ANNALES DE L'I. H. P., SECTION B

KEISHI KOMORIYA

Hydrodynamic limit for asymmetric mean zero exclusion processes with speed change

Annales de l'I. H. P., section B, tome 34, nº 6 (1998), p. 767-797

<http://www.numdam.org/item?id=AIHPB_1998__34_6_767_0>

Ann. Inst. Henri Poincaré,

Vol. 34, n° 6, 1998, p. 767-797 *Probabilités et Statistiques*

Hydrodynamic limit for asymmetric mean zero exclusion processes with speed change

by

Keishi KOMORIYA
Department of Mathematical Sciences, University of Tokyo
3-8-1 Komaba, Tokyo 153, Japan
E-mail: komoriya@ms406ss5.ms.u-tokyo.ac.jp

ABSTRACT. – We consider the hydrodynamic behavior of asymmetric mean zero exclusion processes with speed change. The model discussed in this paper is of non-gradient type and so is its associated symmetric process. We derive a nonlinear diffusion equation for the macroscopic density field obtained in the diffusive scaling limit by estimating the relative entropy with respect to the local equilibrium state of second order approximation. The estimation of the asymmetric part is carried out by using the strong sector condition. The diffusion coefficient is bigger than that of the associated symmetric process in the sense of matrix. © Elsevier, Paris

RÉSUMÉ. – Nous considérons le comportement hydrodynamique des processus d'exclusion asymétrique de moyenne nulle. Le modèle discuté dans cet article est nongradient, de même que le processus symétrique associé. Nous obtenons une équation de diffusion non linéaire pour le champ de densité macroscopique dans la limite de diffusion. La méthode repose sur l'estimation de l'entropie relative par rapport à l'état d'équilibre local de l'approximation du deuxième ordre. L'estimation de la partie asymétrique est obtenue à l'aide de la condition sectorielle forte. Le coefficient de diffusion est plus grand au sens matriciel que celui du processus symétrique associé. © Elsevier, Paris

AMS classification: 60K35

1. INTRODUCTION

We consider the asymmetric mean zero exclusion processes with speed change whose invariant probability measures are Bernoulli measures. The model we discuss in this paper is of so-called non-gradient type and so is its associated symmetric process. We consider the hydrodynamic behavior of this model and derive a nonlinear diffusion equation for the macroscopic density by passing to the hydrodynamic limit. Strong sector condition (see the condition (e) below), which we assume to control the asymmetric part, will play a key role in our discussion.

Now we describe the model. Let Γ_N be the d-dimensional periodic lattice $(\mathbf{Z}/N\mathbf{Z})^d$ whose points are represented by $x = (x_1, ..., x_d)$. The exclusion process with speed change on Γ_N is a Markov process with the state space $\mathcal{X}_N = \{0,1\}^{\Gamma_N} = \{\eta = (\eta_z)_{z\in\Gamma_N}; \eta_z \in \{0,1\}\}$ whose generator is given by

$$L_N f(\eta) = \frac{1}{2} \sum_{x,y\in\Gamma_N} c(x,y,\eta)[f(\eta^{xy}) - f(\eta)] \quad \text{for} \quad \eta \in \mathcal{X}_N, \tag{1.1}$$

where $c : \mathbf{Z}^d \times \mathbf{Z}^d \times \mathcal{X} \to [0,\infty)$ satisfies $c(x,y,\eta) = c(y,x,\eta)$ and η^{xy} is defined by

$$(\eta^{xy})_u = \begin{cases} \eta_y & (\text{ if } u = x \text{ }) \\ \eta_x & (\text{ if } u = y \text{ }) \\ \eta_u & (\text{ otherwise }) \text{ }. \end{cases}$$

Here $\mathcal{X} = \{0,1\}^{\mathbf{Z}^d}$ and $\eta \in \mathcal{X}_N$ is identified with its periodic extention to $\mathcal{X}$. The domain of L_N, denoted by $\mathcal{F}_N$, is the set of all functions on $\mathcal{X}_N$. We consider that the site x is occupied if $\eta_x = 1$ and free if $\eta_x = 0$.

We also define L by

$$Lf(\eta) = \frac{1}{2} \sum_{x,y\in\mathbf{Z}^d} c(x,y,\eta)[f(\eta^{xy}) - f(\eta)] \tag{1.2}$$

for $f \in \mathcal{F}_0$ and $\eta \in \mathcal{X}$, where $\mathcal{F}_0$ denotes the set of all local functions on $\mathcal{X}$, namely the set of all functions depending only on finite coordinates.

Throughout this paper, we assume the following conditions:

(a) Positive and local: $c(x,y,\eta) > 0$ if and only if $|x-y| = 1$ and $\eta_x \neq \eta_y$. $c(x,y,\eta)$ depends only on $\{\eta_z; |z-x| \leq r\}$ for some $r > 0$.

(b) Translation invariance: $c(x,y,\eta) = c(0,y-x,\tau_x\eta)$ for all $x,y \in \mathbf{Z}^d$ and $\eta \in \mathcal{X}$.

(c) Mean zero: $\langle \sum_{|y|=1} c(0,y,\eta)\eta_{0y}\rangle_\rho = 0$ for $0 \le \rho \le 1$.

(d) Stationarity: Bernoulli measures ν_ρ with $0 \le \rho \le 1$ are invariant measures of the process associated with L_N.

(e) Strong sector condition: There exists a constant C_s such that

$$| - \int_{\mathcal{X}} Lf \cdot g d\nu_\rho | \le C_s \{ - \int_{\mathcal{X}} Lf \cdot f d\nu_\rho \}^{1/2} \{ - \int_{\mathcal{X}} Lg \cdot g d\nu_\rho \}^{1/2} \quad (1.3)$$

for all $f, g \in \mathcal{F}_0$ and $0 \le \rho \le 1$.

(f) Smoothness of diffusion coefficient:

$$a(\rho) = \{a_{ij}(\rho)\}_{1 \le i,j \le d} \in C^2([0,1]).$$

($a(\rho)$ will be defined in Section 3.)

Here τ_x, $x \in \mathbf{Z}^d$, denote the shift operators acting on $\mathcal{X}$ defined by $(\tau_x \eta)_y = \eta_{y+x}$ and $\langle \cdot \rangle_\rho$ stands for the expectation with respect to the Bernoulli measure ν_ρ. τ_x, $x \in \Gamma_N$, act also on $\mathcal{X}_N$ by $(\tau_x\eta)_y = \eta_{y+x}$ for $y \in \Gamma_N$, with addition being modulo N. They also act on $\mathcal{F}_0$ or $\mathcal{F}_N$ by $\tau_x f(\eta) = f(\tau_x \eta)$.

In view of (a), we restrict ourselves to the nearest jumps. But the methods used in this paper are valid for more general cases. The fifth condition (e) is the key to control the asymmetric part. As for symmetric process, the condition (e) is automatically satisfied with $C_s = 1$. An asymmetric example which satisfies the conditions (a)–(e) will be given at the end of this section. We will make some remarks about the condition (f) after Theorem 1.1.

Let $\eta^N(t) = \{\eta_x^N(t), x \in \Gamma_N\}$ denote the Markov process on $\mathcal{X}_N$ governed by the generator $N^2 L_N$. Its macroscopic empirical-mass distribution is the measure-valued process defined by

$$\rho^N(t, d\theta) = N^{-d} \sum_{x \in \Gamma_N} \eta_x^N(t) \delta_{x/N}(d\theta), \quad \theta \in \mathbf{T}^d,$$

where $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$ is the d-dimensional torus identified with $[0,1)^d$ and δ_θ is the delta measure at θ.

Now we consider the following nonlinear diffusion equation:

$$\frac{\partial \rho}{\partial t}(t,\theta) = \sum_{i,j=1}^{d} \frac{\partial}{\partial \theta_i} \{ a_{ij}(\rho(t,\theta)) \frac{\partial \rho}{\partial \theta_j}(t,\theta) \}. \quad (1.4)$$

Here $a(\rho) = \{a_{ij}(\rho)\}_{1\le i,j\le d}$ is the diffusion coefficient given in the condition (f).

We state the main result of the present paper.

THEOREM 1.1. – *If the nonlinear diffusion equation (1.4) has a smooth solution $\rho(t,\theta)$ with initial data $\rho_0(\theta) \in (0,1)$ and $H_N(f_0|\bar{\psi}_0) = o(N^d)$ as $N \to \infty$, then $\rho^N(t,d\theta)$ converges in probability to $\rho(t,\theta)d\theta$ for every t. ($H_N(f_0|\bar{\psi}_0)$ is the relative entropy defined by (2.3), f_0 denotes the initial density, $\bar{\psi}_0(\eta) = Z_N^{-1}\exp\{\sum_{x\in\Gamma_N}\bar{\lambda}(\rho_0(x/N))\eta_x\}$ and $\bar{\lambda}(\rho) = \log\{\rho/(1-\rho)\}$; see Section 2 for detail.)*

Hydrodynamic behavior for exclusion processes has been studied by many authors. It is well known that the entropy approach initiated by Guo, Papanicolaou and Varadhan [3] is quite useful for analyzing the hydrodynamic behavior of microscopic systems and the method has been applied to various gradient type models. For non-gradient models, Varadhan [10] proposed an effective approach and it also has been applied to various non-gradient models. In fact, Funaki, Uchiyama and Yau [2] proved the hydrodynamic limit for symmetric non-gradient exclusion processes with speed change with the help of the arguments in [10]. Another method used in [2] is the relative entropy method proposed by Yau [13]. They modified it in order to treat the non-gradient models and introduced the local equilibrium state of second order approximation. The framework of our proof is essentially the same as that of [2]. But, since our model is asymmetric, we have to modify them and some more estimates are needed. For asymmetric models, main technical difficulty is how to control the asymmetric part. For asymmetric mean zero simple exclusion process, Xu [12] proposed "loop decomposition" method to control the asymmetric part. The method, however, depends strongly on the property of the simple exclusion. In this paper, we use the strong sector condition (see the condition (e)) instead of loop decomposition method to control the asymmetric part and we extend to general speed change processes. This condition is quite essential to treat the asymmetric part and simplify the arguments caused by the asymmetry of the process. In fact, the strong sector condition is also used for proving central limit theorems for the tagged particles (cf. [4],[8], or [11]). In addition our model is of non-gradient type and so is its associated symmetric process. It also makes the arguments complicated. For the condition (f), we can show that the diffusion coefficient is Lipschitz continuous in $(0,1)$ in the same manner as in [6]. But the smoothness of the diffusion coefficient has not been verified in any case of interest even if the model is symmetric.

In Section 2, we prove Theorem 1.1 by estimating the relative entropy. Several results in Section 2 can be shown similarly to [2], so we outline the arguments and refer to [2] for details. In Sections 3 and 4, we prove Theorem 2.1 and Lemma 2.2 whose proofs are postponed in Section 2. Theorem 2.1 is quite useful for non-gradient type models. This method, often called "gradient replacement", was proposed by Varadhan [10]. We can prove Theorem 2.1 by computing the central limit theorem variances (see Lemma 3.1). In order to compute the variances of asymmetric terms, we formulate a fundamental estimate on the variances of functions contained in a suitable function space in Section 3 (see Theorem 3.1). We also define the diffusion coefficient in Section 3. In Section 4, we prove Lemma 2.2 by using the strong sector condition. This lemma shows that, in particular, we can take a function F which does not depend on local densities in the local equilibrium state of second order approximation (see (2.5)).

To conclude this section, we explain the 2-dimensional discrete vortex model given in [4] as one of the examples which satisfy the conditions (a)–(e). We consider a discrete vortex model in terms of exclusion process as follows. Each particle moves on $\mathbf{Z}^2$ like vortex with the same vorticity. If two particles are at the neighboring sites, one particle is effected by other particle's presence and the jump rate to a special direction increases. They thus effect each other. Mathematical description of this model is given by the generator $\tilde{L}$, corresponding to (1.2),

$$\tilde{L}f(\eta) = \frac{1}{8} \sum_{x,y \in \mathbf{Z}^2} [f(\eta^{xy}) - f(\eta)] + \sum_{x \in \mathbf{Z}^2} \sum_{i=1}^{8} A_{x,a_i} f(\eta). \tag{1.5}$$

Here the first term on the right-hand side stands for the motion of the 2-dimensional simple random walk with exclusion and the second term is defined by

$$A_{x,a_i} f(\eta) = \eta_x \eta_{x+a_i} (1 - \eta_{x+a_{i+1}}) [f(\eta^{x+a_i, x+a_{i+1}}) - f(\eta)].$$

Here we label eight lattice points on the boundary of the square $[-1,1]^2$ counterclockwise $a_1, a_2, \cdots, a_8$, with $a_1 = (1,0)$. We set $a_9 = a_1$. We refer to [4] for details.

We can easily check that the discrete vortex model satisfies the conditions (a)–(d). For the condition (e), it suffices to check the strong sector condition with respect to each $\sum_{i=1}^{8} A_{x,a_i}$. But this is essentially the same as Observation 2 in [11]. We can also check that the constant C_s does not depend on ρ.

2. PROOF OF THEOREM 1.1

In this section, we prove our main theorem by computing the relative entropy with respect to the local equilibrium state of second order approximation. The computations are essentially the same as those of [2]. But we have to modify some parts since our model is asymmetric. Before proving our main theorem, we prepare some notations.

For $\Lambda \subset \Gamma_N$ or $\subset \mathbf{Z}^d$, $(\Lambda)^*$ denotes the set of all bonds $b = \{x, y\}$ inside Λ. Now we can rewrite the generator of our model as follows:

$$L_N = \sum_{b \in (\Gamma_N)^*} c(b, \eta)\pi_b,$$

where $c(b, \eta) = c(x, y, \eta)$ for $b = \{x, y\}$ and $\pi_b = \pi_{x,y}$ is the operator on $\mathcal{F}_N$ defined by

$$\pi_b f(\eta) = f(\eta^b) - f(\eta) \quad \text{for} \quad f \in \mathcal{F}_N$$

with $\eta^b = \eta^{xy}$.

In the following, we will use the adjoint operator and the symmetrization of L_N. The adjoint operator with respect to ν_ρ is given by

$$L_N^* = \frac{1}{2} \sum_{x,y \in \Gamma_N} c^*(x, y, \eta)\pi_{x,y} = \sum_{b \in (\Gamma_N)^*} c^*(b, \eta)\pi_b, \tag{2.1}$$

where $c^*(x, y, \eta) = c(x, y, \eta^{xy})$. The symmetrization of L_N with respect to ν_ρ is given by

$$L_N^s = \frac{1}{2} \sum_{x,y \in \Gamma_N} c^s(x, y, \eta)\pi_{x,y} = \sum_{b \in (\Gamma_N)^*} c^s(b, \eta)\pi_b, \tag{2.2}$$

where $c^s(x, y, \eta) = \frac{1}{2}\{c(x, y, \eta) + c^*(x, y, \eta)\}$. We also define L^* and L^s acting on $\mathcal{F}_0$ in the same manner as (1.2).

We also define the relative entropy and the local equilibrium state. Let ν^N be the uniform probability measure on $\mathcal{X}_N$. The relative entropy $H_N(f|g)$ of two probability densities f and g relative to ν^N is defined by

$$H_N(f|g) = \int_{\mathcal{X}_N} f \log(f/g) d\nu^N. \tag{2.3}$$

The local equilibrium state $\bar{\psi}_t(\eta) d\nu^N$ is defined by

$$\bar{\psi}_t(\eta) = \bar{Z}_t^{-1} \exp\{\sum_{x \in \Gamma_N} \lambda(t, x/N)\eta_x\} \quad \text{for} \quad \eta \in \mathcal{X}_N, \tag{2.4}$$

where $\bar{Z}_t$ is the normalization constant. For the present model, we take $\lambda(t,\theta) = \bar{\lambda}(\rho(t,\theta))$. Here, $\bar{\lambda}(\rho) = \log\{\rho/(1-\rho)\}$ for $\rho \in (0,1)$ and $\rho(t,\theta)$ is the smooth solution of the nonlinear diffusion equation (1.4).

We also define the local equilibrium state $\psi_t(\eta)d\nu^N$ of second order approximation by

$$\psi_t(\eta) = Z_t^{-1} \exp\Big\{ \sum_{x\in\Gamma_N} \lambda(t,x/N)\eta_x + \frac{1}{N}\sum_{x\in\Gamma_N} (\partial\lambda(t,x/N), \tau_x F(\eta))\Big\} \quad \text{for} \quad \eta \in \mathcal{X}_N. \tag{2.5}$$

Here Z_t is the normalization constant, $\partial\lambda = \{\partial_i\lambda\}_{1\le i\le d}, \partial_i = \frac{\partial}{\partial\theta_i}$, $F = (F_1,\cdots,F_d) \in \mathcal{F}_0^d$ and $(\cdot,\cdot)$ stands for the inner product of $\mathbf{R}^d$. We shall also write $\dot{\lambda} = \frac{\partial\lambda}{\partial t}$ and $\partial^2\lambda = \{\partial_i\partial_j\lambda\}_{1\le i,j\le d} = \{\partial_{ij}\lambda\}_{1\le i,j\le d}$ for $\lambda = \lambda(t,\theta)$. We remark that F, in the right-hand side of (2.5), does not depend on local densities $\rho(t,\theta)$.

Now we prove Theorem 1.1. The proof is divided into three steps.

Step 1. – First we estimate the relative entropy with respect to the local equilibrium state of second order approximation. Let $h_N(t) = N^{-d}H_N(f_t|\psi_t)$, where $f_t(\eta)$ denotes the density of the distribution of $\eta^N(t)$ on $\mathcal{X}_N$ with respect to ν^N.

Then by Lemma 3.1 in [13],

$$\frac{\partial h_N(t)}{\partial t} \le N^{-d}\int_{\mathcal{X}_N} \psi_t^{-1}(N^2L_N^*\psi_t - \frac{\partial\psi_t}{\partial t})\cdot f_t d\nu^N. \tag{2.6}$$

Now we compute the right-hand side of (2.6). We note that

$$\begin{aligned}
&N^{-d}\psi_t^{-1}N^2L_N^*\psi_t(\eta)\\
&= \frac{N^{2-d}}{2}\sum_{x,y\in\Gamma_N} c^*(x,y,\eta)[\exp\{(\lambda(t,x/N)-\lambda(t,y/N))(\eta_y-\eta_x)\\
&\quad + \frac{1}{N}\pi_{x,y}\sum_{z\in\Gamma_N}(\partial\lambda(t,z/N),\tau_z F)\} - 1]\\
&= -\frac{N^{1-d}}{2}\sum_{x,y\in\Gamma_N} c^*(x,y,\eta)\tilde{\Omega}_{x,y}(\eta) + \frac{N^{-d}}{4}\sum_{x,y\in\Gamma_N} c^*(x,y,\eta)\tilde{\Omega}^2_{x,y}(\eta)\\
&\quad - \frac{N^{-d}}{4}\sum_{x,y\in\Gamma_N} c^*(x,y,\eta)\\
&\quad \times \sum_{i,j}\partial^2_{ij}\lambda(t,x/N)(y_i-x_i)(y_j-x_j)(\eta_y-\eta_x) + o(1),
\end{aligned}$$

where

$$\tilde{\Omega}_{x,y}(\eta) = (\partial\lambda(t,x/N),(y-x)(\eta_y-\eta_x)) - \pi_{x,y}\sum_{z\in\Gamma_N}(\partial\lambda(t,z/N),\tau_z F).$$

Set

$$\Omega_{x,y}(\eta) = (\partial\lambda(t,x/N),(y-x)(\eta_y-\eta_x) - \pi_{x,y}(\sum_{z\in\Gamma_N}\tau_z F)).$$

Then we have

$$N^{-d}\psi_t^{-1}N^2L_N^*\psi_t(\eta) = \Omega_1(\eta)+\Omega_2(\eta)+o(1) \quad \text{as} \quad N\to\infty, \tag{2.7}$$

where

$$\begin{aligned}\Omega_1(\eta) &= -\frac{N^{1-d}}{2}\sum_{x,y\in\Gamma_N}c^*(x,y,\eta)\tilde{\Omega}_{x,y}(\eta)\\ &= -\frac{N^{1-d}}{2}\sum_{x,y\in\Gamma_N}c^*(x,y,\eta)(\partial\lambda(t,x/N),(y-x)(\eta_y-\eta_x))\\ &+N^{1-d}\sum_{x\in\Gamma_N}(\partial\lambda(t,x/N),L_N^*(\tau_x F)),\\ \Omega_2(\eta) &= \frac{N^{-d}}{4}\sum_{x,y\in\Gamma_N}c^*(x,y,\eta)\Omega_{x,y}^2(\eta)\\ &-\frac{N^{-d}}{4}\sum_{x,y\in\Gamma_N}c^*(x,y,\eta)\sum_{i,j}\partial_{ij}^2\lambda(t,x/N)(y_i-x_i)(y_j-x_j)(\eta_y-\eta_x).\end{aligned}$$

On the other hand,

$$\begin{aligned}N^{-d}\psi_t^{-1}\frac{\partial\psi_t}{\partial t} &= -N^{-d}Z_t^{-1}\frac{\partial Z_t}{\partial t}\\ &+N^{-d}\frac{\partial}{\partial t}\{\sum_{x\in\Gamma_N}\lambda(t,x/N)\eta_x+\frac{1}{N}\sum_{x\in\Gamma_N}(\partial\lambda(t,x/N),\tau_x F)\}\\ &= -E^{\psi_t}[N^{-d}\sum_{x\in\Gamma_N}\dot{\lambda}(t,x/N)\eta_x]\\ &+N^{-d}\sum_{x\in\Gamma_N}\dot{\lambda}(t,x/N)\eta_x+o(1).\end{aligned}\tag{2.8}$$

The law of large numbers with respect to $\psi_t d\nu$ verifies

$$\lim_{N\to\infty}E^{\psi_t}[N^{-d}\sum_{x\in\Gamma_N}\dot{\lambda}(t,x/N)\eta_x] = \int_{\mathbf{T}^d}\dot{\lambda}(t,\theta)\rho(t,\theta)d\theta. \tag{2.9}$$

Summarizing (2.6)–(2.9), we have

$$\frac{\partial h_N(t)}{\partial t} \leq E^{f_t}[\Omega_1(\eta) + \Omega_3(\eta)] + \int_{\mathbf{T}^d} \dot{\lambda}(t,\theta)\rho(t,\theta)d\theta + o(1) \quad (2.10)$$

as $N \to \infty$, where $\Omega_3(\eta) = \Omega_2(\eta) - N^{-d} \sum_{x\in\Gamma_N} \dot{\lambda}(t, x/N)\eta_x$.

Step 2. – We next estimate the right-hand side of (2.10). We first estimate Ω_3 by using so-called one-block estimate (cf. [1]).

By the translation invariance of c and the mean zero assumption (see the conditions (b) and (c) in Section 1, respectively), we have

$$\langle c^*(x,y,\eta) \sum_{i,j} \partial_{ij}^2 \lambda(t,x/N)(y_i - x_i)(y_j - x_j)(\eta_y - \eta_x)\rangle_\rho = 0 \quad (2.11)$$

for all $x, y \in \Gamma_N$ and $0 \leq \rho \leq 1$.

So by one-block estimate (cf. [1]),

$$\begin{aligned} \lim_{K\to\infty} \overline{\lim_{N\to\infty}} \int_0^t & E^{f_t}[\Omega_3(\eta) + N^{-d} \sum_{x\in\Gamma_N} \dot{\lambda}(t,x/N)\bar{\eta}_{x,K} \\ & - \frac{N^{-d}}{2} \sum_{x\in\Gamma_N} (\partial\lambda(t,x/N), \hat{c}(\bar{\eta}_{x,K}; F)\partial\lambda(t,x/N))] = 0, \end{aligned} \quad (2.12)$$

where $\bar{\eta}_{x,K} = (2K+1)^{-d} \sum_{y\in[x-K,x+K]^d\cap\mathbf{Z}^d} \eta_y$ and $\hat{c}(\rho;F)$ denotes the symmetric $d \times d$ matrix corresponding to the quadratic form

$$\begin{aligned} \big(l, \hat{c}(\rho;F)l\big) = \frac{1}{2} \sum_{|y|=1} \langle c^s(0,y,\eta)\big(l, y(\eta_y - \eta_0) \\ - \pi_{0,y}(\sum_{z\in\mathbf{Z}^d} \tau_z F)\big)^2\rangle_\rho \quad \text{for} \quad l \in \mathbf{R}^d. \end{aligned}$$

For the flux term $\Omega_1(\eta)$, the next theorem plays an important role. The proof will be given at the end of Section 3.

THEOREM 2.1. – *There exists $C > 0$ such that*

$$\begin{aligned} \overline{\lim_{K\to\infty}}\ \overline{\lim_{N\to\infty}} \int_0^T E^{f_t}[\Omega_1(\eta) + N^{1-d} \sum_{x\in\Gamma_N} \Big(a(\bar{\eta}_{x,K})\partial\lambda(t,x/N), \frac{1}{|\Lambda(K)|}\tau_x A_K(\eta)\Big) \\ - \frac{\beta N^{-d}}{2} \sum_{x\in\Gamma_N} (\partial\lambda(t,x/N), Z(\bar{\eta}_{x,K};F)\partial\lambda(t,x/N))]dt \leq \frac{C}{\beta} \end{aligned}$$

for all $\beta > 0$, *where*

$$\Lambda(K) = [-K, K]^d \cap \mathbf{Z}^d, \quad |\Lambda(K)| = (2K+1)^d,$$
$$A_K = \sum_{b=\{x,y\} \in (\Lambda(K))^*} (\eta_y - \eta_x)(y - x).$$

The $d \times d$ *matrix* $Z(\rho; F)$ *will be defined in Section 3 (see (3.25)).*

We can show in the same manner as in Lemma 3.4 in [2] that

$$\lim_{K \to \infty} \overline{\lim_{N \to \infty}} \int_0^T E^{f_t} \Big[N^{1-d} \sum_{x \in \Gamma_N} (a(\bar{\eta}_{x,K}) \partial \lambda(t, x/N), \frac{1}{|\Lambda(K)|} \tau_x A_K(\eta)) + N^{-d} \sum_{x \in \Gamma_N} \sum_{i,j} \partial_{ij}^2 \lambda(t, x/N) P_{ij}(\bar{\eta}_{x,K}) \Big] dt = 0, \quad (2.13)$$

where

$$P(\rho) = \int_0^\rho a(m) dm.$$

We remark that the condition (f) (see Section 1) is used to show (2.13) (cf. Lemma 3.4 in [2]). On the other hand, by the integration by parts,

$$- \int_{\mathbf{T}^d} \mathrm{Tr}(\partial^2 \lambda(t, \theta) P(\rho(t, \theta))) d\theta = \int_{\mathbf{T}^d} (\partial \lambda(t, \theta), \chi(\rho(t, \theta)) a(\rho(t, \theta)) \partial \lambda(t, \theta)) d\theta. \quad (2.14)$$

Here Tr denotes the trace of a matrix and $\chi(\rho)$ is the compressibility defined by

$$\chi(\rho) = \rho - \rho^2. \quad (2.15)$$

Collecting these observations, we have shown that

$$h_N(T) - h_N(0) \leq \int_0^T E^{f_t}[\Omega_4(\eta)] dt + \frac{C}{\beta} + \frac{1}{2} \sup_{\rho \in [0,1]} \{\beta \parallel Z(\rho; F) \parallel + \parallel 2\chi(\rho) a(\rho) - \hat{c}(\rho; F) \parallel\} \times \int_0^T \parallel \partial \lambda(s, \cdot) \parallel_{L^2(\mathbf{T}^d)}^2 ds + o(1), \quad (2.16)$$

as $N \to \infty$ and then $K \to \infty$, where

$$\begin{aligned}\Omega_4(\eta) = &-N^{-d} \sum_{x \in \Gamma_N} \dot{\lambda}(t, x/N)\{\bar{\eta}_{x,K} - \rho(t, x/N)\} \\ &+ N^{-d} \sum_{x \in \Gamma_N} \mathrm{Tr}(\partial^2 \lambda(t, x/N)\{P(\bar{\eta}_{x,K}) - P(\rho(t, x/N))\}) \\ &+ N^{-d} \sum_{x \in \Gamma_N} (\partial \lambda(t, x/N), \{\chi(\bar{\eta}_{x,K}) a(\bar{\eta}_{x,K}) \\ &- \chi(\rho(t, x/N)) a(\rho(t, x/N))\} \partial \lambda(t, x/N)),\end{aligned}$$

and $\|\cdot\|$ denotes the operator norm of matrix.

Step 3. Finally we estimate the first term on the right-hand side in (2.16) and complete the proof of Theorem 1.1. By the entropy inequality, we have

$$E^{f_t}[\Omega_4] \leq \frac{1}{\delta N^d} \log E^{\psi_t}[e^{\delta N^d \Omega_4}] + \frac{1}{\delta} h_N(t) \quad \text{for} \quad \delta > 0. \tag{2.17}$$

Now we recall the large deviation type estimate of local Gibbs states (cf. Theorem 3.3 in [2]). For $\lambda(\cdot) \in C^1(\mathbf{T}^d)$ and $F \in \mathcal{F}_0^d$, we define the local equilibrium state $\psi_{\lambda,F}^N(\eta) d\nu^N$ of second order approximation by

$$\begin{aligned}\psi_{\lambda,F}^N(\eta) = Z^{-1} \exp\Big\{ &\sum_{x \in \Gamma_N} \lambda(x/N) \eta_x \\ &+ \frac{1}{N} \sum_{x \in \Gamma_N} (\partial \lambda(x/N), \tau_x F(\eta))\Big\} \quad \text{for} \quad \eta \in \mathcal{X}_N,\end{aligned}$$

where $Z = Z_{\lambda,F,N}$ is the normalization constant. Then for every $G(\theta, \rho) \in C(\mathbf{T}^d \times [0,1])$,

$$\begin{aligned}&\overline{\lim_{K \to \infty}}\ \overline{\lim_{N \to \infty}} N^{-d} \log E^{\psi_{\lambda,F}^N}[\exp \tilde{G}(\eta)] \\ &\leq \sup_{\rho(\theta) \in C(\mathbf{T}^d; [0,1])} \int_{\mathbf{T}^d} \{G(\theta, \rho(\theta)) - I(\rho(\theta); \lambda(\theta))\} d\theta,\end{aligned} \tag{2.18}$$

where

$$\tilde{G}(\eta) = \sum_{\alpha} (2K+1)^d G(\alpha/N, \bar{\eta}_{\alpha,K}),$$

$$I(\rho; \lambda) = -\lambda\rho + \{\rho \log \rho + (1-\rho) \log(1-\rho)\} + \log(e^{\lambda} + 1).$$

Here we divide Γ_N into disjoint boxes of size $(2K+1)$ and α index such boxes or their center sites.

Let $h(t) = \overline{\lim_{N\to\infty}} h_N(t)$. Noting that $H_N(f_0|\bar{\psi}_0) = o(N^d)$ implies $h(0) = 0$, the next lemma follows from (2.16)–(2.18).

LEMMA 2.1. – *Under the assumptions in Theorem 1.1, there exist* $\delta_0, C > 0$ *such that for* $0 < \delta < \delta_0$ *and* $\beta > 0$,

$$\begin{aligned} h(t) &\leq \frac{1}{\delta}\int_0^t g(s)ds + \frac{1}{\delta}\int_0^t h(s)ds + \frac{C}{\beta} \\ &+ \frac{1}{2}\sup_{\rho\in[0,1]}\{\beta \parallel Z(\rho;F) \parallel + \parallel 2\chi(\rho)a(\rho) - \hat{c}(\rho;F) \parallel\} \\ &\times \int_0^t \parallel \partial\lambda(s,\cdot) \parallel^2_{L^2(\mathbf{T}^d)} ds, \end{aligned} \tag{2.19}$$

where

$$g(t) \equiv g_\delta(t) = \sup_{u(\theta)\in C(\mathbf{T}^d;[0,1])} \int_{\mathbf{T}^d} \{\delta \cdot \sigma(u(\theta);t,\theta) - I(u(\theta);\lambda(t,\theta))\}d\theta,$$

$$\begin{aligned} \sigma(u;t,\theta) &= -\dot{\lambda}(t,\theta)\{u - \rho(t,\theta)\} + \mathrm{Tr}(\partial^2\lambda(t,\theta)\{P(u) - P(\rho(t,\theta))\}) \\ &+ (\partial\lambda(t,\theta), \{\chi(u)a(u) - \chi(\rho(t,\theta))a(\rho(t,\theta))\}\partial\lambda(t,\theta)). \end{aligned}$$

We will prove the next lemma in Section 4.

LEMMA 2.2.

$$\inf_{F\in\mathcal{F}_0^d}\sup_{\rho\in[0,1]}\{\beta \parallel Z(\rho;F) \parallel + \parallel 2\chi(\rho)a(\rho) - \hat{c}(\rho;F) \parallel\} = 0 \quad \textit{for} \quad \beta > 0$$

We can show that $g_\delta(t) \leq 0$ by choosing $\delta > 0$ suitably (cf. Corollary 2.1 in [2]). So the next corollary is deduced from Lemmas 2.1 and 2.2 with the help of Gronwall's inequality.

COROLLARY 2.1. – *There exists a function* $F \in \mathcal{F}_0^d$ *such that for every* $\varepsilon > 0$, $0 \leq h(t) \leq \varepsilon$ *for* $t \in [0,T]$.

Now we return to the proof of Theorem 1.1. For $J \in C^\infty(\mathbf{T}^d)$, $\delta > 0$ and $t > 0$, set

$$\mathcal{A} = \{\eta \in \mathcal{X}_N; |N^{-d}\sum_{x\in\Gamma_N} J(x/N)\eta_x - \int_{\mathbf{T}^d} J(\theta)\rho(t,\theta)d\theta| > \delta\}.$$

By the large deviation estimate on $\psi_t d\nu^N$, we have

$$\overline{\lim_{N\to\infty}} N^{-d}\log P^{\psi_t}(\mathcal{A}) = -C(\delta) < 0.$$

On the other hand, by the entropy inequality (cf. [3]),

$$P^{f_t}(\mathcal{A}) \le \frac{\log 2 + H_N(f_t|\psi_t)}{\log\{1 + 1/P^{\psi_t}(\mathcal{A})\}}.$$

So by Corollary 2.1, we have $\lim_{N\to\infty} P^{f_t}(\mathcal{A}) = 0$, completing the proof of Theorem 1.1. □

3. THE DIFFUSION COEFFICIENT

In this section, we prove Theorem 2.1 whose proof was postponed in the previous section. The proof will be given at the end of this section. We use the method called "gradient replacement" proposed by Varadhan [10] and compute the central limit theorem variances. To compute those of asymmetric process, we prepare some notations and lemmas. The most part of the arguments in this section rely on the strong sector condition (e) (see Section 1). We also define the diffusion coefficient $a(\rho)$ in this section.

Let $L^s_{\Lambda(K),\zeta}$ denote the operator acting on functions $f = f(\xi)$ on $\mathcal{X}_{\Lambda(K)} = \{0,1\}^{\Lambda(K)}$ by

$$L^s_{\Lambda(K),\zeta} f(\xi) = \sum_{b\in(\Lambda(K))^*} c^s(b, \xi\cdot\zeta)\pi_b f(\xi).$$

Here $\eta \in \mathcal{X}$ is decomposed into $\eta = \xi\cdot\zeta$ where $\xi = \eta|_{\Lambda(K)} \in \mathcal{X}_{\Lambda(K)}$ and $\zeta = \eta|_{\Lambda(K)^c} \in \mathcal{X}_{\Lambda(K)^c} = \{0,1\}^{\Lambda(K)^c}$. We define the linear space of local functions $\mathcal{G}$ by

$$\mathcal{G} = \{g | g \in \mathcal{F}_0, \langle g\rangle_{\Lambda(s(g)),m} = 0 \ \text{ for } {}^\forall m \in \{0, 1, \cdots, |\Lambda(s(g))|\}\}.$$

Here $\langle\cdot\rangle_{\Lambda(K),m}$ stands for the expectation with respect to the uniform probability measure on $\mathcal{X}_{\Lambda(K),m} = \{\xi \in \mathcal{X}_{\Lambda(K)}; \sum_{x\in\Lambda(K)} \xi_x = m\}$ and $s(g)$ denotes the size of the support of g, namely

$$s(g) = \min\{K \in \mathbf{N}; g \in \mathcal{F}_{\Lambda(K)}\}.$$

Here $\mathcal{F}_{\Lambda(K)}$ denotes the set of all $\sigma\{\eta_x : x \in \Lambda(K)\}$-measurable functions on $\mathcal{X}$. We remark that if $g \in \mathcal{G}$, then $\langle g\rangle_{\Lambda(K),m} = 0$ for all $K \ge s(g)$ and $m \in \{0, 1, \cdots, |\Lambda(K)|\}$. We also remark that if $g \in \mathcal{G}$, then $\langle g\rangle_\rho = 0$ for $0 \le \rho \le 1$. For $g, h \in \mathcal{G} \cap \mathcal{F}_{\Lambda(K)}$, we define

$$\triangle_{K,m,\zeta}(g,h) = -\langle g(L^s_{\Lambda(K),\zeta})^{-1} h\rangle_{\Lambda(K),m},$$
$$\triangle_{K,m,\zeta}(g) = \triangle_{K,m,\zeta}(g,g).$$

The next lemma is the key to prove Theorem 2.1. It shows that Theorem 2.1 is proved by computing the central limit theorem variances of functions in $\mathcal{G}$.

LEMMA 3.1. – *Let* $J(t,\theta) = \{J_i(t,\theta)\}_{i=1}^d \in C^\infty([0,T] \times \mathbf{T}^d, \mathbf{R}^d), G(\eta) = \{G_i(\eta)\}_{i=1}^d \in \mathcal{F}^d_{\Lambda(K)} \cap \mathcal{G}^d$ *and* $M(\rho) = \{M_{ij}(\rho)\}_{1\le i,j\le d} \in C([0,1], \mathbf{R}^d \otimes \mathbf{R}^d)$. *Then there exists* $C > 0$ *such that for every* $\beta > 0$,

$$\begin{aligned}\overline{\lim_{N\to\infty}} \int_0^T & E^{f_t}[N^{1-d} \sum_{x\in\Gamma_N} \big(J(t,x/N), \tau_x G(\eta)\big) \\ & - \beta N^{-d} \sum_{x\in\Gamma_N} \big(J(t,x/N), M(\bar\eta_{x,K}) J(t,x/N)\big)]dt \\ & \le \beta T \sup_{|l|\le\|J\|_\infty} \sup_{m,\zeta} [d|\Lambda(K)| \langle (l,G), (-L^s_{\Lambda(K),\zeta})^{-1}(l,G)\rangle_{\Lambda(K),m} \\ & - \big(l, M(m/|\Lambda(K)|)l\big)] + \frac{C}{\beta}.\end{aligned}$$

Proof. – The proof is based on Lemma 6.1 in [2]. Put

$$\begin{aligned}K_{N,t}(\eta) = N & \sum_{x\in\Gamma_N} \big(J(t,x/N), \tau_x G(\eta)\big) \\ & - \beta \sum_{x\in\Gamma_N} \big(J(t,x/N), M(\bar\eta_{x,K}) J(t,x/N)\big).\end{aligned}$$

By the entropy inequality,

$$\int_0^T E^{f_t}[N^{-d} K_{N,t}(\eta)]dt \le \frac{1}{\beta} N^{-d} \log E^{\nu^N}[e^{\int_0^T \beta K_{N,t}(\eta_t)dt}] + \frac{C}{\beta},$$

where ν^N is the uniform probability measure on $\mathcal{X}_N$. Set

$$u_t(\eta) = E^\eta[e^{\int_0^t \beta K_{N,s}(\eta_s)ds}].$$

Here E^η stands for the expectation with respect to the probability measure on the path space corresponding to $\eta^N(t)$ starting from η. By Kac formula we have

$$\begin{aligned}\frac{d}{dt} E^{\nu^N}[u_t^2] &= 2E^{\nu^N}[u_t \cdot (N^2 L_N^* + \beta K_{N,t}) u_t] \\ &= 2E^{\nu^N}[u_t \cdot (N^2 L_N^s + \beta K_{N,t}) u_t] \\ &\le 2E^{\nu^N}[u_t]^2 \cdot \Omega_{N,\beta}(t),\end{aligned}$$

where $\Omega_{N,\beta}(t)$ stands for the largest eigenvalue of the symmetric operator $N^2 L_N^s + \beta K_{N,t}$. So we have

$$\int_0^T E^{f_t}[N^{-d} K_{N,t}(\eta)]dt \le \frac{1}{\beta} N^{-d} \int_0^T \Omega_{N,\beta}(t)dt + \frac{C}{\beta}.$$

The rest can be shown in the same manner as in Lemma 6.1 in [2]. □

Now we compute the central limit theorem variance of $g \in \mathcal{G}$. The next lemma, the integration by parts formula for functions in $\mathcal{G}$, is useful in computing the variances. It is essentially the same as Lemma 4.1 in [6] and therefore the proof is omitted.

LEMMA 3.2. – *For $g \in \mathcal{G}$, $K \ge s(g)$ and for x, i such that $\{x, x+e_i\} \in (\Lambda(K))^*$, set*

$$\Psi_{K,x,i}(g) = \frac{1}{2} \sum_{y: y \in \Lambda(s(g)), x-y \in \Lambda(K-s(g))} \tau_{-y} \pi_{y,y+e_i} (-\bar{L}^s_{s(g)})^{-1} g,$$

where $\bar{L}^s_{s(g)} = \sum_{b \in (\Lambda(s(g)))^*} \pi_b$. *Then*

$$\sum_{x \in \Lambda(K-s(g))} \langle \tau_x g \cdot u \rangle_{\Lambda(K),m} = \sum_{i=1}^d \sum_{x; \{x,x+e_i\} \in (\Lambda(K))^*} \langle \tau_x \Psi_{K,x,i}(g) \cdot \pi_{x,x+e_i} u \rangle_{\Lambda(K),m}.$$

We consider the following $\mathbf{R}^d$-valued functions $\Phi(\eta)$, $W^s(\eta)$, and $W^*(\eta)$:

$$\Phi(\eta) = \{\Phi_i(\eta)\}_{i=1}^d = \{\eta_{e_i} - \eta_0\}_{i=1}^d,$$
$$W^s(\eta) = \{W_i^s(\eta)\}_{i=1}^d = \{c^s(0, e_i, \eta)(\eta_{e_i} - \eta_0)\}_{i=1}^d,$$
$$W^*(\eta) = \{W_i^*(\eta)\}_{i=1}^d = \{c^*(0, e_i, \eta)(\eta_{e_i} - \eta_0)\}_{i=1}^d,$$

where $e_i \in \mathbf{Z}^d$ denotes the unit vector to the i-th direction. We remark that all these functions belong to $\mathcal{G}^d$.

For $g \in \mathcal{G}$, set

$$V_{K,m,\zeta}(g) = \frac{1}{|\Lambda(K)|} \triangle_{K,m,\zeta}\Big(\sum_{x \in \Lambda(K-s(g))} \tau_x g\Big), \tag{3.1}$$

and

$$V_\rho(g) = \sup_{l\in\mathbf{R}^d, f\in\mathcal{F}_0} \frac{\{\langle g \sum_{x\in\mathbf{Z}^d} (l,x)\eta_x\rangle_\rho + \langle g \sum_{x\in\mathbf{Z}^d} \tau_x f\rangle_\rho\}^2}{\frac{1}{2}\sum_{i=1}^d \langle c^s(0,e_i,\eta)\big(l_i(\eta_{e_i}-\eta_0) - \pi_{0,e_i}(\sum_{x\in\mathbf{Z}^d}\tau_x f)\big)^2\rangle_\rho}. \tag{3.2}$$

We remark that the numerator of $V_\rho(g)$ is well defined for $g \in \mathcal{G}$.

The next theorem essentially follows from the arguments in [10]. We remark that the computations of the central limit theorem variances depend only on the symmetric part of the underlying dynamics.

THEOREM 3.1. – *For $g \in \mathcal{G}$,*

$$\lim_{K,m\to\infty, \frac{m}{|\Lambda(K)|}\to\rho} V_{K,m,\zeta}(g) = V_\rho(g)$$

uniformly in $\rho \in [0,1]$ and $\zeta \in \mathcal{X}$.

Proof. – We first prove the lower bound. By Schwarz inequality,

$$\begin{aligned}
&\triangle_{K,m,\zeta}\Big(\sum_{x\in\Lambda(K-s(g))}\tau_x g,\ -L^s_{\Lambda(K),\zeta}\big(\sum_{x\in\Lambda(K)}(l,x)\xi_x \\
&+ \sum_{x\in\Lambda(K-s(L^s f))}\tau_x f\big)\Big) \\
&\le \sqrt{\triangle_{K,m,\zeta}\Big(\sum_{x\in\Lambda(K-s(g))}\tau_x g\Big)} \\
&\sqrt{\triangle_{K,m,\zeta}\Big(-L^s_{\Lambda(K),\zeta}\big(\sum_{x\in\Lambda(K)}(l,x)\xi_x + \sum_{x\in\Lambda(K-s(L^s f))}\tau_x f\big)\Big)}.
\end{aligned}$$

We note that

$$\begin{aligned}
&\lim_{K,m\to\infty, \frac{m}{|\Lambda(K)|}\to\rho} \frac{1}{|\Lambda(K)|}\triangle_{K,m,\zeta}\Big(-L^s_{\Lambda(K),\zeta}\big(\sum_{x\in\Lambda(K)}(l,x)\xi_x \\
&+ \sum_{x\in\Lambda(K-s(L^s f))}\tau_x f\big)\Big) \\
&= \frac{1}{2}\sum_{i=1}^d \langle c^s(0,e_i,\eta)\big(l_i(\eta_{e_i}-\eta_0) - \pi_{0,e_i}(\sum_{x\in\mathbf{Z}^d}\tau_x f)\big)^2\rangle_\rho,
\end{aligned}$$

and

$$\lim_{K,m\to\infty,\frac{m}{|\Lambda(K)|}\to\rho} \frac{1}{|\Lambda(K)|}\triangle_{K,m,\zeta}(\sum_{x\in\Lambda(K-s(g))} \tau_x g,$$
$$- L^s_{\Lambda(K),\zeta}(\sum_{x\in\Lambda(K)} (l,x)\xi_x + \sum \tau_x f))$$
$$= \langle g \sum_{x\in\mathbf{Z}^d} (l,x)\eta_x\rangle_\rho + \langle g \sum_{x\in\mathbf{Z}^d} \tau_x f\rangle_\rho$$

uniformly in $\rho \in [0,1]$ and $\zeta \in \mathcal{X}$ (cf. [2]). So we have

$$\liminf_{K,m\to\infty,\frac{m}{|\Lambda(K)|}\to\rho} V_{K,m,\zeta}(g) \geq V_\rho(g). \tag{3.3}$$

For the proof of upper bound, assume

$$v < \limsup_{K,m\to\infty,\frac{m}{|\Lambda(K)|}\to\rho} V_{K,m,\zeta}(g). \tag{3.4}$$

Set

$$\tilde{u}_{K,\zeta}(\xi) = -(L^s_{\Lambda(K),\zeta})^{-1}(\sum_{x\in\Lambda(K-s(g))} \tau_x g) \quad \text{for} \quad \xi \in \mathcal{X}_{\Lambda(K)}.$$

Then we have

$$V_{K,m,\zeta}(g) = \frac{1}{|\Lambda(K)|}\langle \sum_{x\in\Lambda(K-s(g))} \tau_x g, \tilde{u}_{K,\zeta}\rangle_{\Lambda(K),m}$$
$$= \frac{1}{2|\Lambda(K)|} \sum_{b\in(\Lambda(K))^*} \langle c^s_b(\xi\cdot\zeta)(\pi_b \tilde{u}_{K,\zeta})^2\rangle_{\Lambda(K),m}.$$

By using this relation, for $K \geq s(g)$, one can arrange $\tilde{u}_{K,\zeta}$ and find a function $u_K \in \mathcal{F}_{\Lambda(K)}$ such that

$$\frac{1}{|\Lambda(K)|}\langle \sum_{x\in\Lambda(K-s(g))} \tau_x g \cdot u_K\rangle_\rho = 1,$$
$$\frac{1}{2|\Lambda(K)|} \sum_{b\in(\Lambda(K))^*} \langle c^s(b,\eta)(\pi_b u_K)^2\rangle_\rho \leq \frac{1}{v} + o(1),$$
$$\frac{1}{|\Lambda(K)|} \sum_{b\in(\Lambda(K)\setminus\Lambda(K-s(g)))^*} \langle(\pi_b u_K)^2\rangle_\rho = o(1) \tag{3.5}$$

(cf. [5] or [10]). Here $o(1)$ terms come from the contributions near the boundary and one can arrange that they are negligible. Set $\tilde{\Psi}_i(g) = \frac{1}{2} \sum_{y: y\in\Lambda(s(g))} \tau_{-y}\pi_{y,y+e_i}(-\bar{L}^s_{s(g)})^{-1} g$. By using Lemma 3.2, we have

$$\frac{1}{|\Lambda(K)|}\langle \sum_{x\in\Lambda(K)} \tau_x g \cdot u_K\rangle_\rho = \frac{1}{|\Lambda(K)|}\sum_{i=1}^{d}\sum_{x}\langle \tau_x \Psi_{K,x,i}(g)\cdot \pi_{x,x+e_i}u_K\rangle_\rho$$
$$= \sum_{i=1}^{d}\langle \tilde{\Psi}_i(g)\cdot \phi_i^K\rangle_\rho + o(1),$$

where

$$\phi_i^K = \frac{1}{|\Lambda(K)|}\sum_{x\in\Lambda(K)} \tau_{-x}\pi_{x,x+e_i}u_K = \pi_{0,e_i}\Big(\frac{1}{|\Lambda(K)|}\sum_{x\in\Lambda(K)} \tau_x u_K\Big).$$

We easily see that $\{\phi_i^K\}_{K\in\mathbf{N}}$ is bounded in $L^2(\nu_\rho)$. So it has a subsequence converging weakly to $\phi_i \in L^2(\nu_\rho)$. Letting $K\to\infty$, we have

$$\sum_{i=1}^{d}\langle \tilde{\Psi}_i(g)\cdot\phi_i\rangle_\rho = 1,$$
$$\frac{1}{2}\sum_{i=1}^{d}\langle c^s(0,e_i,\eta)(\phi_i)^2\rangle_\rho \le \frac{1}{v}. \tag{3.6}$$

From the way of construction, $\phi = \{\phi\}_{i=1}^d$ is the germ of a closed form (see Section 4 in [2]). So ϕ_i is well approximated by functions of the form $l_i(\eta_{e_i} - \eta_0) + \pi_{0,e_i}\sum_x \tau_x f$. However, since

$$\sum_{i=1}^{d}\langle\tilde{\Psi}_i(g)\cdot\{l_i(\eta_{e_i}-\eta_0)+\pi_{0,e_i}\sum_x \tau_x f\}\rangle_\rho = \langle g\sum_{x\in\mathbf{Z}^d}(l,x)\eta_x\rangle_\rho + \langle g\sum_{x\in\mathbf{Z}^d}\tau_x f\rangle_\rho, \tag{3.7}$$

we have $v \le V_\rho(g)$, which combined with (3.4) yields the upper bound. (The uniformity of the convergence automatically follows from the manner the limit is taken.) □

For $g, h\in\mathcal{G}$, we define the inner product $\ll g,h\gg_\rho$ by

$$\ll g,h\gg_\rho = \frac{1}{4}[V_\rho(g+h) - V_\rho(g-h)].$$

Set $\| g \|_\rho = \ll g,g\gg_\rho^{1/2}$. Let $\sim$ be the equivalence relation such that $g\sim h$ if and only if $\| g-h\|_\rho = 0$. The completion of $\mathcal{G}/\sim$ with the inner product $\ll \cdot, \cdot \gg_\rho$ is denoted by $\mathcal{H}$, where $\mathcal{G}/\sim$ is the quotient space. In the following, $g \in \mathcal{G}$ is identified with the element of $\mathcal{H}$.

Let us introduce four subspaces of $\mathcal{H}$:

$$L^s\mathcal{F}_0 = \{L^s f; f \in \mathcal{F}_0\}, \quad L^*\mathcal{F}_0 = \{L^* f; f \in \mathcal{F}_0\},$$
$$\mathcal{G}_c^s = \{(l, W^s(\eta)); l \in \mathbf{R}^d\}, \quad \mathcal{G}_g = \{(l, \Phi(\eta)); l \in \mathbf{R}^d\}.$$

The subscripts of $\mathcal{G}_c^s$ and $\mathcal{G}_g$ indicate "current" and "gradient", respectively.

The next lemma is easily obtained from the definition of the inner product of $\mathcal{H}$.

LEMMA 3.3. – *For $f \in \mathcal{F}_0$, $g \in \mathcal{G}$ and $l \in \mathbf{R}^d$,*

$$\ll L^s f, g \gg_\rho = -\langle g \sum_{x\in\mathbf{Z}^d} \tau_x f\rangle_\rho, \tag{3.8}$$

$$\ll (l, W^s(\eta)), g \gg_\rho = \langle g \sum_{x\in\mathbf{Z}^d} (l, x)\eta_x\rangle_\rho. \tag{3.9}$$

Proof. – We first note that

$$\begin{aligned}
&\frac{1}{4}\{V_{K,m,\zeta}(L^s f + g) - V_{K,m,\zeta}(L^s f - g)\}\\
&= \frac{1}{|\Lambda(K)|}\langle \sum_{x\in\Lambda(K-s(L^s f+g))} \tau_x(L^s f)\\
&\quad \times (-L^s_{\Lambda(K),\zeta})^{-1} \sum_{x\in\Lambda(K-s(L^s f+g))} \tau_x g\rangle_{\Lambda(K),m}\\
&= \frac{1}{|\Lambda(K)|}\langle L^s_{\Lambda(K),\zeta}(\sum_{x\in\Lambda(K-s(L^s f+g))} \tau_x f)\\
&\quad \times (-L^s_{\Lambda(K),\zeta})^{-1} \sum_{x\in\Lambda(K-s(L^s f+g))} \tau_x g\rangle_{\Lambda(K),m}\\
&= -\frac{1}{|\Lambda(K)|}\langle \sum_{x\in\Lambda(K-s(L^s f+g))} \tau_x f \cdot \sum_{x\in\Lambda(K-s(L^s f+g))} \tau_x g\rangle_{\Lambda(K),m}.
\end{aligned}$$

Since $g \in \mathcal{G}$, letting $K, m \to \infty$, $\frac{m}{|\Lambda(K)|} \to \rho$, we have (3.8).

For (3.9),

$$\begin{aligned}
&\frac{1}{4}\{V_{K,m,\zeta}((l, W^s(\eta)) + g) - V_{K,m,\zeta}((l, W^s(\eta)) - g)\}\\
&= \frac{1}{|\Lambda(K)|}\langle \sum_{x\in\Lambda(K-s((l,W^s(\eta))+g))} \tau_x(l, W^s(\eta))\\
&\quad \times (-L^s_{\Lambda(K),\zeta})^{-1} \sum_{x\in\Lambda(K-s((l,W^s(\eta))+g))} \tau_x g\rangle_{\Lambda(K),m}
\end{aligned}$$

Here we can replace $\sum_{x \in \Lambda(K - s((l, W^s(\eta)) + g))} \tau_x(l, W^s(\eta))$ by $-L^s_{\Lambda(K),\zeta}(\sum_{x \in \Lambda(K)} (l, x)\xi_x)$, since the contribution near the boundary goes to zero as $K, m \to \infty$ and $\frac{m}{|\Lambda(K)|} \to \rho$. So we have (3.9).

The following relations, which we will use later, can be shown in the same manner as in Lemma 3.3. These relations are valid for every $1 \leq i \leq d$, $l \in \mathbf{R}^d$ and $f \in \mathcal{F}_0$.

$$\ll \Phi_i(\eta), W^s_j(\eta) \gg_\rho = \delta_{ij}\chi(\rho), \tag{3.10}$$

$$\ll (l, \Phi(\eta)), L^s f \gg_\rho = 0, \tag{3.11}$$

$$\begin{aligned} \| (l, W^s(\eta)) \|^2_\rho &= \ll (l, W^*(\eta)), (l, W^s(\eta)) \gg_\rho \\ &= \frac{1}{2}\sum_{i=1}^{d} \langle c(0, e_i, \eta)(l_i(\eta_{e_i} - \eta_0))^2 \rangle_\rho. \end{aligned} \tag{3.12}$$

Here δ_{ij} stands for the delta of Kronecker. Note that $\int L^* f \cdot f d\nu_\rho = \int L^s f \cdot f d\nu_\rho$. This implies

$$\begin{aligned} \| L^s f \|^2_\rho &= \ll L^* f, L^s f \gg_\rho \\ &= \frac{1}{2}\sum_{i=1}^{d} \langle c(0, e_i, \eta)(\pi_{0,e_i}(\sum_{x \in \mathbf{Z}^d} \tau_x f))^2 \rangle_\rho, \end{aligned} \tag{3.13}$$

$$\| L^s f \|_\rho \leq \| L^* f \|_\rho . \tag{3.14}$$

COROLLARY 3.1. – $\mathcal{H} = \overline{\mathcal{G}^s_c + L^s \mathcal{F}_0}$.

Proof. – Let Pr denote the orthogonal projection to $\overline{\mathcal{G}^s_c + L^s \mathcal{F}_0}$. Then by Theorem 3.1 and Lemma 3.3,

$$\begin{aligned} \| g \|_\rho = \{V_\rho(g)\}^{1/2} &= \sup_{l \in \mathbf{R}^d, f \in \mathcal{F}_0} \frac{\ll g, (l, W^s(\eta)) - L^s f \gg_\rho}{\| (l, W^s(\eta)) - L^s f \|_\rho} \\ &= \| \operatorname{Pr}(g) \|_\rho \quad \text{for all} \quad g \in \mathcal{G}. \end{aligned}$$

This completes the proof.

LEMMA 3.4. – $\mathcal{H} = \overline{\mathcal{G}_g + L^* \mathcal{F}_0}$.

Proof. – The assertion obviously holds for $\rho = 0, 1$. So suppose $0 < \rho < 1$. First we prove $\mathcal{H} = \overline{\mathcal{G}_g + L^s \mathcal{F}_0}$. Note that $\mathcal{G}_g \perp \overline{L^s \mathcal{F}_0}$ (see (3.11)). So by corollary 3.1, if $\mathcal{H} \neq \overline{\mathcal{G}_g + L^s \mathcal{F}_0}$ then there exists $\alpha \in \mathbf{R}^d \setminus \{0\}$ such that

$$\ll (\alpha, W^s(\eta)), (\beta, \Phi(\eta)) \gg_\rho = 0$$

for all $\beta \in \mathbf{R}^d$. Hence by taking $\beta = \alpha$, we have

$$\ll \big(\alpha, W^s(\eta)\big), \big(\alpha, \Phi(\eta)\big) \gg_\rho = 0.$$

By (3.10),

$$\ll \big(\alpha, W^s(\eta)\big), \big(\alpha, \Phi(\eta)\big) \gg_\rho = \sum_{i=1}^{d} \alpha_i^2 \cdot \chi(\rho).$$

Combining last two equalities yields $\alpha = 0$. Hence we have $\mathcal{H} = \overline{\mathcal{G}_g + L^s\mathcal{F}_0}$.

We next show $\overline{\mathcal{G}_g + L^s\mathcal{F}_0} = \overline{\mathcal{G}_g + L^*\mathcal{F}_0}$. This holds if we check that the orthogonal projection from $\overline{L^*\mathcal{F}_0}$ to $\overline{L^s\mathcal{F}_0}$ is onto. Here we have to modify above argument since the dimension of $\overline{L^s\mathcal{F}_0}$ is infinite (This fact was pointed out in [7].). Suppose that there exists an element $g \in \overline{L^s\mathcal{F}_0}$ such that $\ll g, h \gg_\rho = 0$ for all $h \in \overline{L^*\mathcal{F}_0}$. By the definition of $\overline{L^s\mathcal{F}_0}$, there exists a local function f_ε such that

$$\| g - L^s f_\varepsilon \|_\rho \leq \varepsilon \tag{3.15}$$

for $\varepsilon > 0$. By taking $h = L^* f_\varepsilon$ and by (3.13),

$$\begin{aligned} \ll g, L^* f_\varepsilon \gg_\rho &= \ll (g - L^s f_\varepsilon) + L^s f_\varepsilon, L^* f_\varepsilon \gg_\rho \\ &= \ll (g - L^s f_\varepsilon), L^* f_\varepsilon \gg_\rho + \| L^s f_\varepsilon \|_\rho^2 \\ &= 0. \end{aligned} \tag{3.16}$$

In Lemma 3.5, we will prove

$$\| L^* f_\varepsilon \|_\rho \leq C_s \| L^s f_\varepsilon \|_\rho .$$

Combining this with (3.15), we obtain

$$| \ll (g - L^s f_\varepsilon), L^* f_\varepsilon \gg_\rho | \leq C_s \varepsilon \| L^s f_\varepsilon \|_\rho . \tag{3.17}$$

Combining (3.16) and (3.17) yields

$$\| L^s f_\varepsilon \|_\rho \leq C_s \varepsilon. \tag{3.18}$$

Consequently, by (3.15), (3.18) and the triangle inequality, we have

$$\begin{aligned} \| g \|_\rho &\leq \| g - L^s f_\varepsilon \|_\rho + \| L^s f_\varepsilon \|_\rho \\ &\leq \varepsilon + C_s \varepsilon = (C_s + 1)\varepsilon. \end{aligned} \tag{3.19}$$

Since ε is arbitrary, $\| g \|_\rho = 0$ and this proves the onto-property of the projection. □

Now we are at the position to define the diffusion coefficient $a(\rho)$ of the nonlinear diffusion equation (1.4) for the limiting macroscopic density field. Since $W_i^* \in \mathcal{H}$ for $1 \leq i \leq d$, from Lemma 3.4, there exists a matrix $a(\rho) = \{a_{ij}(\rho)\}_{1 \leq i,j \leq d}$ such that

$$\sum_{j=1}^{d} a_{ij}(\rho)\Phi_j(\eta) - W_i^*(\eta) \in \overline{L^* \mathcal{F}_0} \tag{3.20}$$

for $0 \leq \rho \leq 1$. We next show the uniqueness of $a(\rho)$. We first remark that

$$\| \big(l, \Phi(\eta)\big) \|_\rho = 0 \ \text{ if and only if } \ l = 0 \tag{3.21}$$

for $0 < \rho < 1$ (cf. Theorem 5.1 in [2]).

We will use the next lemma not only for showing the uniqueness of $a(\rho)$, but also for proving Lemma 2.2 (see Section 4). It gives a bound to control the asymmetric part in terms of the symmetric part.

LEMMA 3.5. – *For $f \in \mathcal{F}_0$,*

$$\| L^* f \|_\rho \leq C_s \| L^s f \|_\rho \, . \tag{3.22}$$

Here the constant C_s is given in the condition (e) in Section 1.

Proof. – (3.22) obviously holds for $\rho = 0, 1$. So suppose $0 < \rho < 1$. We first remark that if $\| L^s f \|_\rho = 0$ then $\| L^* f \|_\rho = 0$, since $\| L^s f \|_\rho = 0$ if and only if f =const. by (3.13). So we also suppose $\| L^* f \|_\rho \neq 0$. Then as in (3.5), for each $K \in \mathbf{N}$, we can find a function $u_K \in \mathcal{F}_{\Lambda(K+s(L^*f))}$ such that

$$\frac{1}{|\Lambda(K)|}\langle L^*(\sum_{x\in\Lambda(K)} \tau_x f) u_K\rangle_\rho = 1,$$
$$\frac{1}{2|\Lambda(K)|} \sum_{b\in(\Lambda(K))^*} \langle c^s(b,\eta)(\pi_b u_K)^2\rangle_\rho \leq \frac{1}{\| L^* f \|_\rho^2} + o(1),$$
$$\frac{1}{|\Lambda(K)|} \sum_{b\in(\Lambda(K+s(L^*f))\setminus\Lambda(K))^*} \langle(\pi_b u_K)^2\rangle_\rho = o(1). \tag{3.23}$$

By the strong sector condition and the third relation in (3.23),

$$\frac{1}{|\Lambda(K)|}\langle L^*(\sum_{x\in\Lambda(K)}\tau_x f)u_K\rangle_\rho$$
$$\leq C_s\{-\frac{1}{|\Lambda(K)|}\langle L^s(\sum_{x\in\Lambda(K)}\tau_x f)$$
$$\cdot(\sum_{x\in\Lambda(K)}\tau_x f)\rangle_\rho\}^{\frac{1}{2}}\{-\frac{1}{|\Lambda(K)|}\langle L^s u_K\cdot u_K\rangle_\rho\}^{\frac{1}{2}}$$
$$= C_s\{-\frac{1}{|\Lambda(K)|}\langle L^s(\sum_{x\in\Lambda(K)}\tau_x f)\cdot(\sum_{x\in\Lambda(K)}\tau_x f)\rangle_\rho\}^{\frac{1}{2}}$$
$$\times\{\frac{1}{2|\Lambda(K)|}\sum_{b\in(\Lambda(K))^*}\langle c^s(b,\eta)(\pi_b u_K)^2\rangle_\rho\}^{\frac{1}{2}}+o(1).$$

Noting (3.23) and the relation

$$\{-\frac{1}{|\Lambda(K)|}\langle L^s(\sum_{x\in\Lambda(K)}\tau_x f)\cdot(\sum_{x\in\Lambda(K)}\tau_x f)\rangle_\rho\}^{\frac{1}{2}}\to\| L^s f\|_\rho,$$

as $K\to\infty$, we have $\| L^* f\|_\rho\leq C_s\| L^s f\|_\rho$ as desired. □

In order to prove the uniqueness of $a(\rho)$, we prepare the next lemma.

LEMMA 3.6. – $\overline{L^*\mathcal{F}_0}\cap\overline{\mathcal{G}_g}=0$.

Proof. – The assertion obviously holds for $\rho=0,1$. So suppose $0<\rho<1$. Suppose for each $\varepsilon>0$, there exists a function $\tilde{f}_\varepsilon\in\mathcal{F}_0$ so that

$$\|\big(l,\Phi(\eta)\big)-L^*\tilde{f}_\varepsilon\|_\rho\leq\varepsilon. \tag{3.24}$$

By using (3.11) and (3.13),

$$\| L^s\tilde{f}_\varepsilon\|_\rho^2=\ll\big(l,\Phi(\eta)\big)-L^*\tilde{f}_\varepsilon, L^s\tilde{f}_\varepsilon\gg_\rho\leq\varepsilon\| L^s\tilde{f}_\varepsilon\|_\rho\,.$$

So we have $\| L^s\tilde{f}_\varepsilon\|_\rho\leq\varepsilon$. By Lemma 3.5, we also have $\| L^*\tilde{f}_\varepsilon\|_\rho\leq C_s\varepsilon$. Since ε is arbitrary, $\big(l,\Phi(\eta)\big)=0$ in $\mathcal{H}$ by (3.24). Therefore lemma follows from (3.21). □

The uniqueness of $a(\rho)$ for $0<\rho<1$ follows from (3.20), (3.21) and Lemma 3.6. The continuity of $a(\rho)$ (see the condition (f) in Section 1) implies the uniqueness of $a(\rho)$ for $\rho=0,1$.

Now we prove Lemma 2.2 except for the uniformity with respect to ρ. Let $Z(\rho; F)$ denote the symmetric $d \times d$ matrix corresponding to the quadratic form

$$(l, Z(\rho; F)l) = \| \big(l, a(\rho)\Phi(\eta) - W^*(\eta) + L^*F\big) \|_\rho^2 \tag{3.25}$$

for $l \in \mathbf{R}^d$ and $F \in \mathcal{F}_0^d$. In the next lemma, as in Section 2, $\| \cdot \|$ denotes the operator norm of matrix.

LEMMA 3.7. – (i) $\inf_{F \in \mathcal{F}_0^d} \| Z(\rho; F) \| = 0$ *for* $0 \leq \rho \leq 1$.

(ii) *There exists a constant* C, *which does not depend on* ρ, *such that*

$$\| 2\chi(\rho)a(\rho) - \hat{c}(\rho; F) \| \leq C \| Z(\rho; F) \| .$$

Proof. – The first assertion follows from (3.20). Hence we only have to prove (ii). From (3.20), there exists a function $F^\varepsilon = \{F_i^\varepsilon\}_{1 \leq i \leq d} \in \mathcal{F}_0^d$ such that

$$\big(e_i, Z(\rho; F^\varepsilon)e_i\big)^{1/2} = \| \sum_{k=1}^d a_{ik}(\rho)\Phi_k(\eta) - W_i^*(\eta) + L^*F_i^\varepsilon \|_\rho \leq \varepsilon \tag{3.26}$$

for $\varepsilon > 0$ and $1 \leq i \leq d$. By Schwarz inequality,

$$| \ll \sum_{k=1}^d a_{ik}(\rho)\Phi_k(\eta) - W_i^*(\eta) + L^*F_i^\varepsilon, L^sF_j^\varepsilon \gg_\rho | \leq \varepsilon \| L^sF_j^\varepsilon \|_\rho . \tag{3.27}$$

We note that

$$\| L^sF_j^\varepsilon \|_\rho \leq \| L^*F_j^\varepsilon \|_\rho \leq \varepsilon + \| \sum_{k=1}^d a_{jk}(\rho)\Phi_k(\eta) - W_j^*(\eta) \|_\rho .$$

Since $a(\rho)$ and $\| \cdot \|_\rho$ are continuous in [0,1] (see the condition (f) in Section 1 and Section 4, respectively), the right-hand side of (3.27) is bounded above by $C_1\varepsilon$ with some constant C_1, which does not depend on ρ, for $1 \leq i \leq d$ and $0 < \varepsilon < 1$.

On the other hand, by Schwarz inequality and the continuity of $\| \cdot \|_\rho$,

$$| \ll \sum_{k=1}^d a_{ik}(\rho)\Phi_k(\eta) - W_i^*(\eta) + L^*F_i^\varepsilon, W_j^s(\eta) \gg_\rho | \leq {}^\exists C_2\varepsilon. \tag{3.28}$$

Using the relations (3.8)–(3.13), for $l \in \mathbf{R}^d$, we obtain from (3.27) and (3.28) that

$$\begin{aligned}
&|(l, 2\chi(\rho)a(\rho)l) - (l, \hat{c}(\rho; F^\varepsilon)l)| \\
&= |2\chi(\rho) \sum_{i,j=1}^{d} a_{ij}(\rho) l_i l_j \\
&\quad - \sum_{i=1}^{d} \langle c^s(0, e_i, \eta)\big(l, e_i(\eta_{e_i} - \eta_0) - \pi_{0,e_i}(\sum_{z \in \mathbf{Z}^d} \tau_z F^\varepsilon)\big)^2 \rangle_\rho| \\
&= |2 \sum_{i,j=1}^{d} l_i l_j \{\ll \sum_{k=1}^{d} a_{ik}(\rho)\Phi_k(\eta) - W_i^*(\eta) + L^* F_i^\varepsilon, L^s F_j^\varepsilon \gg_\rho \\
&\quad + \ll \sum_{k=1}^{d} a_{ik}(\rho)\Phi_k(\eta) - W_i^*(\eta) + L^* F_i^\varepsilon, W_j^s(\eta) \gg_\rho\}| \\
&\le 2d(C_1 + C_2)\varepsilon |l|^2 \qquad (3.29)
\end{aligned}$$

for $0 \le \rho \le 1$. This completes the proof. □

Now we recall the definition of the diffusion coefficient of the associated symmetric process (cf. [9]). The diffusion coefficient of the associated symmetric process, denoted by $a^s(\rho)$, is given by

$$(l, a^s(\rho)l) = \inf_{F \in \mathcal{F}_0^d} \frac{(l, \hat{c}(\rho; F)l)}{2\chi(\rho)} \quad \text{for } l \in \mathbf{R}^d. \qquad (3.30)$$

On the other hand, we have

$$(l, a(\rho)l) = \lim_{\varepsilon \to 0} \frac{(l, \hat{c}(\rho; F^\varepsilon)l)}{2\chi(\rho)} \quad \text{for } l \in \mathbf{R}^d \qquad (3.31)$$

with a sequence $F^\varepsilon \in \mathcal{F}_0^d$ defined in the proof of Lemma 3.7. So the following assertion follows from (3.30) and (3.31).

COROLLARY 3.2. – $(l, a^s(\rho)l) \le (l, a(\rho)l)$ *for all* $l \in \mathbf{R}^d$ *and* $0 \le \rho \le 1$.

To conclude this section, we give the proof of Theorem 2.1.

Proof of Theorem 2.1. – Apply Lemma 3.1 with $J(t,\theta) = \partial\lambda(t,\theta)$, $M(\rho) = Z(\rho; F)$ and

$$\begin{aligned}
G(\eta) = \frac{1}{|\Lambda(K)|}\{&a(\bar{\eta}_{x,K})A_K(\eta) \\
&- \sum_{x \in \Lambda(K - s(W^*))} \tau_x W^*(\eta) + \sum_{x \in \Lambda(K - s(L^*F))} \tau_x (L^*F)(\eta)\}.
\end{aligned}$$

Then Theorem 2.1 follows from above arguments. □

4. PROOF OF LEMMA 2.2

Finally we prove Lemma 2.2 which was proved in Lemma 3.7 except for the uniformity with respect to the density ρ. The key idea is that, though our model is asymmetric, the uniformity follows from the arguments of the symmetric case by using the strong sector condition (e) (see Section 1).

We need the continuity of $\| g \|_\rho$ with respect to ρ for the proof of Lemma 2.2.

LEMMA 4.1. – *For $g \in \mathcal{G}$ and $\delta > 0$, $\| g \|_\rho^2 = V_\rho(g)$ is Lipschitz continuous in $[\delta, 1-\delta]$.*

Proof. – We modify the proof of Lemma 4.2 in [6], so we refer to it for details. From the variational principle and Lemma 3.2, we have

$$\begin{aligned}
V_{K,m,\zeta}(g) &= \frac{1}{|\Lambda(K)|} \sup_{u \in \mathcal{F}_{\Lambda(K)}} \{2 \sum_{x \in \Lambda(K-s(g))} \langle \tau_x g \cdot u \rangle_{\Lambda(K),m} \\
&- \frac{1}{4} \sum_{x,y \in \Lambda(K)} \langle c^s_{x,y,\zeta}(\xi)(\pi_{x,y}u)^2 \rangle_{\Lambda(K),m} \} \\
&= \frac{1}{|\Lambda(K)|} \sup_{u \in \mathcal{F}_{\Lambda(K)}} \{ \sum_{i=1}^{d} \Big(2 \sum_x \langle \tau_x \Psi_{K,x,i}(g) \cdot \pi_{x,x+e_i} u \rangle_{\Lambda(K),m} \\
&- \frac{1}{2} \sum_{x:\{x,x+e_i\} \in (\Lambda(K))^*} \langle c^s_{x,x+e_i,\zeta}(\xi)(\pi_{x,x+e_i}u)^2 \rangle_{\Lambda(K),m} \Big) \}. \qquad (4.1)
\end{aligned}$$

Here $\xi \in \mathcal{X}_{\Lambda(K)}$, $\zeta \in \mathcal{X}_{\Lambda(K)^c}$ and $c^s_{x,y,\zeta}$ is a function on $\mathcal{X}_{\Lambda(K)}$ defined by $c^s_{x,y,\zeta}(\xi) = c^s(x,y,\xi \cdot \zeta)$. We define the operators $\sigma^-_{K,m}$ and $\sigma^+_{K,m}$ acting on $\mathcal{F}_{\Lambda(K)}$ by

$$(\sigma^-_{K,m}u)(\xi) = \frac{1}{m} \sum_{x \in \Lambda(K)} u(\sigma_x \xi)\xi_x,$$

$$(\sigma^+_{K,m}u)(\xi) = \frac{1}{|\Lambda(K)| - m} \sum_{x \in \Lambda(K)} u(\sigma_x \xi)(1 - \xi_x)$$

for $u \in \mathcal{F}_{\Lambda(K)}$, where

$$(\sigma_x \xi)_y = \begin{cases} 1 - \xi_x & (\text{ if } y = x \text{ }) \\ \xi_y & (\text{ if } y \neq x \text{ }). \end{cases}$$

We can easily see that $\sigma^-_{K,m} \circ \pi_{x,y} = \pi_{x,y} \circ \sigma^-_{K,m}$ and

$$\langle \sigma^-_{K,m+1} u \cdot h \rangle_{\Lambda(K),m+1} = \langle u \cdot \sigma^+_{K,m} h \rangle_{\Lambda(K),m},$$

for $u, h \in \mathcal{F}_{\Lambda(K)}$. So by using Schwarz inequality, we have

$$\begin{aligned}
&\langle c^s_{x,y,\zeta}(\pi_{x,y}\sigma^-_{K,m+1}u)^2 \rangle_{\Lambda(K),m+1} = \langle c^s_{x,y,\zeta}(\sigma^-_{K,m+1}\pi_{x,y}u)^2 \rangle_{\Lambda(K),m+1} \\
&\leq \langle c^s_{x,y,\zeta}\sigma^-_{K,m+1}(\pi_{x,y}u)^2 \rangle_{\Lambda(K),m+1} = \langle \sigma^+_{K,m} c^s_{x,y,\zeta} \cdot (\pi_{x,y}u)^2 \rangle_{\Lambda(K),m}.
\end{aligned} \tag{4.2}$$

Therefore

$$\begin{aligned}
&2\langle \tau_x \Psi_{K,x,i}(g) \cdot \pi_{x,x+e_i} u \rangle_{\Lambda(K),m} - \frac{1}{2}\langle c^s_{x,x+e_i,\zeta}(\pi_{x,x+e_i}u)^2 \rangle_{\Lambda(K),m} \\
&\leq 2\langle \tau_x \Psi_{K,x,i}(g) \cdot \pi_{x,x+e_i}\sigma^-_{K,m+1} u \rangle_{\Lambda(K),m+1} \\
&\quad - \frac{1}{2}\langle c^s_{x,x+e_i,\zeta}(\pi_{x,x+e_i}\sigma^-_{K,m+1}u)^2 \rangle_{\Lambda(K),m+1} \\
&\quad - 2\langle (\sigma^+_{K,m} - 1)(\tau_x \Psi_{K,x,i}(g)) \cdot \pi_{x,x+e_i} u \rangle_{\Lambda(K),m} \\
&\quad + \frac{1}{2}\langle (\sigma^+_{K,m} - 1)(c^s_{x,x+e_i,\zeta}) \cdot (\pi_{x,x+e_i}u)^2 \rangle_{\Lambda(K),m}.
\end{aligned}$$

Since $\Psi_{K,x,i}(g)$ is a local function,

$$\begin{aligned}
&|(\sigma^+_{K,m} - 1)(\tau_x \Psi_{K,x,i}(g))(\xi)| \\
&= \frac{1}{|\Lambda(K)| - m} \sum_y |(\tau_x \Psi_{K,x,i}(g))(\sigma_y \xi) - (\tau_x \Psi_{K,x,i}(g))(\xi)|(1 - \xi_y) \\
&\leq \frac{1}{|\Lambda(K)|} \frac{C_1(g)}{1-n},
\end{aligned}$$

where $n = \frac{m}{|\Lambda(K)|}$. We also have

$$|(\sigma^+_{K,m} - 1)(c^s_{x,x+e_i,\zeta})(\xi)| \leq \frac{1}{\Lambda(K)} \frac{C_2}{1-n}.$$

Since we may restrict the supremum on the right-hand side of (4.1) to functions satisfying

$$\sum_{x,y \in \Lambda(K)} \langle (\pi_{x,y}u)^2 \rangle_{\Lambda(K),m} \leq C_3(g)|\Lambda(K)|,$$

we have

$$2 \sum_{x;\{x,x+e_i\}\in(\Lambda(K))^*} |\langle(\sigma^+_{K,m} - 1)(\tau_x \Psi_{K,x,i}(g)) \cdot \pi_{x,x+e_i} u\rangle_{\Lambda(K),m}| \\ \leq C_4(g) \frac{1}{1-n}.$$

We also see that

$$\frac{1}{2} \sum_{x;\{x,x+e_i\}\in(\Lambda(K))^*} |\langle(\sigma^+_{K,m} - 1)(c^s_{x,x+e_i,\zeta})(\pi_{x,x+e_i} u)^2\rangle_{\Lambda(K),m}| \leq C_5 \frac{1}{1-n}.$$

By collecting above arguments, we have

$$V_{K,m,\zeta}(g) \leq V_{K,m+1,\zeta}(g) + \frac{1}{|\Lambda(K)|} \frac{C_6(g)}{1-n}.$$

We can show in the same manner that

$$V_{K,m+1,\zeta}(g) \leq V_{K,m,\zeta}(g) + \frac{1}{|\Lambda(K)|} \frac{C_7(g)}{n}.$$

This completes the proof. □

LEMMA 4.2. – *For* $g \in \mathcal{G}$, $\lim_{\rho\to 0} \| g \|_\rho = \lim_{\rho\to 1} \| g \|_\rho = 0$.

Proof. – From the arguments in Theorem 3.1, we have

$$V_\rho(g) = \sup_{l\in\mathbf{R}^d, f\in\mathcal{F}_0} \frac{\{\sum_{i=1}^d \langle \tilde{\Psi}_i(g) \cdot (l_i(\eta_{e_i} - \eta_0) + \pi_{0,e_i} \sum_{x\in\mathbf{Z}^d} \tau_x f)\rangle_\rho\}^2}{\frac{1}{2}\sum_{i=1}^d \langle c^s(0,e_i,\eta)\big(l_i(\eta_{e_i} - \eta_0) - \pi_{0,e_i}(\sum_{x\in\mathbf{Z}^d} \tau_x f)\big)^2\rangle_\rho}.$$

By Schwarz inequality and the condition (a) (see Section 1), the right-hand side is bounded above by $C_0 \sum_{i=1}^d \langle \tilde{\Psi}_i(g)^2\rangle_\rho$ with some constant C_0. Since $\tilde{\Psi}_i(g)$'s are local functions and $\langle \tilde{\Psi}_i(g)^2\rangle_0 = \langle \tilde{\Psi}_i(g)^2\rangle_1 = 0$, the proof is completed. □

Under these preparations, we complete the proof of Lemma 2.2.

Proof of Lemma 2.2. – By Lemma 3.7, it suffices to show that

$$\inf_{F\in\mathcal{F}_0^d} \sup_{\rho\in[0,1]} \| Z(\rho; F) \| = 0. \tag{4.3}$$

From Lemma 3.4, for $\varepsilon > 0$ and $\rho_0 \in [\delta, 1-\delta]$, there exists a function $\tilde{F}_{\rho_0} = \{\tilde{F}_{\rho_0,i}\}_{i=1}^d \in \mathcal{F}_0^d$ such that

$$(e_i, Z(\rho_0; \tilde{F}_{\rho_0})e_i)^{\frac{1}{2}} = \| \sum_{j=1}^d a_{ij}(\rho_0)\Phi_j(\eta) - W_i^*(\eta) + L^*\tilde{F}_{\rho_0,i} \|_{\rho_0} \leq \varepsilon \quad (4.4)$$

for $1 \leq i \leq d$. Then for $\rho \in [0,1]$,

$$\begin{aligned}(e_i, Z(\rho; \tilde{F}_{\rho_0})e_i)^{\frac{1}{2}} &= \| \sum_{j=1}^d a_{ij}(\rho)\Phi_j(\eta) - W_i^*(\eta) + L^*\tilde{F}_{\rho_0,i} \|_{\rho} \\ &\leq | \, \| \sum_{j=1}^d a_{ij}(\rho)\Phi_j(\eta) - W_i^*(\eta) + L^*\tilde{F}_{\rho_0,i} \|_{\rho} \\ &\quad - \| \sum_{j=1}^d a_{ij}(\rho)\Phi_j(\eta) - W_i^*(\eta) + L^*\tilde{F}_{\rho_0,i} \|_{\rho_0} \, | \\ &\quad + \| \sum_{j=1}^d a_{ij}(\rho_0)\Phi_j(\eta) - W_i^*(\eta) + L^*\tilde{F}_{\rho_0,i} \|_{\rho_0} \\ &\quad + \| \sum_{j=1}^d (a_{ij}(\rho) - a_{ij}(\rho_0))\Phi_j(\eta) \|_{\rho_0} .\end{aligned}$$

From Lemma 4.1 and the continuity of $a(\rho)$, there exists a neighborhood N_{ρ_0} of ρ_0 such that above expression is bounded by 3ε for all $\rho \in N_{\rho_0}$. The family $\{N_{\rho_0}, \rho_0 \in [\delta, 1-\delta]\}$ constitutes an open covering of $[\delta, 1-\delta]$. So we have a finite subcovering family $\{N_i\}_{1 \leq i \leq n_0}$ such that $\cup_{i=1}^{n_0} N_i \supset [\delta, 1-\delta]$. To include 0 and 1, by using Lemma 4.2 and the continuity of $a(\rho)$, we observe that

$$\| \sum_{j=1}^d a_{ij}(\rho)\Phi_j(\eta) - W_i^*(\eta) \|_{\rho} \leq \varepsilon \quad (4.5)$$

for every $\rho \in [0, \delta] \cup [1-\delta, 1]$ by choosing $\delta > 0$ suitably. Therefore by interpolation, for large n, we can find $F_\rho = \{F_{\rho,i}\}_{i=1}^d \in \mathcal{F}_{\Lambda(n)}^d$ such that $F_\rho(\eta)$ is continuously differentiable with respect to $\rho \in [0,1]$ for each η and

$$\| \sum_{j=1}^d a_{ij}(\rho)\Phi_j(\eta) - W_i^*(\eta) + L^*F_{\rho,i} \|_{\rho} \leq \varepsilon \quad (4.6)$$

for $\rho \in [0,1]$ and $1 \leq i \leq d$.

In order to remove the dependence on ρ, we define F by $F(\eta) = \{F_i(\eta)\}_{i=1}^d = \{F_{\bar{\eta}_{0,m},i}(\eta)\}_{i=1}^d$ with sufficiently large m. In fact we have

$$\begin{aligned}&\|\sum_{j=1}^d a_{ij}(\rho)\Phi_j(\eta) - W_i^*(\eta) + L^*F_i\|_\rho\\ &\quad\leq \|\sum_{j=1}^d a_{ij}(\rho)\Phi_j(\eta) - W_i^*(\eta) + L^*F_{\rho,i}\|_\rho + \|L^*(F_i - F_{\rho,i})\|_\rho .\end{aligned}$$

By Lemma 3.5 and (3.13),

$$\begin{aligned}\|L^*(F_i - F_{\rho,i})\|_\rho &\leq C_s \|L^s(F_i - F_{\rho,i})\|_\rho\\ &= C_s\{\langle\frac{1}{2}\sum_{i=1}^d c^s(0, e_i, \eta)(\pi_{0,e_i}(\sum_{x\in\mathbf{Z}^d}\tau_x(F_i - F_{\rho,i}))^2\rangle_\rho\}^{\frac{1}{2}}.\end{aligned}$$

Now the estimate required for F follows from computing the symmetric term and it is essentially the same as the arguments in Lemma 2.1 in [2]. □

ACKNOWLEDGMENTS

The author wishes to thank Professors T. Funaki and H. Osada for very helpful discussions and encouragement. The author also wishes to thank Professor L. Xu. The proof of Lemma 3.1 is suggested by him.

REFERENCES

[1] T. Funaki, K. Handa and K. Uchiyama, Hydrodynamic limit of one-dimensional exclusion processes with speed change, Ann. Probab., **19**, 1991, pp. 245-265.

[2] T. Funaki, K. Uchiyama and H.T. Yau Hydrodynamic limit for lattice gas reversible under Bernoulli measures, in: Nonlinear Stochastic PDE's: Hydrodynamic Limit and Burgers' Turbulence (eds. Funaki and Woyczynski), IMA volume **77**, 1995, pp. 1-40.

[3] M.Z. Guo, G.C. Papanicolaou and S.R.S. Varadhan, Nonlinear diffusion limit for a system with nearest neighbor interactions, Commun. Math. Phys., **118**, 1988, pp. 31-59.

[4] K. Komoriya, An asymmetric exclusion process related to vortex flow in viscous planar fluid, in: Probability Theory and Mathematical Statistics (eds. Watanabe, Fukushima, Prohorov and Shiryaev), World Scientific, 1996, pp. 220-228.

[5] C. Kipnis, C. Landim and S. Olla Hydrodynamical limit for a nongradient system: the generalized symmetric exclusion process, Commun. Pure Appl. Math., **47**, 1994, pp. 1475-1545.

[6] C. Landim, S. Olla and H.T. Yau Some properties of the diffusion coefficient for asymmetric simple exclusion processes, Ann. Probab., **24**, 1996, pp. 1779-1808.

[7] C. Landim and H.T. Yau Fluctuation–dissipation equation of asymmetric simple exclusion processes, Prob. Th. Rel. Fields, **108**, 1997, pp. 321-356.

[8] H. OSADA and T. SAITOH, An invariance principle for non-symmetric Markov processes and reflecting diffusions in random domains, Prob. Th. Rel. Fields, **101**, 1995, pp. 45-63.
[9] H. SPOHN Large Scale Dynamics of Interacting Particles, Springer, 1991.
[10] S.R.S. VARADHAN, Nonlinear diffusion limit for a system with nearest neighbor interactions - II, in: Asymptotic Problems in Probability Theory: Stochastic Models and Diffusions on Fractals (eds. Elworthy and Ikeda), Longman, 1993, pp. 75-128.
[11] S.R.S. VARADHAN, Self diffusion of a tagged particle in equilibrium for asymmetric mean zero random walk with simple exclusion, Ann. Inst. Henri Poincaré, **31**, 1995, pp. 273-285.
[12] L. XU, Diffusive scaling limit for mean zero asymmetric simple exclusion processes, Courant thesis, 1993.
[13] H.T. YAU, Relative entropy and hydrodynamics of Ginzburg-Landau models, Letters Math. Phys., **22**, 1991, pp. 63-80.

(Manuscript received October 10, 1997;
Revised version received February 2, 1998.)