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MARIO V. WÜTHRICH

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Geodesics and crossing Brownian motion in a soft Poissonian potential

by

Mario V. WÜTHRICH

ETH Zürich, Department of Mathematics, ETH Zentrum, CH-8092 Zürich, Switzerland

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ABSTRACT. – We compare the model of crossing Brownian motion in a soft Poissonian potential with the model of continuum first-passage percolation among soft Poissonian obstacles. In both models we construct (via a shape theorem) a deterministic norm on $\mathbb{R}^d$, called the Lyapounov coefficient in the former and time-constant in the latter. The main theorem of this article claims that the properly rescaled Lyapounov coefficient converges to the time-constant as the strength of the potential tends to infinity. © Elsevier, Paris

RÉSUMÉ. – Nous comparons le modèle d'un mouvement Brownien traversant un potentiel Poissonnien, au modèle continu de percolation de premier passage associé à des obstacles Poissonniens. Dans ces deux modèles, la distance naturelle d'un point à l'origine se comporte asymptotiquement comme une norme déterministe sur $\mathbb{R}^d$, appelée dans le premier modèle coefficient de Liapounov, et dans le second constante-temps. Le théorème principal de cet article montre que le coefficient de Liapounov convenablement normalisé converge vers la constante-temps lorsque la force du potentiel tend vers l'infini. © Elsevier, Paris

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0. INTRODUCTION AND RESULTS

In this article we consider crossing Brownian motion evolving in a soft Poissonian potential. Crossing Brownian motion describes the evolution of Brownian motion in a Poissonian potential conditioned to reach a remote location. Various properties of this model have been studied in the literature (see, e.g., Sznitman [10,11]). The exponential decay of the normalizing constant of crossing Brownian motion, as its goal is tending to infinity, is described by a deterministic norm $\alpha_{\lambda,\beta}(\cdot)$, called the Lyapounov coefficient or norm. If we increase the strength of the Poissonian potential by a factor β, we observe that "non-optimal" paths lose weight in probability. This strengthening of the potential leads to the idea of comparing the Lyapounov coefficient (after rescaling with $\beta^{-1/2}$) with the random Riemannian distance associated with the Poissonian potential, where only optimal paths (geodesics) survive. Considering this second model, we observe (as in first-passage percolation on $\mathbb{Z}^d$) that the random distance $\varrho_\lambda(0,x) \sim \mu_\lambda(x)$, as $|x| \to \infty$, where μ_λ is a deterministic norm on $\mathbb{R}^d$. Our main theorem (see Theorem 0.3 below) states that $\beta^{-1/2}\alpha_{\lambda,\beta} \to \mu_\lambda$, as $\beta \to \infty$.

Let us precisely describe the setting. For $x \in \mathbb{R}^d$ ($d \geqslant 1$), we denote by P_x the Wiener measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ starting at site x, Z denotes the canonical process on $C(\mathbb{R}_+, \mathbb{R}^d)$. We denote by $\mathbb{P}$ the Poissonian law with fixed intensity $\nu > 0$ on the space Ω of locally finite, simple, pure point measures on $\mathbb{R}^d$. For a cloud configuration $\omega = \sum_i \delta_{x_i} \in \Omega$, $\lambda > 0$ and $x \in \mathbb{R}^d$ the soft Poissonian potential is defined as

$$q(x,\omega) = \lambda + V(x,\omega) = \lambda + \sum_i W(x - x_i) = \lambda + \int_{\mathbb{R}^d} W(x-y)\omega(dy), \tag{0.1}$$

where the shape function $W(\cdot) \geqslant 0$ is bounded, continuous, compactly supported and not a.e. equal to zero. For $\lambda, \beta > 0$, $\omega \in \Omega$, $x, y \in \mathbb{R}^d$ our main object of interest will be the normalizing constant for crossing Brownian motion in the soft Poissonian potential:

$$e_{\lambda,\beta}(x,y,\omega) = E_x\left[\exp\left\{-\int_0^{H(y)} \beta q(Z_s,\omega)\,ds\right\}, H(y) < \infty\right], \tag{0.2}$$

where $H(y)$ denotes the entrance time of Z into the closed ball $\overline{B}(y,1)$. β will be the parameter measuring the strength of the soft potential, which we let tend to infinity; λ will be kept fixed.

The function $u(x)=e_{\lambda,\beta}(x,0,\omega)$ appears as the $\beta q(\cdot,\omega)$-equilibrium potential of the set $\overline{B}(0,1)$, which satisfies in a weak sense the following second order equation (see Proposition 2.3.8 of [11] or Proposition 2.3 below):

$$\begin{cases} -\frac{1}{2}\Delta u+\beta q u=0 & \text{in } \overline{B}(0,1)^c,\\ u=1 & \text{on } \partial B(0,1),\\ u=0 & \text{at infinity } (\lambda>0). \end{cases} \tag{0.3}$$

Furthermore, $-\log e_{\lambda,\beta}(x,y,\omega)$ has the nice property that it measures the distance between x and y with respect to the potential $\beta q(\cdot,\omega)=\beta(\lambda+V(\cdot,\omega))$ for our crossing Brownian motion: $-\log e_{\lambda,\beta}(\cdot,\cdot,\omega)$ is up to a small correction term a distance function on $\mathbb{R}^d$ which increases if we add additional points to the Poissonian cloud ω (for more details see Sznitman [11], formula (5.2.3), and Proposition 5.2.2).

From Sznitman [10] we have a shape theorem, describing for typical cloud configurations ω the principal behavior of $-\log e_{\lambda,\beta}(x,0,\omega)$ as x tends to infinity:

THEOREM 0.1. – *For $\lambda\geqslant 0$ and $\beta>0$, there exists a deterministic norm $\alpha_{\lambda,\beta}(\cdot)$ on $\mathbb{R}^d$ such that, on a set of full $\mathbb{P}$-measure, we have*

$$\lim_{x\to\infty}\frac{1}{|x|}\left|-\log e_{\lambda,\beta}(x,0,\omega)-\alpha_{\lambda,\beta}(x)\right|=0. \tag{0.4}$$

The convergence takes place in $L^1(\mathbb{P})$ as well.

We call $\alpha_{\lambda,\beta}(\cdot)$ the Lyapounov coefficients. We now consider the following continuum first-passage percolation model on $\mathbb{R}^d$, obtained by considering the Riemannian distance between x and y with respect to the metrics $ds^2=2q(\cdot,\omega)dx^2$:

$$\varrho_\lambda(x,y,\omega)=\inf_{\gamma\in\mathcal{P}(x,y,1)}\left\{\int_0^1\sqrt{2q(\gamma_s,\omega)}\,|\dot{\gamma}_s|\,ds\right\}, \tag{0.5}$$

where $\mathcal{P}(x,y,1)$ is the set of Lipschitz paths γ leading in time 1 from x to y, i.e., $\gamma_0=x$ and $\gamma_1=y$. Of course, ϱ_λ also fulfills a shape theorem:

THEOREM 0.2. – *For $\lambda > 0$, there exists a deterministic norm $\mu_\lambda(\cdot)$ on $\mathbb{R}^d$ such that, on a set of full $\mathbb{P}$-measure, we have*

$$\lim_{x\to\infty} \frac{1}{|x|}\left|\varrho_\lambda(x,0,\omega) - \mu_\lambda(x)\right| = 0. \tag{0.6}$$

The convergence also holds in $L^1(\mathbb{P})$.

Following the terminology of the first-passage percolation model on the lattice $\mathbb{Z}^d$ (see Hammersley–Welsh [5], Kesten [6]) we call $\mu_\lambda(\cdot)$ the time-constant. The connection between the two models comes in the following theorem, which is our main result:

THEOREM 0.3. – *For $\lambda > 0$ fixed,*

$$\lim_{\beta\to\infty} \frac{1}{\sqrt{\beta}}\alpha_{\lambda,\beta}(e) = \mu_\lambda(e), \quad e \in \mathbb{R}^d, \tag{0.7}$$

where the above convergence is uniform on the unit-sphere and hence uniform on every compact subset of $\mathbb{R}^d$.

Let us point out that we obtain a much sharper lower bound than upper bound on the difference between $-\beta^{-1/2}\log e_{\lambda,\beta}(x,0,\omega)$ and $\varrho_\lambda(x,0)$ (compare Theorem 1.1 to Corollary 2.7).

Zerner [12] has studied a discrete related model (see Proposition 9 of [12]). He considers a random walk on $\mathbb{Z}^d$ with random potentials $\omega(x)$ at the sites $x \in \mathbb{Z}^d$. He obtains a similar result to Theorem 0.3, for increasing strength of the potentials, however, the normalizing factor turns out to be β instead of $\beta^{1/2}$ (in the continuous model Brownian motion can choose the velocity at which it passes the obstacles). The proof in the discrete model is much simpler, the quantity replacing the left member of (0.7) turns out to decrease to its limit as β tends to infinity. We do not know whether this is the case in our model.

This article is organised as follows: In Section 1 we prove the upper bound of Theorem 0.3 (see Corollary 1.3 below). The main idea is to consider "nearly optimal tubes", along which the crossing paths should move. For Brownian motion restricted to these tubes, we use the Cameron–Martin–Girsanov transformation (see Freidlin–Wentzell [3] and Carmona–Simon [2]). This classical construction gives a bound, which is not sharp, but which is sufficient for our purpose.

In Section 2 we prove the lower bound of Theorem 0.3 (see Corollary 2.7 below). The main control comes from the use of estimates of Agmon [1] for $u(\cdot) = e_{\lambda,\beta}(\cdot,0,\omega)$, a non-negative, bounded weak solution of the second order elliptic equation (0.3). It provides for all $\varepsilon \in (0,1)$ an L^2-bound on the product $f = ue^{(1-\varepsilon)\sqrt{\beta}\varrho_\lambda(\cdot,0)}$. Having this L^2-bound we use Harnack type inequalities to get pointwise upper bounds on $u(\cdot) = e_{\lambda,\beta}(\cdot,0,\omega)$ in terms of $\sqrt{\beta}\varrho_\lambda(\cdot,0)$. From this we will easily deduce the proof of Corollary 2.7.

Finally in Section 3 we give a short proof of the shape theorem in continuum first-passage percolation (see Theorem 0.2). In Appendix A we provide the proof of the estimates of Agmon (see Lemma 2.2) for the reader's covenience.

1. THE UPPER BOUND ON $\alpha_{\lambda,\beta}$

From Sznitman [11], Lemma 4.5.2, we know that there exists Ω_0, a subset of Ω with full $\mathbb{P}$-measure, such that for all $\omega \in \Omega_0$

$$\sup_{x\in[-l,l]^d} V(x,\omega) = \mathrm{o}(\log l), \quad \text{as } l \to \infty. \tag{1.1}$$

For $\delta > 0$, $\omega \in \Omega$ and $x \in \mathbb{R}^d$, we define

$$V_\delta(x,\omega) = \sup_{y\in B(x,\delta)} V(y,\omega). \tag{1.2}$$

The geodesic distance $\varrho_{\lambda,\delta}$ is then defined, for $\lambda > 0$, $\omega \in \Omega$ and $x, z \in \mathbb{R}^d$, as

$$\varrho_{\lambda,\delta}(x,z,\omega) = \inf_{\gamma\in\mathcal{P}(x,z,1)} \left\{ \int_0^1 \sqrt{2(\lambda + V_\delta(\gamma_s,\omega))}\,|\dot{\gamma}_s|\,ds \right\}. \tag{1.3}$$

The distance function $\varrho_{\lambda,\delta}(\cdot,\cdot)$ will play an important role, because if we consider Brownian motion moving in a tube around the geodesic path γ from x to z (with respect to $\varrho_\lambda(\cdot,\cdot)$), this motion will typically experience the potential $q(\cdot,\omega) = \lambda + V(\cdot,\omega)$ in a neighborhood of γ.

THEOREM 1.1. – *There exists $c_1 = c_1(d) \in (0,\infty)$ such that for all $\delta \in (0,1)$, $\beta \geqslant 1$, $x \in \mathbb{R}^d$ and $\omega \in \Omega_0$,*

$$e_{\lambda,\beta}(x,0,\omega) \geqslant c_1 \exp\left\{ -\sqrt{\beta}\varrho_{\lambda,\delta}(x,0,\omega)\left(1 + \frac{\lambda_D}{2\lambda\beta\delta^2}\right)\right\}, \tag{1.4}$$

where λ_D is the first Dirichlet eigenvalue of $-\frac{1}{2}\Delta$ in the ball $B(0,1)$.

Proof. – Choose $\delta \in (0,1)$, $\beta \geqslant 1$ and $x \in \mathbb{R}^d$ fixed. Then pick $t > 0$ and $\phi \in \mathcal{P}(x,0,t)$. We define the following tube around ϕ with radius δ up to time t:

$$T(\phi,\delta,t) = \{w \in C(\mathbb{R}_+,\mathbb{R}^d);\ |w(s) - \phi_s| < \delta \text{ for all } s \in [0,t]\}. \quad (1.5)$$

Of course, having chosen $\delta < 1$, if $Z_. \in T(\phi,\delta,t)$ then $Z_t \in B(0,1)$, hence we see that on the event $T(\phi,\delta,t)$, $H(0) \leqslant t$. Therefore we have for $\omega \in \Omega_0$,

$$\begin{aligned} e_{\lambda,\beta}(x,0,\omega) &= E_x\left[\exp\left\{-\int_0^{H(0)} \beta q(Z_s,\omega)\,ds\right\},\ H(0) < \infty\right] \\ &\geqslant E_x\left[\exp\left\{-\int_0^t \beta(\lambda + V)(Z_s,\omega)\,ds\right\},\ T(\phi,\delta,t)\right] \\ &\geqslant \exp\left\{-\int_0^t \beta(\lambda + V_\delta)(\phi_s,\omega)\,ds\right\} \times P_x\left[T(\phi,\delta,t)\right]. \quad (1.6) \end{aligned}$$

Consider the last term of the above inequality. For Brownian motion remaining in the tube $T(\phi,\delta,t)$ we use Cameron–Martin–Girsanov's formula to describe the density of the law of $Z_. - \phi$ with respect to P_0 (see, e.g., [3]). We obtain the following lower bound (with the obvious notation that $T(0,\delta,t) = \{w \in C(\mathbb{R}_+,\mathbb{R}^d);\ Z_s \in B(0,\delta)$ for all $s \in [0,t]\}$),

$$\begin{aligned} P_x\left[T(\phi,\delta,t)\right] &= E_0\left[T(0,\delta,t),\ \exp\left\{-\int_0^t 1/2|\dot\phi_s|^2\,ds - \int_0^t \dot\phi_s\,dZ_s\right\}\right] \\ &\geqslant \exp\left\{-\int_0^t 1/2|\dot\phi_s|^2\,ds\right\} \\ &\qquad \times P_0\left[T(0,\delta,t)\right]\exp\left\{-E_0\left[\int_0^t \dot\phi_s\,dZ_s \,\middle|\, T(0,\delta,t)\right]\right\} \\ &= \exp\left\{-\int_0^t 1/2|\dot\phi_s|^2\,ds\right\} P_0\left[T(0,\delta,t)\right] \\ &\geqslant c_1 \exp\left\{-\int_0^t 1/2|\dot\phi_s|^2\,ds\right\} \times \exp\{-t\lambda_D/\delta^2\}, \quad (1.7) \end{aligned}$$

where we have used Jensen's inequality in the second step and Brownian symmetry in the third step. Hence, we have for all $t > 0$ and $\phi \in \mathcal{P}(x,0,t)$:

$$e_{\lambda,\beta}(x,0,\omega) \geqslant c_1 \exp\left\{ -\int_0^t \beta(\lambda + V_\delta)(\phi_s,\omega)\,ds - \int_0^t 1/2|\dot{\phi}_s|^2\,ds - \frac{t\lambda_D}{\delta^2}\right\}. \tag{1.8}$$

Next we consider the two functionals

$$L_2(\phi) = \int_0^t \beta(\lambda + V_\delta)(\phi_s,\omega)\,ds + \int_0^t 1/2|\dot{\phi}_s|^2\,ds, \tag{1.9}$$

$$L_1(\phi) = \int_0^t \sqrt{2\beta(\lambda + V_\delta)(\phi_s,\omega)}\,|\dot{\phi}_s|\,ds. \tag{1.10}$$

LEMMA 1.2. – *For $\beta > 0$, $\lambda > 0$, $\delta \geqslant 0$, $x, y \in \mathbb{R}^d$ and $\omega \in \Omega_0$ we have: There exists at least one minimizing element $\psi \in \mathcal{P}(x,y,1)$ such that*

$$\sqrt{\beta}\varrho_{\lambda,\delta}(x,y) = \inf_{\phi\in\mathcal{P}(x,y,1)} L_1(\phi) = L_1(\psi). \tag{1.11}$$

Furthermore, we have

$$\inf_{t>0}\inf_{\phi\in\mathcal{P}(x,y,t)} L_2(\phi) = L_1(\psi). \tag{1.12}$$

Proof of Lemma 1.2. – Choose $t > 0$, $\beta > 0$, $\lambda > 0$, $\delta \geq 0$, $x, y \in \mathbb{R}^d$ and $\omega \in \Omega_0$. Using the inequality $2ab \leqslant a^2 + b^2$, we see that for all Lipschitz paths ϕ, with $\phi_0 = x$ and $\phi = 0$,

$$L_2(\phi) \geqslant L_1(\phi). \tag{1.13}$$

Using Theorem 1, p. 261 of [4], Vol. II, we know that there exists at least one minimizing element $\psi \in \mathcal{P}(x,0,t)$ such that

$$\inf_{\phi\in\mathcal{P}(x,0,t)} L_1(\phi) = L_1(\psi), \quad \text{and} \tag{1.14}$$

$$\sqrt{2\beta(\lambda + V_\delta)(\psi_s,\omega)}\,|\dot{\psi}_s| = L_1(\psi)/t = \text{const.} \quad \text{for a.e. } s \in [0,t]. \tag{1.15}$$

We can now find a continuous, strictly increasing Lipschitz reparametrization $s = \sigma(u)$, such that for $\widetilde{\psi}(u) = \psi(\sigma(u)) \in \mathcal{P}(x, 0, \sigma^{-1}(t))$:

$$|\dot{\widetilde{\psi}}_u| = \sqrt{2\beta(\lambda + V_\delta)(\widetilde{\psi}_u, \omega)} \quad \text{for a.e. } u \in [0, \sigma^{-1}(t)]. \tag{1.16}$$

In view of (1.13), and because L_1 is invariant under reparametrization:

$$\inf_{t>0} \inf_{\mathcal{P}(x,0,t)} L_2(\phi) \geqslant \inf_{t>0} \inf_{\mathcal{P}(x,0,t)} L_1(\phi) = L_1(\psi) = L_1(\widetilde{\psi}) = L_2(\widetilde{\psi}), \tag{1.17}$$

where in the last step we have used that $2ab = a^2 + b^2$ for $a = b$. This finishes the proof of Lemma 1.2. □

Let us now see, how the claim of the theorem follows. We choose a minimizing element $\widetilde{\psi}$ of L_1 satisfying (1.16), where $\tilde{t}$ denotes the first time that $\widetilde{\psi}$ reaches 0. We have

$$\sqrt{\beta}\varrho_{\lambda,\delta}(x, 0) = L_1(\widetilde{\psi}) = \int_0^{\tilde{t}} 2\beta(\lambda + V_\delta)(\widetilde{\psi}_s)\, ds \geqslant 2\beta\lambda\tilde{t}. \tag{1.18}$$

Therefore, coming back to (1.8) with $\widetilde{\psi}$ in place of ϕ,

$$\begin{aligned} e_{\lambda,\beta}(x, 0, \omega) &\geqslant c_1 \exp\left\{-\sqrt{\beta}\varrho_{\lambda,\delta}(x, 0, \omega) - \frac{\tilde{t}\lambda_D}{\delta^2}\right\} \\ &\geqslant c_1 \exp\left\{-\sqrt{\beta}\varrho_{\lambda,\delta}(x, 0, \omega)\left(1 + \frac{\lambda_D}{2\lambda\beta\delta^2}\right)\right\}. \end{aligned} \tag{1.19}$$

This finishes the proof of Theorem 1.1. □

COROLLARY 1.3. – *For* $\lambda > 0$,

$$\limsup_{\beta\to\infty} \sup_{e\in\partial B(0,1)} \frac{1}{\beta^{1/2}} \alpha_{\lambda,\beta}(e) - \mu_\lambda(e) \leqslant 0. \tag{1.20}$$

Proof. – Choose a unit vector $e \in \mathbb{R}^d$, $n \in \mathbb{N}$ and $\delta > 0$. Using Theorem 1.1, we see that for all $\omega \in \Omega_0$,

$$\begin{aligned} &-\frac{1}{\beta^{1/2} n} \log e_{\lambda,\beta}(ne, 0, \omega) \\ &\quad\leqslant \frac{1}{n}\varrho_{\lambda,\delta}(ne, 0, \omega)\left(1 + \frac{\lambda_D}{2\lambda\beta\delta^2}\right) - \frac{\log c_1}{\beta^{1/2} n}. \end{aligned} \tag{1.21}$$

Using the shape theorems (see Theorems 0.1 and 0.2) on both sides of the above inequality, we see that for $n \to \infty$

$$\frac{1}{\beta^{1/2}}\alpha_{\lambda,\beta}(e) \leqslant \mu_{\lambda,\delta}(e)\left(1+\frac{\lambda_D}{2\lambda\beta\delta^2}\right), \tag{1.22}$$

where $\mu_{\lambda,\delta}(\cdot)$ denotes the time-constant for the continuum first-passage percolation model with respect to the potential V_δ. On the right-hand side of (1.22) only the second factor depends on β, and the left-hand side is independent of δ, passing to the limit we conclude

$$\limsup_{\beta\to\infty}\frac{1}{\beta^{1/2}}\alpha_{\lambda,\beta}(e) \leqslant \inf_{\delta>0}\mu_{\lambda,\delta}(e) = \lim_{\delta\to 0}\mu_{\lambda,\delta}(e). \tag{1.23}$$

For $m \in \mathbb{N}$, choose $\delta = 1/m$. Of course, $\mu_{\lambda,1/m}(e)$ is non-increasing in m, hence the claim of our corollary follows by a Dini type argument, if we manage to prove that

$$\lim_{m\to\infty}\mu_{\lambda,1/m}(e) = \mu_\lambda(e). \tag{1.24}$$

From Kingman's subadditive ergodic theorem (see, for instance, Liggett [7, p. 277]), we know that

$$\mu_{\lambda,1/m}(e) = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}\big[\varrho_{\lambda,1/m}(ne,0)\big] = \inf_{n\geqslant 1}\frac{1}{n}\mathbb{E}\big[\varrho_{\lambda,1/m}(ne,0)\big]. \tag{1.25}$$

Therefore, we can exchange the following infima

$$\begin{aligned}\lim_{m\to\infty}\mu_{\lambda,1/m}(e) &= \inf_{m\geqslant 1}\mu_{\lambda,1/m}(e)\\ &= \inf_{m\geqslant 1}\inf_{n\geqslant 1}\frac{1}{n}\mathbb{E}\big[\varrho_{\lambda,1/m}(ne,0)\big]\\ &= \inf_{n\geqslant 1}\frac{1}{n}\inf_{m\geq 1}\mathbb{E}\big[\varrho_{\lambda,1/m}(ne,0)\big]\\ &= \inf_{n\geqslant 1}\frac{1}{n}\mathbb{E}\big[\varrho_\lambda(ne,0)\big] = \mu_\lambda(e),\end{aligned} \tag{1.26}$$

to apply Lebesgue's dominated convergence theorem in the last step of (1.26), we have used that $\varrho_{\lambda,1/m}(ne,0) \leqslant \varrho_{\lambda,1}(ne,0) \in L^1(\mathbb{P})$ and that $\varrho_{\lambda,1/m}(ne,0) \to \varrho_\lambda(ne,0)$ $\mathbb{P}$-a.s., as $m \to \infty$ (W has been chosen to be continuous). This finishes the proof of Corollary 1.3. □

2. THE LOWER BOUND ON $\alpha_{\lambda,\beta}$

In this section we always work with typical cloud configurations $\omega \in \Omega_0$ (see (1.1)). As a result of our assumptions on W, $V(\cdot,\omega)$ is continuous for all $\omega \in \Omega_0$. We pick a fixed $\omega \in \Omega_0$, $\lambda > 0$, then $q(x,\omega)$ is a strictly positive, continuous potential on $\mathbb{R}^d$. Therefore, using Lemma 1.3 and Theorem 1.4 of Agmon [1], we see that

$$\varrho_\lambda(\cdot,0) \text{ is locally Lipschitz,} \tag{2.1}$$

$$\frac{1}{2}\left|\nabla\varrho_\lambda(x,0)\right|^2 \leqslant q(x), \quad \text{a.e.} \tag{2.2}$$

For $D = \overline{B}(0,1)^c$ an open subset of $\mathbb{R}^d$, $H^1(D)$ denotes the Sobolev space of functions $f \in L^2(D)$ such that the distributional derivatives of f are in $L^2(D)$. The next theorem gives us an L^2-bound for a non-negative, bounded weak solution of the second order equation (2.3) with q as above. The key estimate to prove the theorem is a simplified version of Theorem 1.5 given in Agmon [1] (see Lemma 2.2 below).

THEOREM 2.1. – *Take $\beta \geqslant 1$ and suppose that $u \in H^1_{loc}(D)$ is a weak solution of*

$$-\frac{1}{2}\Delta u + \beta q u = 0 \quad \text{in } D = \overline{B}(0,1)^c, \tag{2.3}$$

in the sense that $qu \in L^1_{loc}(D)$ and

$$\int_D \frac{1}{2}\nabla u \nabla\phi + \beta q u \phi \, dx = 0 \tag{2.4}$$

for every $\phi \in C_c^\infty(D)$. Furthermore, suppose that there exists $b < \infty$ such that

$$0 \leqslant u(x) \leqslant b \quad \text{for all } x \in D. \tag{2.5}$$

Then there exists a constant $c(q,\lambda) > 0$ such that for all $\varepsilon \in (0,1)$, $\delta \in (0,1)$,

$$\int_D |u(x)|^2 e^{2(1-\varepsilon)\sqrt{\beta}\varrho_\lambda(x,0)}\,dx \leqslant \frac{b^2 c(q,\lambda)}{2\varepsilon-\varepsilon^2}\left(\frac{1+\delta}{\delta}\right)^2 \sup_{B(0,1+\frac{1}{\sqrt{2\lambda}})} e^{2\sqrt{\beta}\varrho_\lambda(\cdot,0)}. \tag{2.6}$$

The above theorem is a consequence of Agmon's Theorem 1.5 in [1]. We state here a simplified version, which is sufficient in our case of strictly positive potential q and positive and bounded weak solutions u. As we shall see, Theorem 2.1 follows from

LEMMA 2.2. – *Let β, D and q be as above. For $\delta > 0$ we define*

$$D_\delta = \{x \in D;\ \varrho_\lambda(x, \partial D) > \delta\}. \tag{2.7}$$

Choose $\varepsilon \in (0,1)$ and define for $x \in D$:

$$h(x) = (1-\varepsilon)\sqrt{\beta}\varrho_\lambda(x, 0). \tag{2.8}$$

For $u \in H^1_{loc}(D)$, a weak solution of (2.3) *satisfying* (2.5), *we have*

$$\int_{D_{\delta/\sqrt{\beta}}} |u|^2\left(\beta q - \frac{1}{2}|\nabla h|^2\right)e^{2h}\,dx \leqslant \frac{1+2\delta}{\delta^2}\int_{D\setminus D_{\delta/\sqrt{\beta}}} |u|^2 \beta q e^{2h}\,dx. \tag{2.9}$$

We provide the proof of Lemma 2.2 in Appendix A.

Proof of Theorem 2.1. – In view of Lemma 2.2, we consider $h(x) = (1-\varepsilon)\sqrt{\beta}\varrho_\lambda(x,0)$. Of course $h(\cdot)$ is locally Lipschitz (see (2.1) and (2.2)) with

$$\frac{1}{2}|\nabla h(x)|^2 \leqslant (1-\varepsilon)^2 \beta q(x), \quad \text{a.e.} \tag{2.10}$$

Therefore, we see that

$$\beta q(x) - \frac{1}{2}|\nabla h(x)|^2 \geqslant (2\varepsilon - \varepsilon^2)\beta q(x) \geqslant (2\varepsilon - \varepsilon^2)\beta\lambda, \quad \text{a.e.} \tag{2.11}$$

Using Lemma 2.2, we conclude that

$$\int_{D_{\delta/\sqrt{\beta}}} |u|^2 e^{2h}\,dx \leqslant \frac{1}{(2\varepsilon-\varepsilon^2)\beta\lambda}\,\frac{1+2\delta}{\delta^2}\int_{D\setminus D_{\delta/\sqrt{\beta}}} |u|^2\beta q e^{2h}\,dx. \tag{2.12}$$

On the other hand, because $2\varepsilon - \varepsilon^2 < 1$ for $\varepsilon \in (0,1)$, and because $q \geqslant \lambda$, we have that

$$\int_{D\setminus D_{\delta/\sqrt{\beta}}} |u|^2 e^{2h}\,dx \leqslant \frac{1}{(2\varepsilon-\varepsilon^2)\beta\lambda}\int_{D\setminus D_{\delta/\sqrt{\beta}}} |u|^2\beta q e^{2h}\,dx. \tag{2.13}$$

In view of (2.12) and (2.13) we obtain:

$$\int_D |u|^2 e^{2h}\,dx \leqslant \left(\frac{1+\delta}{\delta}\right)^2 \frac{1}{(2\varepsilon-\varepsilon^2)\lambda} \int_{D\setminus D_{\delta/\sqrt{\beta}}} |u|^2 q e^{2h}\,dx$$

$$\leqslant \left(\frac{1+\delta}{\delta}\right)^2 \frac{b^2}{(2\varepsilon-\varepsilon^2)\lambda}$$

$$\times \int_{B(0,1+\frac{1}{\sqrt{2\lambda}})} q(x)\,dx \sup_{z\in B(0,1+\frac{1}{\sqrt{2\lambda}})} e^{2\sqrt{\beta}\varrho_\lambda(z,0)}. \quad (2.14)$$

Now the claim follows choosing

$$c(q,\lambda) = \lambda^{-1} \int_{B(0,1+\frac{1}{\sqrt{2\lambda}})} q(x)\,dx. \qquad \square$$

Our next step is to prove that for $x \in D = \overline{B}(0,1)^c$, $u(x) = e_{\lambda,\beta}(x, 0,\omega)$ is a non-negative, bounded weak solution of (2.3).

PROPOSITION 2.3. – *$u(\cdot) = e_{\lambda,\beta}(\cdot, 0, \omega)$ is a non-negative, bounded weak solution of* (2.3) *on $D = \overline{B}(0,1)^c$ with bound $b = 1$ (see* (2.5)*).*

Proof. – Set $K = \overline{B}(0,1)$. For $\lambda > 0$, $e_{\lambda,\beta}(\cdot, 0, \omega)$ appears as the $\beta(\lambda + V(\cdot,\omega)) = \beta q(\cdot)$-equilibrium potential of K (see (2.3.26) of [11]). Using Proposition 2.3.8 of [11], we see that $e_{\lambda,\beta}(\cdot, 0, \omega)$ is continuous on $\mathbb{R}^d$, equals 1 on K and is $\beta q(\cdot)$-harmonic on D. Using Proposition 2.5.1 and Theorem 1.4.9 of [11], we see that $e_{\lambda,\beta}(\cdot, 0, \omega) \in H^1(\mathbb{R}^d)$, which finishes the proof of Proposition 2.3. $\square$

Applying Theorem 2.1 to $e_{\lambda,\beta}(\cdot, 0, \omega)$, we conclude the following corollary:

COROLLARY 2.4 (L^2-bound). – *For all $\omega \in \Omega_0$, there exists $c(\omega, W, \lambda) < \infty$ such that for all $\beta \geqslant 1$, $\varepsilon \in (0,1)$ and $\delta \in (0,1)$ we have*

$$\int_D |e_{\lambda,\beta}(x,0,\omega)|^2 e^{2(1-\varepsilon)\sqrt{\beta}\varrho_\lambda(x,0,\omega)}\,dx$$

$$\leqslant \frac{c(\omega,W,\lambda)}{2\varepsilon-\varepsilon^2}\left(\frac{1+\delta}{\delta}\right)^2 \sup_{z\in B(0,1+\frac{1}{\sqrt{2\lambda}})} e^{2\sqrt{\beta}\varrho_\lambda(z,0,\omega)}. \quad (2.15)$$

To derive a pointwise upper bound on $e_{\lambda,\beta}(\cdot, 0, \omega)$ we use Harnack type inequalities:

LEMMA 2.5. – *There exist constants* $c_2, c_3 \in (0, \infty)$ (*depending only on the dimension* d) *such that for all* $\omega \in \Omega$ *and all* $z \in \mathbb{R}^d \setminus \overline{B}(0,3)$ *the following estimate holds*:

$$\frac{\sup_{z_1 \in B(z,1)} e_{\lambda,\beta}(z_1, 0, \omega)}{\inf_{z_2 \in B(z,1)} e_{\lambda,\beta}(z_2, 0, \omega)} \leqslant c_2 \exp\Big\{c_3 \beta\Big(\lambda + \sup_{x \in B(z,2)} V(x, \omega)\Big)\Big\}. \tag{2.16}$$

For the proof of Lemma 2.5 we refer the reader to Sznitman [11], formula (5.2.22). This Harnack type inequality leads directly to the desired pointwise upper bound:

THEOREM 2.6 (Pointwise bound). – *There exist constants* $c_2, c_3 \in (0, \infty)$ *such that for all* $\omega \in \Omega_0$ *we find a constant* $c(\omega, W, \lambda) < \infty$ *such that for all* $\beta \geqslant 1$, $\varepsilon \in (0,1)$, $\delta \in (0,1)$ *and* $x \in \overline{B}(0,3)^c$ *the following is true*

$$\begin{aligned} &-\log e_{\lambda,\beta}(x, 0, \omega) \\ &\quad \geqslant (1-\varepsilon)\sqrt{\beta}\varrho_\lambda(x, 0, \omega) - \log\left(\frac{a_1 c_2}{v_d^{1/2}}\right) - c_3\beta\Big(\lambda + \sup_{z \in B(x,2)} V(z, \omega)\Big) \\ &\quad\quad - \sup_{z \in B(0, 1+\frac{1}{\sqrt{2\lambda}})} \sqrt{\beta}\varrho_\lambda(z, 0, \omega) - \sup_{y,z \in B(x,1)} \sqrt{\beta}\varrho_\lambda(y, z, \omega), \end{aligned} \tag{2.17}$$

where

$$a_1^2 = a_1(\omega, \varepsilon, \delta)^2 = \frac{c(\omega, W, \lambda)}{2\varepsilon - \varepsilon^2}\left(\frac{1+\delta}{\delta}\right)^2,$$

and v_d *is the volume of the unit ball in* $\mathbb{R}^d$.

Proof. – Choose $x \in \overline{B}(0,3)^c$. Using the Cauchy–Schwarz inequality, we see that

$$\begin{aligned} &\inf_{z_1 \in B(x,1)} e_{\lambda,\beta}(z_1, 0, \omega) \\ &\quad \leqslant \frac{1}{v_d} \int_{B(x,1)} e_{\lambda,\beta}(z, 0, \omega)\, dz \leqslant \frac{1}{v_d^{1/2}} \left(\int_{B(x,1)} e_{\lambda,\beta}(z, 0, \omega)^2\, dz \right)^{1/2} \\ &\quad \leqslant \frac{a_1}{v_d^{1/2}} \sup_{z \in B(0, 1+\frac{1}{\sqrt{2\lambda}})} e^{\sqrt{\beta}\varrho_\lambda(z,0,\omega)} \sup_{y \in B(x,1)} e^{-(1-\varepsilon)\sqrt{\beta}\varrho_\lambda(y,0,\omega)}, \end{aligned} \tag{2.18}$$

in the last step we have used Corollary 2.4. We easily obtain (2.17) with the help of Lemma 2.5 on the left-hand side of (2.18) and the triangle inequality for ϱ_λ. □

COROLLARY 2.7. – *For* $\lambda > 0$ *and* $\beta \geqslant 1$ *we have*:

$\mathbb{P}$*-a.s. for* $|x|$ *large,*

$$-\log e_{\lambda,\beta}(x,0,\omega) \geqslant \sqrt{\beta}\varrho_\lambda(x,0,\omega) - \mathrm{o}(\log|x|), \tag{2.19}$$

and

$$\inf_{e\in\partial B(0,1)} \frac{1}{\beta^{1/2}}\alpha_{\lambda,\beta}(e) - \mu_\lambda(e) \geqslant 0. \tag{2.20}$$

Proof. – Choose $\delta = 1/2$ fixed, and $\varepsilon = |x|^{-1}$. For $\omega \in \Omega_0$ and $|x|$ large we have that (using Theorem 2.6)

$$-\log e_{\lambda,\beta}(x,0,\omega) \geqslant \sqrt{\beta}\varrho_\lambda(x,0,\omega) - \mathrm{o}\big(\log|x|\big), \tag{2.21}$$

here we have used the control on the growth of V (see (1.1)). Now we apply the shape theorems on both sides of (2.21) to conclude that our corollary holds true (the correction term $\mathrm{o}(\log|x|)$ in (2.21) does not depend on the direction of x). □

Remark. – In view of (2.21) we see that the lower bound on the distance $-\log e_{\lambda,\beta}(x,0,\omega)$ in terms of the geodesic distance $\varrho_\lambda(x,0,\omega)$ differs only by a logarithmic term in $|x|$. On the other hand the upper bound (1.4) is much less sharp. Hence, we have to improve (1.4) if we want to define "quasi geodesics" for the crossing Brownian motion model, where a "quasi geodesic" is a tube of small radius (compared with $|x|$), such that with sufficiently large probability crossing Brownian motion moves along that tube to its goal $B(0,1)$ (as $|x| \to \infty$).

3. SHAPE THEOREMS

In this section we prove Theorem 0.2. For the proof of Theorem 0.1 we refer the reader to Theorem 1.4 of Sznitman [10].

To prove Theorem 0.2 we use the technique given in Proposition 1.2 of [10] (see also Kesten [6, p. 158]). The first step is to see that $\varrho_\lambda(x,y,\omega)$ is a positive, subadditive, translation invariant, ergodic random variable which is in $L^1(\mathbb{P})$. Hence, we can apply Kingman's subadditive ergodic theorem. The second step is to patch up the limits for different directions.

Proof of Theorem 0.2. – Choose $\lambda > 0$ and $v \in \mathbb{R}^d \setminus \{0\}$ fixed, the doubly indexed sequence

$$X^\lambda_{m,n} = \varrho_\lambda(mv, nv, \omega), \quad 0 \leqslant m \leqslant n, \tag{3.1}$$

satisfies the triangle inequality $X^{\lambda}_{0,n} \leqslant X^{\lambda}_{0,m} + X^{\lambda}_{m,n}$. It is easy to verify that this family satisfies the hypothesis of Kingman's subadditive ergodic theorem (see Liggett [7, p. 277]) giving for $v \in \mathbb{R}^d$ (the case $v = 0$ is trivial):

$$\mu_\lambda(v) = \lim_{n\to\infty} \frac{X^{\lambda}_{0,n}}{n} = \lim_{n\to\infty} \frac{\mathbb{E}[X^{\lambda}_{0,n}]}{n} = \inf_{n\geqslant 1} \frac{\mathbb{E}[X^{\lambda}_{0,n}]}{n}$$

$$\text{exists } \mathbb{P}\text{-a.s. and in } L^1(\mathbb{P}). \tag{3.2}$$

For $x, y \in \mathbb{R}^d$, consider the straight line r: $x + s(y - x)$, $s \in [0, 1]$. Then

$$\begin{aligned}\varrho_\lambda(x, y) &\leqslant \int_0^1 \sqrt{2q(r_s)}\,|\dot{r}_s|\,ds \\ &\leqslant |y - x| \int_0^1 \sqrt{2(\lambda + \|W\|_\infty \omega(B(r_s, a)))}\,ds. \end{aligned}\tag{3.3}$$

Hence, from (3.2) and (3.3) we have for an appropriate choice of $k_1 = k_1(d, \nu, W, \lambda)$:

$$\sqrt{2\lambda}\,|v| \leqslant \mu_\lambda(v) \leqslant k_1 |v|. \tag{3.4}$$

Using the triangle inequality and translation invariance, it is easy to conclude that for $v, v' \in \mathbb{R}^d$, $p \geqslant 0$ and $q \geqslant 1$ integers (see Sznitman [11], (5.2.36)–(5.2.39)):

$$\mu_\lambda(v + v') \leqslant \mu_\lambda(v) + \mu_\lambda(v'), \tag{3.5}$$

$$\mu_\lambda(v) = \mu_\lambda(-v), \tag{3.6}$$

$$\frac{p}{q}\mu_\lambda(v) = \mu_\lambda\left(\frac{p}{q}v\right). \tag{3.7}$$

As a result we can extend $\mu_\lambda(\cdot)$ to a norm on $\mathbb{R}^d$:

$$\mu_\lambda(\gamma v) = |\gamma|\mu_\lambda(v), \quad \gamma \in \mathbb{R}, v \in \mathbb{R}^d, \tag{3.8}$$

here we use $\lambda > 0$ to guarantee that we obtain a norm (see (3.4)).

The next step is to patch up the limits. Therefore we need the following maximal lemma (see also [11], Lemma 5.2.6):

LEMMA 3.1. – *There exists a constant* $A(d, \nu, W, \lambda) > 0$, *such that for large* r *and for* $\eta > 0$,

$$\mathbb{P}\Big[\sup_{\|x\|<r} \varrho_\lambda(x, 0) > \eta\Big] \leqslant (4r + 1)^d \exp\{-\eta + Ar\}. \tag{3.9}$$

We will prove this lemma below. Now we choose $\eta = 2A\varepsilon|p|$, $r = \varepsilon|p|$, $p \in \mathbb{Z}^d$, $\varepsilon \in (0,1)$ fixed, so we have for large p:

$$\mathbb{P}\Big[\sup_{\|y-p\|<\varepsilon|p|} \varrho_\lambda(p,y) > 2A\varepsilon|p|\Big] \leqslant (4\varepsilon|p|+1)^d \exp\{-A\varepsilon|p|\}, \quad (3.10)$$

which is summable in p. It thus follows: on a set of full $\mathbb{P}$-measure, for any $\varepsilon \in \mathbb{Q} \cap (0,1)$, for large p,

$$\sup_{\|y-p\|<\varepsilon|p|} \varrho_\lambda(p,y,\omega) \leqslant 2A\varepsilon|p|. \quad (3.11)$$

Now we consider a fixed ω in a set of full $\mathbb{P}$-measure such that (3.11) and (3.2) for all $v \in \mathbb{Q}^d$ hold. It suffices to show that for any sequence $x_k \to \infty$, with

$$\frac{x_k}{|x_k|} \to e \in \partial B(0,1), \quad (3.12)$$

we have

$$\lim_{k\to\infty} \frac{1}{|x_k|}\big|\varrho_\lambda(x_k,0,\omega) - \mu_\lambda(x_k)\big| = 0. \quad (3.13)$$

Choose $\varepsilon \in \mathbb{Q} \cap (0,1)$, $v \in \mathbb{Q}^d$, and an integer M, with

$$|v-e| < \varepsilon/2 \quad \text{and} \quad Mv \in \mathbb{Z}^d. \quad (3.14)$$

Define

$$y_k \left\lfloor \frac{|x_k|}{M} \right\rfloor \cdot Mv \in \mathbb{Z}^d. \quad (3.15)$$

Then, as in [11], formulas (5.2.47)–(5.2.48), we have for large k

$$|y_k - x_k| \leqslant \varepsilon|x_k|/2 \quad \text{and} \quad |y_k - x_k| \leqslant \varepsilon|y_k|. \quad (3.16)$$

Hence, for large k (using the triangle inequality)

$$\begin{aligned}
&\left|\frac{1}{|x_k|}\varrho_\lambda(x_k,0,\omega) - \mu_\lambda\left(\frac{x_k}{|x_k|}\right)\right| \\
&\quad\leqslant \frac{1}{|x_k|}\varrho_\lambda(y_k,x_k,\omega) + \frac{1}{|y_k|}\varrho_\lambda(y_k,0,\omega)\left|\frac{|y_k|}{|x_k|} - 1\right| \\
&\qquad + \left|\frac{1}{|y_k|}\varrho_\lambda(y_k,0,\omega) - \mu_\lambda(v)\right| + \left|\mu_\lambda(v) - \mu_\lambda\left(\frac{x_k}{|x_k|}\right)\right|. \quad (3.17)
\end{aligned}$$

In view of (3.11), (3.16), (3.2) and (3.14), we obtain

$$\limsup_{k\to\infty}\left|\frac{1}{|x_k|}\varrho_\lambda(x_k,0,\omega)-\mu_\lambda\left(\frac{x_k}{|x_k|}\right)\right|$$
$$\leqslant 2A\varepsilon+\left(\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\right)\sup_{|e|=1}\mu_\lambda(e). \tag{3.18}$$

Letting ε tend to 0 finishes the proof of Theorem 0.2. □

Proof of Lemma 3.1. – Once we have an inequality of the form (5.2.49)–(5.2.50) in [11], we prove Lemma 3.1 exactly by the same method as Lemma 5.2.6 of [11]. So our goal is to find such an inequality. Choose $x, y \in \mathbb{R}^d$ such that $|x-y| \leqslant \sqrt{d}$. In view of (3.3) we have

$$\varrho_\lambda(x,y) \leqslant B(x,\omega), \tag{3.19}$$

with

$$B(x,\omega)=\sqrt{2d\big(\lambda+||W||_\infty\omega(B(x,\sqrt{d}+a))\big)}.$$

Thus, $B(x,\omega)$ and $B(y,\omega)$ are independent for $|x-y| \geqslant L(d,a) = 2(\sqrt{d}+a+1)$. Therefore the claim of our lemma follows as in Lemma 5.2.6 of [11], if we choose c (defined in (5.2.53), [11]) to be

$$c=\log\mathbb{E}\big[\exp\{dLB(0,\omega)\}\big]<\infty. \tag{3.20}$$

□

Remark. – We have only treated the case $\lambda > 0$. The case $\lambda = 0$ is more delicate. If the intensity ν of the Poissonian point process is small, then the probability that the origin belongs to an infinite cluster of the complement of the support of $V(\cdot,\omega)$ is strictly positive (see Sarkar [9], Theorem 1 for $d \geqslant 3$, and Meester–Roy [8], Theorems 4.4 and 3.10 for $d = 2$). In that case ($\lambda = 0$, ν small) we expect, as in discrete first-passage percolation, that μ_λ is not positive (see Theorem 6.1, p. 218, of Kesten [6]).

APPENDIX A. PROOF OF LEMMA 2.2

In this appendix we provide for the reader's convenience a proof of Lemma 2.2, which follows Theorem 1.5 of Agmon [1]. In our special setting (we assume that we have a bounded solution) we manage to simplify Agmon's proof substantially.

Proof of Lemma 2.2. – We choose $u \in H^1_{loc}(D)$ to be a non-negative, bounded weak solution of (2.3). $L^\infty_c(D)$ denotes the space of essentially bounded functions with compact support in D. By a density argument, we see that the integral equation (2.4) still holds true, if we choose $\phi \in H^1(D) \cap L_c^\infty(D)$. (We smoothen ϕ with a non-negative function $\rho_\varepsilon \in C_c^\infty(D)$ with suppρ_ε in $B(0,\varepsilon)$ and $|\rho_\varepsilon|_{L^1} = 1$. Letting ε tend to 0 yields the claim.) For ψ is a Lipschitz function with compact support in D we see that $u\psi^2 \in H^1(D) \cap L_c^\infty(D)$, hence we will choose $u\psi^2$ as ϕ in the integral equation (2.4):

$$\begin{aligned} 0 &= \int_D \frac{1}{2}\nabla u \nabla(u\psi^2) + \beta q u^2 \psi^2 \, dx \\ &\geqslant \int_D -\frac{1}{2}u^2 |\nabla\psi|^2 + \beta q u^2 \psi^2 \, dx, \end{aligned} \tag{A.1}$$

where the last expression follows from

$$\nabla u \nabla(u\psi^2) = |\nabla(u\psi)|^2 - u^2|\nabla\psi|^2 \geqslant -u^2|\nabla\psi|^2.$$

Our aim is to choose a function ψ, which will prove the claim. Choose

$$\psi(x) = e^{g(x)}\chi(x), \tag{A.2}$$

where we assume that g and χ are Lipschitz functions, χ is compactly supported with values in $[0,1]$, and g satisfies $1/2|\nabla g(x)|^2 < \beta q(x)$ for a.e. $x \in D$. g will play the role of the function h in (2.8), and χ will be a smoothened characteristic function, such that ψ lives only on a compact subset of $\mathbb{R}^d$. Using (A.1) we have

$$\begin{aligned} 0 &\geqslant \int_D \beta q u^2 e^{2g} \chi^2 - \frac{1}{2}u^2 |\nabla(e^g\chi)|^2 \, dx \\ &\geqslant \int_D (u\chi)^2 \left(\beta q - \frac{1}{2}|\nabla g|^2\right) e^{2g} \, dx \\ &\quad - \int_D \frac{1}{2}u^2 e^{2g} \left(|\nabla\chi|^2 + 2\chi|\nabla g \nabla\chi|\right) dx. \end{aligned} \tag{A.3}$$

Now let $e^g\chi$ approximate e^h. For $i \in \mathbb{N}$ we choose

$$K_i = \overline{B}(0, 2+i) \setminus B^\circ(0, 1+1/i),$$

then $\{K_i\}_i$ is a sequence of compact subsets in $D = \overline{B}(0,1)^c$, such that $K_i \subset K_j$, for all $i < j$, and $\bigcup_{i \geqslant 1} K_i = D$. For $\delta > 0$ we define

$$\chi_i(x) = \begin{cases} \dfrac{\sqrt{\beta}\varrho_\lambda(x, D \setminus K_i)}{\delta} & \text{if } \sqrt{\beta}\varrho_\lambda(x, D \setminus K_i) \in [0, \delta], \\ 1 & \text{otherwise.} \end{cases} \tag{A.4}$$

χ_i is a compactly supported Lipschitz function, which is 0 on K_i^c. Using Theorem 1.4 of [1] and the triangle inequality for the distance function ϱ_λ, we see that

$$\frac{1}{2}|\nabla \chi_i|^2 \leqslant \frac{\beta}{\delta^2} q, \quad \text{a.e.} \tag{A.5}$$

Next we choose g. For $j \geqslant 1$, we define

$$g_j(x) = h(x) \wedge \left(-\frac{1}{2}\sqrt{\beta}\varrho_\lambda(x, 0) + j \right). \tag{A.6}$$

The truncation of h will be important to obtain an integrable function (in order to apply Lebesgue's dominated convergence theorem). We remark that g_j is Lipschitz with

$$\frac{1}{2}|\nabla g_j|^2 < \beta q, \quad \text{a.e.} \tag{A.7}$$

This is true, of course, for $x \in D$ with $h(x) \neq -\frac{1}{2}\sqrt{\beta}\varrho_\lambda(x,0) + j$, but on the other hand, if $h(x) = -\frac{1}{2}\sqrt{\beta}\varrho_\lambda(x,0) + j$, then for $z \in \mathbb{R}^d$, and $t > 0$ small, $g_j(x + tz) - g_j(x) \leqslant h(x + tz) - h(x)$, from which follows that for all $z \in \mathbb{R}^d$

$$\langle z, (\nabla g_j(x) - \nabla h(x)) \rangle \leqslant 0, \quad \text{a.e.} \tag{A.8}$$

This implies $\nabla g_j(x) = \nabla h(x)$ a.e. in the second case. Therefore (A.7) also holds true in this second case. So $\psi = e^{g_j}\chi_i$ fulfills all the assuptions after (A.2). Using (A.3)

$$\int_D (u\chi_i)^2 \left(\beta q - \frac{1}{2}|\nabla g_j|^2 \right) e^{2g_j}\, dx \leqslant \int_D f_{i,j}\, dx, \tag{A.9}$$

with

$$f_{i,j}(x) = \frac{1}{2} u(x)^2 e^{2g_j(x)} \left(|\nabla \chi_i(x)|^2 + 2\chi_i(x) |\nabla g_j(x) \nabla \chi_i(x)| \right). \tag{A.10}$$

Next, we define the "inner" of K_i (with respect to $\delta > 0$):

$$K_i^I = \{x \in K_i;\ \sqrt{\beta}\varrho_\lambda(x, D \setminus K_i) > \delta\}. \tag{A.11}$$

We remark that $\nabla\chi_i = 0$ a.e. on $K_i^c \cup K_i^I$. Therefore, if we define the functions κ_i:

$$\kappa_i(x) = \begin{cases} 1 & \text{if } x \in K_i \setminus K_i^I, \\ 0 & \text{otherwise,} \end{cases} \tag{A.12}$$

it follows that for all x with $K_i(x) = 0$: $f_{i,j}(x) = 0$. Hence, we can split the right member of (A.9) into two parts:

$$\int_D f_{i,j}\,dx = \int_{D\setminus D_{\delta/\sqrt{\beta}}} f_{i,j}\,dx + \int_{D_{\delta/\sqrt{\beta}}} \kappa_i f_{i,j}\,dx. \tag{A.13}$$

We claim that the rightmost term in the above equation tends to 0 for $i \to \infty$: on $D_{\delta/\sqrt{\beta}}$ we know that κ_i tends to 0 (as $i \to \infty$), and

$$\kappa_i f_{i,j} \leqslant u^2 e^{2g_j} q\beta\left(\frac{1+2\delta}{\delta^2}\right) \leqslant e^{2j} u^2 e^{-\sqrt{\beta}\varrho_\lambda(x,0)} q\beta\left(\frac{1+2\delta}{\delta^2}\right). \tag{A.14}$$

Because u is bounded, this last term is in L^1 for every j and every $\omega \in \Omega_0$. Applying Lebesgue's dominated convergence theorem to (A.9) and (A.13), we see that

$$\begin{aligned}
&\limsup_{i\to\infty} \int_D (u\chi_i)^2\left(\beta q - \frac{1}{2}|\nabla g_j|^2\right) e^{2g_j}\,dx \\
&\quad\leqslant \limsup_{i\to\infty} \int_{D\setminus D_{\delta/\sqrt{\beta}}} f_{i,j}\,dx \leqslant \frac{1+2\delta}{\delta^2} \int_{D\setminus D_{\delta/\sqrt{\beta}}} u^2 e^{2h} \beta q\,dx. \qquad \text{(A.15)}
\end{aligned}$$

To conclude the claim of the lemma, we remark that on $D_{\delta/\sqrt{\beta}}$ we know that $\chi_i \to 1$ as $i \to \infty$, $g_j \nearrow h$ as $j \to \infty$ and we know that $\beta q - \frac{1}{2}|\nabla g_j|^2 \geqslant 0$ on D, hence, using Fatou's lemma twice, we see that

$$\begin{aligned}
&\int_{D_{\delta/\sqrt{\beta}}} u^2\left(\beta q - \frac{1}{2}|\nabla h|^2\right) e^{2h}\,dx \\
&\quad\leqslant \liminf_{j\to\infty} \liminf_{i\to\infty} \int_D (u\chi_i)^2\left(\beta q - \frac{1}{2}|\nabla g_j|^2\right) e^{2g_j}\,dx
\end{aligned}$$

$$\leqslant \frac{1+2\delta}{\delta^2} \int_{D \setminus D_{\delta/\sqrt{\beta}}} u^2 e^{2h} \beta q \, dx. \tag{A.16}$$

This finishes the proof of Lemma 2.2. □

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