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# **A continuity property of the dimension of the harmonic measure of Cantor sets under perturbations**

by

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**ABSTRACT.** – We investigate the behaviour of the dimension of harmonic measure of the complementary of Cantor sets as a function of parameters determining these sets, and we establish continuity results.  
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*Key words:* Harmonic measure, Cantor sets, Fractals, Hausdorff dimension, Entropie

**RÉSUMÉ.** – Nous étudions la dimension de la mesure harmonique du complémentaire de certains ensembles de Cantor comme fonction de paramètres associés à ces ensembles et nous établissons des résultats de continuité. © 2000 Éditions scientifiques et médicales Elsevier SAS

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## **1. INTRODUCTION**

The purpose of this work is to study the dimension of the harmonic measure of the complementary of (not necessarily self-similar) Cantor

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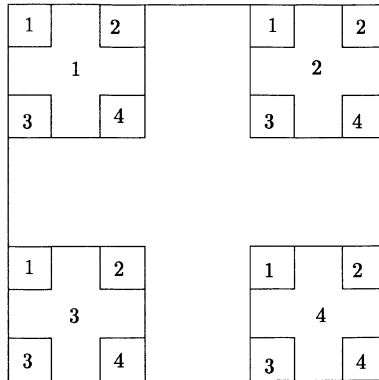


Fig. 1. A 4-corner Cantor set and its enumeration.

sets as a function of parameters assigned to these sets, and to establish some continuity properties. We develop our method on a particular kind of Cantor sets in the plane for convenience, even though the proof can be applied to all “self-similar” Cantor sets in  $\mathbb{R}^n$ ,  $n \geq 2$  (see Theorem 1.2).

A 4-corner Cantor set will be a compact set constructed in the following way: let  $\underline{A}$ ,  $\overline{A}$  be two constants with  $0 < \underline{A} \leq \overline{A} < 1/2$  and let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of real numbers with  $\underline{A} \leq a_n \leq \overline{A}$  for all  $n \in \mathbb{N}$ . We replace the square  $[0, 1]^2$  by four squares of sidelength  $a_1$  situated in the four corners of  $[0, 1]^2$ . Each of these squares is then replaced by four squares of sidelength  $a_1 a_2$  situated in its four corners. At the  $n$ th stage of the construction every square of the  $(n - 1)$ th generation will be replaced by four squares of sidelength  $a_1 \dots a_n$  situated in its four corners (see Fig. 1). Let  $\mathbb{K}$  be the Cantor set constructed by repeating the procedure.

Let us recall that for a probability measure  $\mu$  on  $\mathbb{R}^n$  the dimension  $\dim \mu$  of  $\mu$  is the smallest Hausdorff dimension of sets of measure 1. Carleson [7] has shown that for self-similar 4-corner Cantor sets (the sequence  $(a_n)_{n \in \mathbb{N}}$  is constant) the dimension of the harmonic measure of their complementary is strictly smaller than 1. His proof, involving ergodic theory techniques, was improved by Makarov and Volberg [12] who showed that the dimension of the harmonic measure of any self-similar 4-corner Cantor set is strictly smaller than the dimension of the Cantor set. Finally, Volberg [14,15] extended these results to a class of dynamic Cantor repellers. Other comparisons of harmonic and maximal measures for dynamical systems are proposed in [2,11].

In [3] it is shown that the dimension of the harmonic measure of the complementary of 4-corner Cantor sets is strictly smaller than the

Hausdorff dimension of the Cantor set, even when the sequence  $(a_n)_{n \in \mathbb{N}}$  is not constant. In [4] we prove that small perturbations of the sidelength of the squares of the construction of  $\mathbb{K}$  do not alterate this property.

It is therefore natural to ask whether the dimension of harmonic measure is continuous as a function of the sequence  $(a_n)_{n \in \mathbb{N}}$  with respect to the “sup” norm. We show that constant sequences are continuity “points” of this function. A general continuity statement seems much more difficult to prove as we will point out in Section 4.

**THEOREM 1.1.** – *Let  $\mathbb{K}$  be the 4-corners Cantor set associated to a constant sequence  $a_n = a$ . Let  $\mathbb{K}_{(a'_n)}$  be another Cantor set of the 4-corners type associated to the (not necessarily constant) sequence  $(a'_n)_{n \in \mathbb{N}}$  and let  $\omega$  and  $\omega'$  be the harmonic measures of  $\mathbb{K}$  and  $\mathbb{K}_{(a'_n)}$  respectively. Then for all  $\varepsilon > 0$  there exists a  $\delta = \delta(a, \varepsilon) > 0$  such that if  $|a'_n - a| < \delta$  for all  $n \in \mathbb{N}$  then  $|\dim \omega - \dim \omega'| < \varepsilon$ .*

This result is also valid for general self-similar Cantor sets: Let  $D$  be an open simply connected bounded set in the plane and let  $p_1, \dots, p_k$  be  $k$  affine functions. Let  $p_i(D) = D_i$  for  $i = 1, \dots, k$  and suppose that the sets  $D_i$  are open simply connected subsets of  $D$  with disjoint closures (see Fig. 2). A self-similar Cantor set  $\mathbb{K}$  will be the compact set defined by

$$\mathbb{K} = \bigcap_{n \in \mathbb{N}} \bigcup_{i_1, \dots, i_n} p_{i_1} \circ \dots \circ p_{i_n}(D).$$

The following known result (which can also be proved with classical techniques of the thermodynamical formalism) can be obtained using the method presented in this paper:

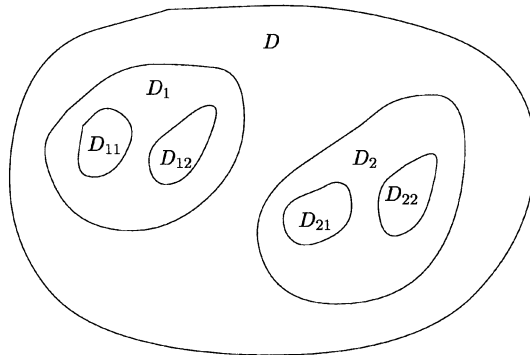


Fig. 2. A general Cantor type set.

**THEOREM 1.2.** – *Let  $p_1, \dots, p_k$  be  $k$  affine functions and  $\mathbb{K}$  be the self-similar Cantor set associated to these functions. Take  $\mathbb{K}'$  to be a self similar Cantor set associated to the functions  $p'_1, \dots, p'_k$ . Then for all  $\varepsilon > 0$  there exists a  $\delta = \delta(p_1, \dots, p_k, \varepsilon) > 0$  such that if  $\|p_i - p'_i\|_\infty < \delta$  for all  $i = 1, \dots, k$  then  $|\dim \omega - \dim \omega'| < \varepsilon$ , where  $\omega$  and  $\omega'$  are the harmonic measures of  $\mathbb{R}^2 \setminus \mathbb{K}$  and  $\mathbb{R}^2 \setminus \mathbb{K}'$ , respectively.*

The following sections are entirely devoted to the proof of Theorem 1.1.

## 2. PRELIMINARY RESULTS

In this section we establish some estimates on the harmonic measure of a Cantor set under perturbation, and recall some known results on the harmonic measure of Cantor-type sets. We also introduce the tools needed, such as the Hausdorff dimension and the entropy of a probability measure on a Cantor set.

*Notation 2.1.* – Let  $\mathbb{K}$  be a 4-corner Cantor set as described in the introduction. We enumerate  $\mathbb{K}$  by identifying it to the abstract Cantor set  $\{1, \dots, 4\}^{\mathbb{N}}$ . We denote  $I_{i_1 \dots i_n}$ , where  $i_j \in \{1, 2, 3, 4\}$  for  $1 \leq j \leq n$ , the  $4^n$  squares of the  $n$ th generation of the construction of  $\mathbb{K}$  with the enumeration shown in the Fig. 1 and the usual condition that  $I_{i_1 \dots i_n}$  is the “father” of the sets  $I_{i_1 \dots i_n i}$ ,  $i \in \{1, 2, 3, 4\}$ . It is clear that

$$\overline{A} \geq \frac{\text{diam } I_{i_1 \dots i_n i}}{\text{diam } I_{i_1 \dots i_n}} = a_{n+1} \geq \underline{A}, \quad i = 1, \dots, 4.$$

The collection of the squares of the  $n$ th generation of the construction of  $\mathbb{K}$  will be  $\mathcal{F}_n = \{I_{i_1 \dots i_n}; i_1, \dots, i_n = 1, \dots, 4\}$ , for  $n \in \mathbb{N}$ . For a square  $I \in \mathcal{F}_n$  we note  $P_k(I)$  the unique square of the  $(n - k)$ th generation containing  $I$ ; in particular we note  $\hat{I} = P_1(I)$  the “father” of  $I$ . If  $I = I_{i_1 \dots i_k} \in \mathcal{F}_k$  and  $J = I_{j_1 \dots j_n} \in \mathcal{F}_n$  we will note  $IJ = I_{i_1 \dots i_k j_1 \dots j_n} \in \mathcal{F}_{n+k}$ . Finally, for  $x \in \mathbb{K}$  and  $n \in \mathbb{N}$  let  $I_n(x)$  be the unique square of  $\mathcal{F}_n$  containing  $x$ .

For a domain  $\Omega$ , a point  $x \in \Omega$  and a Borel set  $F \subset \mathbb{R}^2$  we denote by  $\omega(x, F, \Omega)$  the harmonic measure of  $F \cap \partial\Omega$  (for the domain  $\Omega$ ) assigned to the point  $x$ . Clearly,  $F$  carries no measure if it does not intersect  $\partial\Omega$ . If  $\Omega$  is not specified it will be  $\mathbb{R}^2 \setminus \mathbb{K}$  and if  $x$  is the point at infinity we will simply note  $\omega(F)$ . Finally, for a Borel set  $E \subset \mathbb{R}^2$  we note  $\dim E$  the Hausdorff dimension of the set  $E$ .

## 2.1. Dimension of measures and entropy

In this section we recall some known results on the dimensions of measures (see also [13]).

DEFINITION 2.2. – For a probability measure  $\mu$  in  $\mathbb{R}^n$  we note  $\dim \mu$  the dimension of  $\mu$

$$\dim \mu = \inf \{ \dim E; E \text{ measurable, } \mu(E) = 1 \}.$$

We say that the measure  $\mu$  is monodimensional if  $\mu(E) = 0$  for all measurable sets  $E$  of Hausdorff dimension  $\dim E < \dim \mu$ .

One can prove that (see for instance [4,8]) if  $\mu$  is monodimensional then

$$\dim \mu = \liminf_{r \rightarrow 0} \frac{\log \mu B(x, r)}{\log r}, \quad \mu\text{-almost everywhere.} \quad (1)$$

If the probability measure  $\mu$  is supported by a 4-corner Cantor set, the balls  $B(x, r)$  can be replaced by the squares of the construction of the Cantor set (see [6]):

$$\dim \mu = \liminf_{n \rightarrow \infty} \frac{\log \mu(I_n(x))}{\log l(I_n(x))}, \quad \mu\text{-almost everywhere,} \quad (2)$$

where  $l(I_n(x))$  is the sidelength of the square  $I_n(x)$  and  $\underline{A}^n \leq l(I_n) \leq \overline{A}^n$ .

Remark 2.3. – If  $\mu$  is an arbitrary (not necessarily monodimensional) probability measure we get

$$\liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \leq \dim \mu, \quad \mu\text{-almost everywhere.} \quad (3)$$

Moreover

$$\dim \mu = \sup_{\mu} \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r}.$$

DEFINITION 2.4. – The entropy of a probability measure  $\mu$  supported by a Cantor set,  $h(\mu)$ , is defined

$$h(\mu) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{I \in \mathcal{F}_n} |\log \mu(I)| \mu(I),$$

whenever this limit exists.

For more information on entropy of measures see for instance [16].

For a self-similar Cantor set and an invariant ergodic measure  $\mu$  on the Cantor set one gets  $\dim \mu = h(\mu)/\chi(\mu)$ , where  $\chi(\mu)$  is the Lyapounov exponent of  $\mu$ . If  $\mathbb{K}$  is a 4-corners self-similar Cantor set (i.e.,  $a_n = a$  for all  $n \in \mathbb{N}$ ), then for all invariant ergodic probability measures  $\mu$  on  $\mathbb{K}$  we have  $\chi(\mu) = |\log a|$  (see also [12]).

**2.2. Estimating perturbations of the harmonic measure**

Suppose that the 4-corner Cantor set  $\mathbb{K}$  is associated to the sequence  $(a_n)_{n \in \mathbb{N}}$  and let  $\mathbb{K}'$  be another Cantor set associated to the sequence  $(a'_n)_{n \in \mathbb{N}}$ . Let  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  be the collections of squares associated to  $\mathbb{K}$  and  $(\mathcal{F}'_n)_{n \in \mathbb{N}}$  those associated to  $\mathbb{K}'$ . For  $I \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$  and  $I' \in \bigcup_{n \in \mathbb{N}} \mathcal{F}'_n$  we will write  $I \overset{\text{cod}}{\sim} I'$  if  $I$  and  $I'$  have the same enumeration (with respect to the identification to the abstract Cantor set  $\{1, 2, 3, 4\}^{\mathbb{N}}$ ).

If  $\omega$  is the harmonic measure of  $\mathbb{R}^2 \setminus \mathbb{K}$  and  $\omega'$  the harmonic measure of  $\mathbb{R}^2 \setminus \mathbb{K}'$  we have the following theorem.

**THEOREM 2.5.** – *For all  $\varepsilon > 0$  there exists a  $\delta > 0$  such that*

$$\sup_{n \in \mathbb{N}} |a_n - a'_n| < \delta \Rightarrow \left| \frac{\omega(I)}{\omega(\hat{I})} : \frac{\omega'(I')}{\omega'(\hat{I}')} - 1 \right| < \varepsilon, \tag{4}$$

for all  $I \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$  and  $I' \in \bigcup_{n \in \mathbb{N}} \mathcal{F}'_n$  with  $I \overset{\text{cod}}{\sim} I'$ .

The remaining of this section is devoted to the proof of Theorem 2.5. Let us first remark that there exists a constant  $c_0 = c_0(\underline{A}, \overline{A}) > 1$  such that for all squares  $I = I_{i_1 \dots i_n} \in \mathcal{F}_n$  of the construction of  $\mathbb{K}$  (or of  $\mathbb{K}'$ ) we have  $(c_0 I) \cap \mathbb{K} = I \cap \mathbb{K}$  (and  $(c_0 I) \cap \mathbb{K}' = I \cap \mathbb{K}'$ ).

We will use the following result which is a variant of a strong Harnack’s principle.

**LEMMA 2.6** ([7,12]). – *Let  $\Omega$  be a domain containing  $\infty$  and let  $A_1 \subset B_1 \subset A_2 \subset B_2 \subset \dots \subset A_n \subset B_n$  be conformal discs such that the annuli  $B_i \setminus A_i$  are contained in  $\Omega$ , for  $1 \leq i \leq n$ . If the moduli of the annuli are uniformly bounded away from zero and if  $\infty \in \Omega \setminus B_n$  then, for all pairs of positive harmonic functions  $u, v$  vanishing on  $\partial\Omega \setminus A_1$  and for all  $x \in \Omega \setminus B_n$  we have*

$$\left| \frac{u(x)}{v(x)} : \frac{u(\infty)}{v(\infty)} - 1 \right| \leq Kq^n, \tag{5}$$

where  $q < 1$  and  $K$  are two constants that depend only on the lower bound of the moduli of the annuli.

We use the previous result to establish the following lemma. Both of them are closely related to the Boundary Harnack principle (see [1]).

LEMMA 2.7. – *Let  $\varepsilon > 0$ . Under the conditions of Theorem 2.5 there exists a  $k_0 = k_0(\underline{A}, \overline{A}) > 0$  such that for all  $k \geq k_0$  and all squares  $I$  of the construction of  $\mathbb{K}$ , if  $Q = c_0 P_k(I)$  then*

$$\left| \frac{\omega(x, I, Q \setminus \mathbb{K})}{\omega(x, \hat{I}, Q \setminus \mathbb{K})} : \frac{\omega(I)}{\omega(\hat{I})} - 1 \right| < \varepsilon \quad \text{for all } x \in \partial \left\{ \frac{1+c_0}{2} \right\} P_k(I). \quad (6)$$

The result applies also to the Cantor set  $\mathbb{K}'$ .

*Proof.* – By Lemma 2.6, if  $k_0 = k_0(\underline{A}, \overline{A})$  is big enough (such that  $Kq^{k_0} < \varepsilon$ , where  $K, q$  are the constants given by the lemma) then

$$\left| \frac{\omega(x, I)}{\omega(x, \hat{I})} : \frac{\omega(I)}{\omega(\hat{I})} - 1 \right| < \varepsilon, \quad \text{for } x \notin \left\{ \frac{1+c_0}{2} \right\} P_k(I). \quad (7)$$

Let  $A = \omega(I)/\omega(\hat{I})$ . We have

$$\omega(x, I, Q \setminus \mathbb{K}) = \omega(x, I) - \int_{\partial Q} \omega(z, I) \omega(x, dz, Q \setminus \mathbb{K}),$$

for  $x \in \partial \left\{ \frac{1+c_0}{2} \right\} P_k(I)$ .

By eq. (7),

$$A\omega(x, \hat{I}) - \varepsilon A\omega(x, \hat{I}) \leq \omega(x, I) \leq A\omega(x, \hat{I}) + \varepsilon A\omega(x, \hat{I}).$$

We get

$$\begin{aligned} \omega(x, I, Q \setminus \mathbb{K}) &\leq A\omega(x, \hat{I}) + \varepsilon A\omega(x, \hat{I}) \\ &\quad - \int_{\partial Q} (A\omega(z, \hat{I}) - \varepsilon A\omega(z, \hat{I})) \omega(x, dz, Q \setminus \mathbb{K}) \\ &= A\omega(x, \hat{I}) - \int_{\partial Q} A\omega(z, \hat{I}) \omega(x, dz, Q \setminus \mathbb{K}) \\ &\quad + \varepsilon \left( A\omega(x, \hat{I}) + \int_{\partial Q} A\omega(z, \hat{I}) \omega(x, dz, Q \setminus \mathbb{K}) \right) \end{aligned}$$



$$\begin{aligned}
&= A\omega(x, \hat{I}, Q \setminus \mathbb{K}) \\
&\quad + \varepsilon \left( A\omega(x, \hat{I}) + \int_{\partial Q} A\omega(z, \hat{I})\omega(x, dz, Q \setminus \mathbb{K}) \right). \quad (8)
\end{aligned}$$

Therefore,

$$\frac{\omega(x, I, Q \setminus \mathbb{K})}{\omega(x, \hat{I}, Q \setminus \mathbb{K})} \leq A + \varepsilon A \frac{\omega(x, \hat{I}) + \int_{\partial Q} \omega(z, \hat{I})\omega(x, dz, Q \setminus \mathbb{K})}{\omega(x, \hat{I}, Q \setminus \mathbb{K})}. \quad (9)$$

It suffices now to show that the quantity

$$\frac{\omega(x, \hat{I}) + \int_{\partial Q} \omega(z, \hat{I})\omega(x, dz, Q \setminus \mathbb{K})}{\omega(x, \hat{I}, Q \setminus \mathbb{K})}$$

is smaller than a given constant. Take  $x_0 \in \partial\left\{\frac{1+c_0}{2}\right\}P_k(I)$  such that

$$\omega(x_0, \hat{I}) = \max \left\{ \omega(x, \hat{I}); x \notin \left\{ \frac{1+c_0}{2} \right\} P_k(I) \right\}.$$

We then have

$$\begin{aligned}
\omega(x_0, \hat{I}, Q \setminus \mathbb{K}) &= \omega(x_0, \hat{I}) - \int_{\partial Q} \omega(z, \hat{I})\omega(x_0, dz, Q \setminus \mathbb{K}) \\
&\geq \omega(x_0, \hat{I}) - \int_{\partial Q} \omega(x_0, \hat{I})\omega(x_0, dz, Q \setminus \mathbb{K}) \\
&= \omega(x_0, \hat{I})(1 - \omega(x_0, \partial Q, Q \setminus \mathbb{K})).
\end{aligned}$$

By standard techniques one can verify (see [3]) that  $(1 - \omega(x_0, \partial Q, Q \setminus \mathbb{K}))$  is greater than a constant  $c > 0$  depending only on  $\underline{A}, \bar{A}$ .

By using Harnack's principle we get

$$1 - \omega(x, \partial Q, Q \setminus \mathbb{K}) \geq c, \quad \text{for all } x \in \partial\left\{\frac{1+c_0}{2}\right\}P_k(I),$$

for a new constant  $c > 0$ .

Hence

$$\frac{\omega(x, \hat{I}) + \int_{\partial Q} \omega(z, \hat{I})\omega(x, dz, Q \setminus \mathbb{K})}{\omega(x, \hat{I}, Q \setminus \mathbb{K})} \leq \frac{2}{c}$$

and therefore, by relation (9),

$$\frac{\omega(x, I, Q \setminus \mathbb{K})}{\omega(x, \hat{I}, Q \setminus \mathbb{K})} \leq A \left( 1 + \frac{2}{c} \varepsilon \right). \tag{10}$$

On the other hand  $A = \omega(I)/\omega(\hat{I})$ ; we obtain

$$\frac{\omega(x, I, Q \setminus \mathbb{K})}{\omega(x, \hat{I}, Q \setminus \mathbb{K})} : \frac{\omega(I)}{\omega(\hat{I})} - 1 < \frac{2}{c} \varepsilon, \quad \text{for all } x \in \partial \left\{ \frac{1+c_0}{2} \right\} P_k(I).$$

The left hand inequality can be established in the same way and the proof is complete.  $\square$

LEMMA 2.8. – *Let  $Q_1 \subset Q_2 \subset Q_3 \subset \dots \subset Q_n$  be squares verifying that the moduli of the annuli  $Q_j \setminus Q_{j-1}$  are greater than  $1/c$  and smaller than  $c > 1$ . Let  $S \subset \frac{1}{c} Q_1$  be the intersection of a Cantor set  $\mathbb{K}$  as above with  $Q_1$ , and suppose that the annuli do not intersect  $\mathbb{K}$ . Then, there exist two constants  $C > 0$  and  $\delta > 0$  depending only on  $c, \bar{A}$  and  $\underline{A}$  such that for all  $x \in \{\frac{2}{1+c}\} Q_1$*

$$\omega(x, S, Q_n \setminus S) > 1 - C \prod_{k=1}^n \left( 1 - \frac{\delta}{k} \right). \tag{11}$$

*Proof.* – We can assume that  $Q_1 = [0, 1]^2$ . Let us recall that there is a constant  $c_4 = c_4(\underline{A}, \bar{A})$  such that for  $x \in \partial\{\frac{2}{1+c}\} Q_j$  we have

$$\omega(x, S, Q_j \setminus S) \geq c_4 \frac{-1}{\log(\text{diam } Q_j)}$$

(see for instance [3]). Let  $\omega_j$  be the harmonic measures of the domains  $Q_j \setminus S, j = 1, \dots, n$ . Take  $x_0 \in \partial\{\frac{2}{1+c}\} Q_1$ .

We have

$$\begin{aligned} \omega_{j-1}(x_0, S) &= \omega_j(x_0, S) - \int_{\partial Q_{j-1}} \omega_j(z, S) \omega_{j-1}(x_0, dz) \\ &\leq \omega_j(x_0, S) + c_4 \frac{1}{\log(\text{diam } Q_j)} \omega_{j-1}(x_0, \partial Q_{j-1}). \end{aligned}$$

Now by the lower bound of the annuli  $Q_j \setminus Q_{j-1}$  we get that

$$\frac{-1}{\log(\text{diam } Q_j)} \geq j^{-1} C_0,$$

where  $C_0 > 1$  is a constant. We get

$$\omega_{j-1}(x_0, \partial Q_{j-1}) \left(1 - \frac{c_4}{jC_0}\right) \geq \omega_j(x_0, \partial Q_j).$$

To finish the proof, it suffices to recall that  $\omega_j(x_0, S) = 1 - \omega_j(x_0, \partial Q_j)$  and to take  $\delta = c_4/C_0$  and  $C = \max\{\omega_1(x, S); x \in \{\frac{1+\alpha}{2}\}Q_1\}$ .  $\square$

LEMMA 2.9. – For  $J \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ ,  $J' \in \bigcup_{n \in \mathbb{N}} \mathcal{F}'_n$  and  $s > 0$  we define

$$U_s(J) = \{x \in (c_0J) \setminus \mathbb{K} \text{ such that } \omega(x, J, (c_0J) \setminus \mathbb{K}) > 1 - s\},$$

$$U'_s(J') = \{x \in (c_0J') \setminus \mathbb{K}' \text{ such that } \omega(x, J', (c_0J') \setminus \mathbb{K}') > 1 - s\}.$$

There is an increasing function  $\eta > 0$  depending only on  $\underline{A}, \bar{A}$ , such that

$$\begin{aligned} \text{dist}(J \cap \mathbb{K}, \partial U_s(J)) &> \eta(s)l(J), \\ \text{dist}(J' \cap \mathbb{K}', \partial U'_s(J')) &> \eta(s)l(J'). \end{aligned} \tag{12}$$

*Proof.* – By dilating the square  $J$  we can assume that  $c_0J = [0, 1]^2 = Q_0$  (idem for the square  $J'$ ). Let  $J_n \subset J$  be a square such that  $P_n(J_n) = J$ . The squares

$$Q_1 = c_0P_1(J_n) \subset Q_2 = c_0P_2(J_n) \subset \dots \subset Q_n = c_0P_n(J_n) = Q_0$$

satisfy the conditions of Lemma 2.8, by construction. Hence there exists a  $\delta = \delta(\underline{A}, \bar{A}) > 0$  and a constant  $C > 0$  such that for all  $x \in \frac{c_0+1}{2}J_n$

$$\omega(x, J_n, Q_n \setminus \{J_n \cap \mathbb{K}\}) > 1 - C \prod_{k=1}^n \left(1 - \frac{\delta}{k}\right).$$

Using the maximum principle, we can easily verify that for all  $x \in \frac{c_0+1}{2}J_n$

$$1 - \omega(x, \partial Q_0, Q_0 \setminus \{J \cap \mathbb{K}\}) \geq 1 - \omega(x, \partial Q_0, Q_0 \setminus \{J_n \cap \mathbb{K}\}).$$

Therefore,

$$\begin{aligned} \omega(x, J, Q_0 \setminus \{J \cap \mathbb{K}\}) &\geq \omega(x, J_n, Q_0 \setminus \{J_n \cap \mathbb{K}\}) \\ &\geq 1 - C \prod_{k=1}^n \left(1 - \frac{\delta}{k}\right). \end{aligned} \tag{13}$$

The square  $J_n$  has been chosen arbitrarily, hence the last equation gives

$$\begin{aligned} \text{dist}(x, J \cap \mathbb{K}) &< \frac{(c_0 - 1)\underline{A}^n}{4} \Rightarrow \\ \Rightarrow \omega(x, J, Q_0 \setminus \{J \cap \mathbb{K}\}) &\geq 1 - C \prod_{k=1}^n \left(1 - \frac{\delta}{k}\right). \end{aligned}$$

We can now choose  $n = n(\underline{A}, \overline{A}, s)$  such that  $C \prod_{k=1}^n (1 - \frac{\delta}{k}) < s$ .  $\square$

*Proof of Theorem 2.5.* – Fix  $\varepsilon > 0$ . Let  $I$  be a square of the construction of the Cantor set  $\mathbb{K}$  and  $I'$  a square of the construction of  $\mathbb{K}'$ , with  $I \overset{\text{cod}}{\sim} I'$ . We note  $Q = c_0 P_k(I)$ . By translating and dilating the Cantor set  $\mathbb{K}'$  we can assume that  $c_0 P_k(I') = Q$  (recall that the harmonic measure is invariant under affine maps).

By Lemma 2.7 it suffices to compare

$$\frac{\omega(x, I, Q \setminus \mathbb{K})}{\omega(x, \hat{I}, Q \setminus \mathbb{K})} \quad \text{and} \quad \frac{\omega(x, I', Q \setminus \mathbb{K}')}{\omega(x, \hat{I}', Q \setminus \mathbb{K}')},$$

if  $k$  is taken sufficiently large. Let us fix  $k \in \mathbb{N}$ . For  $J \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ ,  $J' \in \bigcup_{n \in \mathbb{N}} \mathcal{F}'_n$  with  $J \cup J' \subset Q$  and for  $0 < s < 1$ , put

$$\begin{aligned} \Omega_s(J) &= \{x \in Q \setminus \mathbb{K} \text{ such that } \omega(x, J, Q \setminus \mathbb{K}) > 1 - s\}, \\ \Omega'_s(J') &= \{x \in Q \setminus \mathbb{K}' \text{ such that } \omega(x, J', Q \setminus \mathbb{K}') > 1 - s\}. \end{aligned} \tag{14}$$

Since the harmonic measure is increasing as a function of the domain we have

$$U_s(J) \subset \Omega_s(J) \quad \text{and} \quad U'_s(J') \subset \Omega'_s(J').$$

Therefore,

$$\begin{aligned} \text{dist}(J \cap \mathbb{K}, \partial \Omega_s(J)) &> \eta(s)l(J) \quad \text{and} \\ \text{dist}(J' \cap \mathbb{K}', \partial \Omega'_s(J')) &> \eta(s)l(J'), \end{aligned} \tag{15}$$

for all squares  $J \overset{\text{cod}}{\sim} J'$ .

Now let  $s$  be a positive constant which will be precised later. By eqs. (15) we can choose  $\delta = \delta(s, k) > 0$  such that

$$\sup_{n \in \mathbb{N}} |a'_n - a_n| < \delta \Rightarrow I \cap \mathbb{K} \subset \Omega'_s(I') \quad \text{and} \quad I' \cap \mathbb{K}' \subset \Omega_s(I). \tag{16}$$

The idea is that if the sequences  $(a_n)_{n \in \mathbb{N}}$  and  $(a'_n)_{n \in \mathbb{N}}$  are close enough, then any two squares of the  $k$ th generation of the constructions (with the same encoding) will be close with regard to the Hausdorff metric (where  $k$  is a fixed positive integer).

By the formulas (16) it follows on applying the maximum principle

$$|\omega(\zeta, J, Q \setminus \mathbb{K}) - \omega(\zeta, J', Q \setminus \mathbb{K}')| < s, \quad \text{for all } \zeta \in \partial \left\{ \frac{1+c_0}{2} \right\} P_k(I).$$

On the other hand, Harnack's principle gives a constant  $c_3 = c_3(k) > 0$  such that

$$\min\{\omega(x, J, Q \setminus \mathbb{K}), \omega(x, J', Q \setminus \mathbb{K}')\} > c_3, \quad \forall x \in \partial \left\{ \frac{1+c_0}{2} \right\} P_k(I) \quad (17)$$

for all squares  $J$  of the construction of  $\mathbb{K}$  and all squares  $J'$  of the construction of  $\mathbb{K}'$  of the same generation with  $I$  such that  $J \cup J' \subset Q$ .

Take  $s > 0$  verifying  $s/c_3 < \varepsilon^2/10$ . Then, for all squares  $J$  and  $J'$  of the same generation with  $I$  such that  $J \overset{\text{cod}}{\sim} J' \subset Q$ , we get

$$\left| \frac{\omega(\zeta, J, Q \setminus \mathbb{K})}{\omega(\zeta, J', Q \setminus \mathbb{K}')} - 1 \right| < \varepsilon^2/10, \quad \forall \zeta \in \partial \left\{ \frac{1+c_0}{2} \right\} P_k(I). \quad (18)$$

We deduce

$$\left| \frac{\omega(\zeta, I, Q \setminus \mathbb{K})}{\omega(\zeta, \hat{I}, Q \setminus \mathbb{K})} : \frac{\omega(\zeta, I', Q \setminus \mathbb{K}')}{\omega(\zeta, \hat{I}', Q \setminus \mathbb{K}')} - 1 \right| < \varepsilon, \quad \forall \zeta \in \partial \left\{ \frac{1+c_0}{2} \right\} P_k(I) \quad (19)$$

which completes the proof.  $\square$

### 2.3. Some estimates on harmonic measure when $(a_n)_{n \in \mathbb{N}}$ is constant

Throughout this section  $\mathbb{K}$  is a 4-corners self similar Cantor set associated to a constant sequence  $(a_n)_{n \in \mathbb{N}}$  ( $a_n = a$  for all  $n \in \mathbb{N}$ ,  $0 < a < 1/2$ ) and  $\omega$  will be the harmonic measure of  $\mathbb{K}$ . The following lemma is a corollary of Lemma 2.6.

LEMMA 2.10 ([7,12]). – For every  $I \in \mathcal{F}_n$ ,  $J \in \mathcal{F}_k$  and every  $L \in \mathcal{F}_m$ ,  $n, k, m \in \mathbb{N}$

$$\left| \frac{\omega(IJL)}{\omega(IJ)} : \frac{\omega(JL)}{\omega(J)} - 1 \right| < C q^k, \quad (20)$$

where the constants  $C > 0$  and  $q \in (0, 1)$ , depend only on  $a$ .

Using a slightly weaker version this lemma, Carleson [7] shows that for the self-similar Cantor set  $\mathbb{K}$  there exists an invariant ergodic measure  $\mu$  and a constant  $C > 0$  such that  $\frac{1}{C}\omega \leq \mu \leq C\omega$ . Therefore,

$$\begin{aligned} \dim \omega &= \dim \mu = \frac{h(\mu)}{|\log a|} = \lim_{n \rightarrow \infty} \frac{1}{n |\log a|} \sum_{I \in \mathcal{F}_n} |\log \omega(I)| \omega(I) \\ &= \lim_{n \rightarrow \infty} \frac{\log \omega(I_n(x))}{n \log a}, \end{aligned}$$

for  $\omega$ -a.e.  $x \in \mathbb{K}$ .

With the same notation as before we have the following technical but essential lemma:

LEMMA 2.11. – Take  $\varepsilon > 0$ . There exists  $p_0 \in \mathbb{N}$  big enough such that if  $p \geq p_0$  then

$$\left| \frac{1}{p} \sum_{J \in \mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \left| \log \left( \frac{\omega(IJ)}{\omega(I)} \right) \right| - h(\mu) \right| < \varepsilon \tag{21}$$

for all  $I \in \bigcup \{ \mathcal{F}_n, n \in \mathbb{N} \}$ .

*Proof.* – Take  $\varepsilon > 0$  and  $p \in \mathbb{N}$ . We write  $p = p_1 + p_2$ , with  $p, p_1, p_2$  to be chosen later. We get

$$\begin{aligned} \sum_{J \in \mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \log \left( \frac{\omega(IJ)}{\omega(I)} \right) &= \sum_{J_1 \in \mathcal{F}_{p_1}} \sum_{J_2 \in \mathcal{F}_{p_2}} \frac{\omega(IJ_1J_2)}{\omega(I)} \log \left( \frac{\omega(IJ_1J_2)}{\omega(I)} \right) \\ &= \sum_{J_1 \in \mathcal{F}_{p_1}} \sum_{J_2 \in \mathcal{F}_{p_2}} \frac{\omega(IJ_1J_2)}{\omega(I)} \log \left( \frac{\omega(IJ_1J_2)}{\omega(IJ_1)} \right) \\ &\quad + \sum_{J_1 \in \mathcal{F}_{p_1}} \sum_{J_2 \in \mathcal{F}_{p_2}} \frac{\omega(IJ_1J_2)}{\omega(I)} \log \left( \frac{\omega(IJ_1)}{\omega(I)} \right). \end{aligned}$$

Let us note

$$\begin{aligned} A &= \sum_{J_1 \in \mathcal{F}_{p_1}} \sum_{J_2 \in \mathcal{F}_{p_2}} \frac{\omega(IJ_1J_2)}{\omega(I)} \log \left( \frac{\omega(IJ_1J_2)}{\omega(IJ_1)} \right) \quad \text{and} \\ B &= \sum_{J_1 \in \mathcal{F}_{p_1}} \sum_{J_2 \in \mathcal{F}_{p_2}} \frac{\omega(IJ_1J_2)}{\omega(I)} \log \left( \frac{\omega(IJ_1)}{\omega(I)} \right). \end{aligned}$$

Let us recall that there exists a constant  $c_2 > 0$  such that  $\omega(I_n(x)) \geq c_2 \omega(I_{n-1}(x))$  for all  $x \in \mathbb{K}$  and all  $n \in \mathbb{N}$ . It follows that  $|B| \leq -\log(c_2^{p_1})$ . We show that if  $p_2$  is big enough, then  $-\frac{1}{p}A$  will be close to  $h(\mu)$  for all  $I$ .

By the Shannon–McMillan theorem we get that for  $p_1$  fixed

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \log \left( \frac{\omega(I_n(x))}{\omega(I_{p_1}(x))} \right) \right| = h(\mu) \quad \text{for } \omega\text{-almost every } x \in \mathbb{K}. \quad (22)$$

By the dominated convergence theorem we get that there exists  $N_0 = N_0(p_1) \in \mathbb{N}$  big enough such that for all  $p_2 \geq N_0$  and all  $J_1 \in \mathcal{F}_{p_1}$

$$\left| \frac{1}{p_2} \sum_{J_2 \in \mathcal{F}_{p_2}} \left| \log \left( \frac{\omega(J_1 J_2)}{\omega(J_1)} \right) \right| \frac{\omega(J_1 J_2)}{\omega(J_1)} - h(\mu) \right| < \varepsilon. \quad (23)$$

By Lemma 2.10 we have

$$\left| \frac{\omega(I J_1 J_2)}{\omega(I J_1)} : \frac{\omega(J_1 J_2)}{\omega(J_1)} - 1 \right| < C q^{p_1} \quad \text{with } q < 1. \quad (24)$$

Choose  $p_1$  big enough to have  $C q^{p_1} < \varepsilon$  and take  $p_2 \geq N_0(p_1)$  in a way that (23) remains valid. Then,

$$\begin{aligned} -h(\mu) - \frac{1}{p} A &= -h(\mu) - \frac{1}{p} \sum_{J_1 \in \mathcal{F}_{p_1}} \sum_{J_2 \in \mathcal{F}_{p_2}} \frac{\omega(I J_1 J_2)}{\omega(I)} \log \left( \frac{\omega(I J_1 J_2)}{\omega(I J_1)} \right) \\ &= -h(\mu) - \frac{1}{p} \sum_{J_1 \in \mathcal{F}_{p_1}} \frac{\omega(I J_1)}{\omega(I)} \\ &\quad \times \sum_{J_2 \in \mathcal{F}_{p_2}} \frac{\omega(I J_1 J_2)}{\omega(I J_1)} \log \left( \frac{\omega(I J_1 J_2)}{\omega(I J_1)} \right) \\ &\leq -h(\mu) + (h(\mu) + 2\varepsilon) \frac{p_2}{p} \sum_{J_1 \in \mathcal{F}_{p_1}} \frac{\omega(I J_1)}{\omega(I)} \\ &= -h(\mu) + \frac{p_2}{p} (h(\mu) + 2\varepsilon). \end{aligned} \quad (25)$$

It suffices now to modify the choice of  $p_2$  by taking, if necessary,  $p_2$  even greater so that

$$\frac{p_2}{p} = \frac{p_2}{p_1 + p_2} > 1 - \varepsilon.$$

The lower bound is obtained in the same manner.

To estimate  $B$  remark that

$$\left| \frac{1}{p} B \right| \leq -\frac{p_1}{p} \log c_2.$$

By the choice of  $p_2$  we have  $p_1/p < \varepsilon$  and therefore

$$\left| \frac{1}{p} B \right| \leq -\varepsilon \log c_2.$$

The quantities  $c_2$  and  $h(\mu)$  not depending on  $p_1$  or  $p_2$ , we have shown that

$$\left| \frac{1}{p} \sum_{J \in \mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \left| \log \left( \frac{\omega(IJ)}{\omega(I)} \right) \right| - h(\mu) \right| < \varepsilon, \tag{26}$$

if  $p$  is large enough.  $\square$

### 3. PROOF OF THEOREM 1.1

Take  $\varepsilon > 0$ . By Theorem 2.5 there is a  $\delta > 0$  such that

$$(|a_n - a| < \delta, \forall n \in \mathbb{N}) \Rightarrow \left| \frac{\omega(I)}{\omega(\hat{I})} : \frac{\omega'(I')}{\omega'(\hat{I}')} - 1 \right| < \varepsilon \tag{27}$$

for all  $I \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$  and  $I' \in \bigcup_{n \in \mathbb{N}} \mathcal{F}'_n$  with  $I' \overset{\text{cod}}{\sim} I$ .

By Lemma 2.11 we can find an integer  $p$  big enough for the inequality (21) to be valid for all  $I \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ .

By (27), if  $\delta > 0$  is small one gets

$$\left| \frac{\omega(I)}{\omega(\hat{I})} : \frac{\omega'(I')}{\omega'(\hat{I}')} - 1 \right| < \varepsilon', \tag{28}$$

where  $\varepsilon' > 0$  is small enough to have  $(1 + \varepsilon')^p < 1 + \varepsilon^{10}$ .

We obtain

$$\left| \frac{\omega'(I'J')}{\omega'(I')} : \frac{\omega(IJ)}{\omega(I)} - 1 \right| < \varepsilon^{10},$$

for all  $I \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$  with  $I' \overset{\text{cod}}{\sim} I$  and all  $J \in \mathcal{F}_p$  with  $J' \overset{\text{cod}}{\sim} J$ .

In the same way as before, for  $\delta < a/2$  there is a constant  $c_3 = c_3(a) > 0$  such that for all  $I' \in \mathcal{F}'_n$  and all  $I \in \mathcal{F}_n, n \in \mathbb{N}$ ,

$$\omega(I) \geq c_3^n \quad \text{and} \quad \omega'(I') \geq c_3^n.$$

Then, if we take  $\varepsilon < \inf\{\log^{-1} c_3, 2^{-1}\}$ , we get that for all  $I \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ , with  $I' \overset{\text{cod}}{\sim} I$



$$\begin{aligned}
& \frac{1}{p} \sum_{J' \in \mathcal{F}'_p} \frac{\omega'(I'J')}{\omega'(I')} \left| \log \left( \frac{\omega'(I'J')}{\omega'(I')} \right) \right| \\
& \leq (1 + \varepsilon^{10}) \sum_{\substack{J \in \mathcal{F}_p \\ J \stackrel{\text{cod}}{\sim} J'}} \frac{\omega(IJ)}{\omega(I)} \left| \log \left( \frac{\omega'(I'J')}{\omega'(I')} \right) \right| \\
& \leq \frac{1}{p} (1 + \varepsilon^{10}) \left( \sum_{J \in \mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \left| \log \left( \frac{\omega(IJ)}{\omega(I)} \right) \right| \right. \\
& \quad \left. + \sum_{J \in \mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} |\log(1 - \varepsilon^{10})| \right) \\
& \leq \frac{1}{p} \sum_{J \in \mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \left| \log \left( \frac{\omega(IJ)}{\omega(I)} \right) \right| + \frac{1}{p} \varepsilon^{10} \sum_{J \in \mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \left| \log \left( \frac{\omega(IJ)}{\omega(I)} \right) \right| \\
& \quad + \frac{1}{p} (1 + \varepsilon^{10}) |\log(1 - \varepsilon^{10})| \\
& \leq \frac{1}{p} \sum_{J \in \mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \left| \log \left( \frac{\omega(IJ)}{\omega(I)} \right) \right| + \frac{\varepsilon^{10}}{p} |\log c_3| \\
& \quad + \frac{1}{p} (1 + \varepsilon^{10}) |\log(1 - \varepsilon^{10})|. \tag{29}
\end{aligned}$$

In the same way we obtain

$$\begin{aligned}
& \frac{1}{p} \sum_{J' \in \mathcal{F}'_p} \frac{\omega'(I'J')}{\omega'(I')} \left| \log \left( \frac{\omega'(I'J')}{\omega'(I')} \right) \right| \\
& \geq \frac{1}{p} \sum_{J \in \mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \left| \log \left( \frac{\omega(IJ)}{\omega(I)} \right) \right| - \frac{\varepsilon^{10}}{p} |\log c_3| \\
& \quad - \frac{1}{p} (1 + \varepsilon^{10}) |\log(1 - \varepsilon^{10})|. \tag{30}
\end{aligned}$$

We combine eqs. (26), (29) and (30) to get

$$\begin{aligned}
& \left| \frac{1}{p} \sum_{J' \in \mathcal{F}'_p} \frac{\omega'(I'J')}{\omega'(I')} \left| \log \left( \frac{\omega'(I'J')}{\omega'(I')} \right) \right| - h(\mu) \right| \\
& < \varepsilon + \frac{\varepsilon^{10}}{p} |\log c_3| + \frac{(1 + \varepsilon^{10}) |\log(1 - \varepsilon^{10})|}{p} \tag{31}
\end{aligned}$$

and therefore

$$\left| \frac{1}{p} \sum_{J' \in \mathcal{F}'_p} \frac{\omega'(I'J')}{\omega'(I')} \left| \log \left( \frac{\omega'(I'J')}{\omega'(I')} \right) \right| - h(\mu) \right| < 3\varepsilon. \tag{32}$$

Let us now show that for  $\omega'$ -almost all  $x \in \mathbb{K}'$

$$\left| \liminf_{n \rightarrow \infty} \frac{\log \omega'(I'_n(x))}{n} + h(\mu) \right| < 3\varepsilon. \tag{33}$$

With this relation and with eq. (2) we will finish the proof.

Consider the sequence of random variables  $(X_n)_{n \in \mathbb{N}}$  defined on  $\mathbb{K}'$  in the following way:

For  $x \in \mathbb{K}'$  we put

$$X_n(x) = -\frac{1}{p} \log \left( \frac{\omega'(I'_{np}(x))}{\omega'(I'_{(n-1)p}(x))} \right). \tag{34}$$

We will make use of the following known version of the theorem of large numbers (see for instance [9]).

LEMMA 3.1. – *Let  $X_n$  be a sequence of uniformly bounded real random variables on a probability space  $(\mathbb{X}, \mathcal{B}, P)$  and let  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  be an increasing sequence of  $\sigma$ -subalgebra of  $\mathbb{B}$  such that  $X_n$  is measurable with respect to  $\mathcal{F}_n$  for all  $n \in \mathbb{N}$ . Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (X_k - \mathbb{E}(X_k | \mathcal{F}_{k-1})) = 0 \quad P\text{-almost surely.} \tag{35}$$

Consider the sequence of  $\sigma$ -algebras  $(\mathcal{R}'_n)_{n \in \mathbb{N}}$  where  $\mathcal{R}'_n$  is generated by  $\mathcal{F}'_{np}$ . The hypothesis of Lemma 3.1 can be easily verified to hold for the sequence of random variables  $(X_n)_{n \in \mathbb{N}}$  and the sequence of  $\sigma$ -algebras  $(\mathcal{R}'_n)_{n \in \mathbb{N}}$ .

We get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n [X_k - \mathbb{E}_{\omega'}(X_k | \mathcal{R}'_{k-1})] = 0 \quad \omega'\text{-almost everywhere.} \tag{36}$$

On the other hand, on  $I' \in \bigcup_{n \in \mathbb{N}} \mathcal{R}'_n$ ,

$$\mathbb{E}_{\omega'}(X_n | \mathcal{R}'_{n-1}) = \frac{1}{p} \sum_{J' \in \mathcal{F}'_p} \frac{\omega'(I'J')}{\omega'(I')} \left| \log \left( \frac{\omega'(I'J')}{\omega'(I')} \right) \right|.$$

By relation (32) we obtain

$$|\mathbb{E}_{\omega'}(X_n | \mathcal{R}'_{n-1}) - h(\mu)| < 3\varepsilon,$$

for all  $n \in \mathbb{N}$ . Hence,

$$h(\mu) - 3\varepsilon < \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n X_k < h(\mu) + 3\varepsilon \quad \omega'\text{-almost everywhere.}$$

One can easily verify that,

$$\frac{1}{n} \sum_{k=1}^n X_k = \frac{1}{np} \log \omega'(I'_{np}(x)). \tag{37}$$

Therefore,

$$\left| \liminf_{n \rightarrow \infty} \frac{1}{n} \log \omega'(I'_n(x)) + h(\mu) \right| < 3\varepsilon \quad \omega'\text{-almost everywhere.} \tag{38}$$

We use this relation together with relation (2) to achieve the proof. The measure  $\omega$  is monodimensional (as well as  $\omega'$ , see [3]). Remark that for  $n \in \mathbb{N}$ , an all  $I \in \mathcal{F}_n$  the sidelength of the square  $I$  est  $a^n$ . On the other hand, for each  $I' \in \mathcal{F}'_n$  the sidelength of the square  $I'$  is bounded by  $(a - \delta)^n$  and  $(a + \delta)^n$ .

By relation (2)

$$\dim \omega = \liminf_{n \rightarrow \infty} \frac{\log \omega(I_n(x))}{\log l(I_n(x))} \quad \omega\text{-almost everywhere on } \mathbb{K}.$$

By Remark 2.3 and the doubling property of the measure  $\omega'$

$$\dim \omega' = \supess_{\omega'} \liminf_{n \rightarrow \infty} \frac{\omega'(I'_n(x))}{\log l(I'_n(x))} = \supess_{\omega'} \liminf_{n \rightarrow \infty} \frac{\omega'(I'_{np}(x))}{\log l(I'_{np}(x))}.$$

Using the Shannon–McMilan’s theorem and the fact that  $l(I) = a^n$  for all  $I \in \mathcal{F}_n$ , we get

$$\dim \omega = \frac{h(\mu)}{|\log a|}.$$

On the other hand the relation (38) gives

$$\dim \omega' \in \left] \frac{h(\mu) - 3\varepsilon}{|\log(a - \delta)|}, \frac{h(\mu) + 3\varepsilon}{|\log(a + \delta)|} \right[.$$

It suffices to choose  $\delta$  even smaller to have

$$\dim \omega' \in ]\dim \omega - 5\varepsilon, \dim \omega + 5\varepsilon[$$

which completes the proof.  $\square$

*Remark 3.2.* – Let us point out that we did not need any regularity conditions on the harmonic measure  $\omega'$  during the proof of Theorem 1.1.

Theorem 1.2 is proved with the same arguments: we use the formula  $\dim \omega = h(\mu)/\chi(\mu)$ , valid for the harmonic measure of all self-similar Cantor sets. The entropy  $h(\mu)$  is controlled in the same way as in Section 3 and it is easy to verify that the Lyapounov exponent varies continuously.

#### 4. CONSEQUENCES AND REMARKS

A. Ancona showed that the dimension of the harmonic measure of a 4-corners Cantor set, assigned to a constant sequence, converges to 1 as the value of the sequence tends to  $\frac{1}{2}$  (see [3]). With this and the previously presented results one can easily prove the following.

**COROLLARY 4.1.** – *If  $\omega_\alpha$  is the harmonic measure of the Cantor set assigned to the constant sequence  $a_n = \alpha$ , then the function  $f(\alpha) = \dim \omega_\alpha$  is continuous and takes all values between 0 and 1 as  $\alpha$  varies in  $]0, \frac{1}{2}[$ .*

In [3] we show that for every possible value of the dimension of harmonic measure of a 4-corners Cantor set, there is a Cantor-type set  $\mathbb{K}$  (without symmetry properties) such that the dimension of the harmonic measure of the complementary of  $\mathbb{K}$  equals  $\dim \mathbb{K}$ . We deduce

**COROLLARY 4.2.** – *For all  $0 < \alpha < 1$  there is a Cantor-type set  $\mathbb{K}$  such that the dimension of the harmonic measure of  $\mathbb{R}^2 \setminus \mathbb{K}$  equals  $\dim \mathbb{K} = \alpha$ .*

We should point out that the proof strongly depends on the invariance properties of harmonic measures of self-similar Cantor sets. Therefore, we have not been able to prove the continuity in the neighborhoods of Cantor sets associated to non-constant sequences  $(a_n)_{n \in \mathbb{N}}$ .

It is natural to ask whether the relation (4) suffices to conclude that the dimensions of two measures  $\omega$  and  $\omega'$  (not necessarily harmonic) are close. This is not the case. There are counterexamples (see [4]) even when the measures are doubling on  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  and monodimensional.

The equality between the Hausdorff dimension and the entropy of the harmonic measure of self-similar Cantor sets plays a crucial role in the proof of Theorems 1.1 and 1.2. In a more general case, for a monodimensional probability measure  $\mu$  supported by a Cantor set  $\mathbb{K}$ , let us define its lower and higher information dimensions (we are using the notation introduced previously):

$$h_*(\mu) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{I \in \mathcal{F}_n} |\log \mu(I)| \mu(I),$$

$$h^*(\mu) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{I \in \mathcal{F}_n} |\log \mu(I)| \mu(I).$$

In order to simplify, suppose that  $\mathbb{K}$  is associated to the constant sequence  $a_n = e^{-1}$  and therefore  $\chi(\mu) = 1$ .

We can introduce the random variables  $X_n$ ,  $n \in \mathbb{N}$ , as in Eq. (34). By Fatou's lemma and relation (2) we get

$$\dim \mu = \mathbb{E}_\mu \left\{ \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n X_k \right\} \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_\mu \sum_{k=1}^n X_k,$$

which gives the known formula (see [10])  $\dim \mu \leq h_*(\mu)$ . Some necessary and sufficient conditions in order to have equality are given in [5]. However, we have not been able to prove this equality for the harmonic measure of the Cantor sets of our context.

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