ANNALES DE L'I. H. P., SECTION B

JULIEN BARRAL

Differentiability of multiplicative processes related to branching random walks

Annales de l'I. H. P., section B, tome 36, nº 4 (2000), p. 407-417
<http://www.numdam.org/item?id=AIHPB_2000__36_4_407_0>

Ann. Inst. Henri Poincaré, Probabilités et Statistiques 36, 4 (2000) 407–417

Differentiability of multiplicative processes related to branching random walks

by

Julien BARRAL *

Université Paris-Sud, Equipe d'Analyse Harmonique (URA 757 du CNRS), Mathématiques, bât. 425, 91405 Orsay Cedex, France

Received in 5 February 1999, revised in 7 May 1999

ABSTRACT. – A family of one-dimensional branching random walks indexed by an interval define a martingale taking values in the space of continuous functions. We propose a new approach to study the differentiability of the limit of this martingale. Under suitable conditions, this differentiability is obtained by assuming that the functions defining the martingale are differentiable only once; there is no loss of regularity. In this sense there is a progress with respect to the corresponding result of Biggins (1991).© 2000 Éditions scientifiques et médicales Elsevier SAS

Key words: Branching random walks, Multiplicative cascades, Martingales, Functional equations

RÉSUMÉ. – Etant donnée une famille de marches aléatoires de branchement sur $\mathbb{R}$ indexée par un intervalle, nous proposons une nouvelle façon d'étudier la dérivabilité de la limite de la martingale à valeurs dans les fonctions continues qu'elles définissent. Sous de bonnes hypothèses, cette dérivabilité est obtenue en supposant que les fonctions définissant la martingale sont dérivables une fois seulement ; il n'y a pas de perte de régularité, et en ce sens il y a un progrès par rapport au résultat de Biggins

* E-mail: julien.barral@math.u-psud.fr.

1. INTRODUCTION

Let $T = \bigcup_{n \geqslant 0} \mathbb{N}^n$ be the set of finite words on $\mathbb{N}$, equipped with the concatenation operation (ε stands for the empty word and $\mathbb{N}^0 = \{\varepsilon\}$). Let $(\Omega, \mathcal{B}, \mathbb{P})$ stand for the probability space on which the random variables (r.v.) in this paper are defined.

Let $\mathcal{A}$ be the set of random sequences $A = (A_i)_{i \geqslant 0} \in \mathbb{R}_+^{\mathbb{N}}$ such that almost surely (a.s.) $\sum_{i \geqslant 0} \mathbb{I}_{\{A_i > 0\}} < \infty$ and $\mathbb{E}(\sum_{i \geqslant 0} A_i) = 1$.

If $A \in \mathcal{A}$ and $(A(a))_{a \in T}$ are independent copies of A then the following sequence

$$Y_{A,n} = \sum_{a_1 \ldots a_n \in \mathbb{N}^n} A_{a_1}(\varepsilon) A_{a_2}(a_1) \ldots A_{a_n}(a_1 \ldots a_{n-1})$$

is a non-negative martingale with mean 1, which converges with probability one to a r.v. $Y_A \geqslant 0$ and such that $\mathbb{E}(Y_A) \leqslant 1$. This martingale is introduced in particular by Mandelbrot in a model for turbulence [12,13] in the case where there exists $c \geqslant 2$ such that a.s. $A_i = 0$ for all $i \geqslant c$. In a different notation it was used by Kingman to study a general branching process [8], and it can also be found in [1], where Biggins constructs it from the branching random walk with points $\{-\log A_i; i \geqslant 0\}$ at the first generation (by convention $\log 0 = -\infty$). Necessary and sufficient conditions for Y_A to be non degenerate and to have finite moments of orders greater than 1 are given in the following result

THEOREM 0. – (1) *Assume that* $\mathbb{E}(\sum_{i \geqslant 0} A_i \log A_i)$ *exists and is finite. The following assertions are equivalent*:

(i) $\mathbb{P}(Y_A = 0) < 1$;

(ii) $\mathbb{E}(Y_A) = 1$;

(iii) $\mathbb{E}(Y_{A,1} \log^+(Y_{A,1})) < \infty$ *and* $\mathbb{E}(\sum_{i \geqslant 0} A_i \log A_i) < 0$.

$(\log^+(.) = \max(0, \log(.)))$.

(2) *Assume the hypothesis of* (1) *and one of the assertions* (i), (ii) *or* (iii). *Then, for each* $p > 1$, $\mathbb{E}(Y_A^p) < \infty$ *if and only if*

$$\mathbb{E}\Big(\sum_{i\geqslant 0} A_i^p\Big) < 1 \quad \textit{and} \quad \mathbb{E}(Y_{A,1}^p) < \infty.$$

Parts (1) and (2) are due to Lyons [11] and Liu [10] respectively, after similar results due to other authors under stronger hypotheses or in particular cases (Kingman [8], Kahane [7], Biggins [1,2], Durrett and Liggett [5], Liu [9]).

Given I an open subinterval of $\mathbb{R}$, $t \mapsto A_t$ is a random process from I to $\mathbb{R}_+^{\mathbb{N}}$ such that for every $t \in I$, $A_t \in \mathcal{A}$, and $(t \mapsto A_t(a))_{a\in T}$ a sequence of independent copies of $t \mapsto A_t$. Now one obtains for every $t \in I$ a martingale $Y_{A_t,n}$ and its limit Y_{A_t}. By the regularity of $t \mapsto A_t$, we mean the regularity of its components, the $t \mapsto A_{t,i}$'s. Then, it is natural to ask whether the martingales $Y_{A_t,n}$ converge simultaneously, and if so, whether some regularity of $t \mapsto A_t$ implies that $t \mapsto Y_{A_t}$ has some related regularity. These problems were studied by Joffe et al. [6] and Biggins [3,4] for particular cases of the previous construction, and for related processes by Watanabe [18]. Biggins [3] considers the random walk $\{-\log A_i;\ i \geqslant 0\}$ and the following family: $\{-\log A_{t,i} = -t\log A_i + \log \mathbb{E}(\sum_{i\geqslant 0} A_i^t);\ i \geqslant 0\}$. It is not difficult to show that his results hold in the general case studied in this paper. His Theorem 2 [3] claims that under suitable conditions, if $t \mapsto A_t$ is a. s. continuously differentiable, then the sequence $t \mapsto Y_{A_t,n}$ converges uniformly a.s. on compact sets to $t \mapsto Y_{A_t}$, and so $t \mapsto Y_{A_t}$ is continuous (he also extends his result to the case where $t \mapsto A_t$ is a. s. C^{n+1} and obtains that $t \mapsto Y_{A_t}$ is of class C^n). If $t \mapsto A_t$ has an analytic extension in a complex neighbourhood of I, Biggins gives also conditions for $t \mapsto Y_{A_t}$ to have an analytic extension in this neighbourhood (Theorem 1 [3]), and in [4] he extends his result to the case of branching random walks indexed by a parameter taking values in an open subset of $\mathbb{C}^n$. The aim of this paper is to give conditions under which if $t \mapsto A_t$ is a.s. continuously differentiable, then $t \mapsto Y_{A_t,n}$ converges uniformly a.s. on compact sets and $t \mapsto Y_{A_t}$ is also continuously differentiable (we also give conditions to extend our result to the case of processes n times continuously differentiable). Our approach has elements in common with [3] and also [6], where Joffe et al. use a result on the convergence of martingales taking values in a Banach space. The new ideas here are the use of a criterion on the second differences of a function to show this function is continuously differentiable, and to exploit the functional equation satisfied for all $t \in I$ by Y_{A_t} a.s.: $Y_{A_t} = \sum_{i\geqslant 0} A_{t,i} Y_{A_t}(i)$, where the $Y_{A_t}(i)$'s are independent copies of Y_{A_t}, and the σ-algebra that they generate is independent of the one generated by A_t.

The main result is the following one.

THEOREM 1. – *Let I be an open subinterval of $\mathbb{R}$. For every $i \in \mathbb{N}$, let $t \mapsto A_{t,i}$ be a random continuously differentiable mapping from I to $\mathbb{R}_+$. Assume that for every $t \in I$, $A_t = (A_{t,i})_{i\geqslant 0}$ is in $\mathcal{A}$ and let $(t \mapsto A_t(a))_{a\in T}$ be a sequence of independent copies of $t \mapsto A_t$. Denote by $t \mapsto Y_{t,n}$ the sequence of functions $t \mapsto Y_{A_t,n}$, and for every $t \in I$, denote by Y_t the almost sure limit of $Y_{t,n}$. Suppose the following two conditions hold.*

(i) *For every $t \in I$, A_t satisfies the assumption and condition* (i) *of Theorem* 0.

(ii) *For every compact subinterval K of I, there exists $p \in]1,2]$ such that*

(a) $\sup_{t\in K} \mathbb{E}(\sum_{i\geqslant 0} A_{t,i}^p) < 1$ *and* $\sup_{t\in K} \mathbb{E}([\sum_{i\geqslant 0} A_{t,i}]^p) < \infty$;

(b) *there exists $C > 0$ such that for all $(t,h) \in K \times \mathbb{R}_+$ such that $t+h \in K$*

$$\mathbb{E}\Big(\sum_{i\geqslant 0} |A_{t+h,i} - A_{t,i}|^p\Big) + \mathbb{E}\Big(\Big|\sum_{i\geqslant 0} A_{t+h,i} - A_{t,i}\Big|^p\Big) \leqslant Ch^p; \tag{1.1}$$

(c) *there exist two positive functions φ and γ on $\mathbb{R}_+^*$, both monotonically decreasing in a neighbourhood of* 0 *such that $\varphi(h)/h$ and $\max(\gamma(h), h^{2p})/(h^{p+2}\varphi^p(h/2))$ are integrable near* 0*, and a constant $C > 0$ such that for all $(t,h) \in K \times \mathbb{R}_+$ such that $t+h \in K$ and $t-h \in K$*

$$\mathbb{E}\Big(\sum_{i\geqslant 0} |A_{t+h} + A_{t-h} - 2A_t|^p\Big) + \mathbb{E}\Big(\Big|\sum_{i\geqslant 0} A_{t+h} + A_{t-h} - 2A_t\Big|^p\Big) \leqslant C\gamma(h). \tag{1.2}$$

Then, with probability one, $Y_{t,n}$ converges uniformly towards Y_t on compact sets and $t \mapsto Y_t$ is continuously differentiable on I.

Remark. – (1) Hypotheses (ii)(a)–(b) are stronger than those of [3, Theorem 2]. However hypotheses (ii)(c) is weaker than assuming that the process is twice differentiable as in [3].

(2) If the mappings $t \mapsto A_{t,i}$'s are only supposed to be continuous, and in (1.1) h^p is replaced by h^α with $\alpha > 1$, then the process $t \mapsto Y_t$, possesses a continuous modification.

(3) If the $t \mapsto A_{t,i}$'s are just supposed to be continuous, (1.1) and (1.2) imply that they are continuously differentiable, so the theorem could be reformulated to account for this.

(4) If one chooses suitably $A \in \mathcal{A}$, $\varepsilon > 0$ and for all $i \in \mathbb{N}$, f_i a continuously differentiable function from $I =]-\varepsilon, \varepsilon[$ to $\mathbb{R}$ such that $f_i(0) = 1$, then defining for every $t \in I$, $A_t = (A_i^{f_i(t)}/\mathbb{E}(\sum_{i\geqslant 0} A_i^{f_i(t)}))_{i\geqslant 0}$ yields examples to which Theorem 1 applies.

(5) For a function f from a subinterval of $\mathbb{R}$ to $\mathbb{R}$ and $h \in \mathbb{R}$, define (where it is possible) $\Delta_h f : t \mapsto f(t+h) - f(t)$ and $\Delta_h^2 f : t \mapsto \Delta_h \circ \Delta_h f(t) = f(t+2h) + f(t) - 2f(t+h)$.

In Theorem 1, if one replaces (ii)(b) and (ii)(c) by the following:
There exists $n \geqslant 2$ such that

(ii)(b) for all $0 \leqslant k \leqslant n-1$, there exists $C_k > 0$ such that for all $(t, h_1, \ldots, h_{k+1})$ in $K \times \mathbb{R}^{k+1}$ such that the $\Delta_{h_{k+1}} \circ \cdots \circ \Delta_{h_1} A_{.,i}(t)$'s, $i \geqslant 0$, are defined,

$$\mathbb{E}\sum_{i\geqslant 0} \left|\Delta_{h_{k+1}} \circ \cdots \circ \Delta_{h_1} A_{.,i}(t)\right|^p + \mathbb{E}\left|\sum_{i\geqslant 0} \Delta_{h_{k+1}} \circ \cdots \circ \Delta_{h_1} A_{.,i}(t)\right|^p \leqslant C_k |h_1|^p \cdots |h_{k+1}|^p.$$

(ii)(c) There exist two positive functions φ and γ on $\mathbb{R}_+^*$, both monotonically decreasing in a neighbourhood of 0, such that $\varphi(h)/h$ and $\max(\gamma(h), h^{2p})/(h^{p+2}\varphi^p(h/2))$ are integrable near 0, and a constant $C > 0$ such that for all $(t, h_1, \ldots, h_n)$ in $K \times \mathbb{R}^n$ such that the $\Delta_{h_n} \circ \cdots \circ \Delta_{h_2} \circ \Delta_{h_1}^2 A_{.,i}(t)$'s, $i \geqslant 0$, are defined,

$$\mathbb{E}\sum_{i\geqslant 0} \left|\Delta_{h_n} \circ \cdots \circ \Delta_{h_2} \circ \Delta_{h_1}^2 A_{.,i}(t)\right|^p + \mathbb{E}\left|\sum_{i\geqslant 0} \Delta_{h_n} \circ \cdots \circ \Delta_{h_2} \circ \Delta_{h_1}^2 A_{.,i}(t)\right|^p \leqslant C\gamma\left(|h_1|\right)|h_2|^p \cdots |h_n|^p.$$

Then, with probability one, $Y_{t,n}$ converges uniformly on the compact sets towards Y_t and the mappings $t \mapsto A_{t,i}$, $i \geqslant 0$, and $t \mapsto Y_t$ are n times continuously differentiable.

2. PROOF OF THEOREM 1

We need a series of lemmas. The first one is a generalization of a result of Von Bahr and Esseen [17], which is also used in a refined form in [3].

LEMMA 2.1. – *Let $(U_i)_{i\geqslant 0}$ and $(V_i)_{i\geqslant 0}$ be two sequences of real r.v.'s such that $\sigma(U_i,\ i \geqslant 0)$ and $\sigma(V_i,\ i \geqslant 0)$ are independent and the V_i's are mutually independent. Assume that for all $i \geqslant 0$, V_i is integrable and $\mathbb{E}(V_i) = 0$. Then, for every $p \in]1,2]$,*

$$\mathbb{E}\left(\left|\sum_{i\geqslant 0} U_i V_i\right|^p\right) \leqslant 2^p \sum_{i\geqslant 0} \mathbb{E}(|U_i|^p)\mathbb{E}(|V_i|^p).$$

If $B = (B_i)_{i\geqslant 0}$ is a random vector taking values in $\mathbb{R}$, denote by ψ_B and S_B the functions from $\mathbb{R}_+$ to $[0,\infty]$ defined by

$$\psi_B(x) = \mathbb{E}\left(\sum_{i\geqslant 0} |B_i|^x\right)$$

and

$$S_B(x) = \mathbb{E}\left[\left|\sum_{i\geqslant 0} B_i\right|^x\right].$$

LEMMA 2.2. – *Let A_1, A_2 and A_3 be elements of $\mathcal{A}$ and let $(A_1(a), A_2(a), A_3(a))_{a\in T}$ be a sequence of independent copies of (A_1, A_2, A_3). Assume that A_1, A_2 and A_3 satisfy the assumption and condition* (i) *of Theorem* 0*, and also that there exists $p' > 1$ such that for every $j \in \{1,2,3\}$, $Y_j = Y_{A_j}$ has a finite moment of order p'. Define $p = \min(p', 2)$. By Theorem* 0*, for $j \in \{1,2,3\}$, $\psi_{A_j}(p) < 1$ and $S_{A_j}(p) < \infty$. Choose an integer m sufficiently large that $\psi_{A_3}(p) < 2^{-p/m+1}$. Then*

$$\|Y_1 + Y_2 - 2Y_3\|_p \leqslant \frac{T_1(p) + T_2(p) + T_3(p)}{(1 - \psi_{A_3}^{1/p}(p))(1 - 2\psi_{A_3}^{(m+1)/p}(p))}$$

with

$$T_1(p) = \|Y_2 - Y_1\|_p \left(\psi_{A_2-A_3}^{1/p}(p) + \psi_{A_1-A_3}^{1/p}(p)\right),$$

$$T_2(p) = \left(\|Y_1 - 1\|_p + \|Y_2 - 1\|_p\right)\psi^{1/p}_{A_1+A_2-2A_3}(p)$$

and $T_3(p) = 2S^{1/p}_{A_1+A_2-2A_3}(p)$.

Proof. – First, it is easily seen that for $n \geqslant 1$ and $j \in \{1, 2, 3\}$:

$$Y_j = \sum_{a=a_1\dots a_n \in \mathbb{N}^n} A_{j,a_1}(\varepsilon)A_{j,a_2}(a_1)\cdots A_{j,a_n}(a_1\dots a_{n-1})Y_j(a),$$

where

$$Y_j(a) = \lim_{p\to\infty} \sum_{a'_1\dots a'_p \in \mathbb{N}^p} A_{j,a'_1}(a)A_{j,a'_2}(aa'_1)\cdots A_{j,a'_p}(aa'_1\dots a'_{p-1}),$$

and the r.v.'s $(Y_1(a), Y_2(a), Y_3(a))$, $a \in \mathbb{N}^n$, are independent copies of (Y_1, Y_2, Y_3), which are also independent of the $(A_1(b), A_2(b), A_3(b))$, $b \in \bigcup_{k=0}^{n-1} \mathbb{N}^k$ and satisfy a.s.

$$Y_j(a) = \sum_{i\geqslant 0} A_{j,i}(a)Y_j(ai).$$

So

$$\begin{aligned} Y_1 + Y_2 - 2Y_3 = &\sum_{i\geqslant 0} A_{3,i}\left(Y_1(i) + Y_2(i) - 2Y_3(i)\right) \\ &+ \sum_{i\geqslant 0}(A_{2,i} - A_{3,i})\left(Y_2(i) - Y_1(i)\right) \\ &+ \sum_{i\geqslant 0}(A_{1,i} + A_{2,i} - 2A_{3,i})\left(Y_1(i) - 1\right) \\ &+ \sum_{i\geqslant 0}(A_{1,i} + A_{2,i} - 2A_{3,i}), \end{aligned}$$

and using this decomposition repeatedly in the first term on the right, one obtains for every integer $m \geqslant 0$

$$Y_1 + Y_2 - 2Y_3 = Q_m + \sum_{k=0}^{m} R_k + S_k + S'_k$$

with

$$\begin{aligned} Q_m = &\sum_{a\in\mathbb{N}^m} \sum_{i\geqslant 0} \left[\prod_{\ell=0}^{m-1} A_{3,a_{\ell+1}}(a_1\dots a_\ell)\right] \\ &\times A_{3,i}(a)\left(Y_1(ai) + Y_2(ai) - 2Y_3(ai)\right), \end{aligned}$$

$$R_k = \sum_{a\in\mathbb{N}^k}\sum_{i\geqslant 0}\left[\prod_{\ell=0}^{k-1} A_{3,a_{\ell+1}}(a_1\dots a_\ell)\right] \times \big(A_{2,i}(a)-A_{3,i}(a)\big)\big(Y_2(ai)-Y_1(ai)\big),$$

$$S_k = \sum_{a\in\mathbb{N}^k}\sum_{i\geqslant 0}\left[\prod_{\ell=0}^{k-1} A_{3,a_{\ell+1}}(a_1\dots a_\ell)\right] \times \big(A_{1,i}(a)+A_{2,i}(a)-2A_{3,i}(a)\big)\big(Y_1(ai)-1\big)$$

and

$$S'_k = \sum_{a\in\mathbb{N}^k}\left[\prod_{\ell=0}^{k-1} A_{3,a_{\ell+1}}(a_1\dots a_\ell)\right]\left[\sum_{i\geqslant 0} A_{1,i}(a)+A_{2,i}(a)-2A_{3,i}(a)\right].$$

Now the fact that A_1, A_2, and A_3 are in $\mathcal{A}$, the equalities $\mathbb{E}(Y_1)=\mathbb{E}(Y_2)=\mathbb{E}(Y_3)=1$ and the independences between r.v.'s allow the application of Lemma 2.1 successively with, instead of the (U_i, V_i)'s, the

$$\left(\left[\prod_{\ell=0}^{m-1} A_{3,a_{\ell+1}}(a_1\dots a_\ell)\right]A_{3,i}(a),\ Y_1(ai)+Y_2(ai)-2Y_3(ai)\right)\text{'s}$$

in Q_m, the

$$\left(\left[\prod_{\ell=0}^{k-1} A_{3,a_{\ell+1}}(a_1\dots a_\ell)\right]\big(A_{2,i}(a)-A_{3,i}(a)\big),\ Y_2(ai)-Y_1(ai)\right)\text{'s}$$

in R_k, the

$$\left(\left[\prod_{\ell=0}^{k-1} A_{3,a_{\ell+1}}(a_1\dots a_\ell)\right]\big(A_{1,i}(a)+A_{2,i}(a)-2A_{3,i}(a)\big),\ Y_1(ai)-1\right)\text{'s}$$

in S_k and the

$$\left(\prod_{\ell=0}^{k-1} A_{3,a_{\ell+1}}(a_1\dots a_\ell),\ \sum_{i\geqslant 0} A_{1,i}(a)+A_{2,i}(a)-2A_{3,i}(a)\right)\text{'s}$$

in S'_k.

Then, standard calculus yields

$$\|Q_m\|_p \leqslant 2\psi_{A_3}^{(m+1)/p}(p)\|Y_1+Y_2-2Y_3\|_p,$$
$$\|R_k\|_p \leqslant 2\psi_{A_3}^{k/p}(p)\psi_{A_2-A_3}^{1/p}(p)\|Y_1-Y_2\|_p,$$
$$\|S_k\|_p \leqslant 2\psi_{A_3}^{k/p}(p)\psi_{A_1+A_2-2A_3}^{1/p}(p)\|Y_1-1\|_p$$

and

$$\|S'_k\|_p \leqslant 2\psi_{A_3}^{k/p}(p) S_{A_1+A_2-2A_3}^{1/p}(p).$$

Now, the conclusion comes from the inequality

$$\|Y_1 + Y_2 - 2Y_3\|_p \leqslant \frac{\sum_{k=0}^m (\|R_k\|_p + \|S_k\|_p + \|S'_k\|_p)}{1 - 2\psi_{A_3}^{(m+1)/p}(p)},$$

and a symetrization of the right-hand side. □

The following lemma is a slightly stronger form of a well known result [15].

LEMMA 2.3. – *Let $a < b$ be in $\mathbb{R}$. Let f be a continuous function from $[a, b]$ to $\mathbb{R}$. Assume that there exists a positive function φ on $\mathbb{R}_+$, monotonically decreasing in a neighbourhood of* 0*, such that $\varphi(h)/h$ is integrable near* 0*, and for some constant $C > 0$: $\forall j \in \mathbb{N}$ and $0 \leqslant k < 2^j$*

$$\left| f\left(a + \frac{k(b-a)}{2^j}\right) + f\left(a + \frac{(k+1)(b-a)}{2^j}\right) \right.$$
$$\left. - 2f\left(a + \frac{(k+\frac{1}{2})(b-a)}{2^j}\right)\right| \leqslant C\frac{b-a}{2^{j+1}}\varphi\left(\frac{b-a}{2^{j+1}}\right).$$

Then, f is continuously differentiable.

Proof of Theorem 1. – Fix $K = [a, b]$ a non-trivial compact subinterval of I and choose an integer $m_K \geqslant 0$ such that $\sup_{t\in K} \psi_{A_t}(p) < 2^{-(m_K+1)/p}$. By the proof of Theorem 5.1 of [9] and the hypothesis (ii)(a), for all $t \in K$,

$$\mathbb{E}(Y_t^p) \leqslant \frac{\sup_{t\in K} S_{A_t}(p)}{1 - \sup_{t\in K} \psi_{A_t}(p)}.$$

So, hypothesis (ii)(b) together with Lemma 2.2 applied with $A_1 = A_2 = A_{t+h}$, $A_3 = A_t$ and $m = m_K$ yield a constant C_1 such that for all $(t, h) \in K \times \mathbb{R}_+$ such that $t + h \in K$

$$\mathbb{E}(|Y_{t+h} - Y_t|^p) \leqslant C_1 h^p. \tag{2.1}$$

Then, the Kolmogorov–Tchentov Theorem [16] yields a continuous modification $t \mapsto \widetilde{Y}_t$ of $t \mapsto Y_t$.

Now, hypothesis (ii)(c), (2.1) and Lemma 2.2 applied with $A_1 = A_{t+h}$, $A_2 = A_{t-h}$, $A_3 = A_t$ and $m = m_K$ yield a constant C_2 such that for all $(t, h) \in K \times \mathbb{R}_+$ such that $t + h \in K$ and $t - h \in K$,

$$\mathbb{E}(|\widetilde{Y}_{t+h} + \widetilde{Y}_{t-h} - 2\widetilde{Y}_t|^p) \leqslant C_2 \max(\gamma(h), h^{2p}).$$

Thus, for all $j \in \mathbb{N}$,

$$
\begin{aligned}
p_j &= \mathbb{P}\biggl(\max_{0\leqslant k<2^j}\Bigl|\widetilde{Y}_{a+\frac{k(b-a)}{2^j}} + \widetilde{Y}_{a+\frac{(k+1)(b-a)}{2^j}} - 2\widetilde{Y}_{a+\frac{(k+1/2)(b-a)}{2^j}}\Bigr| \\
&\qquad > \frac{b-a}{2^{j+1}}\varphi\Bigl(\frac{b-a}{2^{j+1}}\Bigr)\biggr) \\
&\leqslant 2^j\,\frac{2^{(j+1)p}}{(b-a)^p}\varphi^{-p}\Bigl(\frac{b-a}{2^{j+1}}\Bigr)C_2\max\Bigl(\gamma\Bigl(\frac{b-a}{2^{j+1}}\Bigr),\frac{(b-a)^{2p}}{2^{2p(j+1)}}\Bigr),
\end{aligned}
$$

and as $\max(\gamma(h), h^{2p})/(h^{p+2}\varphi^p(h/2))$ is integrable near 0 and φ and γ are monotonically decreasing in a neighborhood of 0, $\sum_{j\geqslant 0} p_j < \infty$. Then, by the Borel–Cantelli Lemma, with probability one, $t \mapsto \widetilde{Y}_t$ satisfies the hypothesis of Lemma 2.3 with the function φ, so it is continuously differentiable on K.

Recall that each random continuous function $t \mapsto Y_{t,n}$ take values in the separable Banach space $(C^0(K,\mathbb{R}_+), \|\,\|_\infty)$. For $n \geqslant 1$, denote by $\mathcal{B}_n$ the σ-algebra generated in Ω by the $t \mapsto Y_{t,k}$'s, $1 \leqslant k \leqslant n$, and define $\mathcal{B}_\infty = \bigcup_{n\geqslant 1}\mathcal{B}_n$. The r.v. $t \mapsto \widetilde{Y}_t$ is $\mathcal{B}_\infty$-measurable, and if we can show that it is integrable, that is $\mathbb{E}(\max_{t\in K}\widetilde{Y}_t) < \infty$, then (proceeding as Joffe et al. [6]), it is easily verified that for all $n \geqslant 1$, $\mathbb{E}(\widetilde{Y}_.|\mathcal{B}_n) = Y_{.,n}$, and by Proposition V-2-6 of [14], with probability one, $t \mapsto Y_{t,n}$ converges uniformly towards $t \mapsto \widetilde{Y}_t$. So with probability one, $\widetilde{Y}_t = Y_t$ for all $t \in K$ and one has the conclusion of the theorem, but with K instead of I.

The fact that $\mathbb{E}(\max_{t\in K}\widetilde{Y}_t) < \infty$ comes from (2.1) together with the differentiability of $t \mapsto \widetilde{Y}_t$, which yield $\sup_{t\in K}\mathbb{E}(|\frac{d}{dt}Y_t|^p) \leqslant C_1$ by the Fatou Lemma.

One ends the proof by writing I as a countable union of compact subintervals. □

ACKNOWLEDGEMENTS

The author thanks J. Peyrière for helpful conversations, and is also indebted to J.D. Biggins for his remarks and advice which improved this text considerably.

REFERENCES

[1] J.D. Biggins, Martingale convergence in the branching random walk, J. Appl. Probab. 14 (1977) 25–37.

[2] J.D. Biggins, Growth rates in the branching random walk, Z. Wahrsch. Verw. Gebiete 48 (1979) 17–34.
[3] J.D. Biggins, Uniform convergence of martingales in the one-dimensional branching random walk, in: I.V. Basawa, R.L. Taylor (Eds.), Selected Proceedings of the Sheffield Symposium on Applied Probability, 1989, IMS Lectures Notes—Monograph Series, Vol. 18, 1991, pp. 159–173.
[4] J.D. Biggins, Uniform convergence of martingales in the branching random walk, Ann. Probab. 20 (1992) 137–151.
[5] R. Durrett, Th. Liggett, Fixed points of the smoothing transformation, Z. Wahrsch. Verw. Gebiete 64 (1983) 275–301.
[6] A. Joffe, L. Le Cam, J. Neveu, Sur la loi des grands nombres pour les variables aléatoires de Bernouilli attachées à un arbre dyadique, C. R. Acad. Sci. Paris 278 (1973) 963–964.
[7] J.-P. Kahane, J. Peyrière, Sur certaines martingales de Benoît Mandelbrot, Adv. Math. 22 (1976) 131–145.
[8] J.F.C. Kingman, The first birth problem for an age-dependent branching process, Ann. Probab. 3 (1975) 790–801.
[9] Q. Liu, Sur une équation fonctionnelle et ses applications: une extension du théorème de Kesten–Stigum concernant des processus de branchement, Adv. Appl. Probab. 29 (2) (1997) 353–373.
[10] Q. Liu, Self-similar cascades and the branching random walk, Preprint, Univ. Rennes 1, 1997.
[11] R. Lyons, A simple path to Biggins' martingale convergence, in: K.B. Athreya, P. Jagers (Eds.), Classical and Modern Branching Processes, IMA Volumes in Mathematics and its Applications, Vol. 84, Springer, Berlin, 1997, pp. 217–222.
[12] B. Mandelbrot, Intermittent turbulence in self-similar cascades, divergence of hight moments and dimension of the carrier, J. Fluid. Mech. 62 (1974) 331–358.
[13] B. Mandelbrot, Multiplications aléatoires itérées et distributions invariantes par moyennes pondérées, C. R. Acad. Sci. Paris 278 (1974) 289–292, 355–358.
[14] J. Neveu, Martingales à Temps Discret, Masson et C^{ie}, Paris, 1972.
[15] E.M. Stein, A. Zygmund, Smoothness and differentiability of functions, Annales Univ. Sci. Budapest. III–IV (1960–1961) 295–307.
[16] N.N. Tchentov, Weak convergence of stochastic processes whoose trajectories have no discontinuity of the second kind and the "heuristic" approach to the Kolmogorov–Smirnov tests, Theory Probab. Appl. 1 (1956) 140–144.
[17] B. Von Bahr, C.-G. Esseen, Inequalities for the rth absolute moment of sum of random variables, $1 \leqslant r \leqslant 2$, Ann. Math. Statist. 36 (1965) 299–303.
[18] S. Watanabe, Limit theorem for a class of branching processes, in: J. Chover (Ed.), Markov Processes and Potential Theory, Wiley, New York, 1967, pp. 205–232.