ \varepsilon_v^\pm = 0$ for $\mu_\lambda \times (N + \widehat{N}_\lambda)$-almost every (ω, u, v).*

3. Gradient arising from Poincaré inequalities

In this section we deduce a Poincaré inequality from the study of cylindrical functionals. In the first part we prove a Poincaré inequality for $F(B) = f(B_{t_1}, \ldots, B_{t_n})$ in which the energy involves the times $t_1, \ldots, t_n$. Then in the second part we express this energy in terms of the gradient $D_t F$. This expression is well defined, not only for cylindrical functionals, but also for all bounded ones (for instance). The Poincaré inequality may thus be extended to this class of functionals.

3.1. Poincaré inequalities for cylindrical functionals

In dimension 1 the law of B_t starting at $B_0 = x$ is given by

$$\mathbb{P}^x\{B_t = y\} = \mathbf{p}_\mathbf{t}^1(x, y) = \frac{1 + xy\mathrm{e}^{-2t}}{2}.$$

More generally,

$$\mathbb{P}^x\{B_t = y\} = \mathbf{p}_\mathbf{t}^d(x, y) = \prod_{j=1}^{d} \mathbf{p}_\mathbf{t}^1\big(x^{(j)}, y^{(j)}\big).$$

Therefore one may tensorize the Poincaré inequality (1) to obtain, for all cylindrical functionals $F = f(B_t)$, and for all dimensions $d \in \mathbb{N}^*$

$$\mathbb{E}\big(F^2\big) - \mathbb{E}(F)^2 \leqslant \frac{1}{4}\big(1 - \mathrm{e}^{-4t}\big)\mathbb{E}\left(\sum_{j=1}^{d}\big(D^{(j)}F\big)^2\right), \tag{16}$$

where $D^{(j)}$ acts on the jth coordinate as $(D^{(j)}F)(B) = (f \circ \tau^{(j)} - f)(B_t)$.

Now we extend (16) to all cylindrical functionals. In dimension 1 we recover inequality (2) mentioned in the introduction. This Poincaré inequality was already presented in [1].

PROPOSITION 3.1. – *If $F(B) = f(B_{t_1}, \ldots, B_{t_n})$ with $0 = t_0 < t_1 < \cdots < t_n$, then*

$$\begin{aligned}\mathbb{E}\big(F^2\big) - \mathbb{E}(F)^2 &\leqslant \mathcal{E}_{\text{cyl}}(F)\\ &= \frac{1}{4}\sum_{j=1}^{d}\sum_{i=1}^{n}\big(1 - \mathrm{e}^{-4(t_i - t_{i-1})}\big)\mathbb{E}\left(\left(\sum_{k=i}^{n} \mathrm{e}^{-2(t_k - t_i)}\widetilde{D}_k^{(j)} F \circ \tau_{i,k-1}^{(j)}\right)^2\right),\end{aligned} \tag{17}$$

where the discrete derivatives of F are defined by

$$\big(\widetilde{D}_k^{(j)} F \circ \tau_{i,k-1}^{(j)}\big)(B) = \big(\widetilde{D}_k^{(j)} f\big)\big(B_{t_1}, \ldots, B_{t_{i-1}}, \tau^{(j)} B_{t_i}, \ldots, \tau^{(j)} B_{t_{k-1}}, B_{t_k}, \ldots, B_{t_n}\big),$$

and

$$\big(\widetilde{D}_k^{(j)} f\big)(x_1, \ldots, x_n) = x_k^{(j)} \big(f(x_1, \ldots, \tau^{(j)} x_k, \ldots, x_n) - f(x_1, \ldots, x_n)\big).$$

Proof. – Since the d coordinates of the process are independent, (17) follows from the one-dimensional case. Thus we assume $d = 1$. The law of $(B_{t_1}, \ldots, B_{t_n})$ is the measure on $\{-1, +1\}^n$ given by

$$dP(x_1, x_n) = \mathbf{p}^1_{\mathbf{t_1}}(x_0, x_1) \cdots \mathbf{p}^1_{\mathbf{t_n - t_{n-1}}}(x_{n-1}, x_n)\, dx_1 \ldots dx_n,$$

where dx denotes the counting measure on $\{-1, +1\}$. By induction on (16) we get that

$$\mathbb{E}F^2 = \int f^2 dP \leqslant \left(\int f\, dP\right)^2 + \sum_{i=1}^n \frac{1 - \mathrm{e}^{-4(t_i - t_{i-1})}}{4} \int (D_i f_i)^2 dP,$$

where $f_n = f$ and $f_i(x_1, \ldots, x_i) = \int f_{i+1}(x_1, \ldots, x_{i+1}) \mathbf{p}^1_{\mathbf{t_{i+1} - t_i}}(x_i, x_{i+1}) \cdot dx_{i+1}$ for $i \leqslant n - 1$. Therefore it is enough to prove that

$$\begin{aligned} x_i D_i f_i = \int &\left(\sum_{k=i}^n \mathrm{e}^{-2(t_k - t_i)} x_k D_k f \circ \tau_{i,k-1}\right) \\ &\times \mathbf{p}^1_{\mathbf{t_{i+1} - t_i}}(x_i, x_{i+1}) \ldots \mathbf{p}^1_{\mathbf{t_n - t_{n-1}}}(x_{n-1}, x_n)\, dx_{i+1} \ldots dx_n \end{aligned} \tag{18}$$

since by Jensen's inequality it will follow that

$$\begin{aligned} (D_i f_i)^2 \leqslant \int &\left(\sum_{k=i}^n \mathrm{e}^{-2(t_k - t_i)} x_k D_k f \circ \tau_{i,k-1}\right)^2 \\ &\times \mathbf{p}^1_{\mathbf{t_{i+1} - t_i}}(x_i, x_{i+1}) \cdots \mathbf{p}^1_{\mathbf{t_n - t_{n-1}}}(x_{n-1}, x_n)\, dx_{i+1} \ldots dx_n. \end{aligned}$$

In case $i = n$, both sides of (18) are equal. Then (18) is easily proved by iteration, using the case $i = 1$ and $n = 2$. Thus we only show (18) in that case, that is:

$$x_1 D_1 f_1(x_1) = \int \big(x_1 D_1 f(x_1, x_2) + \mathrm{e}^{-2(t_2 - t_1)} x_2 D_2 f(-x_1, x_2)\big) \mathbf{p}^1_{\mathbf{t_2 - t_1}}(x_1, x_2)\, dx_2.$$

Let $u = t_2 - t_1$. Then

$$\begin{aligned} D_1 f_1(x_1) &= f_1(\tau x_1) - f_1(x_1) \\ &= \mathbf{P}_{\mathbf{u}}^{\tau x_1}\big(f(\tau x_1, \cdot)\big) - \mathbf{P}_{\mathbf{u}}^{x_1}\big(f(x_1, \cdot)\big) \\ &= \big(\mathbf{P}_{\mathbf{u}}^{\tau x_1} - \mathbf{P}_{\mathbf{u}}^{x_1}\big)\big(f(\tau x_1, \cdot)\big) + \mathbf{P}_{\mathbf{u}}^{x_1}\big((D_1 f(x_1, \cdot)\big). \end{aligned}$$

Since $\mathbf{P_u} g = g + \frac{1}{2}(1 - \mathrm{e}^{-2u}) Dg$ for an arbitrary function g,

$$\begin{aligned} \big(\mathbf{P}_{\mathbf{u}}^{\tau x} - \mathbf{P}_{\mathbf{u}}^{x}\big)(g) = (g(\tau x) - g(x)) + \frac{1 - \mathrm{e}^{-2u}}{2} &\big(\big(g\big(\tau(\tau x)\big) - g(\tau x)\big) \\ &- \big(g(\tau x) - g(x)\big)\big) = \mathrm{e}^{-2u} Dg(x). \end{aligned}$$

Applying this to $g(x)=f(\tau x_1,x)$ and using the fact that $xDg(x)$ is constant, we get that

$$x_1\big(\mathbf{P}_{\mathbf{u}}^{\tau x_1}-\mathbf{P}_{\mathbf{u}}^{x_1}\big)(f(\tau x_1,\cdot))=\mathrm{e}^{-2u}x_2D_2f(\tau x_1,x_2)$$
$$=\int\big(\mathrm{e}^{-2u}x_2D_2f(\tau x_1,x_2)\big)\mathbf{p}_{\mathbf{u}}^1(x_1,x_2)\,dx_2.$$

The proof of Proposition 3.1 is thus complete. □

3.2. Extending the cylindrical Poincaré inequalities

The aim of this part is to give an expression of $\mathcal{E}_{\text{cyl}}(F)$ defined by (17) in terms of the gradient D_tF only, so that this expression is well defined for all bounded functionals (for instance). The following proposition corresponds to (6) in the introduction when dimension $d=1$.

PROPOSITION 3.2. – *Let F be a cylindrical functional $F(B)=f(B_{t_1},\dots,B_{t_n})$, and $T=t_n$. Then the energy $\mathcal{E}_{\text{cyl}}(F)$ defined by* (17) *satisfies*

$$\mathcal{E}_{\text{cyl}}(F)=\mathbb{E}\int_{[0,T]\times J}\big(D_{(t,j)}F-W_{(t,j)}\big)^2d\widehat{N}(t,j),\tag{19}$$

where

$$W_{(t,j)}=\int_t^T 2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s[\times\{j\})}\big(D_{(s,j)}F\circ\varepsilon_{(t,j)}^+\big)\widetilde{N}(ds,j).$$

Remark 3.3. – This equality shows that $\operatorname{var}(F)\leqslant\mathcal{E}_{\text{cyl}}(F)$ holds at least for bounded $\mathcal{F}_T$-measurable functionals. In the next section we will recover this inequality from a more general statement.

Proof of Proposition 3.2. – We may take $d=1$ since the argument is the same for each term of the sum over j. Recall that

$$\mathcal{E}_{\text{cyl}}(F)=\sum_{i=1}^n\frac{1-\mathrm{e}^{-4(t_i-t_{i-1})}}{4}\mathbb{E}\Bigg(\bigg(\sum_{k=i}^n\mathrm{e}^{-2(t_k-t_i)}\widetilde{D}_kF\circ\tau_{i,k-1}\bigg)^2\Bigg)$$

with $\widetilde{D}_kF\circ\tau_{i,k-1}=\widetilde{D}_kf\circ\tau_{i,k-1}(B_{t_1},\dots,B_{t_n})$ and

$$\widetilde{D}_kf\circ\tau_{i,k-1}(x_1,\dots,x_n)=x_k\big(f(x_1,\dots,-x_i,\dots,-x_k,\dots,x_n)$$
$$-f(x_1,\dots,-x_i,\dots,-x_{k-1},x_k,\dots,x_n)\big).$$

We prove (19) by induction on the number n of times. Fix n and $F=f(B_{t_1},\dots,B_{t_n})$. Fix also the process $\omega_{|[0,t_1]}$ up to time t_1, and consider the point process $\bar{\omega}_{|]t_1,\infty[}$. Then F is now an $(n-1)$-cylindrical functional of $\bar{\omega}$. Hence the induction assumption implies that

$$\mathbb{E}\bigg(\int_{t_1}^T(D_tF-W_t)^2dt\,\big|\mathcal{F}_{t_1}\bigg)=\mathbb{E}\Bigg(\sum_{i=2}^n\frac{1-\mathrm{e}^{-4(t_i-t_{i-1})}}{4}\bigg(\sum_{k=i}^n\mathrm{e}^{-2(t_k-t_i)}\widetilde{D}_kF\circ\tau_{i,k-1}\bigg)^2\big|\mathcal{F}_{t_1}\Bigg).$$

It remains to show that

$$\int_0^{t_1}\left(D_tF-\int_{]t,T]}2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s[)}D_sF\circ\varepsilon_t^+\widetilde{N}(ds)\right)^2dt$$
$$=\frac{1-\mathrm{e}^{-4t_1}}{4}\left(\sum_{k=1}^{n}\mathrm{e}^{-2(t_k-t_1)}\widetilde{D}_kF\circ\tau_{1,k-1}\right)^2. \tag{20}$$

To this end, fix a time $t\in\,]0,t_1[$ without jump, and notice that the quantity $(-1)^{N(\{s\})}\times D_sF\circ\varepsilon_t^+$ is constant on each $]t_{i-1},t_i[$, namely

$$(-1)^{N(\{s\})}D_sF\circ\varepsilon_t^+=\Delta_i(B_{t_1},\ldots,B_{t_n})\quad\text{if } s\in\,]t_{i-1},t_i[,$$

whith $\Delta_i=f(-B_{t_1},\ldots,-B_{t_{i-1}},B_{t_i},\ldots,B_{t_n})-f(-B_{t_1},\ldots,-B_{t_n})$. We let also $\Delta_{n+1}=0$. Moreover, notice that $D_tF=-\Delta_1=-f(B_{t_1},\ldots,B_{t_n})+f(-B_{t_1},\ldots,-B_{t_n})$. Hence the left-hand side of (20) amounts to

$$\int_0^{t_1}\Bigg(-\Delta_1-\Delta_1\int_{]t,t_1]}2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s])}\widetilde{N}(ds)$$
$$-\sum_{i=2}^{n}\Delta_i\int_{]t_{i-1},t_i]}2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s])}\widetilde{N}(ds)\Bigg)^2dt. \tag{21}$$

Fortunately, we have the exact formula, for all $t\leqslant T$,

$$\int_{]t,T]}2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s])}\widetilde{N}(ds)=-1+(-1)^{N(]t,T])}\mathrm{e}^{-2(T-t)}. \tag{22}$$

This identity is a particular case of the integration by parts formula, valid for any smooth function g on $\mathbb{R}$:

$$\int_{]t,T]}g(s)(-1)^{N(]t,s])}2N(ds)+\int_{]t,T]}g'(s)(-1)^{N(]t,s])}\,ds=(-1)^{N(]t,T])}g(T)-g(t).$$

Thanks to (22), we show that (21) is equal to

$$\int_0^{t_1}\Bigg(-\Delta_1\mathrm{e}^{-2(t_1-t)}(-1)^{N(]t,t_1])}$$
$$+\sum_{i=2}^{n}\Delta_i\mathrm{e}^{-2(t_{i-1}-t)}(-1)^{N(]t,t_{i-1}])}\big(1-\mathrm{e}^{-2(t_i-t_{i-1})}(-1)^{N(]t_{i-1},t_i])}\big)\Bigg)^2dt$$
$$=\int_0^{t_1}\mathrm{e}^{-4(t_1-t)}dt\left(-\Delta_1(-1)^{N_{t_1}}+\sum_{i=2}^{n}\Delta_i\big(\mathrm{e}^{-2(t_{i-1}-t_1)}(-1)^{N_{t_{i-1}}}-\mathrm{e}^{-2(t_i-t_1)}(-1)^{N_{t_i}}\big)\right)^2$$

$$= \frac{1-\mathrm{e}^{-4t_1}}{4}\left(\sum_{i=1}^{n}\mathrm{e}^{-2(t_i-t_1)}(-1)^{N_{t_i}}(\Delta_{i+1}-\Delta_i)\right)^2$$
$$= \frac{1-\mathrm{e}^{-4t_1}}{4}\left(\sum_{i=1}^{n}\mathrm{e}^{-2(t_i-t_1)}B_{t_i}(\Delta_{i+1}-\Delta_i)\right)^2.$$

The last equality follows from the fact that $(-1)^{N_t} = B_0 B_t$ and $B_0^2 = 1$. It just remains now to identify $B_{t_i}(\Delta_{i+1}-\Delta_i)$ with $\widetilde{D}_i F \circ \tau_{1,i-1}$ for $i \leqslant n$. This is immediate from the definition of the Δ_i's. The proof of Proposition 3.2 is thus complete. □

4. Clark–Ocone formula

In this part we prove the Clark–Ocone formula (9) in dimension $d \geqslant 1$.

THEOREM 4.1 (Clark–Ocone formula). – *Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$. Let N_T be the number of jumps $N([0,T] \times J)$ occuring before time T, and let F be an $\mathcal{F}_T$-measurable functional such that $\mathbb{E}(N_T|F|) < \infty$, which implies that $F \in \mathbf{L}^1(\Omega)$. Recall that μ is the measure $\mathbb{P} \times (N + \widehat{N})$ on $\Omega \times (\mathbb{R}^+ \times J)$.*

(i) *Define $Z_s^{(t,j)} = \mathbf{1}_{\{t<s\}} 2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s[\times\{j\})}(D_{(s,j)}F \circ \varepsilon^+_{(t,j)})$. Define also the processes*

$$W_t^j = \int_{]t,T]} Z_s^{(t,j)}(N-\widehat{N})(ds, j).$$

Then the predictable projection ${}^p|W^j|_t(\omega)$ is finite μ-almost everywhere (a.e.), and ${}^pW_t^j(\omega) = 0$ μ-almost everywhere.

(ii) *Consequently,*

$$F - \mathbb{E}(F) = \int_{[0,T]\times J} H_u^\alpha \widetilde{N}(du), \tag{23}$$

where $H^\alpha_{(\cdot,j)} = (\mathbb{E}^t(D_{(t,j)}F + \alpha_j W_t^j))(\omega_{|[0,t[\times J})$ is defined μ-a.e. It is the predictable projection of $D_{(\cdot,j)}F + \alpha_j W^j$ on the set $\{{}^p|D_{(\cdot,j)}F|_t < \infty, {}^p|W^j|_t < \infty\}$.

Remark 4.2. – It would be possible to replace the constants $\alpha_1, \ldots, \alpha_d$ by deterministic functions $\alpha_1(t), \ldots, \alpha_d(t)$ (bounded for example), or even by predictable processes. It would also be possible to replace the factor $2\mathrm{e}^{-2(s-t)}$ by $\alpha\mathrm{e}^{-\alpha(s-t)}$ for example, because as far as it remains deterministic (and reasonable), it doesn't modify the proof. This last choice would lead to the Clark–Ocone formula (23) were, in dimension 1 for instance, the process H^α would be the predictable projection of

$$DF - \int_0^T \alpha\mathrm{e}^{-\alpha(s-t)}(-1)^{N(]t,s[)}(D_sF \circ \varepsilon_t^+)\widetilde{N}(dt).$$

It is this last formula that is very similar to the Clark–Ocone formula for a diffusion with constant Ricci curvature equal to 2α. The Poincaré inequality coming from the cylindrical method would suggest that the "curvature" of the process is 4. However, the latter formula indicates that this intuition might not be the right one.

As explained in the introduction, introducing the parameter α seems artificial and Theorem 4.1 seems to be odd since we just add zero to the already known martingale representation. Actually the applications prove the contrary. That is why we start by the following corollary. The proof of Theorem 4.1 will follow.

COROLLARY 4.3 (Poincaré inequality). – *Let $F \in \mathbf{L}^1(\Omega)$ be an $\mathcal{F}_T$-measurable functional such that $\mathbb{E}(N_T|F|) < \infty$, and let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$. Define the energy of parameter α by*

$$\mathcal{E}_\alpha(F) = \sum_{j=1}^{d} \mathbb{E}\left(\int_0^T \left(D_{(t,j)}F + \alpha_j \int_t^T 2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s[\times\{j\})} \times \left(D_{(s,j)}F \circ \varepsilon^+_{(t,j)}\right)\widetilde{N}(ds,j)\right)^2 dt\right). \tag{24}$$

Then

$$\mathbb{E}(F^2) - \mathbb{E}(F)^2 \leqslant \mathcal{E}_\alpha(F). \tag{25}$$

Proof. – It follows by a by now standard method. As developed in [3,11] and [1], representation formulas enable to deduce Poincaré inequalities, logarithmic Sobolev inequalities, or modified logarithmic Sobolev inequalities.

If $\mathcal{E}_\alpha(F) = \infty$ then (25) is trivial, and otherwise the martingale $M_t = \mathbb{E}(F|\mathcal{F}_t)$ meets

$$\langle M\rangle_t = \int_{[0,t]\times J} \left({}^p(D_{(\cdot,j)}F + \alpha_j W^j)_t\right)^2 \mathbf{1}_{\{{}^p|D_{(\cdot,j)}F|_t<\infty,\, {}^p|W^j|_t<\infty\}}\, d\widehat{N}(t,j),$$

so that

$$\begin{aligned}
\mathbb{E}\left(M_T^2 - M_0^2\right) &= \mathbb{E}\langle M\rangle_T \\
&= \mathbb{E}\int_{[0,T]\times J} \left({}^p\left(D_{(\cdot,j)}F + \alpha_j W^j\right)_t\right)^2 \mathbf{1}_{\{{}^p|D_{(\cdot,j)}F|_t<\infty,\, {}^p|W^j|_t<\infty\}}\, d\widehat{N}(t,j) \\
&\leqslant \mathbb{E}\int_{[0,T]\times J} {}^p\left(\left(D_{(t,j)}F + \alpha_j W_t^j\right)^2\right) d\widehat{N}(t,j) \\
&= \mathbb{E}\int_{[0,T]\times J} \left(D_{(t,j)}F + \alpha_j W_t^j\right)^2 d\widehat{N}(t,j).
\end{aligned}$$

In the third line we used the Jensen's inequality (12), and in the last one Lemma 2.1. As $\mathbb{E}(M_T^2) = \mathbb{E}(F^2)$ and $\mathbb{E}(M_0^2) = (\mathbb{E}F)^2$, we get (25). □

Proof of Theorem 4.1. – The representation formula (23) is already known with $\alpha = 0$ [1]. Furthermore, it was shown there that ${}^p|D_{(\cdot,j)}F|_t$ is finite μ-a.e. as soon as F is integrable, so that H^0 is well defined. Thus we only show (i).

This proof is divided into two parts. We first prove an integrability property, corresponding somehow to the finiteness of ${}^p|W^j|$, and then we deduce that ${}^pW^j = 0$.

Let V be the non-negative process

$$V_t^j = \int_{]t,T]} \left|Z_s^{(t,j)}\right| (N + \widehat{N})(ds, j).$$

We start by proving that ${}^pV^j$ is finite μ-almost everywhere. Notice that $|W_t^j| \leqslant V_t^j$, so that it will give the finiteness of ${}^p|W^j|$ μ-almost everywhere. Recall that the predictable projection of V^j is given by

$${}^pV_t^j(\omega) = \int V_t^j(\omega_{|[0,t[\times J} + \bar{\omega}_{|[t,\infty[\times J})\, d\mathbb{P}^t(\bar{\omega}) = \left(\mathbb{E}^t V_t^j\right)(\omega_{|[0,t[\times J}),$$

where $\mathbb{P}^t$ is the the law of a Poisson point process $\bar{\omega}$ on $[t, \infty[\times J$ with intensity (or compensator) $d\widehat{P} = ds$ on $[t, \infty[\times \{j'\}$, and where $\mathbb{E}^t$ is the expectation with respect to $\mathbb{P}^t$. Now, for fixed $u = (t, j)$ and $\omega \in \Omega$, using the isometry formula (13) twice for this translated Poisson point process P on $[t, \infty[\times J$ and for $Z_s(\bar{\omega}) = |Z_s^{(t,j)}|(\omega_{|[0,t[\times J} + \bar{\omega}_{|[t,\infty[\times J})$ (for which $D_s Z_s = 0$) yields that

$$\begin{aligned}
{}^pV_t^j(\omega_{|[0,t[\times J}) &= 2\mathbb{E}^t \int_t^T \left|Z_s^u\right| ds \\
&\leqslant 2\mathbb{E}^t \int_t^T 2\mathrm{e}^{-2(s-t)} \left(\left|F \circ \varepsilon_{(t,j)}^+ \circ \varepsilon_{(s,j)}^+\right| + \left|F \circ \varepsilon_{(t,j)}^+ \circ \varepsilon_{(s,j)}^-\right|\right) ds \\
&= 2\mathbb{E}^t \int_t^T 2\mathrm{e}^{-2(s-t)} \left(\left|F \circ \varepsilon_{(t,j)}^+ \circ \varepsilon_{(s,j)}^+\right| P(ds, j) + \left|F \circ \varepsilon_{(t,j)}^+ \circ \varepsilon_{(s,j)}^-\right| ds\right) \\
&= 2\mathbb{E}^t \int_t^T 2\mathrm{e}^{-2(s-t)} \left(\left|F \circ \varepsilon_{(t,j)}^+\right| P(ds, j) + \left|F \circ \varepsilon_{(t,j)}^+\right| ds\right) \\
&\leqslant 2\mathbb{E}^t \left(\left(2\bar{\omega}(]t, T] \times \{j\}\right) + 1\right) \left|F \circ \varepsilon_{(t,j)}^+\right|\right).
\end{aligned}$$

Since $\mathbf{1}_{\{\omega([0,t[\times J) \leqslant n\}}\, {}^pV_t^j$ is exactly ${}^pV_t^j$ for large n, it is enough to show that for all $n \in \mathbb{N}$, this quantity is finite μ-almost everywhere. To this end, we evaluate

$$\begin{aligned}
\mathbb{E}\left(\int_0^T \mathbf{1}_{\{\omega([0,t[\times J) \leqslant n\}}\, {}^pV_t^j\, dt\right) &\leqslant 2\int_0^T \mathbb{E}\left(\mathbf{1}_{\{\omega([0,t[\times J) \leqslant n\}} \mathbb{E}^t\left(\left(2\bar{\omega}\left(]t, T] \times \{j\}\right)\right.\right.\right. \\
&\quad \left.\left.\left. +1\right) \left|F \circ \varepsilon_{(t,j)}^+\right| (\omega_{|[0,t[\times J} + \bar{\omega}_{|]t,T]\times J})\right)\right) dt \\
&= 2\int_0^T \mathbb{E}\left(\mathbf{1}_{\{\omega([0,t[\times J) \leqslant n\}} \left(2\omega\left(]t, T] \times \{j\}\right) + 1\right) \left|F \circ \varepsilon_{(t,j)}^+\right|\right) dt.
\end{aligned}$$

Using now the isometry formula (13) for the original point process, we deduce that

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^T \mathbf{1}_{\{\omega([0,t[\times J)\leqslant n\}}{}^pV_t^j\,dt\right)\\
&\quad\leqslant 2\mathbb{E}\int_0^T \mathbf{1}_{\{\omega([0,t[\times J)\leqslant n\}}\bigl(2\omega\bigl(]t,T]\times\{j\}\bigr)+1\bigr)\bigl|F\circ\varepsilon^+_{(t,j)}\bigr|N(dt,j)\\
&\quad= 2\mathbb{E}\int_0^T \mathbf{1}_{\{t<T_{n+1}\}}\bigl(2\omega\bigl(]t,T]\times\{j\}\bigr)+1\bigr)|F|N(dt,j)\\
&\quad\leqslant 2\mathbb{E}(2N_T+1)|F|\int_0^T \mathbf{1}_{\{t<T_{n+1}\}}N(dt,j)\\
&\quad\leqslant 2n\mathbb{E}\bigl((2N_T+1)|F|\bigr)<\infty.
\end{aligned}
$$

It follows that ${}^pV^j$ is finite $\mathbb{P}\times dt$-almost everywhere. Moreover, since $\mathbf{1}_{\{\omega([0,t[\times J)\leqslant n\}}$ ${}^pV_t^j(\omega_{|[0,t[\times J})$ does not depend on $\omega(\{t\}\times J)$, we may apply the isometry formula (13) once again to get that

$$
\mathbb{E}\left(\int_0^T \mathbf{1}_{\{\omega([0,t[\times J)\leqslant n\}}{}^pV_t^j\,dt\right)=\mathbb{E}\left(\int_0^T \mathbf{1}_{\{\omega([0,t[\times J)\leqslant n\}}{}^pV_t^j\,N(dt,j)\right),
$$

which is thus finite. This completes the proof that ${}^pV^j$ is finite μ-almost everywhere.

Now we fix (ω,u), $u=(t,j)$, such that ${}^pV_t^j(\omega)<\infty$. Then ${}^pW_t^j(\omega)$ is well defined by

$$
{}^pW_t^j(\omega)=\int W_t^j(\omega_{|[0,t[\times J}+\bar\omega_{|[t,\infty[\times J})\,d\mathbb{P}^t(\bar\omega)=\bigl(\mathbb{E}^tW_t^j\bigr)(\omega_{|[0,t[\times J}).
$$

The process $Z_s(\bar\omega)=Z_s^{(t,j)}(\omega_{|[0,t[\times J}+\bar\omega_{|[t,\infty[\times J})$ satisfies $D_sZ_s=0$. Moreover it is $\mathbb{P}^t\times\widehat{P}$ integrable since we exactly have $2\mathbb{E}^t\int_t^T|Z_s(\bar\omega)|\,ds={}^pV_t^j(\omega_{|[0,t[\times J})$. Then it is an easy matter to apply the isometry formula (13) to the translated Poisson point process on $[t,\infty[\times J$ and to get ${}^pW_t^j(\omega)=0$, which ends the proof of (i) and of Theorem 4.1. □

5. Examples

In this section we evaluate the energy $\mathcal{E}_\alpha(F)$ for various α's and functionals F.

As $\mathcal{E}_\alpha(F)$ is polynomial of degree 2 in α, it is minimal for $\alpha=\alpha^{\min}$ given by

$$
\alpha_j^{\min}=-\left(\mathbb{E}\int_0^T(D_{(t,j)}F)W_t^j\,dt\right)\left(\mathbb{E}\int_0^T\bigl(W_t^j\bigr)^2dt\right)^{-1},
$$

where W_t^j is defined in Theorem 4.1.

We are interested in $\alpha=\alpha^{\min}$, but also in the way $\mathcal{E}_\alpha(F)$ varies with F. We will thus be looking, as far as possible, for explicit expressions and bounds of $\mathcal{E}_\alpha(F)$.

In the first paragraph we consider the number of jumps $F = N_T = N[0,T]$ up to time T. In that case the energy $\mathcal{E}_0(N_T)$ gives the best estimate of the variance of N_T, since it is equal to $\mathrm{var}(N_T)$. More generally, we recover this behaviour for any functional of the first chaos.

In the second paragraph we give examples where $\mathcal{E}_{-1}(F)$ is the lowest energy. The first example is provided by cylindrical functionals $F = f(B_T)$, for which we exactly have $\mathcal{E}_{-1}(F) = \mathrm{var}(F)$ while all other energies $\mathcal{E}_\alpha(F)$, $\alpha \neq -1$, are much greater. Notice that $(-1)^{N_T} = B_0 B_T$ is such a functional. The second example of this paragraph is the product $F = (-1)^{N_T} N_T$ of two functionals studied before.

Finally, the last paragraph deals with $F = a^{N_T}$ and shows that for that functional $\alpha^{\min} = -1/2 + 1/(2a)$ for all T. Therefore for every $\alpha^0 \in \mathbb{R}$ ($\alpha^0 \neq -1/2$), there is a functional F for which α^0 minimizes $\mathcal{E}_\alpha(F)$.

5.1. Examples for which $\alpha = 0$ gives the lowest energy

Such examples are produced by functionals of the first chaos (in what concerns Fock spaces, see for example [8]). These functionals are of the form $\int_U f(u)\,d\widetilde{N}(u)$ for some function $f \in \mathbf{L}^2(U, d\widehat{N})$. This quantity is just the Stieljes integral $\int f(u)\,dN(u) - \int f(u)\,d\widehat{N}(u)$ if $f \in \mathbf{L}^1[0,T] \times J$. We assume here that $f \in \mathbf{L}^1([0,T] \times J)$ and we let

$$F = \int_{[0,T]\times J} f(u)\,dN(u) = \sum_{n;T_n \leqslant T} f(T_n, j_n).$$

The choice of $f = 1$ leads to the number of jumps $F = N_T$.

Let us examine first the variance of F. We have that

$$\mathbb{E}F = \mathbb{E}\int_{[0,T]\times J} f(u)\,d\widehat{N}(u) = \sum_{j=1}^{d}\int_0^T f(t,j)\,dt$$

and that

$$\mathrm{var}(F) = \mathbb{E}(F - \mathbb{E}F)^2 = \mathbb{E}\left\langle \int f(u)\,d\widetilde{N}(u)\right\rangle = \mathbb{E}\int_{[0,T]\times J} f^2(u)\,d\widehat{N}(u).$$

Hence

$$\mathrm{var}(F) = \sum_{j=1}^{d}\int_0^T f^2(t,j)\,dt.$$

Now we turn to the energies $\mathcal{E}_\alpha(F)$. As $D_{(t,j)}F = f(t,j)$ and $D_{(s,j)}F \circ \varepsilon^+_{(t,j)} = f(s,j)$ we get that

$$\mathcal{E}_\alpha(F) = \mathbb{E}\sum_{j=1}^{d}\int_0^T \big(f(t,j) + \alpha_j M_T^{(t,j)}\big)^2 dt,$$

where $M_u^{(t,j)}$, $u \geqslant t$, is the martingale

$$M_u^{(t,j)} = \int_{]t,u]} 2\mathrm{e}^{-2(s-t)} f(s,j)(-1)^{N(]t,s[\times\{j\})} \widetilde{N}\big(ds \times \{j\}\big).$$

Thus $\mathbb{E}(M_T^{(t,j)}) = 0$ and $d\langle M^{(t,j)}\rangle_s = 4\mathrm{e}^{-4(s-t)} f^2(s,j)\widehat{N}(ds \times \{j\})$, so that

$$\mathbb{E}\big(M_T^{(t,j)^2}\big) = \mathbb{E}\big\langle M^{(t,j)}\big\rangle_T = \int_t^T 4\mathrm{e}^{-4(s-t)} f^2(s,j)\,ds.$$

Finally when $\alpha = 0$ we exactly get

$$\mathcal{E}_0(F) = \sum_{j=1}^d \int_0^T f^2(t,j)\,dt = \mathrm{var}(F)$$

while

$$\begin{aligned}\mathcal{E}_\alpha(F) &= \mathcal{E}_0(F) + \sum_{j=1}^d \alpha_j^2 \int_0^T \int_t^T 4\mathrm{e}^{-4(s-t)} f^2(s,j)\,ds\,dt \\ &= \mathcal{E}_0(F) + \sum_{j=1}^d \alpha_j^2 \int_0^T (1-\mathrm{e}^{-4s}) f^2(s,j)\,ds.\end{aligned}$$

Therefore, all the energies $\mathcal{E}_\alpha(F)$ are larger than or equal to $\mathcal{E}_0(F)$, but still of the same order, in the sense that

$$\mathrm{var}(F) = \mathcal{E}_0(F) \leqslant \mathcal{E}_\alpha(F) \leqslant \Big(1 + \max_{j \leqslant d} \alpha_j^2\Big)\mathrm{var}(F).$$

Remark 5.1. – If we consider now a functional of the nth chaos

$$F = \int_{u_i \neq u_j} f(u_1,\ldots,u_n)\widetilde{N}(du_1)\ldots\widetilde{N}(du_n)$$

for a symmetric f in $\mathbf{L}^1(\mathbb{R}^+)^n$, it is not so easy to evaluate its energies $\mathcal{E}_\alpha(F)$ since its derivative

$$D_t F = n \int_{u_i \neq u_j, u_i \neq t} f(u_1,\ldots,u_{n-1},t)\widetilde{N}(du_1)\ldots\widetilde{N}(du_{n-1})$$

is no more deterministic. However, it is known that

$$\mathrm{var}(F) = \mathbb{E}F^2 = n!\|f\|^2_{\mathbf{L}^2(\mathbb{R}^+)^n}$$

and we get immediately that

$$\mathcal{E}_0(F) = \int_0^T \mathbb{E}\big((D_t F)^2\big)dt = n^2(n-1)!\|f\|^2_{\mathbf{L}^2(\mathbb{R}^+)^n} = n\mathrm{var}(F).$$

Unless $n = 1$, $\mathcal{E}_0(F)$ is greater than the variance of F and may not be the minimal energy.

5.2. Examples for which $\alpha = -1$ gives the lowest energy

5.2.1. One-cylindrical functionals

Let $F = f(B_T)$ be a one-cylindrical functional. We consider the case $d = 1$ to start with. In our setting, $(-1)^{N_T} = B_0 B_T$ (or $= \prod_{j=1}^d B_0^{(j)} B_T^{(j)}$ in general) is a particular case of such a functional. We first evaluate the derivative of F. If there is no jump at time $t \leqslant T$, then

$$D_t F = f(\tau B_T) - f(B_T) = Df(B_T)$$

and for $t \leqslant s \leqslant T$ we have that $D_s F \circ \varepsilon_t^+ = -(-1)^{N(\{s\})} Df(B_T)$. Therefore the quantity W_t is

$$W_t = -Df(B_T) \int_{]t,T]} 2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s])}\widetilde{N}(ds).$$

Recall the exact formula (22)

$$\int_{]t,T]} 2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s])}\widetilde{N}(ds) = -1 + (-1)^{N(]t,T])}\mathrm{e}^{-2(T-t)}.$$

The energy $\mathcal{E}_\alpha(F)$ is now easy to evaluate. Namely,

$$\begin{aligned}
\mathcal{E}_\alpha(F) &= \mathbb{E}\int_0^T (D_t F + \alpha W_t)^2 dt \\
&= \mathbb{E}\int_0^T \big(Df(B_T)\big)^2 \left(1 - \alpha \int_{]t,T]} 2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s])}\widetilde{N}(ds)\right)^2 dt \\
&= \int_0^T \mathbb{E}\Big(\big(Df(B_T)\big)^2\big(1+\alpha-\alpha(-1)^{N(]t,T])}\mathrm{e}^{-2(T-t)}\big)^2\Big)\,dt,
\end{aligned}$$

where we used (22). As $(Df(B_T))^2$ is a constant and $\mathbb{E}((-1)^{N(]t,T])}) = \mathrm{e}^{-2(T-t)}$ we get that

$$\mathcal{E}_\alpha(F) = (Df)^2\left((1+\alpha)^2 T - \alpha(2+\alpha)\frac{1-\mathrm{e}^{-4T}}{4}\right).$$

When $\alpha = -1$ it becomes obviously

$$\mathcal{E}_{-1}(F) = \frac{1-\mathrm{e}^{-4T}}{4}\big(Df(B_T)\big)^2 = \mathrm{var}(F),$$

but when $\alpha \neq -1$ then $\mathcal{E}_\alpha(F) \geqslant \mathcal{E}_{-1}(F)$ and $\mathcal{E}_\alpha(F) \sim 4(1+\alpha)^2 \mathcal{E}_{-1}(F)\,T$ explodes as time T tends to infinity.

Now, for arbitrary dimension $d \geqslant 1$, $\mathcal{E}_\alpha(F)$ is still minimized at $\alpha = (-1, \dots, -1)$, and

$$\mathcal{E}_\alpha(F) = \sum_{j=1}^d \int_0^T \mathbb{E}\Big(\big(D^{(j)} f(B_T)\big)^2 \big(1 + \alpha_j - \alpha_j (-1)^{N(]t,T]\times\{j\})} \mathrm{e}^{-2(T-t)}\big)^2\Big)\, dt.$$

When T is large,

$$\mathcal{E}_{(-1,\dots,-1)}(f(B_T)) = \frac{1 - \mathrm{e}^{-4T}}{4} \sum_{j=1}^d \mathbb{E}\Big(\big(D^{(j)} f(B_T)\big)^2\Big)$$

tends to a constant, But if one of $\alpha_j \neq -1$

$$\mathcal{E}_\alpha\big(f(B_T)\big) \sim \sum_{j=1}^d (1+\alpha_j)^2 \mathbb{E}\Big(\big(D^{(j)} f(B_T)\big)^2\Big) T.$$

5.2.2. Mixing two previous examples

Now, we turn to the functional $F = (-1)^{N_T} N_T$ in dimension $d = 1$ (without loss of generality). We have seen that the energy of $(-1)^{N_T}$ is lowest for $\alpha = -1$, whereas it is lowest for $\alpha = 0$ in case of N_T. We shall see that the behaviour of $(-1)^{N_T}$ is the most important. More precisely, we show that asymptotically $\operatorname{var}(F) \sim T^2$ and $\mathcal{E}_{-1}(F) \sim T^2$ whereas $\mathcal{E}_\alpha(F) \sim 4(1+\alpha)^2\, T^3$ for $\alpha \neq -1$.

Let us start with the variance of F. As N_T is Poissonian with parameter T, the mean of $(-1)^{N_T} N_T$ is $-T\mathrm{e}^{-2T}$, and its variance is

$$\operatorname{var}(F) = T^2\big(1 - \mathrm{e}^{-4T}\big) + T \overset{T\to\infty}{\sim} T^2.$$

Now we want to evaluate the energy of parameter α. If there is no jump at time t we have that $D_t F = (-2N_T - 1)(-1)^{N_T}$ and

$$D_s F \circ \varepsilon_t^+ = \big(2(N_T+1)(-1)^{N(\{s\})} + 1\big)(-1)^{N_T}.$$

Then we have that

$$\begin{aligned}
W_t &= \int_t^T 2\mathrm{e}^{-2(s-t)} (-1)^{N(]t,s[)} D_s F \circ \varepsilon_t^+ \widetilde{N}(ds) \\
&= 2(N_T+1)(-1)^{N_T} \int_t^T 2\mathrm{e}^{-2(s-t)} (-1)^{N(]t,s])} \widetilde{N}(ds) \\
&\quad + (-1)^{N_T} \int_t^T 2\mathrm{e}^{-2(s-t)} (-1)^{N(]t,s[)} \widetilde{N}(ds) \\
&= 2(N_T+1)(-1)^{N_T} \big(-1 + (-1)^{N(]t,T])} \mathrm{e}^{-2(T-t)}\big) + (-1)^{N_T} M_T^{(t)},
\end{aligned}$$

using (22) and the martingale $M_u^{(t)}$ introduced above,

$$M_u^{(t)} = \int_t^u 2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s[)}\widetilde{N}(ds). \tag{26}$$

It follows that

$$\begin{aligned}\mathcal{E}_\alpha(F) &= \mathbb{E}\int_0^T (D_t F + \alpha W_t)^2 dt \\ &= \mathbb{E}\int_0^T \left(-2N_T - 1 - 2\alpha(N_T+1) + 2\alpha(N_T+1)(-1)^{N(]t,T])}\mathrm{e}^{-2(T-t)} + \alpha M_T^{(t)}\right)^2 dt.\end{aligned}$$

Taking $\alpha = -1$, we get that

$$\mathcal{E}_{-1}(F) = \mathbb{E}\int_0^T \left(1 - 2(N_T+1)(-1)^{N(]t,T])}\mathrm{e}^{-2(T-t)} - M_T^{(t)}\right)^2 dt.$$

Developing the expression leads to the following estimates: $\mathbb{E}(N_T^2) \sim T^2$, $\mathbb{E}({M_T^{(t)}}^2) \leqslant 1$ as we have seen in Section 5.1, $\mathbb{E}(N_T) \sim T$, and finally

$$|\mathbb{E} N_T M_T^{(t)}| \leqslant \sqrt{\mathbb{E}(N_T^2)}\sqrt{\mathbb{E}\left({M_T^{(t)}}^2\right)} = \mathrm{O}(T).$$

Thus

$$\mathcal{E}_{-1}(F) \overset{T\to\infty}{\sim} T^2.$$

Assume now $\alpha \neq -1$. New terms appearing in the expression of $\mathcal{E}_\alpha(F)$ are of order $\mathrm{O}(T^2)$, except $\mathbb{E}\int_0^T 4(1+\alpha)^2 N_T^2\,dt$, which is of order $4(1+\alpha)^2 T^3$. Hence

$$\mathcal{E}_\alpha(F) \overset{T\to\infty}{\sim} 4(1+\alpha)^2 T^3.$$

This indicates that $\mathcal{E}_\alpha(F)$ is “much” greater than $\mathcal{E}_{-1}(F)$ when T is large.

5.3. Other examples

In this paragraph the case $F = a^{N_T}$, $a \in \mathbb{R}$, is treated. By convention, if $a = 0$ we set $F = \mathbf{1}_{\{N_T=0\}}$.

PROPOSITION 5.2. – *For all T and all α,*

$$\begin{aligned}\mathcal{E}_\alpha\left(a^{N_T}\right) = \frac{(a-1)^2}{2(a^2+1)}\mathrm{e}^{(a^2-1)T}\Big[&\left((2a\alpha + a - 1)^2 + (a+1)^2\right)T \\ &- \left((2a\alpha + a - 1)^2 - (a-1)^2\right)\frac{1 - \mathrm{e}^{-2(a^2+1)T}}{2(a^2+1)}\Big].\end{aligned} \tag{27}$$

It follows that $\alpha^{\min}$ *is exactly* $(1-a)/(2a)$ *for every* T*, and that there is explosion in the sense that* $\mathcal{E}_\alpha(a^{N_T})/\mathrm{var}(a^{N_T})$ *tends to infinity when* T *becomes large, whenever* $a \neq 1$ *and* $(a,\alpha) \neq (-1,-1)$*. Indeed,*

$$\mathrm{var}\big(a^{N_T}\big) \sim \mathrm{e}^{(a^2-1)T} \quad \textit{and} \quad \mathcal{E}_\alpha\big(a^{N_T}\big) \sim C(a,\alpha)T\mathrm{e}^{(a^2-1)T}$$

for some constant $C(a,\alpha)$ *depending only on* a *and* α*.*

Remark 5.3. – As a by-product, this example shows that for all $\alpha^0 \in \mathbb{R}$, $\alpha^0 \neq -1/2$, there exists a functional F such that α^0 minimizes $\mathcal{E}_\alpha(F)$. More generally, this property is still true in dimension d for all $\alpha^0 \in \mathbb{R}^d$ such that each $\alpha^0_j \neq -1/2$. The restriction $\alpha^0 \neq -1/2$ comes from the examples we have chosen, but there is no reason a priori for which $-1/2 \neq \alpha^{\min}$ for any F.

Proof. – We first evaluate the derivative of F. If there is no jump at time $t \leqslant T$ then $D_tF = (a-1)F$. If moreover $s > t$, then $D_sF \circ \varepsilon_t^+ = (a-1)F$ if the process jumps at time s, and $D_sF \circ \varepsilon_t^+ = (a-1)aF$ if it doesn't. It yields

$$\begin{aligned}
W_t &= \int_t^T 2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s[)}\, D_sF \circ \varepsilon_t^+\, \widetilde{N}(ds) \\
&= (a-1)F \int_t^T 2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s[)}\big(N(ds) - a\,ds\big) \\
&= (a-1)F \left(\begin{array}{l} \dfrac{a-1}{2} \displaystyle\int_t^T 2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s])}\widetilde{N}(ds) \\ + \dfrac{a+1}{2} \displaystyle\int_t^T 2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s[)}\widetilde{N}(ds) \end{array} \right) \\
&= (a-1)F\left(\frac{a-1}{2}\big(-1+(-1)^{N(]t,T])}\mathrm{e}^{-2(T-t)}\big) + \frac{a+1}{2}M_T^{(t)}\right),
\end{aligned}$$

where $M_T^{(t)}$ is the martingale introduced in (26). Therefore the energy becomes

$$\begin{aligned}
\mathcal{E}_\alpha(F) = (a-1)^2 \int_0^T \mathbb{E}\bigg\{F^2\bigg(1 - \alpha\frac{a-1}{2} \\
+ \alpha\frac{a-1}{2}(-1)^{N(]t,T])}\mathrm{e}^{-2(T-t)} + \alpha\frac{a+1}{2}M_T^{(t)}\bigg)^2\bigg\}\,dt. \qquad (28)
\end{aligned}$$

Now, we simply compute the various expectations involved in the expression (28), that is to say $\mathbb{E}(F^2)$, $\mathbb{E}(F^2(-1)^{N(]t,T])})$, $\mathbb{E}(F^2M_T^{(t)})$, $\mathbb{E}(F^2(-1)^{N(]t,T])}M_T^{(t)})$, and finally $\mathbb{E}(F^2{M_T^{(t)}}^2)$.

The first ones are easy: $F^2 = (a^2)^{N_T}$ and $F^2(-1)^{N(]t,T])} = (a^2)^{N([0,t])}(-a^2)^{N(]t,T])}$ so that

$$\mathbb{E}\big(F^2\big) = \mathrm{e}^{(a^2-1)T}$$

and

$$\mathbb{E}\big(F^2(-1)^{N(]t,T])}\big) = \mathrm{e}^{(a^2-1)t}\mathrm{e}^{(-a^2-1)(T-t)} = \mathrm{e}^{(a^2-1)T}\mathrm{e}^{-2a^2(T-t)}.$$

For the next expectations, we set $F_s = a^{N_s}$, so that $F = F_T$. As $d(F_s^2) = (a^2-1)F_{s^-}^2 N(ds)$ and $dM_s^{(t)} = 2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s[)}\widetilde{N}(ds)$ we have that

$$\begin{aligned} d\big(F_s^2 M_s^{(t)}\big) &= F_{s^-}^2 dM_s^{(t)} + M_{s^-}^{(t)} d\big(F_s^2\big) + \Delta\big(F_s^2\big)\Delta\big(M_s^{(t)}\big) \\ &= d(\text{martingale}) + \big(a^2-1\big)F_{s^-}^2 M_{s^-}^{(t)} ds \\ &\quad + \big(a^2-1\big)F_{s^-}^2 2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s[)} ds. \end{aligned}$$

Taking expectation yields

$$\begin{aligned} \frac{d}{ds}\mathbb{E}\big(F_s^2 M_s^{(t)}\big) &= (a^2-1)\mathbb{E}\big(F_s^2 M_s^{(t)}\big) + (a^2-1)2\mathrm{e}^{-2(s-t)}\mathbb{E}\big(F_{s^-}^2(-1)^{N(]t,s[)}\big) \\ &= (a^2-1)\mathbb{E}\big(F_s^2 M_s^{(t)}\big) + 2\big(a^2-1\big)\mathrm{e}^{(a^2-1)s}\mathrm{e}^{-2(a^2+1)(s-t)}. \end{aligned}$$

It follows that

$$\mathbb{E}\big(F^2 M_T^{(t)}\big) = \frac{a^2-1}{a^2+1}\mathrm{e}^{(a^2-1)T}\big(1-\mathrm{e}^{-2(a^2+1)(T-t)}\big).$$

Notice that the latter is also equal to $\mathbb{E}(a^{2N_t}(a^2)^{N(]t,T])}M_T^{(t)}) = \mathbb{E}(a^{2N_t})\mathbb{E}((a^2)^{N(]t,T])}M_T^{(t)})$ by independence. But $F^2(-1)^{N(]t,T])}M_T^{(t)} = a^{2N_t}(-a^2)^{N(]t,T])}M_T^{(t)}$ so that we deduce from the preceding that

$$\begin{aligned} \mathbb{E}(F^2(-1)^{N(]t,T])}M_T^{(t)}) &= \mathrm{e}^{(a^2-1)t}\frac{-a^2-1}{-a^2+1}\mathrm{e}^{(-a^2-1)(T-t)}\big(1-\mathrm{e}^{-2(-a^2+1)(T-t)}\big) \\ &= \frac{a^2+1}{a^2-1}\mathrm{e}^{(a^2-1)T}\big(\mathrm{e}^{-2a^2(T-t)} - \mathrm{e}^{-2(T-t)}\big). \end{aligned}$$

It remains to evaluate $\mathbb{E}(F^2 M_T^{(t)2})$. Using the same method as before, we have that

$$\begin{aligned} d\big(F_s^2 M_s^{(t)2}\big) &= F_{s^-}^2 d\big(M_s^{(t)2}\big) + M_{s^-}^{(t)2} d\big(F_s^2\big) + \Delta\big(F_s^2\big)\Delta\big(M_s^{(t)2}\big) \\ &= d(\text{martingale}) + F_{s^-}^2 d\big\langle M^{(t)}\big\rangle_s + \big(a^2-1\big)F_{s^-}^2 M_{s^-}^{(t)2} N(ds) \\ &\quad + \big(a^2-1\big)F_{s^-}^2\big(2M_{s^-}^{(t)}2\mathrm{e}^{-2(s-t)}(-1)^{N(]t,s[)} + 4\mathrm{e}^{-4(s-t)}\big)N(ds) \\ &= d(\text{martingale}) + \big(a^2-1\big)F_{s^-}^2 M_{s^-}^{(t)2} ds + 4a^2\mathrm{e}^{-4(s-t)}F_{s^-}^2 ds \\ &\quad + 4\big(a^2-1\big)\mathrm{e}^{-2(s-t)}F_{s^-}^2(-1)^{N(]t,s[)}M_{s^-}^{(t)} ds. \end{aligned}$$

Taking expectation shows that

$$\begin{aligned} \frac{d}{ds}\mathbb{E}\big(F_s^2 M_s^{(t)2}\big) &= (a^2-1)\mathbb{E}\big(F_s^2 M_s^{(t)2}\big) + 4a^2\mathrm{e}^{-4(s-t)}\mathrm{e}^{(a^2-1)s} \\ &\quad + 4(a^2-1)\mathrm{e}^{-2(s-t)}\frac{a^2+1}{a^2-1}\,\mathrm{e}^{(a^2-1)s}\big(\mathrm{e}^{-2a^2(s-t)} - \mathrm{e}^{-2(s-t)}\big) \\ &= (a^2-1)\mathbb{E}\big(F_s^2 M_s^{(t)2}\big) - 4\mathrm{e}^{-4(s-t)}\mathrm{e}^{(a^2-1)s} \\ &\quad + 4\big(a^2+1\big)\mathrm{e}^{-2(a^2+1)(s-t)}\mathrm{e}^{(a^2-1)s}. \end{aligned}$$

Therefore,

$$\mathbb{E}\big(F^2 {M_T^{(t)}}^2\big) = \mathrm{e}^{(a^2-1)T}\big(1+\mathrm{e}^{-4(T-t)} - 2\mathrm{e}^{-2(a^2+1)(T-t)}\big).$$

Provided with these expressions, computing the energy $\mathcal{E}_\alpha(F)$ is rather long, but absolutely standard. It is left to the reader. □

6. Poincaré inequalities for asymmetric random walks

The aim of this section is to generalize the Poincaré inequality (17) for cylindrical functionals and its generalization (7) to an asymmetric process on the cube. Recall from Section 2.4 that we consider here the generator

$$\mathbf{L} f(x) = \sum_{j=1}^{d} \lambda_j\big(x^{(j)}\big)\big(f\circ\tau^{(j)} - f\big)(x),$$

where $\lambda_j \geqslant 0$ on $\{+1,-1\}$, and that the random walk has independent coordinates $(B_t^{(j)})_{t\geqslant 0}$, each one beeing an asymmetric continous time random walk on $\{+1,-1\}$, jumping from -1 to $+1$ with rate $\lambda_j(-1)$ and from $+1$ to -1 with rate $\lambda_j(+1)$. Recall that $\Lambda_j = \lambda_j(+1)+\lambda_j(-1)$, and see Section 2.4 for more details.

In Section 2.4 we called $\mathbb{P}_\lambda$ the law of the point process of jumps $\omega = \sum \delta_{(T_n,j_n)}$. For simplicity, we lay down the subscript λ in this section, so that $\mathbb{P}$ will denote $\mathbb{P}_\lambda$, $\widehat{N}$ will denote $\widehat{N}_\lambda$, and so on.

6.1. Poincaré inequalities for cylindrical functionals

In this section we extend the Poincaré inequality (17) to cylindrical functionals. We first consider the case of a one-cylindrical functional $F = f(B_t)$ and we bound its variance. Then we consider the general case.

PROPOSITION 6.1. – *If $d=1$, let $c(t,x_0)$ the Poincaré constant of the law of B_t starting at x_0. Then*

$$c(t,x) = \lambda(x)\left(\frac{1-\mathrm{e}^{-\Lambda t}}{\Lambda}\right)\left(\frac{\lambda(-x)+\lambda(x)\mathrm{e}^{-\Lambda t}}{\Lambda}\right) \leqslant \lambda(x)\frac{1-\mathrm{e}^{-\Lambda t}}{\Lambda} \tag{29}$$

and for all functionals $F = f(B_t)$

$$\mathbb{E}\big(F^2\big) - \mathbb{E}(F)^2 = c(t,x_0)\mathbb{E}\big((DF)^2\big) \tag{30}$$

with $DF = Df(B_t) = f(-B_t) - f(B_t)$. More generally, if $c^{(j)}$ denotes $c(t,x)$ associated to λ_j, then for all functional $F = f(B_t)$

$$\mathbb{E}\big(F^2\big) - \mathbb{E}(F)^2 \leqslant \sum_{j=1}^{d} c^{(j)}\big(t,x_0^{(j)}\big)\mathbb{E}\big(\big(D^{(j)}F\big)^2\big), \tag{31}$$

with $D^{(j)}F = D^{(j)}f(B_t)$.

Proof. – Since the coordinates $B_t^{(j)}$ are mutually independent, the law $\mathbf{P}_\mathbf{t}^x$ of B_t starting at x is the product measure $\bigotimes_{j=1}^d \mathbf{P}_\mathbf{t}^{(x^{(j)},j)}$, and Poincaré inequality (31) follows from the one-dimensional case (30). Now we have to show that for any function f on $\{+1,-1\}$

$$\mathbf{P}_\mathbf{t}^x f^2 - \left(\mathbf{P}_\mathbf{t}^x f\right)^2 \leqslant c(t,x)\mathbf{P}_\mathbf{t}^x\left((Df)^2\right). \tag{32}$$

To this end, notice that $\mathbf{L}^2 = -\Lambda \mathbf{L}$, which implies that

$$\mathbf{P}_\mathbf{t} = \mathbf{I} + \frac{1-\mathrm{e}^{-\Lambda t}}{\Lambda}\mathbf{L}$$

and

$$\begin{aligned}\mathbf{P}_\mathbf{t} f^2 - (\mathbf{P}_\mathbf{t} f)^2 &= \left(f^2 + \frac{1-\mathrm{e}^{-\Lambda t}}{\Lambda}\mathbf{L}\left(f^2\right)\right) - \left(f + \frac{1-\mathrm{e}^{-\Lambda t}}{\Lambda}\mathbf{L}f\right)^2 \\ &= \frac{1-\mathrm{e}^{-\Lambda t}}{\Lambda}\left(\mathbf{L}\left(f^2\right) - 2f\mathbf{L}f\right) - \left(\frac{1-\mathrm{e}^{-\Lambda t}}{\Lambda}\right)^2 (\mathbf{L}f)^2.\end{aligned}$$

As $(Df)^2 = (f(+1) - f(-1))^2$ is a constant, we have that $\mathbf{P}_\mathbf{t}(Df)^2 = (Df)^2$, that $\mathbf{L}(f^2) - 2f\mathbf{L}f = \lambda(Df)^2$ and that $(\mathbf{L}f)^2 = \lambda^2(Df)^2$. Finally,

$$\mathbf{P}_\mathbf{t}^x f^2 - \left(\mathbf{P}_\mathbf{t}^x f\right)^2 = c(t,x)(Df)^2 = c(t,x)\mathbf{P}_\mathbf{t}^x(Df)^2,$$

which implies that the desired inequality (32) is actually an equality. □

The Poincaré inequality (31) is extended to arbitrary cylindrical functionals in Proposition 6.2 below.

PROPOSITION 6.2. – *Let $F = f(B_{t_1},\dots,B_{t_n})$ with $0 = t_0 < t_1 < \cdots < t_n$, then*

$$\begin{aligned}\mathbb{E}\left(F^2\right) - \mathbb{E}(F)^2 \leqslant \mathcal{E}_{\text{cyl}}(F) = \sum_{j=1}^d \sum_{i=1}^n \mathbb{E}\Biggl(c^{(j)}\left(t_i - t_{i-1}, B_{t_{i-1}}^{(j)}\right) \\ \times \left(\sum_{k=i}^n \mathrm{e}^{-\Lambda(t_k - t_i)}\widetilde{D}_k^{(j)} F \circ \tau_{i,k-1}^{(j)}\right)^2\Biggr),\end{aligned} \tag{33}$$

where $(\widetilde{D}_k^{(j)} F \circ \tau_{i,k-1}^{(j)})(B)$ is defined by (3).

Proof. – Here again, it is sufficient to consider the one-dimensional case. (33) is proved by iteration, as it is done in the symmetric case, Section 3.1. We shall prove the key point only, that is: for $n = 2$ and

$$h(x_1) = \mathbf{P}_{\mathbf{t_2}-\mathbf{t_1}}^{x_1}\left(f(x_1,\cdot)\right) = \int f(x_1,x_2)\mathbf{p}_{\mathbf{t_2}-\mathbf{t_1}}(x_1,dx_2),$$

then

$$\widetilde{D}h = \mathbf{P}_{\mathbf{t_2}-\mathbf{t_1}}\left(\widetilde{D}_1 f + \mathrm{e}^{-\Lambda(t_2-t_1)}\widetilde{D}_2 f \circ \tau_{1,1}\right), \tag{34}$$

that is to say

$$x_1 Dh(x_1) = \int \big(x_1 D_1 f(x_1, x_2) + \mathrm{e}^{-\Lambda(t_2-t_1)} x_2 D_2 f(-x_1, x_2)\big)\mathbf{p}_{\mathbf{t_2-t_1}}(x_1, dx_2).$$

Let $u = t_2 - t_1$. Then

$$\begin{aligned} Dh(x_1) &= h(-x_1) - h(x_1) \\ &= \mathbf{P}_{\mathbf{u}}^{-x_1}\big(f(-x_1,\cdot)\big) - \mathbf{P}_{\mathbf{u}}^{x_1}\big(f(x_1,\cdot)\big) \\ &= \big(\mathbf{P}_{\mathbf{u}}^{-x_1} - \mathbf{P}_{\mathbf{u}}^{x_1}\big)\big(f(-x_1,\cdot)\big) + \mathbf{P}_{\mathbf{u}}^{x_1}\big(D_1 f(x_1,\cdot)\big). \end{aligned}$$

Now, for an arbitrary function g,

$$\begin{aligned} \big(\mathbf{P}_{\mathbf{u}}^{-x} - \mathbf{P}_{\mathbf{u}}^{x}\big)(g) &= \big(g(-x) - g(x)\big) + \frac{1-\mathrm{e}^{-\Lambda u}}{\Lambda}\big(\lambda(-x)\big(g(-x) - g(x)\big) \\ &\quad - \lambda(x)\big(g(-x) - g(x)\big)\big) \\ &= Dg(x)\left(1 - \big(\lambda(x) + \lambda(-x)\big)\frac{1-\mathrm{e}^{-\Lambda u}}{\Lambda}\right) \\ &= \mathrm{e}^{-\Lambda u} Dg(x). \end{aligned}$$

Applying this to $g(x) = f(-x_1, x)$ and using the fact that $x Dg(x)$ is a constant, we get that

$$\begin{aligned} x_1\big(\mathbf{P}_{\mathbf{u}}^{-x_1} - \mathbf{P}_{\mathbf{u}}^{x_1}\big)\big(f(-x_1,\cdot)\big) &= \mathrm{e}^{-\Lambda u} x_2 D_2 f(-x_1, x_2) \\ &= \int \big(\mathrm{e}^{-\Lambda u} x_2 D_2 f(-x_1, x_2)\big)\mathbf{p}_{\mathbf{u}}(x_1, dx_2), \end{aligned}$$

which gives (34). □

6.2. Energy arising from Poincaré inequalities

In this section we wish to identify the cylindrical energy $\mathcal{E}_{\text{cyl}}(F)$ with an energy expressed in terms of the derivative $D_u F$ only, and to derive a Poincaré inequality valid for a large class of functionals. Unfortunately, unlike the symmetric case, the energy $\mathcal{E}_{\text{cyl}}(F)$ itself is not equal to the analogue of $\mathcal{E}_{-1}(F)$. The reason is that for a fixed functional F, the cylindrical energies associated to distinct partitions $(t_1^1, \ldots, t_n^1)$ and $(t_1^2, \ldots, t_m^2)$ are no more equal. However, the following theorem shows that these energies converge when the mesh of the partition tends to zero.

Recall that the compensator $\widehat{N}$ is given by

$$\widehat{N}\big(dt, \{j\}\big) = \lambda_j\big(B_{t^-}^{(j)}\big)\,dt.$$

THEOREM 6.3. – *Let F be a cylindrical functional $F(B) = f(B_{s_1}, \ldots, B_{s_p})$, and $T = s_p$. Consider a sequence $(t_1^{(n)}, \ldots, t_n^{(n)})_{n\geqslant p}$ of partitions containing $(s_1, \ldots, s_p)$ and such that $t_1^{(n)} < \cdots < t_n^{(n)} = T$. Assume that the mesh $\delta_n = \max_{i\leqslant n}|t_i^{(n)} - t_{i-1}^{(n)}|$ tends to 0 as n tends to infinity, and denote by $\mathcal{E}_{\text{cyl}}^{(n)}(F)$ the cylindrical energy of F associated to the partition $(t_1^{(n)}, \ldots, t_n^{(n)})$. Then*

$$\mathcal{E}_{\text{cyl}}^{(n)}(F) \xrightarrow{n\to\infty} \mathcal{E}_{-1}(F), \tag{35}$$

where, if $d=1$,

$$\mathcal{E}_{-1}(F)=\mathbb{E}\int_0^T\left(D_tF-\int_t^T \mathrm{e}^{-\Lambda(s-t)}(-1)^{N(]t,s[)}\left(D_sF\circ\varepsilon_t^+\right)\left(2N(ds)-\Lambda ds\right)\right)^2 d\widehat{N}(t).$$

When $d>1$,

$$\mathcal{E}_{-1}(F)=\mathbb{E}\int_{[0,T]\times J}\left(D_{(t,j)}F-W_{(t,j)}\right)^2 d\widehat{N}(t,j)$$

with

$$W_{(t,j)}=\int_t^T \mathrm{e}^{-\Lambda_j(s-t)}(-1)^{N(]t,s[\times\{j\})}\left(D_{(s,j)}F\circ\varepsilon_{(t,j)}^+\right)\left(2N(ds,j)-\Lambda_j ds\right).$$

COROLLARY 6.4 (Poincaré inequality). – *Let $T>0$ be finite and let $F\in\mathbf{L}^2(\Omega)$ be an $\mathcal{F}_T$-measurable functional such that the quantities*

$$\mathbb{E}\int_{[0,T]\times J}(D_uF)^2du,\quad \mathbb{E}\int_0^T\left(\int_t^T \mathrm{e}^{-\Lambda_j(s-t)}\left|D_{(s,j)}F\circ\varepsilon_{(t,j)}^+\right|N(ds,j)\right)^2dt$$
$$\text{and}\quad \mathbb{E}\int_0^T\left(\int_t^T \mathrm{e}^{-\Lambda_j(s-t)}\left|D_{(s,j)}F\circ\varepsilon_{(t,j)}^+\right|ds\right)^2dt \tag{36}$$

are finite. Then

$$\mathbb{E}\left(F^2\right)-\mathbb{E}(F)^2\leqslant\mathcal{E}_{-1}(F). \tag{37}$$

Proof of Theorem 6.3. – Once more, we may assume that $d=1$ since the argument is the same for each term of the sum. The proof is divided in two steps. In the first one we rewrite both energies $\mathcal{E}_{\mathrm{cyl}}^{(n)}(F)$ and $\mathcal{E}_{-1}(F)$ in Lemma 6.5, and in the second step we show the convergence result.

LEMMA 6.5. – *Let $F=f(B_{t_1},\ldots,B_{t_n})$. Then*

$$\mathcal{E}_{-1}(F)=\mathbb{E}\sum_{i=1}^n\left(\int_{t_{i-1}}^{t_i}\mathrm{e}^{-2\Lambda(t_i-t)}\lambda(B_t)\,dt\right)\left(\sum_{j=i}^n\mathrm{e}^{-\Lambda(t_j-t_i)}\widetilde{D}_jF\circ\tau_{i,j-1}\right)^2 \tag{38}$$

and

$$\mathcal{E}_{\mathrm{cyl}}(F)=\mathbb{E}\sum_{i=1}^n\mathbb{E}\left(\int_{t_{i-1}}^{t_i}\mathrm{e}^{-2\Lambda(t_i-t)}\lambda(B_t)\,dt\,\middle|\,\mathcal{F}_{t_{i-1}}\right)\left(\sum_{j=i}^n\mathrm{e}^{-\Lambda(t_j-t_i)}\widetilde{D}_jF\circ\tau_{i,j-1}\right)^2. \tag{39}$$

Proof. – To prove (39) it is enough to show that for all $T>0$

$$c(T,B_0)=\mathbb{E}\int_0^T\mathrm{e}^{-2\Lambda(T-t)}\lambda(B_t)\,dt.$$

It just follows from the equality

$$\begin{aligned}\mathbb{E}\big(\lambda(B_t)\big) &= (\mathbf{P_t}\lambda)(B_0) = \lambda(B_0) + \frac{1-\mathrm{e}^{-\Lambda t}}{\Lambda}(\mathbf{L}\lambda)(B_0)\\ &= \frac{\lambda(B_0)}{\Lambda}\big(2\lambda(-B_0) - \mathrm{e}^{-\Lambda t}\big(\lambda(-B_0)-\lambda(B_0)\big)\big)\\ &= 2\frac{\lambda(1)\lambda(-1)}{\Lambda} - \frac{\lambda(B_0)D\lambda(B_0)}{\Lambda}\mathrm{e}^{-\Lambda t}.\end{aligned}$$

Actually, in the constant $c(t,x)$ only matters its order $\lambda_j(x)t$ as t tends to 0. More precisely, any constant c of this order would lead to Theorem 6.3 (with a different proof of course).

The expression (38) of $\mathcal{E}_{-1}(F)$ is obtained with exactly the same arguments as for the symmetric process in Section 3.2. We just recall these arguments.

- For each $t \in]t_{i-1}, t_i[$ without jump and $s \geqslant t$ the quantity $(-1)^{N(\{s\})} D_s F \circ \varepsilon_t^+$ is constant on $s \in]t_{j-1}, t_j[$ and equal to $\Delta_{i,j}(B_{t_1}, \dots, B_{t_n})$ on this interval, with

$$\begin{aligned}\Delta_{i,j}(x_i,\dots,x_n) = f(x_1,\dots,x_{i-1},-x_i,\dots,-x_{j-1},x_j,\dots,x_n)\\ - f(x_1,\dots,x_{i-1},-x_i,\dots,-x_n).\end{aligned}$$

  Moreover, $D_t F = -\Delta_{i,i} = -f(B_{t_1},\dots,B_{t_n}) + f(B_{t_1},\dots,B_{t_{i-1}},-B_{t_i},\dots,-B_{t_n})$.
- The following exact formula is still available:

$$\int_{]t,T]} \mathrm{e}^{-\Lambda(s-t)}(-1)^{N(]t,s])}\big(2N(ds) - \Lambda\, ds\big) = -1 + (-1)^{N(]t,T])}\mathrm{e}^{-\Lambda(T-t)}.$$

- Finally, $(-1)^{N(]t,T])} = B_t B_T$ so that the terms $\widetilde{D}_j F \circ \tau_{i,j-1} = B_{t_j}(\Delta_{i,j+1} - \Delta_{i,j})$ appear as in the symmetric case. □

Recall now that F is the functional $f(B_{s_1},\dots,B_{s_p})$ and that $(t_1^{(n)},\dots,\ t_n^{(n)})_{n\geqslant p}$ is a partition containing $(s_1,\dots,s_p)$. Let us denote by n_i the index (depending on n) satisfying $t_{n_i} = s_i$. Let f_n be the function on $\{-1,+1\}^n$ defined by $f_n(x_1,\dots,x_n) = f(x_{n_1},\dots,x_{n_p})$, so that $\mathcal{E}_{\mathrm{cyl}}^{(n)}(F)$ is the cylindrical energy of F associated to f_n. Thanks to Lemma 6.5 we may write

$$\mathcal{E}_{\mathrm{cyl}}^{(n)}(F) = \mathbb{E}\sum_{k=1}^{n}\mathbb{E}\left(\int_{t_{k-1}}^{t_k}\mathrm{e}^{-2\Lambda(t_k-t)}\lambda(B_t)\,dt\,\middle|\,\mathcal{F}_{t_{k-1}}\right)\left(\sum_{l=k}^{n}\mathrm{e}^{-\Lambda(t_l-t_k)}\widetilde{D}_l F\circ\tau_{k,l-1}\right)^2.$$

It is clear that the gradient $\widetilde{D}_l f_n = 0$ if l is not an n_j. Fix $i \leqslant p$ and $k \in]n_{i-1}, n_i]$. Then we get easily

$$\begin{aligned}&\sum_{l=k}^{n}\mathrm{e}^{-\Lambda(t_l-t_k)}\widetilde{D}_l f_n\circ\tau_{k,l-1}(x_1,\dots,x_n)\\ &\quad= \sum_{j=i}^{p}\mathrm{e}^{-\Lambda(s_j-t_k)}\widetilde{D}_j f\circ\tau_{i,j-1}(x_{n_1},\dots,x_{n_p})\\ &\quad= \mathrm{e}^{-\Lambda(s_i-t_k)}\sum_{j=i}^{p}\mathrm{e}^{-\Lambda(s_j-s_i)}\widetilde{D}_j f\circ\tau_{i,j-1}(x_{n_1},\dots,x_{n_p}).\end{aligned}$$

Let G_i be the functional $\sum_{j=i}^{p} \mathrm{e}^{-\Lambda(s_j-s_i)} \widetilde{D}_j f \circ \tau_{i,j-1}(B_{t_{n_1}}, \ldots, B_{t_{n_p}})$. We may now simplify the expression of the cylindrical energy

$$\mathcal{E}^{(n)}_{\mathrm{cyl}}(F) = \mathbb{E} \sum_{i=1}^{p} \sum_{k=n_{i-1}+1}^{n_i} \mathbb{E}\left(\int_{t_{k-1}}^{t_k} \mathrm{e}^{-2\Lambda(t_k-t)} \lambda(B_t)\, dt \middle| \mathcal{F}_{t_{k-1}} \right) \mathrm{e}^{-2\Lambda(s_i-t_k)} G_i^2$$

$$= \mathbb{E} \sum_{i=1}^{p} \left(\int_{s_{i-1}}^{s_i} \mathrm{e}^{-2\Lambda(s_i-t)} \mathbb{E}\big(\lambda(B_t) \big| \mathcal{F}_{t_{k(t)}}\big)\, dt \right) G_i^2,$$

where $k(t)$ is the index (depending on n) satisfying $t \in [t_{k(t)}, t_{k(t)+1}[$. Recall from Lemma 6.5 that the expression of $\mathcal{E}_{-1}(F)$ is now very close

$$\mathcal{E}_{-1}(F) = \mathbb{E} \sum_{i=1}^{p} \left(\int_{s_{i-1}}^{s_i} \mathrm{e}^{-2\Lambda(s_i-t)} \lambda(B_t)\, dt \right) G_i^2.$$

When the mesh of the partition tends to 0, the time $t_{k(t)}$ tends to t and $\mathbb{E}(\lambda(B_t)|\mathcal{F}_{t_{k(t)}})$ tends almost everywhere to $\lambda(B_t)$ as n goes to infinity. It then follows from the dominated convergence theorem that $\mathcal{E}^{(n)}_{\mathrm{cyl}}(F)$ tends to $\mathcal{E}_{-1}(F)$, and the proof of Theorem 6.3 is complete. □

Proof of Corollary 6.4. – In case F is cylindrical, Corollary 6.4 is a direct consequence of Theorem 6.3. We first extend (37) to a bounded $\mathcal{F}_T$-measurable functional F. Let $M \geqslant 0$ such that $|F| \leqslant M$ a.s. Then there exist cylindrical $\mathcal{F}_T$-measurable functionals F_n bounded by M such that $\lim F_n = F$ almost surely. Inequality (37) is true for each F_n, and the variance of F_n tends to the variance of F by dominated convergence, so that it remains to prove that $\mathcal{E}_{-1}(F_n)$ also tends to $\mathcal{E}_{-1}(F)$.

Applying Proposition 2.7 with $G = \mathbf{1}_{\{|F| \leqslant M\}}$ or $G = \mathbf{1}_{\{F_n \to F\}}$ etc., we get that

- $|D_u F| \leqslant 2M$, $|D_u F_n| \leqslant 2M$ and $\lim_{n\to\infty} D_u F_n = D_u F$ for μ-almost every (ω, u).
- $|D_{(s,j)} F \circ \varepsilon^+_{(t,j)}| \leqslant 2M$, $|D_{(s,j)} F_n \circ \varepsilon^+_{(t,j)}| \leqslant 2M$ and $\lim_{n\to\infty} D_{(s,j)} F_n \circ \varepsilon^+_{(t,j)} = D_{(s,j)} F \circ \varepsilon^+_{(t,j)}$ for $\mu \times dt$-almost every $(\omega, (s, j), t)$.

We are then able to deduce that $\mathcal{E}_{-1}(F_n)$ tends to $\mathcal{E}_{-1}(F)$ by dominated convergence, so that (37) is true for F.

Now for a general functional F satisfying conditions (36) of Corollary 6.4, we set $F_n = (F \wedge n) \vee (-n)$. As F_n is bounded, it satisfies (37). Moreover,

- $|F_n| \leqslant |F|$ and $F_n \to F$ almost surely,
- $|D_u F_n| \leqslant |D_u F|$ and $D_u F_n \to D_u F$ μ-almost everywhere,
- $|D_{(s,j)} F_n \circ \varepsilon^+_{(t,j)}| \leqslant |D_{(s,j)} F \circ \varepsilon^+_{(t,j)}|$ and $D_{(s,j)} F_n \circ \varepsilon^+_{(t,j)} \to D_{(s,j)} F \circ \varepsilon^+_{(t,j)}$ for $\mu \times dt$-almost every $(\omega, (s, j), t)$.

Conditions on F are equivalent to

$$\mathbb{E} \sum_{j=1}^{n} \int_0^T \left(|D_{(t,j)} F| + \int_t^T \mathrm{e}^{-\Lambda_j(s-t)} \big|D_{(s,j)} F \circ \varepsilon^+_{(t,j)}\big| \big(2N(ds, j) + \Lambda_j\, ds\big) \right)^2 dt < \infty,$$

so that as before, $\mathcal{E}_{-1}(F_n)$ tends to $\mathcal{E}_{-1}(F)$ by dominated convergence, and it implies (37) for F. □

Remark 6.6. – The existence of a representation formula for martingales is well known [7, Theorem 4.37, p. 177]. Every local martingale has the form

$$M = M_0 + \int W\,dN - \int W\,d\widehat{N},$$

where W is a predictable process such that $\int |W|\,d\widehat{N}$ is locally integrable. It would be interesting to know W when M is the martingale $\mathbb{E}(F|\mathcal{F}_t)$. It is shown in Section 2 that Lemmas 2.1 and 2.2 are still true for asymmetric random walks, and one may wonder wether $W = {}^{p}DF$, or not. The answer is *not*. Take $F = N_T$. Then $DF = \mathbf{1}_{[0,T]}(t) = {}^{p}DF$, but $\int_0^T d\widetilde{N} = N_T - \widehat{N}([0,T])$ is no more $N_T - \mathbb{E}N_T$ since $\widehat{N}$ is no more deterministic. Is W equal to some H^α (defined in Theorem 4.1)? Regarding Poincaré inequality (37), H^{-1} seems to be a natural candidate.

7. Examples of functionals for asymmetric random walks

Only one Poincaré inequality has been shown for asymmetric random walks: the Poincaré inequality of parameter -1. Therefore we shall only compare $\mathcal{E}_{-1}(F)$ with the variance of F for some functionals F. However, as for the symmetric random walk we may define

$$\mathcal{E}_0(F) = \mathbb{E}\int (D_u F)^2 d\widehat{N}(u) \in [0,\infty],$$

and we shall also compare this quantity with $\mathcal{E}_{-1}(F)$ and $\operatorname{var}(F)$.

We assume in this section that $d = 1$. In the first paragraph the example of cylindrical functionals $F = f(B_t)$ is treated. It turns out that $\mathcal{E}_{-1}(F)$ gives exactly $\operatorname{var}(F)$, whereas $\mathcal{E}_0(F)/\operatorname{var}(F)$ tends to infinity when t is large and for fixed f. It is the same behaviour than in the symmetric case.

Then we turn to the functional $F = N_T$ in the second paragraph. On the opposite, we have asymptotically that $\mathcal{E}_0(F) < \operatorname{var}(F) < \mathcal{E}_{-1}(F)$, each term being of order KT for some constants K. It means that the Poincaré inequality of parameter 0 does not hold for N_T, yet it is a nice functional with finite moments and bounded derivative. This example shows that the Poincaré inequality $\operatorname{var}(F) \leqslant \mathcal{E}_0(F)$ is false, even for reasonable functionals F.

7.1. Cylindrical functionals

Assume that $F = f(B_T)$ is 1-cylindrical, and that $d = 1$. Then $D_t F = Df(B_T)$ if there is no jump at time t, and $D_s F \circ \varepsilon_t^+ = -(-1)^{N(\{s\})} Df(B_T)$. Notice that $(Df)^2$ is a constant. Then

$$\mathcal{E}_0(F) = (Df)^2 \mathbb{E}\left(\int_0^T \lambda(B_t)\,dt\right) = (Df)^2 \int_0^T (\mathbf{P_t}\lambda)(B_0)\,dt$$

whereas, by Lemma 6.5 and its proof p. 130

$$
\begin{aligned}
\mathcal{E}_{-1}(F) &= (Df)^2 \mathbb{E} \int_0^T \mathrm{e}^{-2\Lambda(T-t)} \lambda(B_t)\, dt \\
&= (Df)^2 \int_0^T \mathrm{e}^{-2\Lambda(T-t)} (\mathbf{P_t}\lambda)(B_0)\, dt = c(T, B_0)(Df)^2.
\end{aligned}
$$

Therefore it gives exactly

$$
\begin{aligned}
\mathcal{E}_{-1}(F) &= c(T, B_0)(Df)^2 = \operatorname{var}(F) \leqslant \mathcal{E}_0(F) \\
\text{and} \quad \mathcal{E}_0(F) &\overset{T\to\infty}{\sim} 2\frac{\lambda(1)\lambda(-1)}{\Lambda} T (Df)^2.
\end{aligned}
$$

Recall that c is the bounded constant arising in Proposition 6.1. We mentionned in Section 6 that any constant $c'(t,x)$ having the same behaviour as c when t is small would lead to the same Poincaré inequality with energy $\mathcal{E}_{-1}(F)$. It is surprising here to recover the best constant $c(t,x)$ through $\mathcal{E}_{-1}(F)$.

7.2. Number of jumps

Recall $N_T = N([0,T])$ is the number of jumps up to time T. We first estimate the energies $\mathcal{E}_0(N_T)$ and $\mathcal{E}_{-1}(N_T)$, and then the variance of N_T. We start with $\mathcal{E}_0(N_T)$.

As for the symmetric random walk in Section 5.2.1 we have that $D_s F = 1$ for $s \leqslant T$, so that

$$
\mathcal{E}_0(F) = \mathbb{E}\left(\int_t^T \lambda(B_t)\, dt \right) = \int_0^T (\mathbf{P_t}\lambda)(B_0)\, dt \overset{T\to\infty}{\sim} 2\frac{\lambda(1)\lambda(-1)}{\Lambda} T\ .
$$

In order to estimate $\mathcal{E}_{-1}(N_T)$, notice that

$$
\begin{aligned}
\lambda(x) &= \frac{1}{2}\big(\lambda(x) + \lambda(-x)\big) + \frac{1}{2}\big(\lambda(x) - \lambda(-x)\big) \\
&= \Lambda/2 + x D\lambda(-1)/2,
\end{aligned}
$$

and that $B_{s^-} = B_t(B_t B_{s^-}) = B_t(-1)^{N(]t,s[)}$ for $t < s$. Therefore the martingale $M_T^{(t)}$ defined in the symmetric case by (26) and here by

$$
M_T^{(t)} = \int_{]t,T]} 2\mathrm{e}^{-\Lambda(s-t)} (-1)^{N(]t,s[)} \big(N(ds) - \lambda(B_{s^-})\, ds\big)
$$

satisfies

$$
\begin{aligned}
M_T^{(t)} &= \int_{]t,T]} \mathrm{e}^{-\Lambda(s-t)} (-1)^{N(]t,s[)} \big(2N(ds) - \Lambda\, ds\big) - B_t D\lambda(-1) \int_t^T \mathrm{e}^{-\Lambda(s-t)} ds \\
&= \int_{]t,T]} \mathrm{e}^{-\Lambda(s-t)} (-1)^{N(]t,s[)} \big(2N(ds) - \Lambda\, ds\big) - \frac{1 - \mathrm{e}^{-\Lambda(T-t)}}{\Lambda} B_t D\lambda(-1).
\end{aligned}
$$

We have then

$$\mathcal{E}_{-1}(F) = \mathbb{E}\int_0^T \left(1 - \int_{]t,T]} \mathrm{e}^{-\Lambda(s-t)}(-1)^{N(]t,s[)}\big(2N(ds) - \Lambda ds\big)\right)^2 \lambda(B_t)\,dt$$
$$= \mathbb{E}\int_0^T \left(1 - M_T^{(t)} - \frac{1-\mathrm{e}^{-\Lambda(T-t)}}{\Lambda} B_t D\lambda(-1)\right)^2 \lambda(B_t)\,dt.$$

Since $\mathbb{E}(M_T^{(t)}|\mathcal{F}_t) = 0$ it yields

$$\mathcal{E}_{-1}(F) = \int_0^T \mathbb{E}\bigg(\bigg(1 + \frac{(1-\mathrm{e}^{-\Lambda(T-t)})^2}{\Lambda^2}(D\lambda)^2 + M_T^{(t)2}$$
$$- 2\frac{1-\mathrm{e}^{-\Lambda(T-t)}}{\Lambda} D\lambda(-1) B_t\bigg)\lambda(B_t)\bigg)dt.$$

Now we have that $\mathbb{E}(\lambda(B_t)) = \mathbf{P}_{\mathbf{t}}^{B_0}\lambda = 2\frac{\lambda(1)\lambda(-1)}{\Lambda} + K\mathrm{e}^{-\Lambda t}$, and that $\mathbb{E}(B_t\lambda(B_t)) = \mathbf{P}_{\mathbf{t}}^{B_0}(\cdot\lambda(\cdot)) = K\mathrm{e}^{-\Lambda t}$, where K is a constant that may differ at each occurence. We have also that

$$\mathbb{E}\big(M_T^{(t)^2}\lambda(B_t)\big) = \mathbb{E}\big(\lambda(B_t)\mathbb{E}\big(M_T^{(t)^2}\big|\mathcal{F}_t\big)\big)$$
$$= \mathbb{E}\left(\lambda(B_t)\mathbb{E}\left(\int_t^T 4\mathrm{e}^{-2\Lambda(s-t)}\lambda(B_s)\,ds\middle|\mathcal{F}_t\right)\right)$$
$$= \mathbb{E}\left(\lambda(B_t)\int_t^T 4\mathrm{e}^{-2\Lambda(s-t)}\big(\mathbf{P}_{\mathbf{s-t}}^{B_t}\lambda\big)\,ds\right)$$
$$= \int_t^T 4\mathrm{e}^{-2\Lambda(s-t)}\mathbb{E}\left(\lambda(B_t)\left(2\frac{\lambda(1)\lambda(-1)}{\Lambda} - \frac{\lambda(B_t)D\lambda(B_t)}{\Lambda}\mathrm{e}^{-\Lambda(s-t)}\right)\right)ds.$$

Since

$$\mathbb{E}\big(\lambda^2(B_t)D\lambda(B_t)\big) = \mathbf{P}_{\mathbf{t}}\big(\lambda^2 D\lambda\big) = -\frac{\lambda(1)\lambda(-1)}{\Lambda}(D\lambda)^2 + K\mathrm{e}^{-\Lambda t},$$

it follows that

$$\mathbb{E}\big(M_T^{(t)2}\lambda(B_t)\big) = \frac{2}{\Lambda}\left(2\frac{\lambda(1)\lambda(-1)}{\Lambda}\right)^2 + \frac{4}{3\Lambda^2}\frac{\lambda(1)\lambda(-1)}{\Lambda}(D\lambda)^2 + K\mathrm{e}^{-\Lambda t} + K\mathrm{e}^{-\Lambda(T-t)}.$$

Therefore,

$$\mathcal{E}_{-1}(F) = \bigg(2\frac{\lambda(1)\lambda(-1)}{\Lambda}\left(1 + \frac{(D\lambda)^2}{\Lambda^2}\right) + \frac{2}{\Lambda}\left(2\frac{\lambda(1)\lambda(-1)}{\Lambda}\right)^2$$
$$+ \frac{4}{3\Lambda^2}\frac{\lambda(1)\lambda(-1)}{\Lambda}(D\lambda)^2\bigg)T + \mathrm{O}(1).$$

Simplifying, we get that

$$\mathcal{E}_{-1}(F) \overset{T\to\infty}{\sim} \frac{2\lambda(1)\lambda(-1)}{\Lambda}\left(2 + \frac{2(D\lambda)^2}{3\Lambda^2}\right)T.$$

Now, we want to estimate the variance of N_T. To start with, note that N_t has mean

$$\mathbb{E}N_t = \mathbb{E}\widehat{N}_t = \int_0^t \left(\mathbf{P}_{\mathbf{s}}^{B_0}\lambda\right) ds = \frac{2\lambda(1)\lambda(-1)}{\Lambda}t - \frac{\lambda(B_0)D\lambda(B_0)}{\Lambda^2} + K\mathrm{e}^{-\Lambda t}.$$

Next we need to estimate $\mathbb{E}(-1)^{N_t}$ and $\mathbb{E}N_t(-1)^{N_t}$ in order to evaluate $\mathbb{E}N_t^2$. Since $(-1)^{N_t} = B_0 B_t$

$$\mathbb{E}(-1)^{N_t} = \mathbf{P}_{\mathbf{t}}^{B_0}(B_0\cdot) = 1 - 2\lambda(B_0)\frac{1-\mathrm{e}^{-\Lambda t}}{\Lambda} = \frac{D\lambda(B_0)}{\Lambda} + K\mathrm{e}^{-\Lambda t}.$$

Then we have that

$$\begin{aligned}
dN_t(-1)^{N_t} &= N_{t^-}d(-1)^{N_t} + (-1)^{N_{t^-}}dN_t + \Delta(-1)^{N_t}\Delta N_t \\
&= \left(-2N_{t^-}(-1)^{N_{t^-}} + (-1)^{N_{t^-}} - 2(-1)^{N_{t^-}}\right) dN_t \\
&= d(\text{martingale}) - (2N_{t^-}+1)(-1)^{N_{t^-}}\lambda(B_{t^-})\,dt \\
&= d(\text{martingale}) - (2N_{t^-}+1)(-1)^{N_{t^-}}\left(\frac{\Lambda}{2} - \frac{D\lambda(B_0)}{2}(-1)^{N_{t^-}}\right)dt,
\end{aligned}$$

which yields

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left(N_t(-1)^{N_t}\right) \\
&\quad = -\Lambda\mathbb{E}\left(N_t(-1)^{N_t}\right) - \frac{\Lambda}{2}\mathbb{E}(-1)^{N_t} + D\lambda(B_0)\mathbb{E}N_t + D\lambda(B_0)/2 \\
&\quad = -\Lambda\mathbb{E}\left(N_t(-1)^{N_t}\right) + \frac{2\lambda(1)\lambda(-1)}{\Lambda}D\lambda(B_0)t - \lambda(B_0)\frac{(D\lambda)^2}{\Lambda^2} + K\mathrm{e}^{-\Lambda t}.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\mathbb{E}\left(N_t(-1)^{N_t}\right) = \frac{2\lambda(1)\lambda(-1)}{\Lambda^2}D\lambda(B_0)t - \Bigg(&\frac{\lambda(B_0)(D\lambda)^2}{\Lambda^3} \\
&+ \frac{2\lambda(1)\lambda(-1)}{\Lambda^3}D\lambda(B_0)\Bigg) + Kt\mathrm{e}^{-\Lambda t} + K\mathrm{e}^{-\Lambda t}.
\end{aligned}$$

Finally, we have that

$$d(N_t)^2 = 2N_{t^-}dN_t + dN_t = d(\text{martingale}) + (2N_{t^-}+1)\lambda(B_t)\,dt,$$

so that

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}N_t^2 &= \mathbb{E}\left((2N_t+1)\left(\frac{\Lambda}{2} - \frac{D\lambda(B_0)}{2}(-1)^{N_t}\right)\right)dt \\
&= \Lambda\mathbb{E}N_t - D\lambda(B_0)\mathbb{E}\left(N_t(-1)^{N_t}\right) + \frac{\Lambda}{2} - \frac{D\lambda(B_0)}{2}\mathbb{E}(-1)^{N_t} \\
&= 2t\left(\frac{2\lambda(1)\lambda(-1)}{\Lambda}\right)^2 + \frac{2\lambda(1)\lambda(-1)}{\Lambda}\left(1 + \frac{(D\lambda)^2}{\Lambda^2} - \frac{2\lambda(B_0)D\lambda(B_0)}{\Lambda^2}\right) \\
&\quad + Kt\mathrm{e}^{-\Lambda t} + K\mathrm{e}^{-\Lambda t}.
\end{aligned}$$

Hence we conclude that

$$\mathbb{E}\big({N_T}^2\big) - \mathbb{E}(N_T)^2 = \frac{2\lambda(1)\lambda(-1)}{\Lambda}\left(1 + \frac{(D\lambda)^2}{\Lambda^2}\right)T + \mathrm{O}(1).$$

Since $1 + \frac{(D\lambda)^2}{3\Lambda^2} \leqslant 1 + \frac{(D\lambda)^2}{\Lambda^2} \leqslant 2$, the energy $\mathcal{E}_{-1}(N_T)$ gives a good estimation of the variance of N_T. Indeed,

$$\frac{1}{2}\mathcal{E}_{-1}(N_T) \leqslant \mathrm{var}(N_T) \leqslant \mathcal{E}_{-1}(N_T)$$

asymptotically. On the opposite, as soon as $(D\lambda)^2 \neq 0$, $\mathcal{E}_0(N_T) < \mathrm{var}(N_T)$. In other words, the Poincaré inequality of parameter 0 is no more true for asymmetric random walks.

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