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# Statistics of return times for weighted maps of the interval

by

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ABSTRACT. – For non markovian, piecewise monotonic maps of the interval associated to a potential, we prove that the law of the entrance time in a cylinder, when renormalized by the measure of the cylinder, converges to an exponential law for almost all cylinders. Thanks to this result, we prove that the fluctuations of $R_n$, first return time in a cylinder, are lognormal. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Pour des applications non markoviennes, monotones par morceaux de l'intervalle, associées à un potentiel, on prouve que la loi du temps d'entrée dans un cylindre, renormalisée par la mesure de ce cylindre, converge vers une loi exponentielle pour presque tous les cylindres. Ce résultat permet ensuite de montrer que les fluctuations de $R_n$, temps de premier retour dans un cylindre, suivent une loi lognormale. © 2000 Éditions scientifiques et médicales Elsevier SAS

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## 1. INTRODUCTION

In this article, we study the asymptotic law of $R_n$, which is, for a stationary stochastic process, the first time when the process repeats its $n$ first symbols. In the same way, for a piecewise monotonic map $T$ of the interval, $R_n$ is the first return time in an interval of continuity of $T^n$. When the dynamical system is ergodic, Ornstein and Weiss [8] have proved that $\lim_{n\to\infty} \frac{1}{n}\log R_n = h$, where the convergence is almost sure and $h$ is the entropy of the system. Results about fluctuations of $\log R_n$ around $nh$ are obtained for systems with the Gibbs property by Collet, Galves and Schmitt [3]. Showing that the non-markov part of the system can be disregarded, and proving something similar to the Gibbs property defined in [1], (third part), we give the same results for piecewise monotonic maps of the interval associated to a bounded variation weight, that is to say: the law of $R_n$, correctly renormalized, converges to a lognormal distribution. This convergence strongly uses the fact that we can approximate the law of the entrance time in a cylinder by an exponential law, which is proved in the fifth part.

Consider the following setting: $T$ is a piecewise monotonic transformation (with $b$ branches). $T$ is piecewise $\mathcal{C}^2$, which means that there is a subdivision $(a_i)_{i=0}^{i=b}$ of [0,1] such that $T$ is monotonic and extends to a $\mathcal{C}^2$ map on each $]a_i, a_{i+1}[$. Denote by $\mathrm{sing}(T)$ the set $\{a_i, i = 0, \ldots, b\}$ of the points where $T$ is not continuous and let $A_i = ]a_i, a_{i+1}[$. We call $n$-cylinder a set as follows: $A_{i_1}^{i_n} = A_{i_1} \cap T^{-1}A_{i_2} \cap \cdots \cap T^{-n+1}A_{i_n}$. Denote by $\mathcal{P}^n$ the set of $n$-cylinders. For all $x$ in $[0,1] \setminus \bigcup_0^\infty T^{-n}(\mathrm{sing}(T))$ and all $n$, there is a unique $n$-cylinder containing $x$, called $\mathcal{P}^n(x)$.

We assume that the borelian $\sigma$-field $\mathcal{B}$ is generated by the finite partition $]a_i, a_{i+1}]$.

We are going to study the asymptotic law of $R_n$ for a measure $\mu_\varphi$ invariant by $T$, where $\varphi$ is a measurable potential. The study of dynamical systems associated to a potential (different from the inverse of the jacobian of $T$) arise from statistical mechanics, where the potential figures the interaction between the particles (see [1]). Another motivation is when the potential is equal to zero, the equilibrium states are then measures which maximize the entropy.

Given a measurable potential $\varphi$, define the associated transfer operator (for $f$ measurable) by

$$P_\varphi f(x) = \sum_{T(y)=x} \mathrm{e}^{\varphi(y)} f(y).$$

We define the topological pressure of the system as follows:

$$p(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in[0,1]} P_\varphi^n \mathbf{1}(x)$$

($p(\varphi)$ is well defined because the sequence $(\sup_{x\in[0,1]} P_\varphi^n \mathbf{1}(x))_{n\in\mathbb{N}}$ is submultiplicative).

DEFINITION 1.1. – *A measurable function $f$ on* $[0,1]$ *has bounded variation* ($f \in BV([0,1])$) *if* $\operatorname{var}_{[0,1]} f = \operatorname{var} f < \infty$, *where we define the variation on a set $A$ by*

$$\operatorname{var}_A f = \sup \sum_{i=1}^{n} \left| f(x_{i-1}) - f(x_i) \right|,$$

*the supremum is taken over all finite partitions of $A$*: $0 = x_0 < \cdots < x_n = 1$, $n \geqslant 1$.

Recall that a measure is $\mathrm{e}^{p(\varphi)-\varphi}$-conformal (in the sense of Denker and Urbanski [4]) if for all measurable sets $A$ such that $T: A \to T(A)$ is invertible:

$$\nu\big(T(A)\big) = \int_A \mathrm{e}^{p(\varphi)-\varphi} \, \mathrm{d}\nu.$$

Assuming certain hypothesis on the potential (see the next section), Liverani, Saussol and Vaienti [7] prove the existence of a conformal measure $\nu$ and the existence of a unique measure invariant by $T$, $\mu_\varphi$, absolutely continuous with respect to $\nu$ and satisfying exponential decay of correlations. Under the same hypothesis on the weight, we can state our main result:

Define the entrance time in a cylinder $A$ by

$$\tau_A(x) = \inf\{k \geqslant 0, T^k(x) \in A\}.$$

In the same way, we define the return time in a cylinder:

$$R_n(x) = \inf\{k > 0, T^k(x) \in \mathcal{P}^n(x)\}.$$

Define, for $f$ with bounded variation, the quantity that usually appears in the central limit theorem, i.e., the asymptotic variance $\sigma(f)$ (see [6]):

$$C_n(f) = \int f \circ T^n f \, \mathrm{d}\mu_\varphi - \left( \int f \, \mathrm{d}\mu_\varphi \right)^2,$$

$$\sigma^2(f) = C_0(f) + 2\sum_{n=1}^{\infty} C_n(f).$$

$\sigma^2(f)$ is well defined because $C_n(f)$ is the autocorrelation of $f$ and so, it decays exponentially fast.

Let $h = h_{\mu_\varphi}$ be the entropy associated to the measure $\mu_\varphi$, i.e.:

$$h = \lim_{n\to\infty} \frac{1}{n} \log \#\{A \in \mathcal{P}^n, \mu_\varphi(A) > 0\}.$$

THEOREM 1.1. – *Assume $\sigma(\varphi) \neq 0$, then*

$$\left(\frac{\log R_n - nh}{\sigma(\varphi)\sqrt{n}}\right)_{n\in\mathbb{N}}$$

*is a sequence of well defined random variables on the probability space* $([0, 1], \mathcal{B}, \mu_\varphi)$ *and*

$$\frac{\log R_n - nh}{\sigma(\varphi)\sqrt{n}} \Rightarrow \mathcal{N}(0, 1),$$

*where $\Rightarrow$ is a convergence in law (and $\sigma(\varphi) = 0$ if and only if there exists a measurable $g$ such that $\varphi = g - g \circ T$).*

## 2. PIECEWISE MONOTONIC MAPS OF THE INTERVAL

Recall that $T$ is a piecewise monotonic map of the interval. For $x \in [0, 1]$, let

$$S_n(x) = \exp\left(\sum_{i=0}^{n-1} \varphi \circ T^i(x)\right).$$

Let us make the following hypothesis on the potential and the system:

(H1) $\exp(\varphi)$ has bounded variation.

(H2) (distortion) $\sum_{n=1}^{+\infty} \sup_{C\in\mathcal{P}^n} \operatorname{var}_C \varphi < \infty$.

(H3) (dilatation) $\sup \varphi < p(\varphi)$.

(H4) (covering) $\forall I$ interval, $\exists N(I) \in \mathbb{N}^*$, $C(I) > 0$, $\inf P_\varphi^{N(I)} \mathbf{1}_I \geqslant C(I)$.

(H2) is called a distortion hypothesis because it allows us to show the distortion property (see Lemma 2.5).

(H3) is called a dilatation hypothesis because it really plays the same role as the hypothesis $\inf |T'| \geqslant \rho > 1$ when the potential is the logarithm of the inverse of the derivative of $T$.

(H4) is equivalent, when $\varphi$ is bounded from below (for example when $\varphi$ is the logarithm of the inverse of the derivative of $T$ and $T$ is strictly expanding), to the following:

$$\forall I \text{ interval}, \ \exists N(I) \in \mathbb{N}^*, \quad T^{N(I)} I \supset [0,1].$$

### Lasota–Yorke inequality

THEOREM 2.1. – *Under the hypothesis* (H1), (H2), (H3), *there exist* $\alpha < 1$ *and* $\xi > 0$ *such that for all* $f \in BV([0,1])$, $f \geqslant 0$:

$$\frac{1}{\lambda} \operatorname{var}\left(P_\varphi(f)\right) \leqslant \alpha \operatorname{var}(f) + \xi \nu(f).$$

*Proof.* – The proof is deeply based on the Sub-lemma 4.1.1 of [7]:

SUB-LEMMA 4.1.1. – *For all integer m, there exists* $B_m < \infty$ *such that, for all positive function f with bounded variation*

$$\operatorname{var}\left(P_\varphi^m f\right) \leqslant 9 \sup S_m \operatorname{var}(f) + B_m \int f \, \mathrm{d}\nu.$$

By hypothesis: $\sup S_m \leqslant \mathrm{e}^{m \sup \varphi} < \lambda^m$; let $m$ such that $\mathrm{e}^{m(\sup\varphi - p(\varphi))} < 1/9$ (recall that $\lambda = \mathrm{e}^{p(\varphi)}$):

$$\frac{1}{\lambda^m} \operatorname{var}\left(P_\varphi^m f\right) \leqslant \alpha_m \operatorname{var}(f) + B_m \nu(f)$$

with $\alpha_m < 1$ and $B_m < \infty$. It is then sufficient to consider the iterate $P_\varphi^m$ to get the desired inequality. □

### Existence of conformal and invariant measures

THEOREM 2.2 (Liverani, Saussol and Vaienti [7]). – *Under the hypothesis* (H1), ..., (H4), *there exists a non atomic* $\mathrm{e}^{p(\varphi)-\varphi}$*-conformal measure* $\nu$ *and there exists a unique invariant probability measure* $\mu_\varphi$ *absolutely continuous with respect to* $\nu$. $\nu$ *and* $\mu_\varphi$ *are obtained in the following way*:

*there exist* $\lambda > 0$ *and* $h_\varphi$ *such that*:

$$P_\varphi h_\varphi = \lambda h_\varphi, \qquad \nu(h_\varphi) = 1, \qquad P_\varphi^*(\nu) = \lambda \nu$$

$\mu_\varphi = h_\varphi \nu$, *the density* $h_\varphi$ *is positive, has bounded variation and* $\lambda = \mathrm{e}^{p(\varphi)}$. *Moreover,* $\inf(h_\varphi) > 0$.

THEOREM 2.3 ([7]). – *Under the same hypothesis, $\mu_\varphi$ is the unique equilibrium state for $\varphi$, i.e.:*

$$p(\varphi) = h_{\mu_\varphi}(T) + \int \varphi \, \mathrm{d}\mu_\varphi = \sup\left\{ h_m(T) + \int \varphi \, \mathrm{d}m \right\},$$

*where $h_m(T)$ denotes the entropy of the measurable system $(T, m)$ and the supremum is taken over all the $T$-invariant measures $m$.*

The main ingredient to show these theorems is the Lasota–Yorke inequality. The covering hypothesis is needed to get a strictly positive density $h_\varphi$.

## Decay of correlations

THEOREM 2.4 ([7]). – *Assuming the same hypothesis as before, the decay of correlation is exponential: there is $\gamma > 0$ and a constant $K$ such that, if $f$ has bounded variation and $g$ is integrable:*

$$\left| \int f g \circ T^n \, \mathrm{d}\mu_\varphi - \int f \, \mathrm{d}\mu_\varphi \int g \, \mathrm{d}\mu_\varphi \right| \leqslant K \, \mathrm{e}^{-\gamma n} \left( \int |f| \, \mathrm{d}\mu_\varphi + \operatorname{var} f \right) \int |g| \, \mathrm{d}\mu_\varphi.$$

In particular, if $f = \mathbf{1}_A$ and $g = \mathbf{1}_B$ with $A$ interval and $B$ measurable, then $\operatorname{var} f = 2$ and for all $n$

$$\left| \mu_\varphi(A \cap T^{-n}B) - \mu_\varphi(A)\mu_\varphi(B) \right| \leqslant K \, \mathrm{e}^{-\gamma n} \left(2 + \mu_\varphi(A)\right) \mu_\varphi(B).$$

This kind of mixing, which is weaker than $\Phi$-mixing, is a key tool in the following.

CENTRAL LIMIT THEOREM. – *For functions with summable decay of correlation (which is the case for $\varphi_0 = \mu_\varphi(\varphi) - \varphi$ since it has bounded variation and then decays exponentially fast), the central limit theorem is true (see [6]), i.e., recall that*

$$\sigma^2(f) = C_0(f) + 2 \sum_{n=1}^{\infty} C_n(f),$$

*and assume that $\sigma(\varphi) \neq 0$, then we have*

$$\frac{\sum_{i=0}^{n-1} \varphi_0 \circ T^i}{\sigma(\varphi_0)\sqrt{n}} \Rightarrow \mathcal{N}(0, 1),$$

*which is equivalent to*

$$\frac{-\log S_n + n\mu_\varphi(\varphi)}{\sigma(\varphi)\sqrt{n}} \Rightarrow \mathcal{N}(0,1)$$

*(and* $\sigma(\varphi) = 0$ *if and only if there exists a measurable function* $g$ *such that* $\varphi = g - g \circ T$*).*

**Distortion property**

LEMMA 2.5. – *Assume* (H2)*, then there is a constant* $c > 1$ *such that, for all* $n$*, all* $A \in \mathcal{P}^n$*, all* $x$ *and* $y$ *in* $A$

$$\frac{1}{c} \leqslant \frac{S_n(y)}{S_n(x)} \leqslant c.$$

*Proof.* –

$$\frac{S_n(y)}{S_n(x)} = \mathrm{e}^{(\varphi(y)-\varphi(x))+\cdots+(\varphi\circ T^{n-1}(y)-\varphi\circ T^{n-1}(x))}$$

$x$ and $y$ are in the same $n$-cylinder, therefore, for all $k$, $T^{n-k}(x)$ and $T^{n-k}(y)$ are in the same $k$-cylinder and

$$\frac{S_n(y)}{S_n(x)} \leqslant \exp\left(\sum_{k=1}^{n} \mathrm{var}_{C_k(T^{n-k}(x))}\,\varphi\right) \leqslant \exp\left(\sum_{n=1}^{+\infty} \sup_{C\in\mathcal{P}^n} \mathrm{var}_C\,\varphi\right).$$

We get the other inequality by changing $x$ and $y$. □

*Remark* 2.1. – In case when $\mathrm{e}^{\varphi}$ is the inverse of the derivative of the transformation, the bounded distortion property comes from the fact that $T$ is $\mathcal{C}^2$ and from the uniform dilatation hypothesis made for $T$ (see [2]).

## 3. ESTIMATES OF THE MEASURE OF A CYLINDER

In the following, $K$ and $\beta$ are generic positive constants independant from $n$ and $A$. It is proven in this section first that the measure of a $n$-cylinder decays exponentially fast to zero, then that, for most $n$-cylinders, we can give an equivalent for this measure.

LEMMA 3.1. – *There exist* $\theta > 0$ *and a constant* $C$ *such that, for all* $n$ *and all* $n$*-cylinder* $A$

$$\mu_\varphi(A) \leqslant C\,\mathrm{e}^{-\theta n}.$$

*Proof.* – Let $A = A_{i_1}^{i_n}$ be a $n$-cylinder. For all $n_0 < n$ we get:

$$\mu_\varphi(A) \leqslant \mu_\varphi\Big(A_{i_1} \cap T^{-n_0} A_{i_{n_0}} \cap \cdots \cap T^{-\left[\frac{n}{n_0}\right] n_0} A_{i_{\left[\frac{n}{n_0}\right] n_0}}\Big).$$

Let us use the mixing inequality with the interval $A_{i_1}$ and the measurable set $A_{i_{n_0}} \cap \cdots \cap T^{-\left(\left[\frac{n}{n_0}\right]-1\right) n_0} A_{i_{\left[\frac{n}{n_0}\right] n_0}}$:

$$\begin{aligned}
&\mu_\varphi\Big(A_{i_1} \cap \cdots \cap T^{-\left[\frac{n}{n_0}\right] n_0} A_{i_{\left[\frac{n}{n_0}\right] n_0}}\Big) \\
&\quad - \mu_\varphi(A_{i_1})\mu_\varphi\Big(A_{i_{n_0}} \cap \cdots \cap T^{-\left(\left[\frac{n}{n_0}\right]-1\right) n_0} A_{i_{\left[\frac{n}{n_0}\right] n_0}}\Big) \\
&\quad \leqslant K\,\mathrm{e}^{-\gamma n_0}\left(2 + \mu_\varphi(A_{i_1})\right)\mu_\varphi\Big(A_{i_{n_0}} \cap \cdots \cap T^{-\left(\left[\frac{n}{n_0}\right]-1\right) n_0} A_{i_{\left[\frac{n}{n_0}\right] n_0}}\Big) \\
&\quad \leqslant 3K\,\mathrm{e}^{-\gamma n_0}\,\mu_\varphi\Big(A_{i_{n_0}} \cap \cdots \cap T^{-\left(\left[\frac{n}{n_0}\right]-1\right) n_0} A_{i_{\left[\frac{n}{n_0}\right] n_0}}\Big),
\end{aligned}$$

if we call $s = \sup\{\mu_\varphi(A_i),\, i = 0, \ldots, b-1\}$ we have

$$\mu_\varphi(A) \leqslant \left(s + 3K\,\mathrm{e}^{-\gamma n_0}\right)\mu_\varphi\Big(A_{i_{n_0}} \cap \cdots \cap T^{-\left(\left[\frac{n}{n_0}\right]-1\right) n_0} A_{i_{\left[\frac{n}{n_0}\right] n_0}}\Big),$$

and, by induction

$$\mu_\varphi(A) \leqslant \left(s + 3K\,\mathrm{e}^{-\gamma n_0}\right)^{\left[\frac{n}{n_0}\right]+1}.$$

Now, there is $n_0$ such that $s + 3K\,\mathrm{e}^{-\gamma n_0} < 1$ which ends the proof. □

The following lemma gives an equivalent of the measure of almost all $n$-cylinders (which are intervals). We cannot get the equivalent for all cylinders because of the following remark:

*Remark* 3.1. – Let $A$ be a $n$-cylinder whose boundary does not contain any singularity of $T$, then $T(A)$ is a $(n-1)$-cylinder. (When the system is markovian, the image of a $n$-cylinder is always a $(n-1)$-cylinder, that is why we get the equivalent for all cylinders.) Conversely, if the boundary of $A$ contains a singularity of $T$, $T(A)$ can be much smaller than the $(n-1)$-cylinder it is included in.

*Proof of the remark.* – If $A$ is a $n$-cylinder, its boundary is contained in $\bigcup_{i=0}^{n-1} T^{-i}(\operatorname{sing} T)$. If its boundary does not contain any singularity of $T$ then it is included in $\bigcup_{i=1}^{n-1} T^{-i}(\operatorname{sing} T)$. The boundary of $T(A)$ is then included in $\bigcup_{i=0}^{n-2} T^{-i}(\operatorname{sing} T)$ and $T(A)$ is a union of $(n-1)$-cylinders. By an argument of connexity, as $T|_A$ is continuous, $T(A)$ is one $n$-cylinder.

## Example

In this example, $A$ is a 2-cylinder, the boundary of $A$ contains a singularity of $T$ and $T(A)$ is strictly included in the 1-cylinder $B$.

LEMMA 3.2. – *Let $k_0 > 0$ and $n > k_0$. Let $A \in \mathcal{P}^n$ such that, for all $k \leqslant n - k_0$, $T^k(A)$ has no singularity of $T$ in its boundary. Then, there exists a constant $c(k_0) > 1$ such that, for all $x$ in $A$*

$$\frac{1}{c(k_0)} \leqslant \frac{\mu_\varphi(A)}{\lambda^{-n} S_n(x)} \leqslant c(k_0).$$

*Proof.* – Let $A \in \mathcal{P}^n$ such that, for all $k \leqslant n - k_0$, $T^k(A)$ has no singularity of $T$ in its boundary and let $x \in A$:

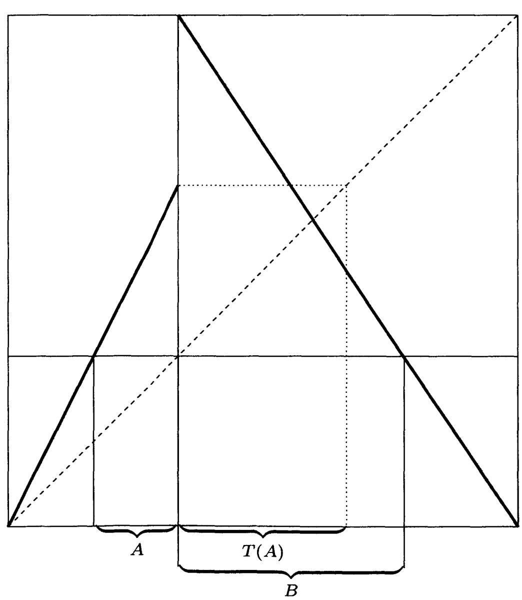


Fig. 1. Non markov map.

$$\mu_\varphi(A) = \nu(h_\varphi \mathbf{1}_A) = \frac{1}{\lambda^{n-k_0}} \nu\big(P_\varphi^{n-k_0}(h_\varphi \mathbf{1}_A)\big),$$
$$P_\varphi^{n-k_0}(h_\varphi \mathbf{1}_A)(z) = \sum_{T^{n-k_0}(y)=z,\, y\in A} S_{n-k_0}(y) h_\varphi(y). \tag{1}$$

Let us take $z \in [0,1] \setminus \bigcup_{n\in\mathbb{N}} T^{-n}(\operatorname{sing} T)$ (we can restrict to such $z$ without changing the integral because $\nu(\bigcup_{n\in\mathbb{N}} T^{-n}(\operatorname{sing} T)) = 0$), $z$ is in a $k_0$-cylinder $C_{k_0}(z)$. $T^{-n+k_0}(C_{k_0}(z))$ is constituted at most by $b^{n-k_0}$ $n$-cylinders and $T^{-n+k_0}(z)$ by at most $b^{n-k_0}$ points. Each of them are in a different $n$-cylinder.

If $A$ is one of these $n$-cylinders then $A \cap T^{-n+k_0}(z) = z_A$, if it's not the case then $A \cap T^{-n+k_0}(z) = \emptyset$. Therefore we get:

$$P_\varphi^{n-k_0}(h_\varphi \mathbf{1}_A)(z) \leqslant S_{n-k_0}(z_A) h_\varphi(z_A) \leqslant \sup(h_\varphi) S_{n-k_0}(z_A).$$

Let $x \in A$, we use the distortion property (since $x$ and $z_A$ are in the same $n-k_0$-cylinder) in order to get:

$$P_\varphi^{n-k_0}(h_\varphi \mathbf{1}_A)(z) \leqslant K S_{n-k_0}(x), \qquad \frac{\mu_\varphi(A)}{\lambda^{-n+k_0} S_{n-k_0}(x)} \leqslant K.$$

Moreover, because of the previous remark, $T^{n-k_0}(A)$ is a $k_0$-cylinder and the sum (1) is not zero when $T^{-n+k_0}(z) \cap A \neq \emptyset$ which occurs when $T^{n-k_0}(A) = C_{k_0}(z)$ hence:

$$\begin{aligned} P_\varphi^{n-k_0}(h_\varphi \mathbf{1}_A)(z) &= \mathbf{1}_{T^{n-k_0}(A)}(z) \sum_{T^{n-k_0}(y)=z,\, y\in A} S_{n-k_0}(y) h_\varphi(y) \\ &\geqslant \mathbf{1}_{T^{n-k_0}(A)}(z) S_{n-k_0}(z_A) h_\varphi(z_A) \\ &\geqslant \frac{1}{c} \mathbf{1}_{T^{n-k_0}(A)}(z) S_{n-k_0}(x) \inf(h_\varphi). \end{aligned}$$

Now we get

$$\mu_\varphi(A) \geqslant \frac{\lambda^{-n+k_0}}{c} \nu\big(T^{n-k_0}(A)\big) S_{n-k_0}(x) \inf(h_\varphi)$$

and $T^{n-k_0}(A)$ is a $k_0$-cylinder; now denoting $c(k_0) = (\frac{1}{c} \inf_{A\in C_{k_0}} \nu(A) \times \inf(h_\varphi))^{-1}$:

$$\frac{\mu_\varphi(A)}{\lambda^{-n+k_0} S_{n-k_0}(x)} \geqslant \frac{1}{c(k_0)}$$

and $S_n(x) = S_{n-k_0}(x) S_{k_0}(T^{n-k_0}(x))$. But $S_{k_0}$ is bounded and multiplying by $\lambda^{k_0}$ we get the result. □

LEMMA 3.3. – *Let*

$$B(n,C)=\left\{A\in\mathcal{P}^n,\forall x\in A\colon\ \frac{1}{C}\leqslant\frac{\mu_\varphi(A)}{\lambda^{-n}S_n(x)}\leqslant C\right\}.$$

*There is $K$ such that, for all $\varepsilon>0$, there exists $D(\varepsilon)$ and $N_\varepsilon$ such that, for $n>N_\varepsilon$*

$$\mu_\varphi\Big(\bigcup_{A\in B(n,D(\varepsilon))}A\Big)\geqslant 1-K\varepsilon.$$

*Proof.* – Let $\rho=\lambda\,\mathrm{e}^{-\sup\varphi}$. We use the hypothesis $\sup\varphi<p(\varphi)$ to state that $\rho>1$ (recall that $\lambda=\mathrm{e}^{p(\varphi)}$). Let $\varepsilon>0$ and $k_0(\varepsilon)$ such that

$$k\geqslant k_0(\varepsilon)\Rightarrow\frac{1}{\rho^k}<\frac{\varepsilon}{k^2}.$$

Let $n>k_0(\varepsilon)$, according to the previous lemma, if $A\in\mathcal{P}^n$ and if, for all $k\leqslant n-k_0(\varepsilon)$, $T^k(A)$ has no singularity of $T$ in its boundary, then $A\in B(n,D(\varepsilon))$; (with $D(\varepsilon)=c(k_0(\varepsilon))$). We show that the measure of this set is close to one by considering its complement:

Let

$$F(n,\varepsilon)=\big\{A\in\mathcal{P}^n,\ \exists k\leqslant n-k_0,\ T^k(A)\text{ has a singularity of }T\text{ in its boundary}\big\}.$$

Let $A\in F(n,\varepsilon)$ and $x$ in $A$: there exists $k\in[k_0,n]$ such that $T^{n-k}(A)$ has one singularity $s$ of $T$ in its boundary; we get then:

$$\nu\big(\big[T^{n-k}(x),s\big]\big)\leqslant\nu\big(T^{n-k}(A)\big).$$

But $\nu$ is a $\lambda\,\mathrm{e}^{-\varphi}$ conformal measure so we get

$$1\geqslant\nu\big(T^n(A)\big)=\int_{T^{n-1}(A)}\lambda\,\mathrm{e}^{-\varphi}\,\mathrm{d}\nu\geqslant\lambda\,\mathrm{e}^{-\sup\varphi}\,\nu\big(T^{n-1}(A)\big)$$
$$>\rho^k\nu\big(T^{n-k}(A)\big)$$

hence $\nu(T^{n-k}(A))\leqslant 1/\rho^k$ and:

$$\nu\big(\big[T^{n-k}(x),s\big]\big)<\frac{\varepsilon}{k^2}$$

$\nu(\{s\})=0$ and the conformal measure $\nu$ is regular and has no atom, therefore, there exists a union of intervals $V_k$ such that each singularity

$s$ is a bound of an interval and $\nu(V_k) = \varepsilon/k^2$. Since the density $h_\varphi$ is bounded, we obtain: $\mu_\varphi(V_k) \leqslant K(\varepsilon/k^2)$ and, using the invariance by $T$ of $\mu_\varphi$:

$$\mu_\varphi\left(\bigcup_{A\in F(n,\varepsilon)} A\right) \leqslant \mu_\varphi\left(\bigcup_{k=k_0}^{n} T^{k-n}(V_k)\right) \leqslant \sum_{k=k_0}^{n} \mu_\varphi(V_k) \leqslant K\varepsilon. \qquad \square$$

## 4. RETURN TIMES AND ENTRANCE TIMES

In this part, we show that, in some sense, the asymptotic law of $R_n$ can be written as a sum of entrance times laws with fluctuating rates (these rates are the mass of the cylinders).

DEFINITION 4.1. – *A $n$-cylinder $A$ is said $k$-recurrent (for $n > k$) if*

$$\forall\, l < k-1,\ A \cap T^{-l}(A) = \emptyset \quad \textit{and} \quad A \cap T^{-k+1}(A) \neq \emptyset.$$

*$E_k$ is the set of the $k$-recurrent cylinders and $E_{<k}$ the set of the cylinders which recur before $k$.*

PROPERTY 4.1. – *If $k < n$:*

$$\#(E_k) \leqslant b^{k-1} \quad \textit{and} \quad \#(E_{<k}) \leqslant b^k.$$

*Proof of the property.* – If $A = A_{i_1}^{i_n} \in E_k$, there exists $x$ in $A$ such that $T^{k-1}(x)$ is in $A$.

$$x \in A \quad \text{and so} \quad x \in A_{i_1}, T(x) \in A_{i_2}, \ldots, T^{n-1}(x) \in A_{i_n};$$
$$T^{k-1}(x) \in A \quad \text{and so} \quad T^{k-1}(x) \in A_{i_1}, \ldots, T^{n+k-2}(x) \in A_{i_n}.$$

Hence $A_{i_k} = A_{i_1}, \ldots, A_{i_n} = A_{i_{n-k+1}}$. For $A$ we only have the choice for $A_{i_1}, \ldots, A_{i_{k-1}}$ and $\#(E_k) \leqslant b^{k-1}$. Finally

$$\#(E_{<k}) \leqslant \sum_{i=1}^{k} \#(E_k) \leqslant b^k.$$

LEMMA 4.1. – *Let $(t_n)$ be a sequence such that* $\lim_{n\to\infty} t_n/n = +\infty$, *then*

$$\lim_{n\to\infty} \left| \mu_\varphi\{R_n > t_n\} - \sum_{A\in\mathcal{P}^n} \mu_\varphi(A)\mu_\varphi\{\tau_A > t_n\} \right| = 0. \qquad (2)$$

*Proof.* – Recall the definition of $R_n$:

$$R_n(x) = \inf\{k > 0,\, T^k(x) \in \mathcal{P}^n(x)\}.$$

For all $t > 0$ we have

$$\mu_\varphi\{R_n > t\} = \sum_{A \in \mathcal{P}^n} \mu_\varphi\{A \cap \tau_A > t\}.$$

For all $r$ with $n < r < t$ we get:

$$\begin{aligned}
&\left|\mu_\varphi\{A \cap \tau_A > t\} - \mu_\varphi(A)\mu_\varphi\{\tau_A > t\}\right| \\
&\quad \leqslant \left|\mu_\varphi\{A \cap \tau_A > t\} - \mu_\varphi\{A \cap T^{-s+1}(A^c),\ r < s \leqslant t\}\right| \\
&\qquad + \big|\mu_\varphi\{A \cap T^{-s+1}(A^c),\ r < s \leqslant t\} \\
&\qquad - \mu_\varphi(A)\mu_\varphi\{T^{-s+1}(A^c),\ r < s \leqslant t\}\big| \\
&\qquad + \mu_\varphi(A)\left|\mu_\varphi\{T^{-s+1}(A^c),\ r < s \leqslant t\} - \mu_\varphi\{\tau_A > t\}\right|.
\end{aligned}$$

Bound for the third term: using the inclusion

$$\left(\bigcap_{r < s \leqslant t} T^{-s+1}A^c\right) \setminus \left(\bigcap_{1 \leqslant s \leqslant t} T^{-s+1}A^c\right) \subset \left(\bigcup_{1 \leqslant s \leqslant r} T^{-s+1}A\right)$$

it comes:

$$\begin{aligned}
&\left|\mu_\varphi\{T^{-s+1}(A^c), r < s \leqslant t\} - \mu_\varphi\{T^{-s+1}(A^c), 1 \leqslant s \leqslant t\}\right| \\
&\quad \leqslant \mu_\varphi\left\{\bigcup_{1 \leqslant s \leqslant r} T^{-s+1}(A)\right\} \leqslant r\mu_\varphi(A)
\end{aligned}$$

so an upper bound for the third term is $r\mu_\varphi(A)^2$. For the second one, the mixing inequality (see Theorem 2.4) gives the following bound: $3K\,\mathrm{e}^{-\gamma r}$. As for the first one, we get the estimate

$$\sum_{i=1}^{r} \mu_\varphi\{A \cap T^{-i+1}(A)\}.$$

It remains to sum over all $n$-cylinders. For the third term, we get:

$$\sum_{A \in \mathcal{P}^n} r\mu_\varphi(A)^2 \leqslant rC\,\mathrm{e}^{-\theta n} \sum_{A \in \mathcal{P}^n} \mu_\varphi(A) \leqslant rC\,\mathrm{e}^{-\theta n}.$$

For the second one, we get (since $\operatorname{card}(\mathcal{P}^n) \leqslant b^n$):

$$\sum_{A\in\mathcal{P}^n} 3K\,\mathrm{e}^{-\gamma r} \leqslant 3K\,\mathrm{e}^{n\operatorname{Log}(b)-r\gamma}.$$

A good choice of $r$ will give the convergence to zero. For the first term, we must set apart the cylinders which recur too fast:

If $A \in E^c_{<k}$ then $\mu_\varphi\{A \cap T^{-i+1}(A)\} \leqslant \mu_\varphi(A) \leqslant C\,\mathrm{e}^{-\theta n}$ and

$$\sum_{A\in E^c_{<k}} \sum_{i=1}^{r} \mu_\varphi\{A \cap T^{-i+1}(A)\} \leqslant \sum_{A\in E^c_{<k}} rC\,\mathrm{e}^{-\theta n} \leqslant rC\,\mathrm{e}^{-\theta n+k\operatorname{Log}(b)}.$$

Besides, if $A \in E^c_{<k}$, $\forall i < k$: $\mu_\varphi\{A \cap T^{-i+1}(A)\} = 0$ and

$$\sum_{A\in E_{<k}} \sum_{i=1}^{r} \mu_\varphi\{A \cap T^{-i+1}(A)\} \leqslant \sum_{A\in E_{<k}} \sum_{i=k}^{r} \mu_\varphi\{A \cap T^{-i+1}(A)\}.$$

And if $i \geqslant k$, the mixing property yields to:

$$\mu_\varphi\{A \cap T^{-i+1}(A)\} \leqslant \left(3K\,\mathrm{e}^{-\gamma k} + C\,\mathrm{e}^{-\theta n}\right)\mu_\varphi(A),$$

$$\sum_{i=k}^{r} \mu_\varphi\{A \cap T^{-i+1}(A)\} \leqslant r\left(3K\,\mathrm{e}^{-\gamma k} + C\,\mathrm{e}^{-\theta n}\right)\mu_\varphi(A),$$

$$\sum_{A\in E^c_{<k}} \sum_{i=k}^{r} \mu_\varphi\{A \cap T^{-i+1}(A)\} \leqslant r\left(3K\,\mathrm{e}^{-\gamma k} + C\,\mathrm{e}^{-\theta n}\right).$$

Now we choose $r = \min(n^2, \sqrt{nt_n})$ and $k = [\theta n/\log(b)]$ (we only have to change $\theta$ to ensure $k < n$) which gives us the convergence of all terms to zero. □

## 5. APPROXIMATION OF THE LAW OF THE ENTRANCE TIME IN A CYLINDER BY AN EXPONENTIAL LAW

This rather technical part is devoted to the control of the law of the entrance times in a cylinder. As it was pointed out in the previous part, this control is needed to estimate the asymptotic law of the return times.

Here the following theorem is proved:

THEOREM 5.1. – *For all* $\varepsilon > 0$, *there exists* $N_\varepsilon$ *such that, for all* $n > N_\varepsilon$ *there exists* $H_{n,\varepsilon} \subset \mathcal{P}^n$ *with*:

$$\mu_\varphi\Big(\bigcup_{A\in H_{n,\varepsilon}} A\Big) > 1 - K\varepsilon.$$

*There exists two strictly positive constants* $\beta$ *and* $K$ *such that, for all n-cylinder* $A\in H_{n,\varepsilon}$:

$$\sup_{t>0}\left|\mu_\varphi\left\{\tau_A > \frac{t}{\mu_\varphi(A)}\right\} - \mathrm{e}^{-t}\right| \leqslant K\,\mathrm{e}^{-\beta n}.$$

In order to prove this theorem, we use the method of Galves and Schmitt [5].

LEMMA 5.2. – *For all* $t > 0$, *we have, if* $A$ *is measurable*:

$$\mu_\varphi\left\{\tau_A \leqslant \frac{t}{\mu_\varphi(A)}\right\} \leqslant t + \mu_\varphi(A).$$

The proof is in [5] (Lemma 2). For all $k$ and $m$ positive real numbers, let:

$$X_k = \sum_{l=0}^{[k]} \chi_A \circ T^l, \qquad X_{[k,m]} = X_{[m]} - X_{[k]}.$$

We have: $\{\tau_A \leqslant k\} = \{X_k \geqslant 1\}$.

LEMMA 5.3. – *There exists* $\gamma_0$ *such that, for all* $\varepsilon$, *there exists* $N_\varepsilon$ *such that, for all* $n > N_\varepsilon$ *there exists* $I_{n,\varepsilon} \subset \mathcal{P}^n$ *such that, for all* $A \in I_{n,\varepsilon}$

$$\mu_\varphi\left\{\tau_A \leqslant \frac{t}{\mu_\varphi(A)}\right\} \geqslant \frac{t^2}{t^2 + \mu_\varphi(A)(1+t) + t(1 + K\,\mathrm{e}^{-n\gamma_0})}.$$

*Moreover,*

$$\mu_\varphi\Big(\bigcup_{A\in I_{n,\varepsilon}} A\Big) > 1 - \varepsilon.$$

*Proof.* – Let $X = X_{[\frac{t}{\mu_\varphi(A)}]}$. Using the Schwarz inequality, we get:

$$E(X)^2 \leqslant E\big(X^2\big)\mu_\varphi(X \geqslant 1)$$

and $E(X)^2 \geqslant t^2$. Moreover,

$$E(X^2) = \sum_{l=0}^{[\frac{t}{\mu_\varphi(A)}]} E(\chi_A \circ T^l) + 2 \sum_{l=1}^{[\frac{t}{\mu_\varphi(A)}]} \left( \left[ \frac{t}{\mu_\varphi(A)} \right] - l + 1 \right) \times \mu_\varphi\{A \cap T^{-l}(A)\}.$$

The first term is $E(X) \leqslant t + \mu_\varphi(A)$. We bound the second for cylinders which don't recur too fast; for $A \in E^c_{<[ns]}$ (where $s$ is positive) we get:

$$\sum_{l=1}^{[\frac{t}{\mu_\varphi(A)}]} \left( \left[ \frac{t}{\mu_\varphi(A)} \right] - l - 1 \right) \mu_\varphi\{A \cap T^{-l}(A)\} = \sum_{l=[ns]}^{[\frac{t}{\mu_\varphi(A)}]} \left( \left[ \frac{t}{\mu_\varphi(A)} \right] - l - 1 \right) \mu_\varphi\{A \cap T^{-l}(A)\}.$$

The mixing property gives for this term:

$$\sum_{l=[ns]}^{[\frac{t}{\mu_\varphi(A)}]} \left( \left[ \frac{t}{\mu_\varphi(A)} \right] - l + 1 \right) \left[ K \,\mathrm{e}^{-\gamma l} \left(2 + \mu_\varphi(A)\right) \mu_\varphi(A) + \mu_\varphi(A)^2 \right]$$

$$\leqslant \mu_\varphi(A)^2 \sum_{l=[ns]}^{[\frac{t}{\mu_\varphi(A)}]} \left( \left[ \frac{t}{\mu_\varphi(A)} \right] - l + 1 \right) + K \mu_\varphi(A) \sum_{l=[ns]}^{[\frac{t}{\mu_\varphi(A)}]} \left( \left[ \frac{t}{\mu_\varphi(A)} \right] - l + 1 \right) \mathrm{e}^{-\gamma l}$$

$$\leqslant \mu_\varphi(A)^2 \left( \frac{t}{\mu_\varphi(A)} \right) \left( \frac{t}{\mu_\varphi(A)} + 1 \right) + K \mu_\varphi(A) \left( \frac{t}{\mu_\varphi(A)} \right) \sum_{l=[ns]}^{[\frac{t}{\mu_\varphi(A)}]} \mathrm{e}^{-\gamma l}$$

$$\leqslant t\left(t + \mu_\varphi(A)\right) + K t \,\mathrm{e}^{-ns\gamma}.$$

We choose now $s = \theta/(2 \log b)$ (where $\theta$ is given by Lemma 3.1) so that, for $n$ big enough:

$$\mu_\varphi\left( \bigcup_{A \in E_{<[ns]}} A \right) \leqslant C b^{ns} \,\mathrm{e}^{-\theta n} \leqslant C \,\mathrm{e}^{-n\theta/2} < \varepsilon.$$

We take $I_{n,\varepsilon} = E^c_{<[ns]}$. □

Let $g_A(t) = \mu_\varphi\{\tau_A > t/\mu_\varphi(A)\} = \mu_\varphi\{X = 0\}$.

**Independence property.** We need to show that $g_A(t)$ is close to $e^{-t}$; for that, we show that this function satisfies some kind of independence property. We will first show that $g_A(t)$ is close to $e^{-t}$ when $t$ is equal to some power of $\mu_\varphi(A)$; then, given $t>0$, we will divide it by this power of $\mu_\varphi(A)$.

Recall that we denote by $K$ any constant independant of $n$ and of the cylinders.

LEMMA 5.4. – *For n big enough and for all n-cylinder A*:

$$\sup_{s\geqslant\sqrt{\mu_\varphi(A)}}\left|g_A\left(\sqrt{\mu_\varphi(A)}+s\right)-g_A\left(\sqrt{\mu_\varphi(A)}\right)g_A(s)\right|\leqslant K\mu_\varphi(A)^{3/4}.$$

*Proof.* – We must estimate $|g_A(t+s)-g_A(t)g_A(s)|$. To begin with, we dig a hole $\Delta$ between $[0,\frac{t}{\mu_\varphi(A)}]$ and $[\frac{t}{\mu_\varphi(A)},\frac{t+s}{\mu_\varphi(A)}]$. This hole, thanks to the mixing inequality, will enable us to express the probability of not being in $A$ during the time $[0,\frac{t}{\mu_\varphi(A)}]\cup[\frac{t+\Delta}{\mu_\varphi(A)},\frac{t+s}{\mu_\varphi(A)}]$ in terms of the product of the probability of not being in $A$ during each of the intervals $[0,\frac{t}{\mu_\varphi(A)}]$ and $[\frac{t+\Delta}{\mu_\varphi(A)},\frac{t+s}{\mu_\varphi(A)}]$.

$$\begin{aligned}
&\left|g_A(t+s)-g_A(t)g_A(s)\right|\\
&\quad\leqslant\left|g_A(t+s)-\mu_\varphi\{X_{[\frac{t}{\mu_\varphi(A)}]}+X_{[\frac{t}{\mu_\varphi(A)}+\Delta,\frac{t+s}{\mu_\varphi(A)}]}=0\}\right|\\
&\qquad+\left|\mu_\varphi\{X_{[\frac{t}{\mu_\varphi(A)}]}+X_{[\frac{t}{\mu_\varphi(A)}+\Delta,\frac{t+s}{\mu_\varphi(A)}]}=0\}\right.\\
&\qquad\left.-g_A(t)\mu_\varphi\{X_{[\Delta,\frac{s}{\mu_\varphi(A)}]}=0\}\right|\\
&\qquad+\left|g_A(t)\right|\left|\mu_\varphi\{X_{[\Delta,\frac{s}{\mu_\varphi(A)}]}=0\}-g_A(s)\right|.
\end{aligned}$$

Bounds for the first term:

$$\begin{aligned}
&\left|g_A(t+s)-\mu_\varphi\{X_{[\frac{t}{\mu_\varphi(A)}]}+X_{[\frac{t}{\mu_\varphi(A)}+\Delta,\frac{t+s}{\mu_\varphi(A)}]}=0\}\right|\\
&\quad=\mu_\varphi\{X_{[\frac{t}{\mu_\varphi(A)},\frac{t}{\mu_\varphi(A)}+\Delta]}>0\}\\
&\quad=\mu_\varphi\{X_{\Delta-1}>0\}\leqslant\Delta\mu_\varphi(A),
\end{aligned}\tag{3}$$

because of the $T$-invariance. For the third term as well:

$$\left|\mu_\varphi\{X_{[\Delta,\frac{s}{\mu_\varphi(A)}]}=0\}-g_A(s)\right|\leqslant\Delta\mu_\varphi(A).\tag{4}$$

For the second term we use the mixing inequality and we denote:

$$P_A(f)=P_\varphi(f\mathbf{1}_A).$$

Let us renormalize $P_\varphi$ with

$$\mathcal{L}_\varphi = \frac{P_\varphi}{\lambda}, \qquad \mathcal{L}_A = \frac{P_A}{\lambda},$$

$$\begin{aligned}
&\left|\mu_\varphi\{X_{[\frac{t}{\mu_\varphi(A)}]} + X_{[\frac{t}{\mu_\varphi(A)}+\Delta,\frac{t+s}{\mu_\varphi(A)}]} = 0\} - g_A(t)\mu_\varphi\{X_{[\Delta,\frac{s}{\mu_\varphi(A)}]} = 0\}\right| \\
&= \left|\int \prod_0^{[\frac{t}{\mu_\varphi(A)}]} \mathbf{1}_{A^c}\circ T^i \prod_{[\frac{t}{\mu_\varphi(A)}]+\Delta+1}^{[\frac{t+s}{\mu_\varphi(A)}]} \mathbf{1}_{A^c}\circ T^i h_\varphi \,\mathrm{d}\nu \right. \\
&\quad \left. - \int \prod_0^{[\frac{t}{\mu_\varphi(A)}]} \mathbf{1}_{A^c}\circ T^i h_\varphi\,\mathrm{d}\nu \int \prod_{\Delta+1}^{[\frac{s}{\mu_\varphi(A)}]} \mathbf{1}_{A^c}\circ T^i\,\mathrm{d}\mu_\varphi \right| \\
&= \left|\int \mathbf{1}_{A^c}\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi) \prod_{\Delta+1}^{[\frac{s}{\mu_\varphi(A)}]} \mathbf{1}_{A^c}\circ T^i\,\mathrm{d}\nu \right. \\
&\quad \left. - \int \mathbf{1}_{A^c}\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)\,\mathrm{d}\nu \int \prod_0^{[\frac{s}{\mu_\varphi(A)}]-\Delta-1} \mathbf{1}_{A^c}\circ T^i\,\mathrm{d}\mu_\varphi \right| \\
&= \left|\int \mathbf{1}_{A^c}\frac{\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)}{h_\varphi}\left(\prod_0^{[\frac{s}{\mu_\varphi(A)}]-\Delta-1} \mathbf{1}_{A^c}\circ T^i\right)\circ T^{\Delta+1}\,\mathrm{d}\mu_\varphi \right. \\
&\quad \left. - \int \mathbf{1}_{A^c}\frac{\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)}{h_\varphi}\,\mathrm{d}\mu_\varphi \int \prod_0^{[\frac{s}{\mu_\varphi(A)}]-\Delta-1} \mathbf{1}_{A^c}\circ T^i\,\mathrm{d}\mu_\varphi \right| \\
&\leqslant K\,\mathrm{e}^{-\gamma(\Delta+1)}\left[\int \mathbf{1}_{A^c}\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)\,\mathrm{d}\nu + \mathrm{var}\left(\mathbf{1}_{A^c}\frac{\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)}{h_\varphi}\right)\right] \\
&\quad \times \int \prod_0^{[\frac{s}{\mu_\varphi(A)}]-\Delta-1} \mathbf{1}_{A^c}\circ T^i\,\mathrm{d}\mu_\varphi \\
&\leqslant K\,\mathrm{e}^{-\gamma(\Delta+1)}\left[\int \mathbf{1}_{A^c}\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)\,\mathrm{d}\nu + \mathrm{var}\left(\mathbf{1}_{A^c}\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)\right)\left\|\frac{1}{h_\varphi}\right\|_\infty \right. \\
&\quad \left. + \mathrm{var}\left(\frac{1}{h_\varphi}\right)\left\|\mathbf{1}_{A^c}\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)\right\|_\infty\right] \qquad (5)
\end{aligned}$$

In (5), we have used the following property of the variation:

$$\mathrm{var}(fg) \leqslant \|f\|_\infty \,\mathrm{var}(g) + \|g\|_\infty \,\mathrm{var}(f).$$

Now, $h_\varphi$ has bounded variation and $\inf(h_\varphi) > 0$. This implies: $1/h_\varphi$ has bounded variation.

Moreover,

$$\left\|\mathbf{1}_{A^c}\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)\right\|_\infty \leqslant K,$$

because $\mathcal{L}_{A^c}$ and $f \mapsto f\mathbf{1}_{A^c}$ are operators with norm less than one.

$$\begin{aligned}(5) &\leqslant K\,\mathrm{e}^{-\gamma(\Delta+1)}\left[\int \mathbf{1}_{A^c}\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)\,\mathrm{d}\nu + K + K\,\mathrm{var}\Big(\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)\Big)\right]\\ &\leqslant K\,\mathrm{e}^{-\gamma(\Delta+1)}\left[g_A(t) + K + K\,\mathrm{var}\Big(\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)\Big)\right]\\ &\leqslant K\,\mathrm{e}^{-\gamma(\Delta+1)}\left[K + K\,\mathrm{var}\Big(\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)\Big)\right]. \qquad (6)\end{aligned}$$

Where we have used again:

$$\mathrm{var}(fg) \leqslant \|f\|_\infty \,\mathrm{var}(g) + \|g\|_\infty \,\mathrm{var}(f).$$

We must estimate $\mathrm{var}(\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi))$, for that, we use the fact that $\mathcal{L}_\varphi = \mathcal{L}_A + \mathcal{L}_{A^c}$.

$$\begin{aligned}\mathcal{L}_{A^c}^N &= (\mathcal{L}_\varphi - \mathcal{L}_A)^N\\ &= \mathcal{L}_\varphi^N - \sum_{r=0}^{N-1}\mathcal{L}_\varphi^r\mathcal{L}_A\mathcal{L}_\varphi^{N-r-1} + \sum_{0\leqslant i+j\leqslant N-2}\mathcal{L}_\varphi^i\mathcal{L}_A\mathcal{L}_{A^c}^{N-i-j-2}\mathcal{L}_A\mathcal{L}_\varphi^j.\end{aligned}$$

Since $\mathcal{L}_\varphi(h_\varphi) = h_\varphi$, we get:

$$\mathcal{L}_{A^c}^N(h_\varphi) = h_\varphi - \sum_{r=0}^{N-1}\mathcal{L}_\varphi^r\mathcal{L}_A(h_\varphi) + \sum_{0\leqslant i+j\leqslant N-2}\mathcal{L}_\varphi^i\mathcal{L}_A\mathcal{L}_{A^c}^{N-i-j-2}\mathcal{L}_A\mathcal{L}_\varphi^j(h_\varphi).$$

A computation gives:

$$\mathcal{L}_\varphi^i\mathcal{L}_A\mathcal{L}_{A^c}^{N-i-j-2}\mathcal{L}_A\mathcal{L}_\varphi^j = \mathcal{L}_\varphi^i\mathcal{L}_{B_{i,j}}\mathcal{L}_\varphi^{N-i-1}$$

with

$$\begin{aligned}B_{i,j} = A \cap T^{-1}(A^c) \cap T^{-2}(A^c) \cap \cdots \cap T^{-(N-i-j-2)}(A^c)\\ \cap T^{-(N-i-j-1)}(A).\end{aligned}$$

Assume that $A$ is a $n$-cylinder with $n > N$ and let $k \leqslant N$: A is completely included in an interval where $T^n$ is monotone. Besides, $T^{-k}(A)$ is made with at most $b^k$ intervals and each of them is included in an interval where

$T^k$ is monotone. As a consequence, $A \cap T^{-k}(A)$ is either empty, or an interval, or the union of two intervals (when two branches of $T^k$ with opposite slope meet in a single point). Moreover, as $k \leqslant N$, $T^{-k}(A^c)$ either contains $A$ or is disjoint from $A$. That is why $B_{i,j}$ is either an interval (empty or not) or the union of two intervals, therefore:

$$\mathcal{L}_{A^c}^N(h_\varphi) = h_\varphi - \sum_{r=0}^{N-1} \mathcal{L}_\varphi^r \mathcal{L}_A(h_\varphi) + \sum_{0 \leqslant i+j \leqslant N-2} \mathcal{L}_\varphi^i \mathcal{L}_{B_{i,j}}(h_\varphi). \tag{7}$$

We shall estimate the variation of each term.

One the one hand, if $A$ is an interval or the union of two intervals, we apply the Lasota–Yorke inequality to the function $\mathbf{1}_A h_\varphi$ to get:

$$\begin{aligned}\operatorname{var} \mathcal{L}_A(h_\varphi) &= \operatorname{var} \mathcal{L}_\varphi(\mathbf{1}_A h_\varphi) \leqslant \alpha \operatorname{var}(\mathbf{1}_A h_\varphi) + \xi \nu(\mathbf{1}_A h_\varphi) \\ &\leqslant \alpha\big(\operatorname{var}(h_\varphi) + \operatorname{var}(\mathbf{1}_A)\|h_\varphi\|_\infty\big) + \xi \nu(A)\|h_\varphi\|_\infty \\ &\leqslant \alpha \operatorname{var}(h_\varphi) + \big(4\alpha + \xi \nu(A)\big)\|h_\varphi\|_\infty. \end{aligned} \tag{8}$$

On the other hand, iterating the Lasota–Yorke inequality and using the conformality of $\nu$ gives: (for $f$ with bounded variation)

$$\operatorname{var} \mathcal{L}_\varphi^N(h_\varphi) \leqslant \alpha^N \operatorname{var}(h_\varphi) + K\nu(h_\varphi). \tag{9}$$

Grouping (8) and (9), we have:

$$\begin{aligned}&\operatorname{var}\big(\mathcal{L}_\varphi^r \mathcal{L}_A(h_\varphi)\big) \\ &\quad\leqslant \alpha^r \operatorname{var}\big(\mathcal{L}_A(h_\varphi)\big) + K\nu\big(\mathcal{L}_A(h_\varphi)\big) \\ &\quad\leqslant \alpha^{r+1} \operatorname{var}(h_\varphi) + \alpha^r\big(4\alpha + \xi\nu(A)\big)\|h_\varphi\|_\infty + K\nu(A)\|h_\varphi\|_\infty. \end{aligned} \tag{10}$$

Since $B_{i,j}$ is either an interval or the union of two intervals (and it is included in $A$), we can apply (10):

$$\begin{aligned}&\operatorname{var}\big(\mathcal{L}_\varphi^i \mathcal{L}_{B_{i,j}}(h_\varphi)\big) \\ &\quad\leqslant \alpha^{i+1} \operatorname{var}(h_\varphi) + \alpha^i\big(4\alpha + \xi\nu(B_{i,j})\big)\|h_\varphi\|_\infty + K\nu(B_{i,j})\|h_\varphi\|_\infty \\ &\quad\leqslant \alpha^{i+1} \operatorname{var}(h_\varphi) + \alpha^i\big(4\alpha + \xi\nu(A)\big)\|h_\varphi\|_\infty + K\nu(A)\|h_\varphi\|_\infty. \end{aligned}$$

As $\alpha < 1$, we can write:

$$\begin{aligned}\operatorname{var}\big(\mathcal{L}_\varphi^r \mathcal{L}_A(h_\varphi)\big) &\leqslant K + K\nu(A) \leqslant K, \\ \operatorname{var}\big(\mathcal{L}_\varphi^i \mathcal{L}_{B_{i,j}}(h_\varphi)\big) &\leqslant K. \end{aligned}$$

Let us now sum over $r$, $i$ and $j$ by using the relation $\sum_{0 \leqslant i+j \leqslant N-2} 1 = N(N-1)/2 \leqslant N^2$

$$\sum_{r=0}^{N-1} \operatorname{var}\big(\mathcal{L}_\varphi^r \mathcal{L}_A(h_\varphi)\big) \leqslant KN,$$

$$\sum_{0\leqslant i+j\leqslant N-2} \operatorname{var}\big(\mathcal{L}_\varphi^i \mathcal{L}_{B_{i,j}}(h_\varphi)\big) \leqslant KN^2,$$

and according to the relation (7), we obtain, for $N$ big enough:

$$\operatorname{var}\big(\mathcal{L}_{A^c}^N(h_\varphi)\big) \leqslant K + KN + KN^2 \leqslant KN^2.$$

Combining (3), (4) and (6), we get (with $N = [\frac{t}{\mu_\varphi(A)}]$):

$$\big|g_A(t+s) - g_A(t)g_A(s)\big| \leqslant K\left(\frac{t}{\mu_\varphi(A)}\right)^2 \mathrm{e}^{-\gamma(\Delta+1)} + 2\Delta\mu_\varphi(A)$$

if $t/\mu_\varphi(A)$ is big enough. Now we choose the size of the hole $\Delta$: the only requirement is $\Delta < s/\mu_\varphi(A)$. Take $\Delta = 1/\mu_\varphi(A)^{1/4}$, $s \geqslant \sqrt{\mu_\varphi(A)}$ and $t = \sqrt{\mu_\varphi(A)}$:

$$\sup_{s\geqslant\sqrt{\mu_\varphi(A)}} \left|g_A\Big(\sqrt{\mu_\varphi(A)}+s\Big) - g_A\Big(\sqrt{\mu_\varphi(A)}\Big)g_A(s)\right|$$
$$\leqslant K\frac{1}{\mu_\varphi(A)}\,\mathrm{e}^{-\gamma(\frac{1}{\mu_\varphi(A)^{1/4}}+1)} + 2\mu_\varphi(A)^{3/4}.$$

If $n$ is big enough:

$$\frac{1}{\mu_\varphi(A)}\,\mathrm{e}^{-\gamma(\frac{1}{2\sqrt{\mu_\varphi(A)}}+1)} \leqslant \mu_\varphi(A)^{3/4},$$

therefore

$$\sup_{s\geqslant\sqrt{\mu_\varphi(A)}} \left|g_A\Big(\sqrt{\mu_\varphi(A)}+s\Big) - g_A\Big(\sqrt{\mu_\varphi(A)}\Big)g_A(s)\right| \leqslant K\mu_\varphi(A)^{3/4}. \qquad \square$$

Define $r = r(A) = \sqrt{\mu_\varphi(A)}$ and $\theta = \theta(A) = -\log g_A(r)$.

LEMMA 5.5. – *For $n$ big enough and for all $n$-cylinder $A$*:

$$\big|g_A\big(kr(A)\big) - \mathrm{e}^{k\theta(A)}\big| \leqslant \frac{K\mu_\varphi(A)^{3/4}}{1-\mathrm{e}^{-\theta(A)}}.$$

*Proof.* – See [5, Lemma 6]. □

LEMMA 5.6. – *There exists $\gamma_1$ and $\gamma_2$ such that, for all $\varepsilon > 0$, there exists $N_\varepsilon$ such that, for all $n > N_\varepsilon$, for all $A \in I_{n,\varepsilon}$ (where $I_{n,\varepsilon}$ is given by Lemma* 5.3):

$$1 - K\,\mathrm{e}^{-\gamma_1 n} \leqslant \frac{\theta(A)}{r(A)} \leqslant 1 + K\,\mathrm{e}^{-\gamma_2 n}.$$

*Proof.* – On the one hand, for $0 \leqslant u \leqslant 1/2$, $-\log(1-u) \leqslant u + u^2$. Now we get, by choosing $n$ big enough and using Lemma 5.2:

$$\begin{aligned}\theta(A) &\leqslant \mu_\varphi\left\{\tau_A \leqslant \frac{r(A)}{\mu_\varphi(A)}\right\} + \left(\mu_\varphi\left\{\tau_A \leqslant \frac{r(A)}{\mu_\varphi(A)}\right\}\right)^2 \\ &\leqslant r(A) + \mu_\varphi(A) + \big(r(A) + \mu_\varphi(A)\big)^2 \\ &\leqslant r(A)\big(1 + K\,\mathrm{e}^{-n\gamma_2}\big)\end{aligned}$$

since, by Lemma 3.1, $\mu_\varphi(A) \leqslant C\,\mathrm{e}^{-n\theta}$. On the other hand, by Lemma 5.3, if $A \in I_{n,\varepsilon}$:

$$\begin{aligned}\theta(A) &\geqslant 1 - \mathrm{e}^{-\theta(A)} \qquad (11)\\ &\geqslant \frac{r(A)^2}{r(A)^2 + \mu_\varphi(A)(1 + r(A)) + r(A)(1 + K\,\mathrm{e}^{-n\gamma_0})}, \\ \frac{\theta(A)}{r(A)} &\geqslant \frac{\sqrt{\mu_\varphi(A)}}{\mu_\varphi(A) + \mu_\varphi(A)(1 + \sqrt{\mu_\varphi(A)}) + \sqrt{\mu_\varphi(A)}(1 + K\,\mathrm{e}^{-n\gamma_0})} \\ &\geqslant \frac{1}{1 + K\,\mathrm{e}^{-n\gamma_1}} \geqslant 1 - K\,\mathrm{e}^{-n\gamma_1}\end{aligned}$$

which concludes the proof. □

*Proof of Theorem 5.1.* – Let $\varepsilon > 0$. We only consider cylinders $A \in I_{n,\varepsilon}$ and $n$ big enough so as to use the previous lemmas. Let $t > 0$, $t = kr(A) + v$ with $k = [\frac{t}{r(A)}]$ and $0 \leqslant v < r(A)$:

$$\begin{aligned}\left|g_A(t) - \mathrm{e}^{-t}\right| &\leqslant \left|g_A(t) - g_A\big(kr(A)\big)\right| + \left|g_A\big(kr(A)\big) - \mathrm{e}^{-\theta(A)k}\right| \\ &\quad + \left|\mathrm{e}^{-\theta(A)k} - \mathrm{e}^{-r(A)k}\right| + \left|\mathrm{e}^{-r(A)k} - \mathrm{e}^{-t}\right|.\end{aligned}$$

In the rest of the proof, we use the Lemma 3.1 which says that the measure of the $n$-cylinders decrease exponentially fast. First term, by Lemmas 5.2 and 5.3:

$$\begin{aligned}\left|g_A(t)-g_A\bigl(kr(A)\bigr)\right| &= \mu_\varphi\left\{\frac{kr(A)}{\mu_\varphi(A)}<\tau_A\leqslant\frac{t}{\mu_\varphi(A)}\right\}\\ &= \mu_\varphi\left\{0<\tau_A\leqslant\frac{v}{\mu_\varphi(A)}\right\}\leqslant\mu_\varphi(A)\left(1+\frac{v}{\mu_\varphi(A)}\right)\\ &\leqslant 2r(A)\leqslant K\,\mathrm{e}^{-\beta n}.\end{aligned}$$

Second term: by Lemma 5.5

$$\left|g_A\bigl(kr(A)\bigr)-\mathrm{e}^{-\theta(A)k}\right|\leqslant K\frac{\mu_\varphi(A)^{3/4}}{1-\mathrm{e}^{-\theta(A)}}$$

and, taking the inverse in the inequality (11)

$$\frac{1}{1-\mathrm{e}^{-\theta(A)}}\leqslant 2+\sqrt{\mu_\varphi(A)}+\frac{1}{\sqrt{\mu_\varphi(A)}}\bigl(1+K\,\mathrm{e}^{-n\gamma_0}\bigr),$$

$$K\frac{\mu_\varphi(A)^{3/4}}{1-\mathrm{e}^{-\theta(A)}}\leqslant K\mu_\varphi(A)^{3/4}+K\mu_\varphi(A)^{1/4}\leqslant K\,\mathrm{e}^{-n\beta}.$$

Fourth term:

$$\left|\mathrm{e}^{-r(A)k}-\mathrm{e}^{-t}\right|\leqslant v\leqslant\sqrt{\mu_\varphi(A)}\leqslant K\,\mathrm{e}^{-n\beta}.$$

Third term: a computation shows that

$$\left|\mathrm{e}^{-\theta(A)k}-\mathrm{e}^{-r(A)k}\right|\leqslant 2k\left|\theta(A)-r(A)\right|\bigl(\mathrm{e}^{-\theta(A)k}+\mathrm{e}^{-r(A)k}\bigr).$$

Lemma 5.6 ensures that

$$-K\,\mathrm{e}^{-n\gamma_1}r(A)\leqslant\theta(A)-r(A)\leqslant K\,\mathrm{e}^{-n\gamma_2}r(A),$$

$$\begin{aligned}\left|\mathrm{e}^{-\theta(A)k}-\mathrm{e}^{-r(A)k}\right| &\leqslant Kr(A)k\,\mathrm{e}^{-n\beta}\bigl(\mathrm{e}^{-\theta(A)k}+\mathrm{e}^{-r(A)k}\bigr)\\ &\leqslant K\,\mathrm{e}^{-n\beta}\left(r(A)k\,\mathrm{e}^{-r(A)k}+\theta(A)k\,\mathrm{e}^{-\theta(A)k}\frac{r(A)}{\theta(A)}\right)\leqslant K\,\mathrm{e}^{-n\beta}\end{aligned}$$

because $u\,\mathrm{e}^{-u}$ and $r(A)/\theta(A)$ are bounded. This ends the proof. □

## 6. PROOF OF THE MAIN THEOREM 1.1.

The mass of the cylinders, on the one hand, and the laws of the entrance times on the other hand, have a different influence on the sum (2). So, we have to determine which of the two is the most important and will give the behaviour of the law of $R_n$.

We have to prove the convergence in law which means the following:

$$\lim_{n\to+\infty} \mu_\varphi\{R_n > \mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}\} = \frac{1}{\sqrt{2\pi}}\int_u^\infty \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x.$$

Let $\varepsilon > 0$. Let us cut this quantity in several parts so as to use the Lemma 5.1 and the approximation of the law of entrance times:

$$\begin{aligned}
&\mu_\varphi\{R_n > \mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}\}\\
&\quad = \mu_\varphi\{R_n > \mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}\} - \sum_{A\in\mathcal{P}^n} \mu_\varphi(A)\mu_\varphi\{\tau_A > \mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}\} \quad (12)\\
&\qquad + \sum_{A\in\mathcal{P}^n} \mu_\varphi(A)\mu_\varphi\{\tau_A > \mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}\}\\
&\qquad - \sum_{A\in H_{n,\varepsilon}\cap G_{n,\varepsilon}} \mu_\varphi(A)\mu_\varphi\{\tau_A > \mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}\} \quad (13)\\
&\qquad + \sum_{A\in H_{n,\varepsilon}\cap G_{n,\varepsilon}} \mu_\varphi(A)\mu_\varphi\{\tau_A > \mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}\}\\
&\qquad - \sum_{A\in H_{n,\varepsilon}\cap G_{n,\varepsilon}} \mu_\varphi(A)\,\mathrm{e}^{-\mu_\varphi(A)\,\mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}} \quad (14)\\
&\qquad + \sum_{A\in H_{n,\varepsilon}\cap G_{n,\varepsilon}} \mu_\varphi(A)\,\mathrm{e}^{-\mu_\varphi(A)\,\mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}}. \quad (15)
\end{aligned}$$

Thanks to the Lemma 4.1, $\lim_{n\to+\infty}(12) = 0$.

By the Lemma 3.3, there exist $N_\varepsilon$ and $D(\varepsilon)$ such that for all $n > N_\varepsilon$, for all $A \in B(n, D(\varepsilon))$ and all $x$ in $A$: (we use the notation $B(n, D(\varepsilon)) = G_{n,\varepsilon}$)

$$\frac{1}{D(\varepsilon)} \leqslant \frac{\mu_\varphi(A)}{\lambda^{-n}S_n(x)} \leqslant D(\varepsilon), \tag{16}$$

$$\mu_\varphi\Big(\bigcup_{A\in G_{n,\varepsilon}} A\Big) \geqslant 1 - K\varepsilon.$$

By the Theorem 5.1, there exists $N'_\varepsilon$ such that, for all $n > N'_\varepsilon$, there exists $H_{n,\varepsilon} \in \mathcal{P}^n$ such that, for all $A$ in this set:

$$\sup_{t>0} \left|\mu_\varphi\{\tau_A > t\} - \mathrm{e}^{-t\mu_\varphi(A)}\right| \leqslant K\,\mathrm{e}^{-\beta n},$$

$$\mu_\varphi\Bigl(\bigcup_{A\in H_{n,\varepsilon}} A\Bigr) \geqslant 1 - K\varepsilon. \tag{17}$$

If $n > \max(N_\varepsilon, N'_\varepsilon)$:

$$|(13)| = \Bigl|\sum_{A\in(H_{n,\varepsilon}\cap G_{n,\varepsilon})^c} \mu_\varphi(A)\mu_\varphi\{\tau_A > \mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}\}\Bigr|$$
$$\leqslant \Bigl|\sum_{A\in(H_{n,\varepsilon}\cap G_{n,\varepsilon})^c} \mu_\varphi(A)\Bigr|$$
$$\leqslant \sum_{A\in G^c_{n,\varepsilon}} \mu_\varphi(A) + \sum_{A\in H^c_{n,\varepsilon}} \mu_\varphi(A) \leqslant K\varepsilon.$$

As for the term (14), by the Theorem 5.1, for all $\varepsilon > 0$:

$$|(14)| \leqslant \sum_{A\in H_{n,\varepsilon}\cap G_{n,\varepsilon}} \mu_\varphi(A)\left|\mu_\varphi\{\tau_A > \mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}\} - \mathrm{e}^{-\mu_\varphi(A)\mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}}\right|$$
$$\leqslant K\,\mathrm{e}^{-\beta n} \sum_{A\in\mathcal{P}^n} \mu_\varphi(A) \leqslant K\,\mathrm{e}^{-\beta n}.$$

We now turn to the term (15), which we can write $\mu_\varphi(Y_{n,\varepsilon})$ if we call $Y_{n,\varepsilon}$ the random variable:

$$Y_{n,\varepsilon} = \sum_{A\in H_{n,\varepsilon}\cap G_{n,\varepsilon}} \mathbf{1}_A\,\mathrm{e}^{-\mu_\varphi(A)\mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}}.$$

Let $\eta > 0$, the Markov inequality will give us some information about $\liminf \mu_\varphi(Y_{n,\varepsilon})$

$$\mu_\varphi(Y_{n,\varepsilon}) \geqslant \mathrm{e}^{-\mathrm{e}^{-\eta\sqrt{n}}}\,\mu_\varphi\Bigl\{\bigl(\log Y_{n,\varepsilon} \geqslant -\mathrm{e}^{-\eta\sqrt{n}}\bigr) \cap \Bigl(\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A\Bigr)\Bigr\}$$

and by the Lemma 3.3, we have the two following inclusions:

$$\Bigl(\Bigl(D(\varepsilon)\lambda^{-n}S_n \leqslant \frac{\mathrm{e}^{-\eta\sqrt{n}}}{\mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}}\Bigr) \cap \Bigl(\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A\Bigr)\Bigr)$$
$$\subset \Bigl(\bigl(\log Y_{n,\varepsilon} \geqslant -\mathrm{e}^{-\eta\sqrt{n}}\bigr) \cap \Bigl(\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A\Bigr)\Bigr),$$

$$\Bigl(\lambda^{-n}S_n \leqslant \frac{\mathrm{e}^{-2\eta\sqrt{n}}}{\mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}}\Bigr) \subset \Bigl(D(\varepsilon)\lambda^{-n}S_n \leqslant \frac{\mathrm{e}^{-\eta\sqrt{n}}}{\mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}}\Bigr)$$

for $n$ big enough. Consequently, we get the inequalities:

$$\mu_\varphi(Y_{n,\varepsilon}) \geqslant \mathrm{e}^{-\mathrm{e}^{-\eta\sqrt{n}}}\,\mu_\varphi\left(\left(\lambda^{-n}S_n \leqslant \frac{\mathrm{e}^{-2\eta\sqrt{n}}}{\mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}}\right)\cap\left(\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A\right)\right)$$
$$\geqslant \mathrm{e}^{-\mathrm{e}^{-\eta\sqrt{n}}}\,\mu_\varphi\left(\left(\frac{-\log S_n + n\log\lambda - nh}{\sigma(\varphi)\sqrt{n}} \geqslant u + \frac{2\eta}{\sigma(\varphi)}\right)\right.$$
$$\left.\cap\left(\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A\right)\right),$$

and $p(\varphi) = \log\lambda = h + \mu_\varphi(\varphi)$ so

$$\mathrm{e}^{\mathrm{e}^{-\eta\sqrt{n}}}\,\mu_\varphi(Y_{n,\varepsilon})$$
$$\geqslant \mu_\varphi\left(\left(\frac{-\log S_n + n\mu_\varphi(\varphi)}{\sigma(\varphi)\sqrt{n}} \geqslant u + \frac{2\eta}{\sigma(\varphi)}\right)\cap\left(\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A\right)\right)$$
$$\geqslant \mu_\varphi\left(\frac{-\log S_n + n\mu_\varphi(\varphi)}{\sigma(\varphi)\sqrt{n}} \geqslant u + \frac{2\eta}{\sigma(\varphi)}\right) - \mu_\varphi\left(\bigcup_{(G_{n,\varepsilon}\cap H_{n,\varepsilon})^c} A\right)$$
$$\geqslant \mu_\varphi\left(\frac{-\log S_n + n\mu_\varphi(\varphi)}{\sigma(\varphi)\sqrt{n}} \geqslant u + \frac{2\eta}{\sigma(\varphi)}\right) - \mu_\varphi\left(\bigcup_{G^c_{n,\varepsilon}} A\right)$$
$$-\mu_\varphi\left(\bigcup_{H^c_{n,\varepsilon}} A\right)$$
$$\geqslant \mu_\varphi\left(\frac{-\log S_n + n\mu_\varphi(\varphi)}{\sigma(\varphi)\sqrt{n}} \geqslant u + \frac{2\eta}{\sigma(\varphi)}\right) - K\varepsilon.$$

By applying the central-limit theorem to the system $(T, \mu_\varphi, \varphi)$, we obtain, letting first $n$ go to infinity, then $\eta$ to zero:

$$\liminf_{n\to\infty}\,\mu_\varphi(Y_{n,\varepsilon}) \geqslant \frac{1}{\sqrt{2\pi}}\int_u^\infty \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x - K\varepsilon.$$

For the lim sup, we use the inequality, for $\eta > 0$ (notice that $Y_{n,\varepsilon} \leqslant 1$):

$$\mu_\varphi(Y_{n,\varepsilon}) \leqslant \mathrm{e}^{-\mathrm{e}^{\eta\sqrt{n}}}\,\mu_\varphi\left\{\left(\log Y_{n,\varepsilon} < -\mathrm{e}^{\eta\sqrt{n}}\right)\cap\left(\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A\right)\right\}$$
$$+\,\mu_\varphi\left\{\left(\log Y_{n,\varepsilon} \geqslant -\mathrm{e}^{\eta\sqrt{n}}\right)\cap\left(\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A\right)\right\}.$$

Using the other inequality in the Lemma 3.3, we get the following inclusions:

$$\left(\left(\frac{\lambda^{-n} S_n}{D(\varepsilon)} \leqslant \frac{\mathrm{e}^{\eta\sqrt{n}}}{\mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}}\right) \cap \left(\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A\right)\right)$$

$$\supset \left(\left(\log Y_{n,\varepsilon} \geqslant -\mathrm{e}^{\eta\sqrt{n}}\right) \cap \left(\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A\right)\right),$$

$$\left(\lambda^{-n} S_n \leqslant \frac{\mathrm{e}^{2\eta\sqrt{n}}}{\mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}}\right) \supset \left(\frac{\lambda^{-n} S_n}{D(\varepsilon)} \leqslant \frac{\mathrm{e}^{-\eta\sqrt{n}}}{\mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}}\right)$$

for $n$ big enough. Consequently, we get the inequalities:

$$\mu_\varphi(Y_{n,\varepsilon}) \leqslant \mathrm{e}^{-\mathrm{e}^{\eta\sqrt{n}}}\,\mu_\varphi\left\{\left(\log Y_{n,\varepsilon} < -\mathrm{e}^{\eta\sqrt{n}}\right) \cap \left(\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A\right)\right\}$$

$$+\,\mu_\varphi\left(\left(\frac{-\log S_n + n\mu_\varphi(\varphi)}{\sigma(\varphi)\sqrt{n}} \geqslant u - \frac{2\eta}{\sigma(\varphi)}\right) \cap \left(\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A\right)\right)$$

$$\leqslant \mathrm{e}^{-\mathrm{e}^{\eta\sqrt{n}}} + \mu_\varphi\left(\frac{-\log S_n + n\mu_\varphi(\varphi)}{\sigma(\varphi)\sqrt{n}} \geqslant u - \frac{2\eta}{\sigma(\varphi)}\right).$$

Letting first $n$ go to infinity, then $\eta$ to zero:

$$\limsup_{n\to\infty}\ \mu_\varphi(Y_{n,\varepsilon}) \leqslant \frac{1}{\sqrt{2\pi}}\int_u^\infty \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x.$$

Gathering all the results about the terms (12), (14), (15), (16):

$$\liminf_{n\to\infty} \mu_\varphi\{R_n > \mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}\} \geqslant \frac{1}{\sqrt{2\pi}}\int_u^\infty \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x - K\varepsilon,$$

$$\limsup_{n\to\infty} \mu_\varphi\{R_n > \mathrm{e}^{nh}\,\mathrm{e}^{u\sigma(\varphi)\sqrt{n}}\} \leqslant \frac{1}{\sqrt{2\pi}}\int_u^\infty \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x + K\varepsilon.$$

This concludes the proof.

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