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PIERRE-LUC MORIEN

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On the density for the solution of a Burgers-type SPDE

by

Pierre-Luc MORIEN [1]

MODAL'X, Université Paris 10, 200 Avenue de la République, 92001 Nanterre Cedex, France

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ABSTRACT. – We study the existence and regularity of densities for the solution of a class of parabolic SPDEs of Burgers type introduced by Gyöngy in a recent paper. In the case of regular, bounded coefficients, we show the existence of a smooth density and, as a consequence, we obtain the existence of the density for the classical stochastic Burgers equation. © Elsevier, Paris

Key words: Stochastic Burgers equation, parabolic SPDEs, Malliavin calculus

RÉSUMÉ. – On étudie ici l'existence et la régularité des densités pour les solutions d'une classe d'EDPS paraboliques de types Burgers introduite par Gyöngy dans un article récent. Dans le cas de coefficients réguliers et bornés, on montre l'existence d'une densité régulière et, comme conséquence, on obtient l'existence de la densité pour l'équation de Burgers classique. © Elsevier, Paris

[1] E-mail: morien@modalx.u-paris10.fr.

1. INTRODUCTION

Consider the following parabolic stochastic partial differential equation:

$$\frac{\partial X}{\partial t}(t,x) = \frac{\partial^2 X}{\partial x^2}(t,x) + \frac{\partial}{\partial x}(g(X(t,x))) + f(X(t,x)) + \sigma(X(t,x))\dot{W}(t,x), \qquad \text{(E)}$$

for $(t,x) \in]0,T[\times]0,1[$, where W is a space-time white noise on $[0,T]\times[0,1]$, with Dirichlet boundary conditions:

$$X(0,x) = X_0(x), \quad X(t,0) = X(t,1) = 0.$$

The solution of such an equation is the process $X(t,x)$ whose evolution equation formulation is the following (cf. Walsh [16] for the case when $g \equiv 0$ and Gyöngy [8] for g with quadratic growth):

$$\begin{aligned} X(t,x) = G_t(x,X_0) &- \int_0^t\int_0^1 \frac{\partial G_{t-s}}{\partial y}(x,y)\, g(X(s,y))\, dy\, ds \\ &+ \int_0^t\int_0^1 G_{t-s}(x,y)\sigma(X(s,y))\, W(dy,ds) \\ &+ \int_0^t\int_0^1 G_{t-s}(x,y) f(X(s,y))\, dy\, ds, \end{aligned} \qquad (1.1)$$

where G_t denotes the Green kernel related to the heat equation on $[0,T]\times[0,1]$ with Dirichlet boundary conditions and where

$$G_t(x,\phi) = \int_0^1 G_t(x,y)\phi(y)\, dy$$

for any continuous function ϕ on $[0,1]$. We remark that if $f=\sigma=0$ and $g(r)=\frac{1}{2}r^2$, then the above equation is called Burgers equation and has been extensively studied in the literature (see Burgers [5], Hopf [9] and the references therein). More recently, the Burgers equation perturbed by space-time white noise (i.e., when $f=0, g(r)=\frac{1}{2}r^2$ and $\sigma \neq 0$) has been studied by various authors (see, e.g., Da Prato, Debussche and Temam [6], Da Prato and Gatarek [7] and the references therein). Finally, the case $g = 0$ has also been studied intensively, we refer to Walsh [16] for the basics on the subject.

In [8] Gyongy has proved the existence and uniqueness of the solution of **(E)**, along with joint continuity of the paths, when f and g are locally Lipschitz with linear growth, and σ is globally Lipschitz and bounded.

The purpose of our work is to prove that, under more restrictive assumptions on the coefficients (namely: f, g, σ sufficiently regular, with bounded derivatives, and σ bounded and satisfying a strong ellipticity condition), the solution $X(t,x)$ of **(E)** admits a Hölder-continuous version and belongs to the space $\mathbb{D}^\infty$ of infinitely differentiable random variables in the sense of the Malliavin calculus associated with W. Furthermore, using a method developed by Bally and Pardoux in [3], we prove that in this case, for all $0 < x_1 < x_2 < \cdots < x_d < 1$, the random vector $(X(t,x_1), \ldots, X(t,x_d))$ admits a C^∞ density with respect to the Lebesgue measure on $\mathbb{R}^d$.

As a consequence, using a localization argument, we prove that, when g satisfies only a condition of quadratic growth (a case which includes the stochastic Burgers equation), the solution of equation **(E)** belongs to the space $\mathbb{D}^{1,p}_{loc}$. Moreover, using the general criterion for absolute continuity proved by Bouleau and Hirsch (see [4]), we show that in this case the law of the random vector $(X(t,x_1), \ldots, X(t,x_d))$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. A similar one-dimensional result has recently been obtained by Lanjri and Nualart [10] for the stochastic Burgers equation in the case of a nondegeneracy at the origin, using the approach developed by Pardoux and Tusheng in [15]. However, in [10] as in our work, the smoothness of the density cannot be obtained in the case considered *via* the localization and remains an open problem.

The proofs of our results rely on new estimates for the Green kernel G, regarding the behaviour of $\frac{\partial G_{t-s}}{\partial y}(x,y)$. These estimates are proved in Section 3. Section 2 is devoted to the statement of the problem and of the results, and Sections 4 and 5 to the proofs of the main results.

2. GENERAL FRAMEWORK AND STATEMENT OF THE RESULTS

Let T be a fixed deterministic time, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and W a space-time white noise on $[0,T] \times [0,1]$ with covariance $ds\,dy$.

Let

$$\mathcal{F}_t = \sigma\big(W(A), A \in \mathcal{B}([0,t]\times[0,1])\big) \vee \mathcal{N},$$

where $\mathcal{N}$ is the class of $\mathbb{P}$-null sets in $\mathcal{F}$ and let $\mathcal{P}$ denote the σ-algebra of $\mathcal{F}_t$-progressively measurable subsets of $\Omega \times [0,T] \times [0,1]$.

Let $X = (X(t,x))$ be the solution of equation (**E**). This means that X satisfies the evolution equation (1.1) where

$$G_t(x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{n=+\infty} \left[\exp\left\{\frac{-(y-x-2n)^2}{4t}\right\} - \exp\left\{\frac{-(y+x-2n)^2}{4t}\right\}\right], \tag{2.1}$$

In all that follows, we assume that the initial condition X_0 satisfies the following:

$$\begin{aligned} &(\mathrm{H}_0.1) \quad \sup_x \mathbb{E}\big[|X_0(x)|^p\big] \leqslant C_p < \infty \quad \text{for every } p \in [1,+\infty[\\ &(\mathrm{H}_0.2) \quad \mathbb{E}\left[\left(\sup_{y\neq x} \frac{|X_0(y)-X_0(x)|}{|y-x|^\alpha}\right)^p\right] \leqslant C_p < \infty, \\ &\qquad\qquad \text{for some } \alpha > 0 \text{ and for all } p \in [1,+\infty[. \end{aligned} \tag{$\mathbf{H}_0$}$$

To study parabolic SPDEs, it is crucial to have estimates of the Green kernel G. In the case $g \equiv 0$, one uses those recalled in Lemma A.1 of Appendix A. However, due to the presence of $\frac{\partial G_{t-s}}{\partial y}(x,y)$ in (1.1), we need estimates for this quantity as well. They are given in the following lemma:

LEMMA 2.1. –

(a) $$\int_0^T \int_0^1 \left|\frac{\partial G_s}{\partial y}(x,y)\right|^\beta dy\,ds \leqslant C_{T,\beta} < \infty$$

for $T > 0$, $1/2 < \beta < 3/2$,

(b) $$\int_t^{t+h} \int_0^1 \left|\frac{\partial G_{t+h-s}}{\partial y}(x,y)\right|^\beta dy\,ds \leqslant C_\beta h^{\frac{3}{2}-\beta}$$

for $h > 0$, $1/2 < \beta < 3/2$,

(c) $$\int_0^t \int_0^1 \left|\frac{\partial G_{t+h-s}}{\partial y}(x,y) - \frac{\partial G_{t-s}}{\partial y}(x,y)\right|^\beta dy\,ds \leqslant C_\beta h^{\frac{3}{2}-\beta}$$

for $h > 0,\ 1 < \beta < 3/2$,

(d) $$\int_0^t \int_0^1 \left| \frac{\partial G_{t-s}}{\partial y}(x+h, y) - \frac{\partial G_{t-s}}{\partial y}(x, y) \right|^\beta dy\, ds \leqslant C_\beta h^{3-2\beta}$$

for $h > 0,\ 1 < \beta < 3/2$.

A simple, yet interesting application of these estimates is the following regularity result:

THEOREM 2.1. – *Under* $(\mathbf{H_0})$, *if* f, g *and* σ *are globally Lipschitz on* $\mathbb{R}$, *then the solution* $X(t, x)$ *of* **(E)** *is* β*-Hölder continuous with respect to* x *and* $\beta/2$*-Hölder continuous with respect to* t, *for every* $\beta < \inf\{\alpha, 1/2\}$.

We remark that the joint continuity in (t, x) of X has been obtained by Gyöngy in [8] for more general coefficients, as mentioned in the introduction. More precisely, the author proved the following result:

PROPOSITION 2.1 (Theorem 2.1, p. 274, [8]). – *If the following hypotheses are satisfied*:

(a) f *and* g *are locally Lipschitz with linearly growing Lipschitz constants,* σ *is globally Lipschitz, i.e., there exists a constant* L *such that*

$$|f(p) - f(q)| \leqslant L(1 + |p| + |q|) \cdot |p - q|,$$

$$|g(p) - g(q)| \leqslant L(1 + |p| + |q|) \cdot |p - q|,$$

$$|\sigma(p) - \sigma(q)| \leqslant L|p - q|;$$

(b) f *satisfies the linear growth condition and* σ *is bounded*;

(c) X_0 *is an* $\mathcal{F}_0$*-measurable,* $L^p([0, 1])$*-valued random element, for some* $p \geqslant 2$,

then equation **(E)** *has a unique solution* X *on the interval* $[0, \infty)$. *Moreover,* X *is an* $L^p([0, 1])$*-valued continuous stochastic process, and if* X_0 *has a continuous modification, then* X *has a modification which is continuous in* $(t; x) \in [0, \infty) \times [0, 1]$.

Now, our main interest in this paper is the existence of the density for the law of $X(t, x)$ with respect to the Lebesgue measure on $\mathbb{R}$, as well as its possible regularity. A result in that direction has been proved by Bally and Pardoux in [3] in the case $g \equiv 0$ under some regularity assumptions on f and σ, provided a non-degeneracy hypothesis holds on σ. We therefore define three sets of hypotheses on the coefficients. We say that f, g, σ satisfy **(H)** if the following assumptions hold:

(H.1) f, g and σ are of class C^1 on $\mathbb{R}$, f and σ have a bounded first derivative, σ is bounded.

(H.2) g satisfies a quadratic growth condition, i.e.,

$$\exists K,\ \forall r, \quad |g(r)| \leqslant C\left(1+r^2\right).$$

(H.3) There exists $c > 0$ such that, for all $x \in \mathbb{R}$, $\sigma^2(x) \geqslant c$.

We say that f, g, σ satisfy the restricted hypotheses **(RH)** if they satisfy (H.1) and (H.3) and moreover g' is bounded.

Finally, we say that f, g, σ satisfy the restrictive smoothness hypotheses **(RSH)** if they are of class C^∞, with bounded derivatives of order $k \geqslant 1$, and if σ is bounded and satisfies (H.3).

Our main result is the following (the spaces $\mathbb{D}^\infty$ and $(\mathbb{D}^\infty)^d$ are defined below):

THEOREM 2.2. – *Under conditions* **(H$_0$)** *and* **(RSH)**, *for all $t > 0$ and $(x_1, \ldots, x_d) \in \,]0, 1[^d$, where the x_j's are distinct, the random vector $(X(t, x_1), \ldots, X(t, x_d))$ belongs to $(\mathbb{D}^\infty)^d$ and its law admits a C^∞ density with respect to the Lebesgue measure on $\mathbb{R}^d$.*

An interesting by-product of Theorem 2.2 is the following:

COROLLARY 2.1. – *Under* **(H$_0$)** *and* **(H)**, *for all $t > 0$ and $(x_1, \ldots, x_d) \in \,]0, 1[^d$, where the x_j's are distinct, the random vector $(X(t, x_1), \ldots, X(t, x_d))$ belongs to $\mathbb{D}^{1,p}_{loc}$ and its law is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.*

We remark that Corollary 2.1 proves the existence of the density for the solution of the stochastic Burgers equation, i.e., when $f = 0$ and $g(r) = \frac{1}{2}r^2$.

Theorem 2.2 and Corollary 2.1 will be obtained by means of the Malliavin calculus with respect to the white noise W. We recall here basic results of this theory (see [14] for a more detailed account on the subject).

Set $H = L^2([0, T] \times [0, 1])$; for $h \in H$, let $W(h)$ be the Gaussian random variable defined by

$$W(h) = \int_0^T \int_0^1 h(s, y) W(dy, ds).$$

We denote by $\mathcal{S}$ the space of smooth functionals, i.e., of real-valued functionals of the form

$$F = f\big(W(h_1), W(h_2), \dots, W(h_m)\big),$$

where $f \in C^\infty(\mathbb{R}^m)$ and has polynomial growth, as well as its derivatives, and $h_1, h_2, \dots, h_m$ is an orthonormal sequence in H. Finally, $\langle \cdot, \cdot \rangle_H$ will denote the inner product on H.

For $F \in \mathcal{S}$, one defines the first-order Malliavin derivative of F as the H-valued random variable

$$D_{t,x} F = \sum_{i=1}^m \partial_i f\big(W(h_1), W(h_2), \dots, W(h_m)\big) h_i(t,x).$$

Similarly, the derivative of order k of F is defined recursively by

$$D^k_{\alpha_1,\dots,\alpha_k} F = D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} F, \quad \alpha_j = (t_j, x_j) \in [0,T] \times [0,1].$$

Then, for $p \geqslant 1$ and $k \in \mathbb{N}$, the space $\mathbb{D}^{k,p}$ is the closure of $\mathcal{S}$ with respect to the seminorm

$$\|F\|_{k,p} = \left[\left(\mathbb{E}|F|^p\right) + \sum_{i=1}^k \left(\mathbb{E}\|D^i F\|^p_{L^2(([0,T]\times[0,1])^i)}\right) \right]^{1/p}, \tag{2.2}$$

and we set $\mathbb{D}^\infty = \bigcap_{p\geqslant 1} \bigcap_{k\in\mathbb{N}} \mathbb{D}^{k,p}$. Finally, a random vector $(F_1, \dots, F_d)$ is said to be in $(\mathbb{D}^\infty)^d$ if $F_j \in \mathbb{D}^\infty$ for any $j \in \{1, \dots, d\}$.

The operator D is local in the space $\mathbb{D}^{1,1}$ which means that for any $F \in \mathbb{D}^{1,1}$, $DF.1_{\{F=0\}} = 0$ almost surely. Denote by $\mathbb{D}^{k;p}_{loc}$ the set of random variables F such that there exists a sequence Ω_n of events and a sequence F_n of random variable in $\mathbb{D}^{k,p}$ such that $\Omega_n \uparrow \Omega$, and for each n, $F = F_n$ almost surely on Ω_n.

The Malliavin calculus gives convenient criteria for the existence and regularity of densities. We will use the following ones in our proofs:

PROPOSITION 2.2 (Existence, [4], or [14] Theorem 2.1.2). – *Let $F = (F_1, \dots, F_m)$ be a random vector satisfying the following conditions*:

(i) $F_i \in \mathbb{D}^{1,p}_{loc}$ *for all* $p > 1$ *and for* $i = 1, \dots, m$;

(ii) *The Malliavin matrix* $\Gamma_F := (\langle DF_i, DF_j \rangle_H)_{1\leqslant i,j\leqslant m}$ *is almost surely invertible.*

Then the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$.

PROPOSITION 2.3 (Regularity, [14], Corollary 2.1.2). – *Let* $(F_1,\ldots,F_m)$ *a random vector satisfying the following conditions*:

(i) $F_i \in \mathbb{D}^\infty$ *for all* $i = 1,\ldots,m$.

(ii) *The Malliavin matrix* $\Gamma_F := (\langle DF_i, DF_j\rangle_H)_{1\leqslant i,j\leqslant m}$ *satisfies*:

$$\left(\det \Gamma_F\right)^{-1} \in \bigcap_{p>1} L^p(\Omega).$$

Then F *has an infinitely differentiable density with respect to the Lebesgue measure on* $\mathbb{R}^m$.

Finally, we shall use in the sequel the following version of the Burkholder–Davis–Gundy inequality for Hilbert- space valued martingales (see Metivier [11], E.2., p. 212):

If $(Q_{s,y})_{(s,y)\in\Lambda_T}$ is an $L^2(\Lambda_t)$-valued predictable process, then

$$\mathbb{E}\left|\int_{\Lambda_t}\left(\int_0^t\int_0^1 Q_{v,u}(r,z)W(dv,du)\right)^2 dz\,dr\right|^p$$
$$\leqslant C_p\mathbb{E}\left|\int_0^t\int_0^1\left(\int_{\Lambda_t} Q^2_{v,u}(r,z)\,dz\,dr\right)dv\,du\right|^p. \tag{2.3}$$

The next sections are devoted to the proofs of Lemma 2.1, Theorems 2.1, 2.2 and Corollary 2.1.

3. PROOF OF LEMMA 2.1

We first remark that the following decomposition holds:

$$G_t(x,y) = H_t^1(x,y) + H_t^2(x,y) + H_t^3(x,y) + \overline{G_t}(x,y),$$

where $(t,x,y)\mapsto \overline{G_t}(x,y)$ belongs to $C^\infty([0,T]\times\mathbb{R}^2)$, and:

$$H_t^1(x,y) = \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-(x-y)^2/4t},\qquad H_t^2(x,y) = \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-(x-y-1)^2/4t},$$

$$H_t^3(x,y) = \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-(x+y-2)^2/4t}.$$

Hence, the behaviour of the integrals in (a), (b), (c), (d) is determined by that of the corresponding integrals with H_t^i, $i=1,2,3$, instead of G.

Sinces the calculations are similar for the three functions, we shall only prove the required estimates for H^1, which will be simply denoted by H in the sequel.

One has:

$$\frac{\partial H_t}{\partial y}(x,y) = \frac{(x-y)}{4\sqrt{\pi}t^{3/2}} \exp\big(-(x-y)^2/4t\big).$$

To prove (a), we evaluate $\int_0^T \int_0^1 |\frac{\partial H_s}{\partial y}(x,y)|^\beta dy\, ds$. We have:

$$\int_0^T \int_0^1 \left|\frac{\partial H_s}{\partial y}(x,y)\right|^\beta dy\, ds \leqslant \int_0^T \frac{C_\beta}{s^{3\beta/2}} \left(\int_{\mathbb{R}} |z^\beta| \exp\left(-\beta z^2/4s\right) dz \right) ds.$$

In the sequel, the following identity will be used repeatedly:

$$\int_{\mathbb{R}} |z|^r e^{-z^2/\sigma^2} dz = C_r \cdot \sigma^{r+1}. \tag{3.1}$$

Then

$$\int_0^T \int_0^1 \left|\frac{\partial H_s}{\partial y}(x,y)\right|^\beta dy\, ds \leqslant C_\beta \int_0^T \frac{1}{s^{3\beta/2}} \cdot s^{\frac{\beta+1}{2}} ds = C_\beta \int_0^T \frac{ds}{s^{\beta-\frac{1}{2}}},$$

and the last integral is convergent iff $\beta < 3/2$.

A similar method is used to obtain (b). We have:

$$\int_t^{t+h} \int_0^1 \left|\frac{\partial H_{t+h-s}}{\partial y}(x,y)\right|^\beta dy\, ds$$
$$\leqslant C_\beta \int_t^{t+h} \int_{\mathbb{R}} \frac{|x-y|^\beta}{(t+h-s)^{3\beta/2}} e^{-\beta(x-y)^2/4(t+h-s)} dy\, ds.$$

Using the following change of variables: $t+h-s = hv$, $x-y=\sqrt{h}z$, of Jacobian $h^{3/2}$, using (3.1) we obtain:

$$\int_t^{t+h} \int_{\mathbb{R}} \frac{|x-y|^\beta}{(t+h-s)^{3\beta/2}} e^{-\frac{\beta(x-y)^2}{4(t+h-s)}} dy\, ds = \int_0^1 \int_{\mathbb{R}} \frac{|z|^\beta \cdot h^{\beta/2}}{h^{3\beta/2} v^{3\beta/2}} e^{-\frac{\beta z^2}{4v}} \cdot h^{3/2} dz\, dv$$

$$= h^{\frac{3}{2}-\beta} \int_0^1 \int_{\mathbb{R}} \frac{|z|^\beta}{\nu^{3\beta/2}} \exp\Big(- \frac{\beta z^2}{4\nu}\Big) dz\, d\nu = C_\beta h^{\frac{3}{2}-\beta} \int_0^1 \frac{1}{\nu^{3\beta/2}} \cdot \nu^{\frac{\beta+1}{2}} d\nu$$

$$= C_\beta h^{\frac{3}{2}-\beta} \int_0^1 \frac{d\nu}{\nu^{\beta-\frac{1}{2}}},$$

and the last integral is convergent iff $\beta < 3/2$.

We now turn to (c); we have

$$\int_0^t \int_0^1 \left| \frac{\partial H_{t+h-s}}{\partial y}(x,y) - \frac{\partial H_{t-s}}{\partial y}(x,y) \right|^\beta dy\, ds \leqslant C_\beta \int_0^t \int_0^1 |x-y|^\beta$$

$$\times \left| \frac{1}{(t+h-s)^{3/2}} e^{-\frac{(x-y)^2}{4(t+h-s)}} - \frac{1}{(t-s)^{3/2}} e^{-\frac{(x-y)^2}{4(t-s)}} \right|^\beta dy\, ds.$$

Setting $t - s = h\nu$ and $x - y = \sqrt{h} z$, we obtain

$$\int_0^t \int_0^1 \left| \frac{\partial H_{t+h-s}}{\partial y}(x,y) - \frac{\partial H_{t-s}}{\partial y}(x,y) \right|^\beta dy\, ds$$

$$\leqslant C_\beta h^{\frac{3}{2}-\beta} \int_0^{t/h} \int_{\mathbb{R}} |z|^\beta \cdot \left| \frac{1}{(\nu+1)^{3/2}} e^{-\frac{z^2}{4(\nu+1)}} - \frac{1}{\nu^{3/2}} e^{-\frac{z^2}{4\nu}} \right|^\beta dz\, d\nu.$$

Hence we have to prove the convergence of

$$\int_0^{+\infty} \int_{\mathbb{R}} |z|^\beta \cdot \left| \frac{1}{(\nu+1)^{3/2}} e^{-\frac{z^2}{4(\nu+1)}} - \frac{1}{\nu^{3/2}} e^{-\frac{z^2}{4\nu}} \right|^\beta dz\, d\nu.$$

The convergence near zero is obtained by majorizing the difference by the sum and then using the method of (a): this yields the condition $\beta < 3/2$. To prove the convergence near infinity, we use the following trick: set $F_z(t) = \frac{1}{t^{3/2}} e^{-\frac{z^2}{4t}}$, then:

$$F_z(t+1) - F_z(t) = \int_0^1 \frac{\partial F_z}{\partial t}(t+\theta)\, d\theta. \tag{3.2}$$

We have

$$\frac{\partial F_z}{\partial t}(t) = -\frac{3}{2t^{5/2}} e^{-\frac{z^2}{4t}} + \frac{z^2}{4t^{7/2}} e^{-\frac{z^2}{4t}}.$$

Hence:

$$\int_{\mathbb{R}} |z|^{\beta} \cdot \left| \frac{1}{(\nu+1)^{3/2}} \mathrm{e}^{-\frac{z^2}{4(\nu+1)}} - \frac{1}{\nu^{3/2}} \mathrm{e}^{-\frac{z^2}{4\nu}} \right|^{\beta} dz$$

$$= \int_{\mathbb{R}} |z|^{\beta} \cdot \left| \int_0^1 \left(\frac{3}{2(\nu+\theta)^{5/2}} \mathrm{e}^{-\frac{z^2}{4(\nu+\theta)}} + \frac{z^2}{(\nu+\theta)^{7/2}} \mathrm{e}^{-\frac{z^2}{4(\nu+\theta)}} \right) d\theta \right|^{\beta} dz$$

$$\leqslant \int_0^1 \underbrace{\int_{\mathbb{R}} \frac{C_{1,\beta}|z|^{\beta}}{(\nu+\theta)^{5\beta/2}} \mathrm{e}^{-\beta \frac{z^2}{4(\nu+\theta)}} \, dz}_{= \frac{(\nu+\theta)^{\frac{\beta+1}{2}}}{(\nu+\theta)^{5\beta/2}}} \, d\theta + \int_0^1 \underbrace{\int_{\mathbb{R}} \frac{C_{2,\beta}|z|^{3\beta}}{(\nu+\theta)^{7\beta/2}} \mathrm{e}^{-\beta \frac{z^2}{4(\nu+\theta)}} \, dz}_{= \frac{(\nu+\theta)^{\frac{3\beta+1}{2}}}{(\nu+\theta)^{7\beta/2}}} \, d\theta,$$

and both quantities above are equal to $1/(\nu+\theta)^{\frac{4\beta-1}{2}}$. Since $\beta > 1$, the exponent is greater than 1 and

$$\int_0^1 \frac{d\theta}{(\nu+\theta)^{\frac{4\beta-1}{2}}} \leqslant \frac{1}{\nu^{\frac{4\beta-1}{2}}}$$

which yields the convergence of the integral on the interval $[1, \infty[$.

For (d), a similar method is used, using the change of variables $x - y = hz, t - s = h^2 \nu$, of Jacobian h^3 and the following identity, similar to (3.2): if $U_t(z) = z\mathrm{e}^{-\frac{z^2}{4t}}$, then

$$U_t(z+1) - U_t(z) = \int_0^1 \frac{\partial U_t}{\partial z}(z+\theta)\, d\theta. \tag{3.3}$$

4. HÖLDER REGULARITY OF THE SOLUTION

This section is devoted to prove Theorem 2.1. The method used here is an adaptation of Walsh's original proof. We start with the following result:

PROPOSITION 4.1. – *Under* $(\mathbf{H_0})$, *if* f, g *and* σ *are globally Lipschitz on* $\mathbb{R}$, *we have for all* $p \in]1, +\infty[$

$$\sup_{(t,x)\in[0,T]\times[0,1]} \mathbb{E}|X(t,x)|^p \leqslant C_p < \infty.$$

Proof of Proposition 4.1. – Using Lemma 2.1(a) with $\beta = 1$ and the evolution equation (1.1), Hölder's and Burkholder–Davis–Gundy's inequalities, along with the linear growth of f, g, σ, we easily deduce that:

$$\mathbb{E}|X(t,x)|^p \leqslant C_p\left\{1 + \int_0^t\int_0^1 \chi_{t-s}(x,y)\big(1+\mathbb{E}|X(s,y)|^p\big)\,dy\,ds\right\},$$

where

$$\chi_{t-s}(x,y) = G^2_{t-s}(x,y) + \left|\frac{\partial G_{t-s}}{\partial y}(x,y)\right|. \tag{4.1}$$

Furthermore (3.1) immediately yields

$$\int_0^1 \chi_{t-s}(x,y)\,dy \leqslant \frac{C}{\sqrt{t-s}}.$$

Hence, setting $\gamma(t) := \sup_{x\in[0,1]} \mathbb{E}|X(t,x)|^p$, we have

$$\gamma(t) \leqslant C_p\left\{1 + \int_0^t \frac{\gamma(s)\,ds}{\sqrt{t-s}}\right\},$$

which by iteration gives

$$\gamma(t) \leqslant C_p\left\{1 + \int_0^t \gamma(s)\,ds\right\}$$

and Gronwall's Lemma yields the result. □

The next step is to prove upper estimates of $\mathbb{E}|X(t+h,x) - X(t,x)|^{2p}$ and $\mathbb{E}|X(t,x+h) - X(t,x)|^{2p}$ for $h \in [0,1]$ such that $t+h \in [0,T]$ and $x+h \in [0,1]$. For instance, using again Hölder's and Burkholder–Davis–Gundy's inequalities, we obtain:

$$\mathbb{E}\big|X(t,x+h) - X(t,x)\big|^{2p}$$
$$\leqslant C_p\Big\{\mathbb{E}\big|G_t(x+h,X_0) - G_t(x,X_0)\big|^{2p}$$

$$+\left(\int_0^t\int_0^1\left|\frac{\partial G_{t-s}}{\partial y}(x+h,y)-\frac{\partial G_{t-s}}{\partial y}(x,y)\right|dy\,ds\right)^{2p-1}$$

$$\times\int_0^t\int_0^1\left|\frac{\partial G_{t-s}}{\partial y}(x+h,y)-\frac{\partial G_{t-s}}{\partial y}(x,y)\right|\cdot\mathbb{E}|X(s,y)|^{2p}dy\,ds$$

$$+\left(\int_0^t\int_0^1\left|G_{t-s}(x+h,y)-G_{t-s}(x,y)\right|^2dy\,ds\right)^{p-1}$$

$$\times\int_0^t\int_0^1\left|G_{t-s}(x+h,y)-G_{t-s}(x,y)\right|^2\cdot\mathbb{E}|X(s,y)|^{2p}dy\,ds\Bigg\}.$$

Using Proposition 4.1 and the classical estimates for G recalled in Lemma A.1 of the Appendix, along with Lemma 2.1(d) with $1<\beta=6/5<3/2$ and $\beta_0:=\inf\{\alpha,1/2\}$, we get

$$\begin{aligned}&\mathbb{E}\left|X(t,x+h)-X(t,x)\right|^{2p}\\&\quad\leqslant C_p\left\{h^{2p\alpha}+h^p+\left(\int_0^t\int_0^1\left|\frac{\partial G_{t-s}}{\partial y}(x+h,y)\right.\right.\right.\\&\qquad\left.\left.\left.-\frac{\partial G_{t-s}}{\partial y}(x,y)\right|^\beta dy\,ds\right)^{\frac{2p}{\beta}}\right\}\\&\quad\leqslant C_p\{h^{2p\alpha}+h^p\}\leqslant C_ph^{2p\beta_0}.\end{aligned}$$

Then, using Kolmogorov's criterion, we see that there exists a version of X which is β-Hölder-continuous with respect to x for all β such that

$$\beta<\inf\left\{\alpha,1/2,\frac{3}{\beta}-2\right\},$$

and the result is proved. The proof of the regularity with respect to t is similar, and the calculations are omitted.

5. EXISTENCE AND REGULARITY OF THE DENSITY

We devote this section to the proofs of Theorem 2.2 and Corollary 2.1.

5.1. Proof of Theorem 2.2

The method used here is inspired from that used in the case $g \equiv 0$ and is based on Proposition 2.3. We then show that $F = (X(t, x_1), \ldots, X(t, x_d))$ satisfies conditions (i) and (ii) of Proposition 2.3. Part (i) is given by the following proposition:

PROPOSITION 5.1. – *Under* $(\mathbf{H_0})$ *and* **(RH)**, *for all* $(x,t) \in [0,1] \times [0,T]$, $X(t,x) \in \mathbb{D}^{1,p}$ *for all* $p \in]1, \infty[$, *and*

$$\sup_{(x,t)} \mathbb{E}\|DX(t,x)\|^p_{L^2([0,T]\times[0,1])} \leqslant C_p < \infty.$$

Moreover, the first derivative of X *satisfies the following evolution equation*:

$$\begin{aligned} D_{r,z}X(t,x) &= G_{t-r}(x,z)\sigma\big(X(r,z)\big) \\ &+ \int_r^t \int_0^1 G_{t-s}(x,y)\sigma'\big(X(s,y)\big) D_{r,z}X(s,y) W(dy,ds) \\ &- \int_r^t \int_0^1 \frac{\partial G_{t-s}}{\partial y}(x,y) g'\big(X(s,y)\big) D_{r,z}X(s,y)\, dy\, ds \\ &+ \int_r^t \int_0^1 G_{t-s}(x,y) f'\big(X(s,y)\big) D_{r,z}X(s,y)\, dy\, ds \qquad (5.1) \end{aligned}$$

(*and* $D_{r,z}X(t,x) = 0$ *if* $r > t$).

Furthermore, under **(RSH)**, *for all* $(x,t) \in [0,1] \times [0,T]$, $X(t,x) \in \mathbb{D}^\infty$, *and for all* $p \in]1, \infty[$ *and* $M \geqslant 1$

$$\sup_{(x,t)} \mathbb{E}\big\|D^M X(t,x)\big\|^p_{L^2(([0,T]\times[0,1])^M)} = C_{M,p} < \infty.$$

Proposition 5.1 is obtained by means of the following Picard approximation scheme: $X_0(t,x) = G_t(x, X_0)$, and

$$\begin{aligned} X_{n+1}(t,x) = X_0(t,x) &+ \int_0^t \int_0^1 G_{t-s}(x,y) f\big(X_n(s,y)\big)\, dy\, ds \\ &- \int_0^t \int_0^1 \frac{\partial G_{t-s}}{\partial y}(x,y) g\big(X_n(s,y)\big)\, dy\, ds \end{aligned}$$

$$+\int_0^t\int_0^1 G_{t-s}(x,y)\sigma\big(X_n(s,y)\big)W(dy,ds). \tag{5.2}$$

Since under **(RH)** f, g and σ are Lipschitz, we have, using the estimates of Lemma 2.1 and Lemma A.1

$$\mathbb{E}\big|X_{n+1}(t,x)-X_n(t,x)\big|^p \leqslant C_p\int_0^t\int_0^1 \mathbb{E}\big|X_n(s,y)-X_{n-1}(s,y)\big|^p\,dy\,ds.$$

Iterating this inequality yields

$$\sum_{n\geqslant 0}\sup_{(t,x)\in[0,T]\times[0,1]}\mathbb{E}\big|X_{n+1}(t,x)-X_n(t,x)\big|^p<\infty,$$

and therefore:

$$X_n(t,x)\longrightarrow X(t,x) \text{ in } L^p \text{ uniformly in } (x,t),$$
$$\sup_n\sup_{(x,t)}\mathbb{E}|X_n(t,x)|^p\leqslant C_p\leqslant\infty. \tag{5.3}$$

To prove Proposition 5.1, we then use the following result, which can be found for instance in [14] (Lemma 1.5.4, p.71):

LEMMA 5.1. – *Let $\{F_n; n\geqslant 1\}$ be a sequence of random variables in $\mathbb{D}^{k,p}$ with $k\geqslant 1$ and $p>1$. Assume that F_n converges to F in $L^p(\Omega)$ and $\sup_n\|F_n\|_{k,p}<\infty$ (where the norm $\|\cdot\|_{k,p}$ is defined by* (2.2)*). Then F belongs to $\mathbb{D}^{k,p}$.*

Thus the proof reduces to establishing the following lemma:

LEMMA 5.2. – *Under* **(RH)**, *for $p\in\,]3,\infty[$, there exists a constant C_p such that*

$$\sup_n\sup_{(x,t)}\mathbb{E}\big\|DX_n(t,x)\big\|^{2p}_{L^2([0,T]\times[0,1])}\leqslant C_p<\infty, \tag{5.4}$$

and, under **(RSH)**, *for all $M\geqslant 1$, $p\in\,]3,\infty[$, there exists a constant $C_{p,M}$ such that*

$$\sup_n\sup_{(x,t)}\mathbb{E}\big\|D^M X_n(t,x)\big\|^{2p}_{L^2(([0,T]\times[0,1])^M)}\leqslant C_{p,M}<\infty. \tag{5.5}$$

Indeed, if (5.4) (respectively (5.5)) holds, since for all $p > 1$ and all $q \geqslant 1$,

$$\sup_n \sup_{(x,t)} \left\| X_n(t,x) \right\|_{q,p} < \infty,$$

using Lemma 5.1 we deduce that $X(t,x) \in \mathbb{D}^{1,p}$ (respectively in $\mathbb{D}^{\infty}$) and Eq. (5.1) is obtained simply by differentiating Eq. (1.1).

The proof of Lemma 5.2 is similar to that of Bally and Pardoux ([3], Proposition 3.3) or that of Morien ([13], Proposition 3.1). First, one sees easily by induction on n that $X_n \in \mathbb{D}^{\infty}$ and satisfies

$$\begin{aligned} D_{r,z} X_{n+1}(t,x) &= G_{t-r}(x,z)\sigma\big(X_n(r,z)\big) \\ &\quad + \int_r^t \int_0^1 G_{t-s}(x,y)\sigma'\big(X_n(s,y)\big) D_{r,z} X_n(s,y) W(dy,ds) \\ &\quad - \int_r^t \int_0^1 \frac{\partial G_{t-s}}{\partial y}(x,y) g'\big(X_n(s,y)\big) D_{r,z} X(s,y)\, dy\, ds \\ &\quad + \int_r^t \int_0^1 G_{t-s}(x,y) f'\big(X_n(s,y)\big) D_{r,z} X(s,y)\, dy\, ds \end{aligned}$$

(and $D_{r,z} X(t,x) = 0$ if $r > t$).

Since under **(RH)** f', g' and σ' are bounded, we have, using (2.3) and Lemmas 2.1 and A.1

$$\begin{aligned} &\mathbb{E}\left\| D X_{n+1}(t,x) \right\|^p_{L^2([0,T]\times[0,1])} \\ &\quad \leqslant C_p \left\{ 1 + \int_0^t \int_0^1 \chi_{t-s}(x,y) \mathbb{E}\left\| D X_n(s,y) \right\|^p_{L^2([0,T]\times[0,1])} dy\, ds \right\}, \end{aligned}$$

and (5.4) is straightforward. As for (5.5), it is proved by induction on M, using the evolution equation for $D^M X_{n+1}$ obtained by differentiating (5.1) M times.

We now show part (ii). More precisely, we prove the following result, which is slightly more general than needed for Theorem 2.2, but which will be fully used in the next paragraph for Corollary 2.1:

PROPOSITION 5.2. – *Let $\Gamma(t; x_1, \ldots, x_d)$ be the Malliavin matrix of $F = (X(x_1,t), \ldots, X(x_d,t))$. Then, under* **(RH)**, *for all $p \in]1, \infty[$, there* *exists a constant* $C_p(t, x_1, \dots, x_d)$ *such that*

$$\mathbb{E}\left(\left|\frac{1}{\det \Gamma(t; x_1, \dots, x_d)}\right|^p\right) \leqslant C_p(t, x_1, \dots, x_d) < \infty. \tag{5.6}$$

Proof of Proposition 5.2. – We follow the same lines as in Bally and Pardoux [3] and Morien [13]. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $\mathbb{R}^d$. We shall use the following result.

LEMMA 5.3 ([14], Lemma 2.3.1, p. 116). – *Let* $C(\omega)$ *be a symmetric nonnegative definite* $m \times m$*random matrix. Assume that the entries* C^{ij} *have moments of all orders and that for any* $p \in [2, +\infty[$ *there exists* $\varepsilon_0(p)$ *such that for all* $\varepsilon < \varepsilon_0$

$$\sup_{\|v\|=1} \mathbb{P}\big(\langle Cv, v\rangle \leqslant \varepsilon\big) \leqslant \varepsilon^p.$$

Then $(\det C)^{-1} \in L^p$ *for all* p.

Then it is easy to see that we only have to prove that there exists $\beta > 1/2$ such that for all $q > 3$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \leqslant \varepsilon_0$

$$\sup_{\|\xi\|=1} \mathbb{P}\big(\langle \Gamma\xi, \xi\rangle \leqslant \varepsilon^\beta\big) \leqslant \varepsilon^q, \tag{5.7}$$

where Γ stands for $\Gamma(t; x_1, \dots, x_d)$.

Let $\xi \in \mathbb{R}^d$ with $\|\xi\| = 1$, and let $0 < \varepsilon < \frac{1}{4}\min_{i \neq j} |x_i - x_j|^2$. Then:

$$\begin{aligned}\langle \Gamma\xi, \xi\rangle &= \int_0^t \int_0^1 \left(\sum_{i=1}^d D_{r,z}X(t, x_i)\xi_i\right)^2 dz\, dr \\ &\geqslant \sum_{j=1}^d \int_{t-\varepsilon}^t \int_{x_j-\sqrt{\varepsilon}}^{x_j+\sqrt{\varepsilon}} \left(\sum_{i=1}^d D_{r,z}X(t, x_i)\xi_i\right)^2 dz\, dr \geqslant \frac{1}{2} I_2(\xi) - I_1(\xi),\end{aligned}$$

with

$$I_1(\xi) = \sum_{j=1}^d \int_{t-\varepsilon}^t \int_{x_j-\sqrt{\varepsilon}}^{x_j+\sqrt{\varepsilon}} \left(\sum_{i \neq j} D_{r,z}X(t, x_i)\xi_i\right)^2 dz\, dr,$$

$$I_2(\xi) = \sum_{j=1}^d \int_{t-\varepsilon}^t \int_{x_j-\sqrt{\varepsilon}}^{x_j+\sqrt{\varepsilon}} \big(D_{r,z}X(t, x_j)\big)^2 \xi_j^2\, dz\, dr.$$

We set

$$H_{r,z}(x,t) = \int_r^t \int_0^1 G_{t-s}(x,y)\sigma'\big(X(s,y)\big) D_{r,z}X(s,y) W(dy,ds)$$
$$+ \int_r^t \int_0^1 G_{t-s}(x,y) f'\big(X(s,y)\big) D_{r,z}X(s,y)\, dy\, ds$$
$$- \int_r^t \int_0^1 \frac{\partial G_{t-s}}{\partial y}(x,y) g'\big(X(s,y)\big) D_{r,z}X(s,y)\, dy\, ds.$$

Then $I_2(\xi) \geqslant \frac{1}{2} I_4(\xi) - I_3(\xi)$, with

$$I_3(\xi) = \sum_{j=1}^d \int_{t-\varepsilon}^t \int_{x_j-\sqrt{\varepsilon}}^{x_j+\sqrt{\varepsilon}} \big(H_{r,z}(x_j,t)\big)^2 \xi_j^2\, dz\, dr,$$

$$I_4(\xi) = \sum_{j=1}^d \int_{t-\varepsilon}^t \int_{x_j-\sqrt{\varepsilon}}^{x_j+\sqrt{\varepsilon}} G_{t-r}^2(x_j,z)\sigma^2\big(X(s,y)\big)\xi_j^2\, dz\, dr.$$

From (RH.2) and Lemma A.2, we deduce $I_4(\xi) \geqslant C\sqrt{\varepsilon}$. Hence:

$$\langle \sigma\xi, \xi\rangle \geqslant C\sqrt{\varepsilon} - \sup_{\|\xi\|=1} \left(I_1(\xi) + \frac{1}{2} I_3(\xi)\right), \tag{5.8}$$

which yields

$$\mathbb{P}\big(\langle \sigma\xi, \xi\rangle \leqslant \varepsilon^\beta\big) \leqslant \mathbb{P}\left(\sup_{\|\xi\|=1} \left(I_1(\xi) + \frac{1}{2} I_3(\xi)\right) \geqslant C\sqrt{\varepsilon} - \varepsilon^\beta\right).$$

Since we chose $\beta > 1/2$, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \leqslant \varepsilon_0$, we have $C\sqrt{\varepsilon} - \varepsilon^\beta \geqslant \varepsilon^\beta$. Therefore, using Chebyshev's inequality, we obtain

$$\mathbb{P}\big(\langle \sigma\xi, \xi\rangle \leqslant \varepsilon^\beta\big) \leqslant \frac{C_q}{\varepsilon^{\beta q}} \mathbb{E}\Big[\sup_{\|\xi\|=1} \big(|I_1(\xi)|^q + |I_3(\xi)|^q\big)\Big].$$

We then check that $\mathbb{E}(\sup_{\|\xi\|=1} |I_k(\xi)|^q) \leqslant C_q \varepsilon^q$, for $k = 1, 3$ and q large enough. Indeed (bounding ξ_j^2 by 1), we have:

$$I_1(\xi) \leqslant C\left\{\sum_{j=1}^{d}\sum_{i\neq j}\left(\int_{t-\varepsilon}^{t}\int_{x_j-\sqrt{\varepsilon}}^{x_j+\sqrt{\varepsilon}} \sigma^2(X(s,y))G_{t-r}^2(x_i,z)\,dz\,dr\right.\right.$$
$$\left.\left.+\int_{t-\varepsilon}^{t}\int_{0}^{1}\big(H_{r,z}(x_i,t)\big)^2\,dz\,dr\right)\right\} := \sum_{j=1}^{d}\sum_{i\neq j}(a_{ij}+b_{ij}),$$

and thanks to Lemma A.3, setting $l = \frac{1}{2}\min_{i\neq j}|x_i - x_j|$, we have $a_{ij} \leqslant Ce^{-l^2/2\varepsilon}$. On the other hand, we easily get:

$$\mathbb{E}|b_{ij}(\xi)|^q \leqslant C_q\left(\int_{t-\varepsilon}^{t}\int_{0}^{1}\chi_{t-s}(x_i,y)\,dy\,ds\right)^{q-1}$$
$$\times\int_{t-\varepsilon}^{t}\int_{0}^{1}\chi_{t-s}(x_i,y)\mathbb{E}\left|\int_{t-\varepsilon}^{s}\int_{0}^{1}\big(D_{r,z}X(y,s)\big)^2\,dz\,dr\right|^q dy\,ds,$$

where χ is defined as in (4.1). We then use the following lemma:

LEMMA 5.4. – *For all $q > 1$, there exists C_q such that for all $t > 0, s \geqslant 0, y \in [0,1]$:*

$$\mathbb{E}\left|\int_{t-\varepsilon}^{t}\int_{0}^{1}\big(D_{r,z}X(s,y)\big)^2\,dz\,dr\right|^q \leqslant C_q\varepsilon^{q/2}.$$

Proof of Lemma 5.4. – It is similar to that of Lemma 4.2.2 in [12]. We define:

$$H(s,y) := \mathbb{E}\left|\int_{t-\varepsilon}^{t}\int_{0}^{1}\big(D_{r,z}X(s,y)\big)^2\,dz\,dr\right|^q,$$

$$H(s) = \sup_y H(s,y), \qquad K_t(s) = \sup_{t\leqslant v\leqslant s} H(v);$$

using Proposition 5.2, we know that $K_t(s)$ is uniformly bounded with respect to ε, s and $t \in [0,T]$. On the other hand, since σ is uniformly bounded, we have:

$$H(s,y) \leqslant C_q\left\{\left(\int_{t-\varepsilon}^{t}\int_{0}^{1} G_{s-r}^2(y,z)\,dz\,dr\right)^q\right.$$

$$
\begin{aligned}
&+\mathbb{E}\left|\int_{t-\varepsilon}^{t}\int_{0}^{1}\left(\int_{r}^{s}\int_{0}^{1} G_{s-\nu}(y,u)\sigma'\big(X(\nu,u)\big)\right.\right.\\
&\qquad\left.\left.\times D_{r,z}X(\nu,u)W(d\nu,du)\right)^{2} dz\,dr\right|^{q}\\
&+\mathbb{E}\left|\int_{t-\varepsilon}^{t}\int_{0}^{1}\left(\int_{r}^{s}\int_{0}^{1} G_{s-\nu}(y,u)f'\big(X(\nu,u)\big)\right.\right.\\
&\qquad\left.\left.\times D_{r,z}X(\nu,u)\,d\nu\,du\right)^{2} dz\,dr\right|^{q}\\
&+\mathbb{E}\left|\int_{t-\varepsilon}^{t}\int_{0}^{1}\left(\int_{r}^{s}\int_{0}^{1} \frac{\partial G_{s-\nu}}{\partial u}(y,u)g'\big(X(\nu,u)\big)\right.\right.\\
&\qquad\left.\left.\left.\times D_{r,z}X(\nu,u)\,d\nu\,du\right)^{2} dz\,dr\right|^{q}\right\}\\
&:= A_1 + A_2 + A_3 + A_4.
\end{aligned}
$$

It is easy to see that $A_1 \leqslant C_q \varepsilon^{q/2}$. On the other hand, using (2.3), Hölder's inequality and standard estimates on G, we get, for $i = 2, 3$

$$
A_i \leqslant C_q\left\{\varepsilon^{q/2} + \int_{t}^{s} K_t(\nu)\,d\nu\right\}.
$$

As for A_4, using the same method and Lemma 2.1(a), we deduce

$$
\begin{aligned}
A_4 \leqslant C_q&\left\{\mathbb{E}\left|\int_{t-\varepsilon}^{t}\int_{0}^{1}\left(\int_{t-\eta}^{\nu}\int_{0}^{1}\left|\frac{\partial G_{s-\nu}}{\partial u}(y,u)\right|\big(D_{r,z}X(\nu,u)\big)^{2} dz\,dr\right)d\nu\,du\right|^{q}\right.\\
&\left.+\mathbb{E}\left|\int_{t}^{s}\int_{0}^{1}\left(\int_{t-\eta}^{t}\int_{0}^{1}\left|\frac{\partial G_{s-\nu}}{\partial u}(y,u)\right|\big(D_{r,z}X(\nu,u)\big)^{2} dz\,dr\right)d\nu\,du\right|^{q}\right\}\\
:= C_q&(A_{41} + A_{42}).
\end{aligned}
$$

Hölder's inequality with the suitable measure and Lemma 2.1(b) with $\beta = 1$ yield

A_{41}

$$\leqslant C_q \int_{t-\varepsilon}^{t}\int_0^1 \left|\frac{\partial G_{s-\nu}}{\partial u}(y,u)\right| \sup_{u,\nu}\left(\mathbb{E}\left|\int_0^T\int_0^1 \left(D_{r,z}X(\nu,u)\right)^2 dz\,dr\right|^q\right) d\nu\,du$$

$$\times\left(\int_{t-\varepsilon}^{t}\int_0^1 \left|\frac{\partial G_{s-\nu}}{\partial u}(y,u)\right| du\,d\nu\right)^{q-1} \leqslant C_p\varepsilon^{\frac{q}{2}}.$$

The upper estimate of A_{42} is proved along the same line; thus

$$A_{42}\leqslant C_q \int_t^s\int_0^1 \left|\frac{\partial G_{s-\nu}}{\partial u}(y,u)\right|\cdot K_t(\nu)\,du\,d\nu.$$

Thus

$$K_t(s)\leqslant C_p\left\{\varepsilon^{p/2}+\int_t^s\int_0^1\left(1+\left|\frac{\partial G_{s-\nu}}{\partial u}(y,u)\right|\right)K_t(\nu)\,du\,d\nu\right\}.$$

Then, since

$$\int_0^1\left|\frac{\partial G_{s-\nu}}{\partial u}(y,u)\right|du\leqslant C/\sqrt{s-\nu},$$

Gronwall's lemma yields the result.

We now conclude the proof of Proposition 5.2: Lemma 5.4 implies that $\mathbb{E}|b_{ij}(\xi)|^q\leqslant C_q\varepsilon^q$, which gives the suitable bound for I_1. For I_3, we notice that $I_3\leqslant C\sum_{j=1}^d b_{jj}$, which finally gives

$$\mathbb{P}\left(\langle\sigma\xi,\xi\rangle\leqslant\varepsilon^\beta\right)\leqslant C_q\varepsilon^{(1-\beta)q}.$$

Hence, choosing $\beta\in]1/2,1[$ and $q>3$, we obtain (5.7), and Proposition 5.2 is proved. □

5.2. Proof of Corollary 2.1

The proof is based on Proposition 2.2, due to Bouleau and Hirsch.

Assume that $(\mathbf{H_0})$ and $(\mathbf{H})$ hold. Since X_0 clearly has a continuous modification, we see, using Proposition 2.1, that the solution $X(t,x)$ of (E) admits a continuous modification in $(t,x)\in[0,T]\times[0,1]$. We then use the following localization. Set

$$\Omega_n = \Big\{\omega \in \Omega \Big| \sup_{(t,x)\in[0,T]\times[0,1]} |X(t,x,\omega)| \leqslant n\Big\},$$

then, by continuity of $X(t,x)$, we have $\Omega_n \uparrow \Omega$. We then define sequences of functions f_n, g_n in $C_b^1(\mathbb{R})$ such that $f_n(r) = f(r)$, $g_n(r) = g(r)$ on $\{|r| \leqslant n\}$, and $f_n(r) = g_n(r) = 0$ on $\{|r| \geqslant n+1\}$. Then, if $X_n(t,x)$ is the solution of **(E)** with f_n, g_n in place of f, g, we see by a uniqueness argument that $X(t,x) = X_n(t,x)$ on the set Ω_n. Now, Proposition 5.1 implies that, since the coefficients satisfy **(RH)**, the random variable $X_n(t,x)$ belongs to $\mathbb{D}^{1,p}$ for all $p > 1$. Hence we can define $DX(t,x)$ without ambiguity by $DX(t,x) = DX_n(t,x)$ on Ω_n and thus $X(t,x) \in \mathbb{D}^{1,p}_{loc}$ for all $p \in]1,+\infty[$, so $(X(t,x_1),\dots,X(t,x_d))$ satisfies part (i) of Proposition 2.2. Besides, using Proposition 5.2, it is easy to see that the covariance matrix of the random vector $(X(t,x_1),\dots,X(t,x_d))$ is almost surely invertible (again by localization with X_n). This concludes the proof of Corollary 2.1. □

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APPENDIX A

The following classical estimates on G can be found in Bally, Millet and Sanz [2] (Lemma B.1):

LEMMA A.1. – (a) *Let h be a 2β-Lipschitz function, with $\beta > 0$. Then, for all x, x', t, t':*

$$\left|\int_0^1 G_{t'}(x',y)h(y)\,dy - \int_0^1 G_t(x,y)h(y)\,dy\right| \leqslant \|h\|_{Lip2\beta}\big(|t'-t|^\beta + |x'-x|^{2\beta}\big),$$

where

$$\|h\|_{Lip2\beta} = \sup_{x\neq y}\left(\frac{|h(y)-h(x)|}{|y-x|^{2\beta}}\right).$$

(b) *For* $\beta \in]\frac{3}{2};3[$, *there exists* $C>0$ *such that for all* x, y, t *we have*:

$$\int_0^t\int_0^1 \left|G_{t-r}(x,z)-G_{t-r}(y,z)\right|^\beta dz\,dr \leqslant C|x-y|^{3-\beta}.$$

(c) *For all* $\beta \in]1;3[$ *there exists* $C>0$ *such that for all* (s,t) *with* $s\leqslant t$ *and for all* x:

$$\int_s^t\int_0^1 \left|G_{t-r}(x,y)\right|^\beta dy\,dr \leqslant C|t-s|^{\frac{3-\beta}{2}},$$

$$\int_0^s\int_0^1 \left|G_{t-r}(x,y)-G_{s-r}(x,y)\right|^\beta dy\,dr \leqslant C|t-s|^{\frac{3-\beta}{2}}.$$

The next result is proved in Bally, Gyöngy, Pardoux [1] (Lemma 3.3):

LEMMA A.2. – *For every* $\alpha \in]0,1[$, *there exists a constant* C_α *such that for all* $\eta \in]0,1[$, $x\in[\alpha,1-\alpha]$ *and* $t>\eta$:

$$\int_{t-\eta}^t\int_0^1 G^2_{t-s}(x,y)\,dy\,ds \geqslant C_\alpha\sqrt{\eta}.$$

Finally, the following estimate is proved in Bally, Pardoux [3] (inequality (A.2)):

LEMMA A.3. – *There exists a constant* C *such that for all* $t,\varepsilon>0$ *such that* $t-\varepsilon>0$, *we have*

$$\int_{t-\varepsilon}^t\int_{[x-l,x+l]^c} G^2_{t-r}(x,y)\,dy\,ds \leqslant C\mathrm{e}^{-l^2/2\varepsilon}.$$

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