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Large deviations for a random walk in dynamical random environment

by

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ABSTRACT. – We consider a random walk $X_t \in \mathbb{Z}^d, t \in Z_+$, in a dynamical random environment $(\xi_t(x), x \in \mathbb{Z}^d), t \in \mathbb{Z}_+$, with a mutual interaction with each other. The Markov process $(X_t, \xi_t(x), x \in \mathbb{Z}^d)$ is a perturbation of a process for which the random walk X_t and the environment $\xi_t(x), x \in \mathbb{Z}^d$ are independent, $X_t, t \in \mathbb{Z}_+$ is a homogeneous random walk in $\mathbb{Z}^d$ and the environment $\xi_t(x), x \in \mathbb{Z}^d$ behaves independently in each site as an ergodic Markov chain. For the perturbated process we assume that

1. The interaction between the position of the particle and the environment is local;
2. The influence of the environment on the particle X_t is small;
3. The particle modifies the environment of its location (it cancels the memory of the environment).

We consider a large deviation problem for the random walk $X_t, t \in \mathbb{Z}_+$. We prove that a large deviation principle holds for this random walk with a good rate function which is analytic with respect to the parameter of interaction in a neighborhood of 0. © Elsevier, Paris

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RÉSUMÉ. – Nous considérons une marche aléatoire $X_t \in \mathbb{Z}^d, t \in Z_+$, dans un environnement aléatoire dynamique $(\xi_t(x), x \in \mathbb{Z}^d), t \in \mathbb{Z}_+$, ces deux processus interagissant l'un avec l'autre. Le processus de Markov $(X_t, \xi_t(x), x \in \mathbb{Z}^d)$ est une perturbation du processus pour lequel la marche aléatoire X_t et l'environnement $\xi_t(x), x \in \mathbb{Z}^d$ sont indépendants : $X_t, t \in \mathbb{Z}_+$ est une marche aléatoire homogène dans $\mathbb{Z}^d$ et l'environnement $\xi_t(x), x \in \mathbb{Z}^d$ évolue indépendamment à chaque site comme une chaîne de Markov ergodique. Pour ce processus perturbé nous supposons que

1. L'interaction entre la position de la particule et l'environnement est local ;
2. L'influence de l'environnement sur la particule X_t est petite ;
3. La particule modifie l'environnement de la position où elle se trouve (elle annule la mémoire de la position où elle se trouve).

Nous nous intéressons au problème de grandes déviations associé à la marche aléatoire $X_t, t \in \mathbb{Z}_+$. Nous montrons un principe de grande déviation pour cette marche aléatoire avec une bonne fonctionnelle d'action qui est analytique par rapport au paramètre d'interaction dans un voisinage de 0.

1. INTRODUCTION

The expression "random walk in random environment" has several meanings and different models of random walk in random environment were introduced. This difference arises from different definitions of the behavior of the environment and the interaction between the particle and the environment.

The class of models which has been mostly studied is the random motion in a fixed realization of the random environment. For this class of models we refer the reader to the papers of Fisher [1], Bricmont [2] and references therein.

Another example of random walk in random environment is studied by Bolthausen in [3], where the environment is dynamic (it is described by some stationary field $\xi_t(x), t \in \mathbb{Z}_+, x \in \mathbb{Z}^d$ independent in space and time), the influence of the environment on the particle is small and the motion of the particle has no influence on the environment.

In the present paper we are interested in random walks in a dynamical random environment with a mutual influence between the particle and the environment. Namely we consider the random walk in random environment, where

- the environment in the case of the absence of the particle behaves independently in each site of $\mathbb{Z}^d$ as an ergodic Markov chain;
- the transition probabilities of the random walk depend on the environment;
- the particle modifies the transition probabilities of the environment on its location.

This class of models was introduced in [4] where the interaction between the particle and the environment is local and one of the following conditions is satisfied:

1. the influence of the environment on the particle X_t is small and the particle cancels the memory of the environment of its location;
2. the mutual interaction between the environment and the random walk is small;
3. the exponential relaxation rate of the environment is large.

The main result of [4] is the central limit theorem for the displacement of the particle. One can get also the law of large numbers for that, using some technical results of this paper and its trivial generalization.

This class of models was studied further by Boldrighini, Minlos and Pellegrinotti in [5]-[7]. The papers [5] and [6] are devoted to the mixing properties of the environment for the case where the mutual interaction between the environment and the random walk is small. In [7] the authors consider a particular case, where the environment is described by a stationary field $\xi_t(x), t \in \mathbb{Z}_+, x \in \mathbb{Z}^d$ independent in space and time, and prove that, for $d \geq 2$, the central limit theorem for the displacement of the particle holds almost surely with respect to the environment.

In the present paper we study the large deviations problem for displacement of the particle. Up to now for the different models of random walk in random environment there are few results in this domain (we refer the reader to the papers of Dembo, Peres and Zeitouni [8] and Greven and Hollander [12] where the results were obtained for the case of fixed environment).

A possible method to obtain the large deviation principle consists in proving hyper-mixing properties of the process (see [10]). But for random walks in dynamical random environment the results of the papers [5] and [6] are not sufficient to get a large deviation principle.

In the present paper we propose another method to prove the large deviation principle for the random walk in dynamical random environment, using the Gärtner-Ellis theorem and the method of Dyson's equations.

The Gärtner-Ellis theorem (see for example [9]) reduces the large deviations problem to the analysis of the limit

$$\lim_{t\to+\infty} \frac{1}{t} \log E(\exp\langle\alpha, X_t - X_0\rangle),$$

where X_t is the position of the particle at the time t. To prove the existence of this limit and to study its properties we derive Dyson's equations for the moment generating functions

$$E(\exp\langle\alpha, X_t - X_0\rangle).$$

The method of Dyson's equations has been investigated in the Gibbs field theory to study the one-particle spectrum of the transfer matrix. A brief review of this method can be found in [11] and [14]. We derive Dyson's equations using cluster expansion techniques developed in [4], and we use these equations to prove that our random walk satisfies the conditions of the Gärtner-Ellis theorem. This implies the large deviation principle with a good rate function. Our method allows us also to prove that the rate function is strictly convex and analytic with respect to the parameter of the interaction in a neighborhood of 0.

The principal property of our random walk being used here is the independent evolution of the environment having an exponential relaxation rate in the case of the absence of the particle and a local interaction between the particle and the environment being small with respect to the relaxation rate of the environment.

We consider the case where the influence of the environment on the particle is small and the particle cancels the memory of the environment. Using cluster expansion techniques and Dyson's equations it seems to be possible to get the same results also for the case where the mutual interaction between the environment and the random walk is small as well as for the case where the exponential relaxation rate of the environment is large.

2. THE MAIN RESULTS

The random process $(X_t, \xi_t(x), x \in \mathbb{Z}^d), t \in \mathbb{Z}_+$ with an interaction between the random environment $(\xi_t(x), x \in \mathbb{Z}^d)$ and the position of the random walk X_t is defined as follows:

1. $X_0 = x_0$ a.s., $x_0 \in Z^d$.

2. $\xi_0(x), x \in \mathbb{Z}^d$, are independent identically distributed random variables with values in a finite state space S such that

$$P(\xi_0(x) = s) = q_0(s), \quad s \in S$$

where q_0 is a probability distribution on S.

3. For any $t \in \mathbb{Z}_+$, X_{t+1} and $\xi_{t+1}(x), x \in \mathbb{Z}^d$ are conditionally independent given the variables $(X_t, \xi_t(x), x \in \mathbb{Z}^d)$ and for any $y \in \mathbb{Z}^d$ and $s \in S$

$$P(\xi_{t+1}(y) = s | X_t, \xi_t(x), x \in \mathbb{Z}^d) = \begin{cases} q(\xi_t(y), s) & \text{if } X_t \neq y, \\ \pi_0(s) & \text{if } X_t = y \end{cases}$$

where π_0 is a probability distribution on S, and

$$Q = (q(s, s'))_{s,s' \in S}$$

is a stochastic matrix;

$$P(X_{t+1} = y | X_t, \xi_t(x), x \in \mathbb{Z}^d) = p(y - X_t)(1 + \delta c(y - X_t, \xi_t(X_t)))$$

where for $x, y \in \mathbb{Z}^d$

$$p(x, y) = p(y - x)$$

and $\big(p(y), y \in \mathbb{Z}^d\big)$ is a probability distribution on $\mathbb{Z}^d$.

Assumptions:

1. For any $s \in S$

$$p(y)(1 + \delta c(y, s)), \quad y \in \mathbb{Z}^d$$

is a probability distribution on $\mathbb{Z}^d$; this implies that the function

$$c : \mathbb{Z}^d \times S \to \mathbb{R}$$

satisfies the following conditions

$$\sum_{y \in \mathbb{Z}^d} p(y) c(y, s) = 0, \quad \text{for any } s \in S, \tag{1}$$

$$1 \geq p(y)(1 + \delta c(y, s)) \geq 0, \quad \text{for any } y \in \mathbb{Z}^d, s \in S; \tag{2}$$

2. the function $c(y, s)$ is bounded:

(3) $$\max_{s} |c(y, s)| \leq 1, \quad \text{for any } y \in \mathbb{Z}^d;$$

3. the stochastic matrix $P = (p(x, y))_{x,y \in \mathbb{Z}^d}$ is irreducible;
4. the stochastic matrix $Q = (q(s, s'))_{s,s' \in S}$ is irreducible and aperiodic;
5. for any $\alpha \in \mathbb{R}^d$

(4) $$R(\alpha) = \sum_{y} p(y) e^{\langle \alpha, y \rangle} < +\infty.$$

We assume also that for any $y \in \mathbb{Z}^d$

(5) $$\sum_{s} \pi(s) c(y, s) = 0$$

where π is the invariant distribution of the Markov chain with state space S and transition probabilities $q(s, s'), s, s' \in S$. (This invariant distribution exists and it is unique since the matrix Q is irreducible and aperiodic.)

Assumption (5) is not restrictive. Indeed, for the case where (5) is not satisfied the transition probabilities of the free random walk can be redefined as

$$p'(x, y) = p(x, y)(1 + \sum_{s \in S} \delta c(y - x, s) \pi(s)).$$

Then for

$$c'(y, s) = \frac{c(y, s) - \sum_{s' \in S} c(y, s') \pi(s')}{1 + \sum_{s' \in S} c(y, s') \pi(s')}$$

we have

$$p'(y)(1 + \delta c'(y, s)) = p(y)(1 + \delta c(y, s)) \text{ for all } y \in Z^d \text{ and } s \in S,$$

and

$$\sum_{s} \pi(s) c'(y, s) = 0.$$

Denote by P_δ the distribution of the process $(X_t, \xi_t(x), x \in \mathbb{Z}^d), t \in \mathbb{Z}_+$, by E_δ an expectation with respect to the measure P_δ and by $\mu_{\delta,t}$ the distribution of $\frac{X_t}{t}$.

Following the usual terminology (see [9] for example) we say that the sequence of measures $(\mu_{\delta,t})$ satisfies the large deviation principle with a good rate function

$$L_\delta : \mathbb{R}^d \to \mathbb{R}_+$$

iff

1. for any $r \in R_+$ the level set $\{v \in \mathbb{R}^d : L_\delta(v) \le r\}$ is a compact subset of $\mathbb{R}^d$;
2. for any closed set $F \subset \mathbb{R}^d$,

$$\limsup_{t\to+\infty} \frac{1}{t}\log((\mu_{\delta,t}(F)) \le -\inf_{x\in F} L_\delta(x);$$

3. for any open set $G \subset \mathbb{R}^d$,

$$\liminf_{t\to+\infty} \frac{1}{t}\log((\mu_{\delta,t}(G)) \ge -\inf_{x\in G} L_\delta(x).$$

The main result of this paper is the following theorem.

THEOREM 1. – *There exists $\delta_0 > 0$ such that for $\delta \in R, |\delta| < \delta_0$*

1. *the sequence of measures $(\mu_{\delta,t})_{t\in\mathbb{Z}_+}$ satisfies the large deviation principle with a good strictly convex rate function L_δ;*
2. *the function $L_\delta(v)$ is analytic with respect to (v,δ) everywhere in $\mathbb{R}^d \times \{\delta \in \mathbb{R} : |\delta| < \delta_0\}$ and it can be analytically continued to the domain $\mathbb{R}^d \times \{\delta \in \mathbb{C} : |\delta| < \delta_0\}$.*

To prove Theorem 1 we derive Dyson's equations for the moment generating functions

$$R_t(\delta,\alpha) = E_\delta(\exp\langle\alpha, X_t - X_0\rangle).$$

Namely, we consider

$$R(\delta,\alpha,z) = \sum_{t=0}^{+\infty} R_t(\delta,\alpha)z^t$$

and we introduce two functions $\widetilde{\mathcal{J}}(\delta,\alpha,z)$ and $\widetilde{\mathcal{J}}^1(\delta,\alpha,z)$ such that for any α and δ

$$R(\delta,\alpha,z) = \Big(1 + \widetilde{\mathcal{J}}^1(\delta,\alpha,z) + R(\delta,\alpha,z)\widetilde{\mathcal{J}}(\delta,\alpha,z)\Big)R(0,\alpha,z)$$

in some neighborhood of $z = 0$. This equation is called a Dyson's equation for the generating function $R(\delta, \alpha, z)$. For $\delta = 0$ the transition probabilities of the random walk do not depend on the environment and so

$$R_t(0, \alpha) = (R_1(0, \alpha))^t = (R(\alpha))^t.$$

Using this we rewrite Dyson's equation in the following way

$$R(\delta, \alpha, z) = \frac{1 + \widetilde{\mathcal{J}}^1(\delta, \alpha, z)}{1 - zR(\alpha) - \widetilde{\mathcal{J}}(\delta, \alpha, z)},$$

and we study the limit

$$\lim_{t \to +\infty} \frac{1}{t} \log R_t(\delta, \alpha) = H_\delta(\alpha)$$

in terms of the poles of the function

$$\frac{1 + \widetilde{\mathcal{J}}^1(\delta, \alpha, z)}{1 - zR(\alpha) - \widetilde{\mathcal{J}}(\delta, \alpha, z)}.$$

To introduce the functions $\widetilde{\mathcal{J}}(\delta, \alpha, z)$ and $\widetilde{\mathcal{J}}^1(\delta, \alpha, z)$ we use the cluster expansion constructed for the random walks in dynamical random environment in [4]. In section 3, we recall the definition of the clusters. In section 4, the cluster expansion is used to define the functions $\widetilde{\mathcal{J}}(\delta, \alpha, z)$ and $\widetilde{\mathcal{J}}^1(\delta, \alpha, z)$ and to derive Dyson's equations. In section 5 we obtain some cluster estimates. We use these estimates in section 6 to show that for δ small enough, the functions $\widetilde{\mathcal{J}}(\delta, \alpha, z)$ and $\widetilde{\mathcal{J}}^1(\delta, \alpha, z)$ can be analytically continued to the disk

$$Q = \left\{ z \in \mathbb{C} : |z| < \frac{1}{R(\alpha)}(1 + \epsilon) \right\},$$

with some $\epsilon > 0$, and to show further that the function

$$\frac{1 + \widetilde{\mathcal{J}}^1(\delta, \alpha, z)}{1 - zR(\alpha) - \widetilde{\mathcal{J}}(\delta, \alpha, z)}.$$

has in Q a unique simple pole $z_0(\alpha, \delta) \in \mathbb{R}_+$, which is a simple solution of the equation

$$1 - zR(\alpha) - \widetilde{\mathcal{J}}(\delta, \alpha, z) = 0. \tag{6}$$

From this we get

$$\lim_{t\to+\infty} \frac{1}{t} \log R_t(\delta,\alpha) = H_\delta(\alpha) = -\log z_0(\alpha,\delta).$$

Using Gärtner-Ellis theorem we prove therefore the first part of our theorem with the function L_δ being the Fenchel-Legendre transform of $H_\delta(\alpha)$:

$$L_\delta(v) = \sup_{\alpha\in\mathbb{R}^d} (\langle\alpha, v\rangle - H_\delta(\alpha)).$$

In section 7 we prove that for δ small enough the function $H_\delta(\alpha)$ is analytic with respect to (v,δ) using implicit function theorem applied to the equation (6). In section 8 we show that for δ small enough, the function $H_\delta(\alpha)$ is strictly convex everywhere in $\mathbb{R}^d$. Using this we complete the proof of our Theorem in section 9.

3. DEFINITION OF THE CLUSTERS

Let P_δ be the distribution of the random process $(X_t, \xi_t(x), x \in \mathbb{Z}^d), t \in \mathbb{Z}_+$, $\{\mathcal{F}_t\}$ be its natural filtration, and let $P_{\delta,t}$ be the restriction of P_δ on $\mathcal{F}_t$.

From the definition of the process $(X_t, \xi_t(x), x \in \mathbb{Z}^d), t \in \mathbb{Z}_+$, it follows that for any $t \in \mathbb{Z}_+$ the measure $P_{\delta,t}$ is absolutely continuous with respect to the measure $P_{0,t}$ with the density

$$\prod_{k=0}^{t-1}(1 + \delta c(X_{k+1} - X_k, \xi_k(X_k))).$$

Denote by $X_{\leq n}$ the trajectory of the random walk up to time n:

$$X_{\leq n} = (X_0, X_1, \ldots, X_n).$$

Then, for any trajectory $\Gamma = (x_0, x_1, \ldots, x_n) \in (\mathbb{Z}^d)^{n+1}$,

$$\begin{aligned}(7)\qquad P_\delta(X_{\leq n} = \Gamma | X_0 = x_0) &= E_\delta(1_{\{X_{\leq n}=\Gamma\}} | X_0 = x_0)\\ &= E_0\left(1_{\{X_{\leq n}=\Gamma\}} \prod_{t=0}^{n-1} (1 + \delta c(X_{t+1} - X_t, \xi_t(X_t))) \big| X_0 = x_0\right),\end{aligned}$$

where $1_{\{X_{\leq n}=\Gamma\}}$ denotes the indicator of $\{X_{\leq n} = \Gamma\}$.

Using

$$\prod_{t=0}^{n-1}(1+\delta c(X_{t+1}-X_t,\xi_t(X_t))) = \sum_{A\subseteq\{0,\dots,n-1\}} \delta^{|A|}\prod_{t\in A} c(X_{t+1}-X_t,\xi_t(X_t))$$

in the right hand side of (7), we get

$$\begin{aligned}(8)\qquad & P_\delta(X_{\leq n}=\Gamma|X_0=x_0) = P_0(X_{\leq n}=\Gamma|X_0=x_0)\\ &+\sum_{k=1}^{n}\delta^k \sum_{A\subseteq\{0,\dots,n-1\}:|A|=k}\\ &\times E_0\left(1_{\{X_{\leq n}=\Gamma\}}\prod_{t\in A} c(X_{t+1}-X_t,\xi_t(X_t))\big|X_0=x_0\right)\end{aligned}$$

DEFINITION 3.1. – *Let* $n\in\mathbb{Z}_+$, $\Gamma=(x_0,\dots,x_n)\in(\mathbb{Z}^d)^{n+1}$ *and*

$$A\subseteq\{0,\dots,n-1\}.$$

The pair (Γ,A) *is called a cluster with an origin in* x_0. *We denote by* $\mathcal{G}_n(x,y)$ *the set of all clusters* $(\Gamma=(x_0,\dots,x_n),A)$ *with* $x_0=x$ *and* $x_n=y$.

For each $\Gamma=(x_0,\dots,x_n)\in(\mathbb{Z}^d)^{n+1}$ we denote

$$P_0^\Gamma = P_0(X_{\leq n}=\Gamma|X_0=x_0)$$

and for any cluster (Γ,A) we define

$$(9)\qquad I_{(\Gamma,A)} = E_0\left(\prod_{t\in A} c(X_{t+1}-X_t,\xi_t(X_t))\big|X_{\leq n}=\Gamma\right).$$

Then from (8) we get

$$(10)\qquad P_\delta(X_n=x|X_0=x_0) = \sum_{(\Gamma,A)\in\mathcal{G}_n(x_0,x)} \delta^{|A|} I_{(\Gamma,A)} P_0^\Gamma.$$

The relation (10) is called a cluster expansion for the probabilities

$$P_\delta\,(\,X_n\;=\;x\,|X_0=x_0).$$

Using (10) we define the values $P_\delta\,(\,X_n\;=\;x\,|X_0=x_0)$ also for $\delta\in\mathbb{C}$.

We give now a more explicit form for $I_{(\Gamma,A)}$.

LEMMA 3.1. – *Let* $\Gamma = (x_0, \ldots, x_n)$ *and* $A \subseteq \{0, \ldots, n-1\}$, *consider*

$$A_\Gamma^0 = \{t \in A : t \neq 0 \text{ and for some } \tau < t \quad x_\tau = x_t\},$$
$$A_\Gamma^1 = A \backslash A_\Gamma^0,$$

and define for $t \in A_\Gamma^0$

$$\tau_t = \tau_t(\Gamma) = \max\{\tau \in \mathbb{Z}_+ : \tau < t, x_\tau = x_t\}.$$

Then

$$(11)\quad I_{(\Gamma,A)} = \prod_{t \in A_\Gamma^1} \left(\sum_{s,s' \in S} q_0(s) q^{(t)}(s,s') c(x_{t+1} - x_t, s') \right) \times \prod_{t \in A_\Gamma^0} \left(\sum_{s,s' \in S} \pi_0(s) q^{(t-\tau_t-1)}(s,s') c(x_{t+1} - x_t, s') \right),$$

where $q^{(t)}(s,s'), s, s' \in S$ *are t-time transition probabilities of the Markov chain governing the environment:*

$$\left(q^{(t)}(s,s') \right)_{s,s' \in S} = Q^t,$$
$$q^{(0)}(s,s) = 1, \text{ and } q^{(0)}(s,s') = 0 \quad \text{for } s \neq s'.$$

Proof of Lemma 3.1. – Let us notice that that for $\delta = 0$, P_0 corresponds to the case where the transition probabilities of the random walk X_t do not depend on the environment:

$$P_0(X_{t+1} = y | X_t = x, \xi_t(z), z \in \mathbb{Z}^d) = p(y - x),$$

but the random environment depends on the random walk for any δ. For $\delta = 0$ as well as for any other $\delta \neq 0$ we have

$$P_0(\xi_{t+1}(y) = s | X_t, \xi_t(x), x \in \mathbb{Z}^d) = \begin{cases} q(\xi_t(y), s) & \text{if } X_t \neq y \\ \pi_0(s) & \text{if } X_t = y. \end{cases}$$

Notice also that for any $t \in A_\Gamma^0$ the particle cancels the memory of the environment in the site x_t at the time τ_t, and does not visit this site during the interval of the time $[\tau_t, t]$. Hence in the site x_t the environment starts at the time $\tau_t + 1$ with the distribution $\pi_0(s), s \in S$, and before the time t it behaves independently as a Markov chain having transition probabilities $q(s,s'), s, s' \in S$.

Similarly for $t \in A_\Gamma^1$ the particle does not visit the site x_t before the time t, and so the environment in this site starts with the distribution $q_0(s), s \in S$, and before the time t it behaves independently as a Markov chain having transition probabilities $q(s, s'), s, s' \in S$. From this (11) follows and Lemma 3.1 is therefore proved. □

The relation (11) implies that for any $\Gamma = (x_0, \ldots, x_n)$ the value $I_{(\Gamma, A)}$ does not depend on x_t if

$$t \notin \bigcup_{t \in A_\Gamma^1} [0, t+1] \bigcup_{t \in A_\Gamma^0} [\tau_t, t+1].$$

This is the principal property of the cluster expansion (10) being used to derive Dyson's equations in the following section.

4. DYSON'S EQUATIONS

In order to introduce Dyson's equations for the moment generating functions

$$R_n(\delta, \alpha) = \sum_{x \in \mathbb{Z}^d} P_\delta(X_n = x | X_0 = x_0) e^{\langle \alpha, x - x_0 \rangle}, \quad \alpha \in \mathbb{R}^d.$$

we define the notion of irreducible clusters.

DEFINITION 4.1. – *Let* $\Gamma = (x_0, \ldots, x_n), A \subseteq \{0, \ldots, n-1\}$ *and*

$$U(\Gamma, A) = \bigcup_{t \in A_\Gamma^1} [0, t] \bigcup_{t \in A_\Gamma^0} [\tau_t, t].$$

A cluster (Γ, A) *is said to be irreducible iff*

$$U(\Gamma, A) = [0, n-1].$$

We denote by $\mathcal{B}_n(x, y)$ the set of all irreducible clusters (Γ, A) such that $\Gamma = (x_0, x_1, \ldots, x_n)$ with $x_0 = x$ and $x_n = y$, and we define

$$\begin{aligned} \mathcal{J}_n(\delta, x, y) &= \sum_{(\Gamma, A) \in \mathcal{B}_n(x,y) : A_\Gamma^1 = \emptyset} \delta^{|A|} I_{(\Gamma, A)} P_0^\Gamma, \\ \mathcal{J}_n^1(\delta, x, y) &= \sum_{(\Gamma, A) \in \mathcal{B}_n(x,y) : A_\Gamma^1 \neq \emptyset} \delta^{|A|} I_{(\Gamma, A)} P_0^\Gamma. \end{aligned} \tag{12}$$

PROPOSITION 4.1. – *For any* $x_0, x \in \mathbb{Z}^d$, $n \in \mathbb{Z}_+$

$$(13)\ P_\delta(X_n = x | X_0 = x_0) = p^{(n)}(x_0, x) + \sum_{k=1}^{n} \sum_{y \in \mathbb{Z}^d} \mathcal{J}_k^1(\delta, x_0, y) p^{(n-k)}(y, x) + \sum_{0 \le k < l \le n} \sum_{y, y' \in \mathbb{Z}^d} P_\delta(X_k = y | X_0 = x_0) \mathcal{J}_{l-k}(\delta, y, y') p^{(n-l)}(y', x).$$

Proof of Proposition 4.1. – Consider the relation (10). Recall that for $A = \emptyset$,

$$I_{(\Gamma, A)} = 1,$$

and so from (10) it follows

$$P_\delta(X_n = x | X_0 = x_0) = p^{(n)}(x_0, x) + \sum_{(\Gamma, A) \in \mathcal{G}_n(x_0, x): A \neq \emptyset} \delta^{|A|} I_{(\Gamma, A)} P_0^\Gamma,$$

where $\mathcal{G}_n(x_0, x)$ is the set of all clusters (Γ, A) such that $\Gamma = (x_0, \ldots, x_n)$ with $x_n = x$. Hence to verify (13) we have to show that

$$(14)\quad \sum_{(\Gamma, A) \in \mathcal{G}_n(x_0, x): A \neq \emptyset} \delta^{|A|} I_{(\Gamma, A)} P_0^\Gamma = \sum_{k=1}^{n} \sum_{y \in \mathbb{Z}^d} \mathcal{J}_k^1(\delta, x_0, y) p^{(n-k)}(y, x) + \sum_{0 \le k < l \le n} \sum_{y, y' \in \mathbb{Z}^d} P_\delta(X_k = y | X_0 = x_0) \mathcal{J}_{l-k}(\delta, y, y') p^{(n-l)}(y', x).$$

Let $0 < k \le n$. Denote by $\mathcal{G}_n^k(x, y)$ the set of all clusters $(\Gamma, A) \in \mathcal{G}_n(x, y)$ for which

$$(15)\quad U(\Gamma, A) = [0, k-1],$$

and consider $(\Gamma, A) \in \mathcal{G}_n^k(x_0, x)$. Then $A \subset \{0, \ldots, k-1\}$, the cluster $(\Gamma' = (x_0, \ldots, x_k), A)$ is irreducible, and from (11) we get

$$I_{(\Gamma, A)} = I_{(\Gamma', A)}.$$

Notice also that for $\Gamma' = (x_0, \ldots, x_k)$ and $\Gamma'' = (x_k, \ldots, x_n)$

$$P_0^\Gamma = P_0^{\Gamma'} \times P_0^{\Gamma''}.$$

We get therefore

$$(16)\qquad \sum_{(\Gamma,A)\in\mathcal{G}_n^k(x_0,x)} \delta^{|A|} I_{(\Gamma,A)} P_0^{\Gamma} = \sum_{y\in\mathbb{Z}^d} \sum_{(\Gamma',A)\in\mathcal{B}_k(x_0,y)} \delta^{|A|} I_{(\Gamma',A)} P_0^{\Gamma'} \times \sum_{\Gamma''=(x_k=y,x_{k+1},\dots,x_n=x)} P_0^{\Gamma''}.$$

In the right hand side of (16) we have

$$\sum_{(\Gamma',A)\in\mathcal{B}_k(x_0,y)} \delta^{|A|} I_{(\Gamma',A)} P_0^{\Gamma'} = \mathcal{J}_k^1(\delta,x_0,y) + \mathcal{J}_k(\delta,x_0,y)$$

and

$$\sum_{\Gamma''=(x_k=y,x_{k+1},\dots,x_n=x)} P_0^{\Gamma''} = p^{(n-k)}(y,x).$$

Thus (16) yields

$$\sum_{(\Gamma,A)\in\mathcal{G}_n^k(x_0,x)} \delta^{|A|} I_{(\Gamma,A)} P_0^{\Gamma} = \sum_{y\in\mathbb{Z}^d} \big(\mathcal{J}_k^1(\delta,x_0,y) + \mathcal{J}_k(\delta,x_0,y)\big) p^{(n-k)}(y,x),$$

and therefore to verify (14) it is sufficient to show that

$$(17)\qquad \sum_{(\Gamma,A)\in\mathcal{G}_n'(x_0,x):A\neq\emptyset} \delta^{|A|} I_{(\Gamma,A)} P_0^{\Gamma} = \sum_{0<k<l\le n} \sum_{y,y'\in\mathbb{Z}^d} P_\delta(X_k=y|X_0=x_0)\mathcal{J}_{l-k}(\delta,y,y')p^{(n-l)}(y',x),$$

where

$$\mathcal{G}_n'(x,y) = \mathcal{G}_n(x,y)\backslash\left(\bigcup_{k=1}^{n}\mathcal{G}_n^k(x,y)\right).$$

To prove (17) we generalize the notion of cluster.

DEFINITION 4.2. – *Let* $0\le k<l$, $x_k,\dots,x_l\in\mathbb{Z}^d$ *and* $A\subset\{k,\dots,l-1\}$. *Then the pair* $(\Gamma=(x_k,\dots,x_l),A)$ *is called a* (k,l)*-cluster if for any* $t\in A$ *there exists* $s\in\{k,\dots,t-1\}$ *such that*

$$x_t = x_s.$$

For a (k,l)*-cluster* $(\Gamma=(x_k,\dots,x_l),A)$ *we consider*

$$\tau_t = \max\{s : k\le s<t, \textit{ and } x_s=x_t\}, \quad \textit{for } t\in A$$

and then we define $I_{(\Gamma,A)}$ *by (11).*

A (k,l)-cluster $(\Gamma = (x_k, \dots, x_l), A)$ is said to be irreducible iff

$$U(\Gamma, A) = \bigcup_{t\in A} [\tau_t, t] = [k, l-1].$$

We denote by $\mathcal{B}_{(k,l)}(x,y)$ the set of all irreducible (k,l)-clusters (Γ, A) where $\Gamma = (x_k, \dots, x_l)$ with $x_k = x$ and $x_l = y$.

Let us notice that

$$\sum_{(\Gamma,A)\in\mathcal{B}_{(k,l)}(y,y')} \delta^{|A|} I_{(\Gamma,A)} P_0^\Gamma = \sum_{(\Gamma,A)\in\mathcal{B}_{l-k}(y,y'):A_\Gamma^1=\emptyset} \delta^{|A|} I_{(\Gamma,A)} P_0^\Gamma = \mathcal{J}_{l-k}(\delta, y, y'), \tag{18}$$

Indeed, there is a one to one correspondence φ between the set of all irreducible (k,l)-clusters $\mathcal{B}_{(k,l)}(y,y')$ and the set of all irreducible clusters $(\Gamma, A) \in \mathcal{B}_{l-k}(y,y')$ such that $A_\Gamma^1 = \emptyset$, where for $(\Gamma = (x_k, \dots, x_l), A) \in \mathcal{B}_{(k,l)}(y,y')$

$$\varphi(\Gamma, A) = (\varphi(\Gamma), \varphi(A))$$

with $\varphi(\Gamma) = (y_0 = x_k, \dots, y_{l-k} = x_l)$
and $\varphi(A) = \{s : 0 \le s < l-k, s+k \in A\}$.
It is clear that for any $(\Gamma, A) \in \mathcal{B}_{(k,l)}(y,y')$

$$P_0^{\varphi(\Gamma)} = P_0^\Gamma,$$

and from (11) it follows that

$$I_{\varphi(\Gamma,A)} = I_{(\Gamma,A)}.$$

Thus the relation (18) holds.

We are ready now to prove (17). Let $(\Gamma = (x_0, \dots, x_n), A) \in \mathcal{G}_n'(x_0, x)$ and $A \ne \emptyset$, that is there is no k, $0 < k \le n$, such that $U(\Gamma, A) = [0, k-1]$. Then there exist $0 < k < l \le n$ such that

$$U(\Gamma, A) = U' \cup [k, l-1], \quad \text{where } U' \subset [0, k-1]. \tag{19}$$

Denote by $\mathcal{G}_n^{(k,l)}(x,y)$ the set of all clusters $(\Gamma, A) \in \mathcal{G}_n(x,y)$ for which (19) holds. We have therefore

$$\sum_{(\Gamma,A)\in\mathcal{G}_n'(x_0,x):A\ne\emptyset} \delta^{|A|} I_{(\Gamma,A)} P_0^\Gamma = \sum_{0<k<l\le n} \sum_{(\Gamma,A)\in\mathcal{G}_n^{(k,l)}(x_0,x)} \delta^{|A|} I_{(\Gamma,A)} P_0^\Gamma. \tag{20}$$

Consider now $(\Gamma, A) \in \mathcal{G}_n^{(k,l)}(x_0, x)$. Let

$$A^* = A \cap [k, l-1] \text{ and } A' = A \backslash A^*,$$

then $A' \subset \{0, \dots, k-1\}$ and the pair $(\Gamma^* = (x_k, \dots, x_l), A^*)$ is an irreducible (k,l)-cluster. Thus $(\Gamma', A') \in \mathcal{G}_k(x_0, x_k)$, $(\Gamma^*, A^*) \in \mathcal{B}_{(k,l)}(x_k, x_l)$ and from (11) it follows that

$$I_{(\Gamma,A)} = I_{(\Gamma',A')} \times I_{(\Gamma^*,A^*)}.$$

Notice also that

$$P_0^{\Gamma} = P_0^{\Gamma'} \times P_0^{\Gamma^*} \times P_0^{\Gamma''}$$

where

$$\Gamma'' = (x_l, \dots, x_n = x).$$

Hence

$$\text{(21)} \quad \sum_{(\Gamma,A)\in\mathcal{G}_n^{(k,l)}(x_0,x)} \delta^{|A|} I_{(\Gamma,A)} P_0^{\Gamma} = \sum_{y,y'\in\mathbb{Z}^d} \sum_{(\Gamma',A')\in\mathcal{G}_k(x_0,y)} \delta^{|A'|} I_{(\Gamma',A')} P_0^{\Gamma'} \times \sum_{(\Gamma^*,A^*)\in\mathcal{B}_{(k,l)}(y,y')} \delta^{|A^*|} I_{(\Gamma^*,A^*)} P_0^{\Gamma^*} \times \sum_{\Gamma''=(x_l=y', x_{l+1},\dots,x_n=x)} P_0^{\Gamma''}.$$

The terms in the right hand side of (21) can be expressed as

$$\sum_{(\Gamma',A')\in\mathcal{G}_k(x_0,y)} \delta^{|A|} I_{(\Gamma',A')} P_0^{\Gamma'} = P_\delta(X_k = y | X_0 = x_0),$$

$$\sum_{\Gamma''=(x_l=y', x_{l+1},\dots,x_n=x)} P_0^{\Gamma''} = p^{(n-l)}(y', x),$$

and using (18) we get therefore

$$\sum_{(\Gamma,A)\in\mathcal{G}_n^{(k,l)}(x_0,x)} \delta^{|A|} I_{(\Gamma,A)} P_0^{\Gamma} = \sum_{y,y'\in\mathbb{Z}^d} P_\delta(X_k = y | X_0 = x_0) \mathcal{J}_{l-k}(\delta, y, y') p^{(n-l)}(y', x).$$

The last identity with (20) gives (17), and therefore our proposition is proved. □

Let us notice now that the Markov process $(X_t, \xi_t(x), x \in \mathbb{Z}^d)$ is invariant with respect to the drifts in $\mathbb{Z}^d$, and therefore for any cluster (Γ, A) and for any $x \in \mathbb{Z}^d$

$$I_{(\Gamma+x,A)} = I_{(\Gamma,A)}$$

and

$$P_0^{\Gamma+x} = P_0^{\Gamma},$$

where for $\Gamma = (x_0, \ldots, x_n)$ we denote

$$\Gamma + x = (x_0 + x, \ldots, x_n + x).$$

This implies that

$$\begin{aligned} \mathcal{J}_n(\delta, x, y) &= \mathcal{J}_n(\delta, x + y', y + y') = \mathcal{J}_n(\delta, 0, y - x), \\ \mathcal{J}_n^1(\delta, x, y) &= \mathcal{J}_n^1(\delta, x + y', y + y') = \mathcal{J}_n^1(\delta, 0, y - x), \end{aligned} \tag{22}$$

for all $x, y, y' \in \mathbb{Z}^d$.

Let us define the functions $\widetilde{\mathcal{J}}_n(\delta, \alpha)$ and $\widetilde{\mathcal{J}}_n^1(\delta, \alpha)$ by setting

$$\begin{aligned} \widetilde{\mathcal{J}}_n(\delta, \alpha) &= \sum_{x \in \mathbb{Z}^d} e^{\langle \alpha, x \rangle} \mathcal{J}_n(\delta, 0, x), \\ \widetilde{\mathcal{J}}_n^1(\delta, \alpha) &= \sum_{x \in \mathbb{Z}^d} e^{\langle \alpha, x \rangle} \mathcal{J}_n^1(\delta, 0, x), \end{aligned} \tag{23}$$

and consider the generating functions

$$\begin{aligned} R(\delta, \alpha, z) &= \sum_{n=0}^{+\infty} z^n R_n(\delta, \alpha), \\ \widetilde{\mathcal{J}}(\delta, \alpha, z) &= \sum_{n=1}^{+\infty} z^n \widetilde{\mathcal{J}}_n(\delta, \alpha), \\ \widetilde{\mathcal{J}}^1(\delta, \alpha, z) &= \sum_{n=1}^{+\infty} z^n \widetilde{\mathcal{J}}_n^1(\delta, \alpha), \end{aligned}$$

for $z \in \mathbb{C}$.

PROPOSITION 4.2. – *For any $\delta \in \mathbb{C}$ and $\alpha \in \mathbb{R}^d$, the functions $R(\delta, \alpha, z)$, $\widetilde{\mathcal{J}}(\delta, \alpha, z)$ and $\widetilde{\mathcal{J}}^1(\delta, \alpha, z)$ are analytic with respect to z in a neighborhood of $z = 0$, and the following equation holds*

$$R(\delta, \alpha, z) = \Big(1 + \widetilde{\mathcal{J}}^1(\delta, \alpha, z) + R(\delta, \alpha, z)\widetilde{\mathcal{J}}(\delta, \alpha, z)\Big) R(0, \alpha, z). \tag{24}$$

Proof of Proposition 4.2. – Let us show first that for any $\delta \in \mathbb{C}$ and $\alpha \in \mathbb{R}^d$ the following equations hold.

$$
\begin{aligned}
(25) \qquad R_n(\delta,\alpha) = R_n(0,\alpha) + \sum_{m,0<m\le n} \tilde{\mathcal{J}}^1_m(\delta,\alpha) R_{n-m}(0,\alpha) \\
+ \sum_{m,l:0\le m<l\le n} R_m(\delta,\alpha)\tilde{\mathcal{J}}_{l-m}(\delta,\alpha)R_{n-l}(0,\alpha),\ n \in \mathbb{Z}_+,
\end{aligned}
$$

with the convention

$$R_0(\delta,\alpha) = R_0(0,\alpha) = 1.$$

Because of Proposition 4.1 and (22) it is sufficient to show that the series in the right hand side of (23) are absolutely convergent. For this, we notice that for any cluster (Γ, A) the relations (9) and (3) imply

$$|I_{(\Gamma,A)}| \le 1$$

and so from (12) it follows that

$$|\mathcal{J}_n(\delta,x,y)| + |\mathcal{J}^1_k(\delta,x,y)| \le \sum_{(\Gamma,A)\in\mathcal{B}_n(x,y)} |\delta|^{|A|} P^\Gamma_0.$$

Because of

$$\mathcal{B}_n(x,y) \subset \mathcal{G}_n(x,y)$$

the last inequality yields

$$(26) \qquad |\mathcal{J}_n(\delta,x,y)| + |\mathcal{J}^1_n(\delta,x,y)| \le \sum_{(\Gamma,A)\in\mathcal{G}_n(x,y)} |\delta|^{|A|} P^\Gamma_0,$$

where right hand side can be expressed as

$$
\begin{aligned}
\sum_{(\Gamma,A)\in\mathcal{G}_n(x,y)} |\delta|^{|A|} P^\Gamma_0 &= \sum_{A\in\{0,\dots,n-1\}} |\delta|^{|A|} \times \sum_{\Gamma=(x_0=x,x_1,\dots,x_n=y)} P^\Gamma_0 \\
&= (1+|\delta|)^n p^{(n)}(x,y).
\end{aligned}
$$

Thus from (26) we get

$$(27) \qquad |\mathcal{J}_n(\delta,x,y)| + |\mathcal{J}^1_k(\delta,x,y)| \le (1+|\delta|)^n p^{(n)}(x,y).$$

Using assumption 5 the series

$$\sum_{x\in\mathbb{Z}^d} e^{\langle\alpha,x\rangle} p^{(n)}(0,x) = \left(\sum_{x\in\mathbb{Z}^d} e^{\langle\alpha,x\rangle} p(0,x)\right)^n < +\infty$$

converge absolutely for any $\alpha \in \mathbb{R}^d$. Hence because of (27) the series

$$\sum_{x\in\mathbb{Z}^d} e^{\langle\alpha,x\rangle} \mathcal{J}_n(\delta,0,x) \text{ and } \sum_{x\in\mathbb{Z}^d} e^{\langle\alpha,x\rangle} \mathcal{J}_n^1(\delta,0,x)$$

converge also absolutely for any $\alpha \in \mathbb{R}^d$, and therefore (25) is verified.

We are ready now to prove (24). For this, it is sufficient to show that the functions $\mathcal{J}(\delta,\alpha,z)$, $\mathcal{J}^1(\delta,\alpha,z)$, $R(0,\alpha,z)$ and $R(\delta,\alpha,z)$ are analytic with respect to z in some neighborhood of $z=0$. Indeed, the inequality (27) implies

$$|\mathcal{J}_n(\delta,\alpha)| + |\mathcal{J}_k^1(\delta,\alpha)| \leq (1+|\delta|)^n R_n(0,\alpha) \tag{28}$$

where

$$R_n(0,\alpha) = (R(\alpha))^n.$$

Similarly

$$R_n(\delta,\alpha) \leq (1+|\delta|)^n (R(\alpha))^n.$$

Hence the functions $\mathcal{J}(\delta,\alpha,z)$, $\mathcal{J}^1(\delta,\alpha,z)$ and $R(\delta,\alpha,z)$ are analytic with respect to z in the disk

$$\left\{z \in \mathbb{C} : |z| < \frac{1}{(1+|\delta|)R(\alpha)}\right\}.$$

The function

$$R(0,\alpha,z) = \sum_{n=0}^{+\infty} (R(\alpha))^n z^n$$

is obviously also analytic with respect to z in the same disk. Therefore (24) is verified, and Proposition 4.2 is proved. □

In the section 6 we shall show that for δ small enough the functions $\mathcal{J}(\delta,\alpha,z)$ and $\mathcal{J}^1(\delta,\alpha,z)$ can be analytically continued to the disk

$$\left\{z \in \mathbb{C} : |z| < \frac{1}{R(\alpha)}(1+\epsilon)\right\}, \quad \epsilon > 0,$$

where the function $R(\delta,\alpha,z)$ has a unique simple pole. For this we need of the more precise estimates of the values $\mathcal{J}_n(\delta,\alpha)$ and $\mathcal{J}_k^1(\delta,\alpha)$ then (28). We get these estimates in the following section.

5. THE CLUSTER ESTIMATES

Consider the Markov chain with state space S and transition probabilities $q(s,s')$, $s,s' \in S$, governing the environment $\xi_t(x), x \in \mathbb{Z}^d$ when the particle does not interfere. This Markov chain is ergodic by assumption. So due to the finiteness of the state space S, there exist C_0 and $\gamma > 0$ such that for any $t \geq 0$,

$$\max_{s\in S} \sum_{s'\in S} |q^{(t)}(s,s') - \pi(s')| \leq C_0 e^{-\gamma t}, \tag{29}$$

$(\pi(s)), s \in S$ being the invariant measure of this Markov chain.

We use (29) to estimate the values of $\widetilde{\mathcal{J}}_n(\delta,\alpha)$ and $\widetilde{\mathcal{J}}_n^1(\delta,\alpha), \alpha \in \mathbb{R}^d$. The main result of this section is the following lemma.

LEMMA 5.1. – *For $n \in \mathbb{Z}_+, x \in \mathbb{Z}^d$,*

$$|\mathcal{J}_n(\delta,0,x)| + |\mathcal{J}_n^1(\delta,0,x)| \leq \theta_\delta(1+\theta_\delta)^{n-1} e^{-(n-1)\gamma} p^{(n)}(0,x), \tag{30}$$

where we denote

$$\theta_\delta = C_0 e^{\gamma} |\delta|.$$

Proof. – For any $x \in \mathbb{Z}^d, n \in \mathbb{Z}_+$, from (12) we get

$$|\mathcal{J}_n(\delta,0,x)| + |\mathcal{J}_n^1(\delta,0,x)| \leq \sum_{(\Gamma,A)\in\mathcal{B}_n(0,x)} |I_{(\Gamma,A)}| P_0^\Gamma |\delta|^{|A|}, \tag{31}$$

where $\mathcal{B}_n(0,x)$ is the set of the irreducible clusters $(\Gamma = (x_0,\ldots,x_n), A)$ such that $x_0 = 0$ and $x_n = x$.
To estimate $I_{(\Gamma,A)}$ we rewrite (11) using (5) as follows

$$I_{(\Gamma,A)} = \prod_{t\in A_\Gamma^1} \left(\sum_{s,s'\in S} q_0(s)(q^{(t)}(s,s') - \pi(s'))c(x_{t+1} - x_t, s') \right)$$
$$\times \prod_{t\in A_\Gamma^0} \left(\sum_{s,s'\in S} \pi_0(s)(q^{(t-\tau_t-1)}(s,s') - \pi(s'))c(x_{t+1} - x_t, s') \right).$$

Thus using (29) we get

$$|I_{(\Gamma,A)}| \leq \prod_{t\in A_\Gamma^1} C_0 e^{-\gamma t} \prod_{t\in A_\Gamma^0} C_0 e^{-\gamma(t-\tau_t-1)}. \tag{32}$$

For the right hand side of (32) we notice that for $(\Gamma, A) \in \mathcal{B}_n(0,x)$,

$$U(\Gamma, A) = \bigcup_{t\in A_\Gamma^1} [0,t] \bigcup_{t\in A_\Gamma^0} [\tau_t, t] = [0, n-1],$$

and hence

$$\sum_{t\in A_\Gamma^1} t + \sum_{t\in A_\Gamma^0} (t-\tau_t) \geq n-1.$$

The relation (32) implies therefore for $(\Gamma, A) \in \mathcal{B}_n(0,x)$

$$|I_{(\Gamma,A)}| \leq (C_0 e^\gamma)^{|A|} \exp\{-\gamma(n-1)\}. \tag{33}$$

From (31) and (33) we get

(34)

$$|\mathcal{J}_n(\delta,0,x)| + |\mathcal{J}_n^1(\delta,0,x)| \leq \sum_{(\Gamma,A)\in\mathcal{B}_n(0,x)} \exp\{-\gamma(n-1)\}|\delta C_0 e^\gamma|^{|A|} P_0^\Gamma.$$

Notice now that

$$\sum_{(\Gamma,A)\in\mathcal{B}_n(0,x)} |\delta C_0 e^\gamma|^{|A|} P_0^\Gamma \leq \sum_{(\Gamma,A)\in\mathcal{G}_n(0,x):n-1\in A} |\delta C_0 e^\gamma|^{|A|} P_0^\Gamma. \tag{35}$$

Indeed, in the left hand side of (35) the summation is over all irreducible clusters $(\Gamma, A) \in \mathcal{B}_n(0,x)$, and in the right hand side of (35) the summation is over all clusters $\Gamma = (x_0 = 0, x_1, \ldots, x_n = x), A \subseteq \{0, \ldots, n-1\}$ such that $n-1 \in A$. But for any irreducible cluster $(\Gamma, A) \in \mathcal{B}_n(0,x)$ we have $n-1 \in A$, and so (35) holds.

Now for the right hand side of (35) we remark that

$$\sum_{(\Gamma,A)\in\mathcal{G}_n(0,x):n-1\in A} |\delta C_0 e^\gamma|^{|A|} P_0^\Gamma$$
$$\leq \sum_{A\subseteq\{0,\ldots,n-1\}:n-1\in A} |\delta C_0 e^\gamma|^{|A|} \times \sum_{\Gamma=(0,x_1,\ldots,x_n):x_n=x} P_0^\Gamma,$$

where

$$\sum_{A\subseteq\{0,\ldots,n-1\}:n-1\in A} |\delta C_0 e^\gamma|^{|A|} = |\delta C_0 e^\gamma|(1+|\delta C_0 e^\gamma|)^{n-1}$$

and

$$\sum_{\Gamma=(0,x_1,\ldots,x_n):x_n=x} P_0^\Gamma = p^{(n)}(0,x) = P_0(X_n = x|X_0 = 0).$$

Thus from (35) we get

$$\sum_{(\Gamma,A)\in\mathcal{B}_n(0,x)} |\delta C_0 e^{\gamma}|^{|A|} P_0^{\Gamma} \leq |\delta C_0 e^{\gamma}|(1+|\delta C_0 e^{\gamma}|)^{n-1} p^{(n)}(0,x) \tag{36}$$

The relation (30) follows from (34) and (36). Our lemma is proved. □

From Lemma 5.1 we immediately get

COROLLARY 5.1. – *For* $n \in \mathbb{Z}_+, \alpha \in \mathbb{R}^d$,

$$|\widetilde{\mathcal{J}}_n(\delta,\alpha)| + |\widetilde{\mathcal{J}}_n^1(\delta,\alpha)| \leq \theta_\delta(1+\theta_\delta)^{n-1} e^{-(n-1)\gamma}(R(\alpha))^n. \tag{37}$$

6. THE EXISTENCE OF $H_\delta(\alpha) = \lim_{t\to+\infty} \frac{1}{t}\log R_t(\delta,\alpha)$

In the case $\delta = 0$ there is no any influence of the environment on the particle, and therefore X_t is a homogeneous random walk in $\mathbb{Z}^d$. Hence for $\delta = 0$, we have

$$R_n(0,\alpha) = (R(\alpha))^n.$$

and

$$R(0,\alpha,z) = \frac{1}{1 - zR(\alpha)}. \tag{38}$$

Using (38) we rewrite (24) as follows

$$R(\delta,\alpha,z) = \frac{1 + \widetilde{\mathcal{J}}^1(\alpha,z)}{1 - zR(\alpha) - \widetilde{\mathcal{J}}(\delta,\alpha,z)}.$$

From Corollary 5.1 it follows that the functions $\widetilde{\mathcal{J}}(\delta,\alpha,z)$ and $\widetilde{\mathcal{J}}^1(\delta,\alpha,z)$ can be analytically continued to the disk

$$Q_\alpha = \{z \in \mathbb{C} : |z| < \frac{1}{(1+\theta_\delta)e^{-\gamma}R(\alpha)}\},$$

where $\theta_\delta = C_0|\delta|e^{\gamma}$ and $C_0, \gamma > 0$ are the constants of the identity (29), and so the function $z \to R(\delta,\alpha,z)$ is meromorphic in Q_α. In order to prove the existence of the limit $\lim_{t\to+\infty} \frac{1}{t}\log R_t(\delta,\alpha)$ we need to study the poles of this function. For this we consider the following proposition.

PROPOSITION 6.1. – *Let* $r \in \mathbb{R}$ *and* $e^{-\gamma} < r < 1$. *Consider*

$$Q_\alpha^r = \left\{ z \in \mathbb{C} : |z| < \frac{r}{(1+\theta_\delta)e^{-\gamma}R(\alpha)} \right\},$$

and set

$$\delta_1(r) = \frac{1}{C_0 e^\gamma} \min\left\{ re^\gamma - 1, \frac{1-r}{r(e^\gamma+1)-1}, \frac{(1-r)(re^\gamma-1)}{r(e^\gamma-1)+1} \right\}. \tag{39}$$

Then for any $\delta \in \mathbb{C}$ *such that* $|\delta| < \delta_1(r)$ *the following propositions hold*

1. *for any* $z \in Q_\alpha^r$

$$1 + \tilde{\mathcal{J}}^1(\delta, \alpha, z) \neq 0;$$

2. *the equation*

$$1 - zR(\alpha) - \tilde{\mathcal{J}}(\delta, \alpha, z) = 0 \tag{40}$$

has a unique simple solution $z_0(\alpha, \delta)$ *in* Q_α^r;

3. *for* $\delta \in \mathbb{R}$ *the solution* $z_0(\alpha, \delta)$ *is real and strictly positive.*

Proof. – For $|z| \leq \frac{r}{(1+\theta_\delta)e^{-\gamma}R(\alpha)}$ Lemma 5.1 implies

$$|\tilde{\mathcal{J}}^1(\delta, \alpha, z)| + |\tilde{\mathcal{J}}(\delta, \alpha, z)| \leq \frac{\theta_\delta r}{(1+\theta_\delta)e^{-\gamma}(1-r)}. \tag{41}$$

But for $|\delta| < \delta_1(r)$, (39) gives $\theta_\delta = C_0 e^\gamma \delta < \frac{1-r}{r(e^\gamma+1)-1}$ which implies

$$\frac{\theta_\delta r}{(1+\theta_\delta)e^{-\gamma}(1-r)} < 1. \tag{42}$$

Hence for $|\delta| < \delta_1(r)$, the relation (41) yields

$$|\tilde{\mathcal{J}}^1(\delta, \alpha, z)| < 1,$$

for any $z \in Q_\alpha^r$, and the first part of Proposition 6.1 is therefore verified.

In order to prove the second part of our proposition we remark that for $|\delta| < \delta_1(r)$, (39) gives

$$\theta_\delta < re^\gamma - 1$$

and therefore

$$(1+\theta_\delta)e^{-\gamma} < r < 1.$$

From the last inequality it follows that $\frac{1}{R(\alpha)} \in Q^r_\alpha$, and so the function $z \to 1 - zR(\alpha)$ has in Q^r_α a unique simple zero $z = \frac{1}{R(\alpha)}$. Thus to prove the second part of Proposition 6.1 it is sufficient, using Rouché's theorem, to show that for $|z| = \frac{r}{(1+\theta_\delta)e^{-\gamma}R(\alpha)}$ the following relation holds

$$|1 - zR(\alpha)| > |\widetilde{\mathcal{J}}(\delta,\alpha,z)|. \tag{43}$$

Let us verify (43). Indeed, for $|z| = \frac{r}{(1+\theta_\delta)e^{-\gamma}R(\alpha)}$

$$|1 - zR(\alpha)| \geq |z|R(\alpha) - 1 \geq \frac{r}{(1+\theta_\delta)e^{-\gamma}} - 1, \tag{44}$$

but for $|\delta| < \delta_1(r)$, (39) gives

$$\theta_\delta < \frac{(1-r)(re^\gamma - 1)}{r(e^\gamma - 1) + 1},$$

which is equivalent to

$$\frac{\theta_\delta r}{(1+\theta_\delta)e^{-\gamma}(1-r)} < \frac{r}{(1+\theta_\delta)e^{-\gamma}} - 1, \tag{45}$$

and comparison of the last inequality with (41) and (44) gives (43). The second part of our proposition is therefore proved.

Let us prove now that for $\delta \in \mathbb{R}$, the solution $z_0(\alpha,\delta)$ is real and strictly positive. Indeed, $z_0(\alpha,\delta)$ is a unique simple solution of the equation (40) in the disk

$$Q^r_\alpha = \left\{ z \in \mathbb{C} : |z| < \frac{r}{(1+\theta_\delta)e^{-\gamma}R(\alpha)} \right\},$$

where

$$1 - zR(\alpha) - \widetilde{\mathcal{J}}(\delta,\alpha,z) = 1 - zR(\alpha) - \sum_{n=1}^{+\infty} \mathcal{J}_n(\delta,\alpha)z^n,$$

and for $\delta \in \mathbb{R}$, $R(\alpha)$ and $\widetilde{\mathcal{J}}_n(\delta,\alpha)$ are real for all $n \in \mathbb{Z}_+$. Suppose that $z_0(\alpha,\delta)$ is not real, then its conjugate $\overline{z_0(\alpha,\delta)}$ is also a solution of the equation (40), and obviously $\overline{z_0(\alpha,\delta)} \in Q^r_\alpha$. We get therefore a contradiction with the second part of our proposition. Hence for $\delta \in \mathbb{R}$, $z_0(\alpha, \delta)$ is real.

To prove that $z_0(\alpha, \delta) > 0$ we use again Rouché's theorem. Consider

$$\varepsilon = \min\left(1, \frac{r}{(1+\theta_\delta)e^{-\gamma}} - 1\right)$$

and

$$D_\alpha^\varepsilon = \{z \in \mathbb{C} : |zR(\alpha) - 1| < \varepsilon\},$$

then $D_\alpha^\varepsilon \subset Q_\alpha^r$, and for $z \in \partial D_\alpha^\varepsilon = \{z \in \mathbb{C} : |1 - zR(\alpha)| = \varepsilon\}$ the relations (41), (42) and (45) yield

$$|\widetilde{\mathcal{J}}(\delta, \alpha, z)| < |1 - zR(\alpha)|.$$

Using Rouché's theorem we get therefore

$$z_0(\alpha, \delta) \in D_\alpha^\varepsilon. \tag{46}$$

Notice now that $Re(z) > 0$ for any $z \in D_\alpha^\varepsilon$. Hence (46) implies that $z_0(\alpha, \delta) > 0$. Proposition 6.1 is proved. □

Using Proposition 6.1 we shall prove now

PROPOSITION 6.2. – *Consider*

$$\delta_1 = \sup_{e^{-\gamma} < r < 1} \delta_1(r).$$

Then for any $\delta \in \mathbb{R}$ for which $|\delta| < \delta_1$ and for any $\alpha \in \mathbb{R}^d$

$$\lim_{t \to +\infty} \frac{1}{t} \log R_t(\delta, \alpha) = -\log(z_0(\alpha, \delta)),$$

Proof. – Indeed, let $\delta \in \mathbb{R}$ and $|\delta| < \delta_1$, then there exists $e^{-\gamma} < r < 1$ such that $|\delta| < \delta_1(r)$, and using Proposition 6.1 it follows that the function

$$R(\delta, \alpha, z) = \frac{1 + \widetilde{\mathcal{J}}^1(\delta, \alpha, z)}{1 - zR(\alpha) - \widetilde{\mathcal{J}}(\delta, \alpha, z)}$$

is meromorphic in the disk $Q_\alpha^r = \{z \in \mathbb{C} : |z| < \frac{r}{(1+\theta_\delta)e^{-\gamma}R(\alpha)}, \}$, and it has in this disk a unique simple pole $z_0(\alpha, \delta)$. Hence

$$R(\delta, \alpha, z) = \frac{c}{z - z_0(\alpha, \delta)} + f(z),$$

where c is the residue of the function $R(\delta, \alpha, z)$ at the point $z_0(\alpha, \delta)$ and the function $f(z)$ is analytic in the disk Q_α^r. Consequently

$$R(\delta, \alpha, z) = \sum_{n=0}^{+\infty} (b_n - \frac{c}{(z_0(\alpha, \delta))^{n+1}}) z^n,$$

where

$$\limsup_{n \to +\infty} \sqrt[n]{|b_n|} < \frac{1}{z_0(\alpha, \delta)}.$$

But

$$R(\delta, \alpha, z) = \sum_{n=0}^{+\infty} R_n(\delta, \alpha) z^n.$$

Hence

$$R_n(\delta, \alpha) = b_n - \frac{c}{(z_0(\alpha, \delta))^{n+1}}, \quad n \in \mathbb{N},$$

and therefore

$$\lim_{n \to +\infty} \frac{1}{n} \log(R_n(\delta, \alpha)) = - \log(z_0(\alpha, \delta)).$$

Proposition 6.2 is proved. □

7. ANALYTICITY OF THE FUNCTION H_δ

LEMMA 7.1. – *Let $\alpha^* \in \mathbb{R}^d$ and $\delta^* \in \mathbb{C}$, such that $|\delta^*| < \delta_1$, where δ_1 is the constant of Proposition 6.2. Then the function*

$$(\delta, \alpha) \to z_0(\alpha, \delta)$$

is analytic in a neighborhood of (δ^*, α^*).

Proof. – Indeed, since $|\delta| < \delta_1$, then there exists $e^{-\gamma} < r < 1$ such that $|\delta| < \delta_1(r)$, and Proposition 6.1 implies therefore that $z_0(\alpha^*, \delta^*)$ is a unique simple zero of the equation

$$(47) \qquad 1 - R(\alpha^*)z - \widetilde{\mathcal{J}}(\delta^*, \alpha^*, z) = 0$$

in the disk $Q_{\alpha^*}^r = \{z \in \mathbb{C} : |z| < \frac{r}{(1+\theta_\delta)e^{-\gamma} R(\alpha^*)}\}$.

Thus to prove our proposition we can use implicit function theorem applied to the equation (47). For this we have to verify that the function

$$(\delta, \alpha, z) \to R(\alpha)z + \widetilde{\mathcal{J}}(\delta, \alpha, z)$$

is analytic in some neighborhood of $(\delta^*, \alpha^*, z_0(\alpha^*, \delta^*))$ and

$$\frac{d}{dz}(R(\alpha)z + \widetilde{\mathcal{J}}(\delta, \alpha, z))\Big|_{z=z_0(\alpha^*,\delta^*)} \neq 0. \tag{48}$$

The relation (48) is verified, because the solution $z_0(\alpha^*, \delta^*)$ of the equation (47) is simple.

Let us show that the function $R(\alpha)z + \widetilde{\mathcal{J}}(\delta, \alpha, z)$ is analytic in a neighborhood of $(\delta^*, \alpha^*, z_0(\alpha^*, \delta^*))$. Indeed, the assumption 5 implies that the series in the right hand side of

$$R(\alpha) = \sum_{x\in\mathbb{Z}^d} e^{\langle\alpha,x\rangle} p(0, x)$$

converges uniformly with respect to $\alpha \in \mathbb{C}^d$ on every compact subset of $\mathbb{C}^d$, and therefore the function $R(\alpha)$ is analytic with respect to α everywhere in $\mathbb{C}^d$. Notice also that

$$R_n(\alpha) = \sum_{x\in\mathbb{Z}^d} e^{\langle\alpha,x\rangle} p^{(n)}(0, x) = (R(\alpha))^n = \left(\sum_{x\in\mathbb{Z}^d} e^{\langle\alpha,x\rangle} p(0, x)\right)^n,$$

and therefore the series

$$\sum_{x\in\mathbb{Z}^d} e^{\langle\alpha,x\rangle} p^{(n)}(0, x)$$

converges uniformly with respect to $\alpha \in \mathbb{C}^d$ on every compact subset of $\mathbb{C}^d$. Using now lemma 5.1 we conclude that the series in the right hand side of

$$\widetilde{\mathcal{J}}_n(\delta, \alpha) = \sum_{x\in\mathbb{Z}^d} e^{\langle\alpha,x\rangle} \mathcal{J}_n(\delta, 0, x)$$

converges also uniformly with respect to $(\delta, \alpha) \in \mathbb{C}^{d+1}$ on every compact subset of $\mathbb{C}^{d+1}$. The functions $\mathcal{J}_n(\delta, 0, x)$ are obviously analytic with respect to δ everywhere in $\mathbb{C}$, and therefore the functions $\widetilde{\mathcal{J}}_n(\delta, \alpha)$, $n \geq 0$, are analytic with respect to (δ, α) everywhere in $\mathbb{C}^{d+1}$. Lemma 5.1 implies now that the function $\widetilde{\mathcal{J}}(\delta, \alpha, z)$ is analytic in (δ, α, z) if

$$(1 + \theta_\delta)e^{-\gamma}|z|R(Re(\alpha)) < 1. \tag{49}$$

where for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{C}^d$ we denote $Re(\alpha) = (Re(\alpha_1), \ldots, Re(\alpha_d)) \in \mathbb{R}^d$. To conclude now that the function

$R(\alpha)z + \widetilde{\mathcal{J}}(\delta, \alpha, z)$ is analytic in a neighborhood of $(\delta^*, \alpha^*, z_0(\alpha^*, \delta^*))$ it is sufficient to notice that for $\delta = \delta^*$, $\alpha = \alpha^*$ and $z = z_0(\alpha^*, \delta^*)$ the relation (49) holds. Proposition 7.1 is therefore proved. □

COROLLARY 7.1. – *For any $\alpha^* \in \mathbb{R}^d$ and $\delta^* \in \mathbb{R}$ such that $|\delta^*| < \delta_1$ the function $H_\delta(\alpha)$ is analytic with respect to (δ, α) in (δ^*, α^*) and for some $\epsilon = \epsilon(\alpha^*, \delta^*) > 0$ it can be analytically continued in the domain $\{(\delta, \alpha) \in \mathbb{C}^{d+1} : |\delta| < \delta_1, |\alpha - \alpha^*| < \epsilon\}$.*

Corollary 7.1 follows from Proposition 7.1, since for $\alpha \in \mathbb{R}^r$ and $\delta \in \mathbb{R}$ such that $|\delta| < \delta_1$,

$$H_\delta(\alpha) = -\log z_0(\alpha, \delta),$$

and $z_0(\alpha, \delta)$ is real and strictly positive (see Proposition 6.1 and Proposition 6.2).

8. CONVEXITY OF THE FUNCTION H_δ

In this section we prove the following proposition.

PROPOSITION 8.1. – *There exists $\delta_2 > 0$ such that for $\delta \in \mathbb{R}$ and $|\delta| < \delta_2$ the function*

$$\alpha \to H_\delta(\alpha)$$

is strictly convex everywhere in $\mathbb{R}^d$.

To prove this proposition we shall use the following three lemmas.

LEMMA 8.1. – *The function $R(\alpha)e^{\alpha_0}$ is strictly convex everywhere in $\mathbb{R}^{d+1}$ and for any $\overline{v} = (v_0, \ldots, v_d)$ and $(\alpha_0, \alpha) \in \mathbb{R}^{d+1}$ the following relation holds*

$$\langle \overline{v}, \partial_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0}) \rangle^2 \le R(\alpha)e^{\alpha_0} \langle \overline{v}, \partial^2_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0})\overline{v}, \rangle \tag{50}$$

where we denote

$$\langle \overline{v}, \partial_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0}) \rangle = \sum_{i=0}^{d} v_i \frac{\partial}{\partial \alpha_i}(R(\alpha)e^{\alpha_0}),$$

and

$$\langle \overline{v}, \partial^2_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0})\overline{v} \rangle = \sum_{i,j=0}^{d} v_i v_j \frac{\partial^2}{\partial \alpha_i \partial \alpha_j}(R(\alpha)e^{\alpha_0}).$$

Proof. – To prove the first part of our lemma we have to show that

$$\langle \overline{v}, \partial^2_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0})\overline{v}\rangle > 0 \tag{51}$$

for all $(\alpha_0, \alpha) \in \mathbb{R}^{d+1}$ and $\overline{v} \in \mathbb{R}^{d+1}\backslash\{0\}$.

Let us consider the set

$$\mathcal{E} = \{(x_0, x) \in \mathbb{Z} \times \mathbb{Z}^d : x_0 = 1 \text{ and } p(0,x) \neq 0\},$$

then

$$R(\alpha)e^{\alpha_0} = \sum_{\overline{x}=(x_0,x)\in\mathcal{E}} p(0,x)e^{\langle\alpha,x\rangle+\alpha_0},$$

where because of the assumption 5 the series converge uniformly with respect to (α_0, α) on every compact subset of $\mathbb{C}^{d+1}$, and therefore for any $\overline{v} = (v_0, \ldots, v_d) \in \mathbb{R}^{d+1}$

$$\langle \overline{v}, \partial^2_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0})\overline{v}\rangle = \sum_{\overline{x}=(x_0,x)\in\mathcal{E}} \langle \overline{v}, \overline{x}\rangle^2 p(0,x)e^{\langle\alpha,x\rangle+\alpha_0}.$$

Hence to verify (51) it is sufficient to show that the set $\mathcal{E}$ contains a basis of $\mathbb{R}^{d+1}$.

Furthermore, by assumption the matrix $(p(x,y))_{x,y\in\mathbb{Z}^d}$ is irreducible, and therefore the set $\{x \in \mathbb{Z}^d : p(0,x) \neq 0\}$ contains a basis of R^d. Thus to verify that the set $\mathcal{E}$ contains a basis of $\mathbb{R}^{d+1}$ it is sufficient to show that the vector $\overline{e} = (1, 0)$ is included to the linear space spanned by $\mathcal{E}$.

Consider now $\overline{y} = (1, y) \in \mathcal{E}$. Because the matrix $(p(x,y))_{x,y\in\mathbb{Z}^d}$ is irreducible it follows that $p^{(n)}(0, -y) \neq 0$ for some $n \geq 1$. This implies that there exist $\overline{x}_1 = (1, x_1), \ldots, \overline{x}_n = (1, x_n) \in \mathcal{E}$ such that $x_1 + \ldots + x_n = -y$ and consequently

$$\overline{y} + \overline{x}_1 + \ldots + \overline{x}_n = (n + 1, 0) = (n + 1)\overline{e}.$$

Hence $\overline{e}$ is included to the linear space spanned by $\mathcal{E}$, and the first part of our lemma is proved. To prove the second one we notice that

$$\langle \overline{v}, \partial_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0})\rangle = \sum_{\overline{x}=(x_0,x)\in\mathcal{E}} \langle \overline{v}, \overline{x}\rangle p(0,x)e^{\langle\alpha,x\rangle+\alpha_0},$$

and Cauchy-Schwartz inequality implies

$$\left(\sum_{\overline{x}=(x_0,x)\in\mathcal{E}} \langle \overline{v}, \overline{x}\rangle p(0,x)e^{\langle\alpha,x\rangle+\alpha_0}\right)^2 \leq \sum_{\overline{x}=(x_0,x)\in\mathcal{E}} p(0,x)e^{\langle\alpha,x\rangle+\alpha_0} \times \sum_{\overline{x}=(x_0,x)\in\mathcal{E}} \langle \overline{v}, \overline{x}\rangle^2 p(0,x)e^{\langle\alpha,x\rangle+\alpha_0}.$$

The last inequality gives (50) and therefore Lemma 8.1 is proved. □

LEMMA 8.2. – *Let* $0 < r < 1$, *consider*

$$\delta_2(r) = \frac{(1-r)^3}{C_0 e^{-\gamma}(1+2r)},$$

then for $\delta \in \mathbb{R}$ *such that* $|\delta| < \delta_2(r)$ *the function* $R(\alpha)e^{\alpha_0} + \widetilde{\mathcal{J}}(\delta, \alpha, e^{\alpha_0})$ *is strictly convex in the domain*

$$V_r = \{(\alpha_0, \alpha) \in \mathbb{R}^{d+1} : e^{\alpha_0}(1+\theta_\delta)e^{-\gamma}R(\alpha) < r\}.$$

Proof. – Recall that

$$\widetilde{\mathcal{J}}(\delta, \alpha, e^{\alpha_0}) = \sum_{n=1}^{+\infty} \sum_{x \in \mathbb{Z}^d} e^{\langle \alpha, x\rangle + \alpha_0 n} \mathcal{J}_n(\delta, 0, x). \tag{52}$$

where because of the Lemma 5.1

$$|\mathcal{J}_n(\delta, 0, x)| < \theta_\delta(1+\theta_\delta)^{n-1} e^{-(n-1)\gamma} p^{(n)}(0, x). \tag{53}$$

Hence using the assumption 5 it follows that the series in the right hand side of (52) converge uniformly with respect to (α_0, α) on every compact subset of

$$\Omega = \{(\alpha_0, \alpha) \in \mathbb{C}^{d+1} : |e^{\alpha_0}(1+\theta_\delta)e^{-\gamma}R(\alpha)| < 1\}.$$

We conclude therefore that the function $\widetilde{\mathcal{J}}(\delta, \alpha, e^{\alpha_0})$ is analytic in Ω and for any $\overline{v} = (v_0, \ldots, v_d) \in \mathbb{R}^{d+1}$ and $(\alpha_0, \alpha) \in \Omega$

(54)

$$\langle \overline{v}, \partial^2_{(\alpha_0,\alpha)}(\widetilde{\mathcal{J}}(\delta, \alpha, e^{\alpha_0}))\overline{v}\rangle = \sum_{n=1}^{+\infty} \sum_{x \in \mathbb{Z}^d} \langle \overline{v}, \partial^2_{(\alpha_0,\alpha)}\left(e^{\langle \alpha, x\rangle + \alpha_0 n}\right)\overline{v}\rangle \mathcal{J}_n(\delta, 0, x).$$

Notice now that for $(\alpha_0, \alpha) \in \mathbb{R}^{d+1}$, for any $n \geq 0$

$$\langle v, \partial^2_{(\alpha_0,\alpha)}\left(e^{\langle \alpha, x\rangle + \alpha_0 n}\right)v\rangle = e^{\langle \alpha, x\rangle + \alpha_0 n}\left(\sum_{i=1}^{d} x_i v_i + n v_0\right)^2 \geq 0,$$

and so for $(\alpha_0, \alpha) \in \Omega \cap \mathbb{R}^{d+1}$ (53) and (54) yield

$$|\langle \overline{v}, \partial^2_{(\alpha_0,\alpha)}(\widetilde{\mathcal{J}}(\delta, \alpha, e^{\alpha_0}))\overline{v}\rangle| \leq \tag{55}$$
$$\sum_{n=1}^{+\infty} \sum_{x \in \mathbb{Z}^d} \langle \overline{v}, \partial^2_{(\alpha_0,\alpha)}\left(e^{\langle \alpha, x\rangle + \alpha_0 n}\right)\overline{v}\rangle \theta_\delta(1+\theta_\delta)^{n-1} e^{-(n-1)\gamma} p^{(n)}(0, x).$$

Consider now the function

$$\Psi(\alpha_0, \alpha) = \frac{R(\alpha)e^{\alpha_0}}{1 - (1+\theta_\delta)e^{-\gamma}R(\alpha)e^{\alpha_0}}.$$

For $(\alpha_0, \alpha) \in \Omega$ obviously

$$\Psi(\alpha_0, \alpha) = \sum_{n=1}^{+\infty} \sum_{x \in \mathbb{Z}^d} e^{\langle \alpha, x\rangle + \alpha_0 n}(1+\theta_\delta)^{n-1} e^{-(n-1)\gamma} p^{(n)}(0, x),$$

where the series converge uniformly with respect to (α_0, α) on every compact subset of Ω, and consequently

$$\text{(56)} \quad \langle \overline{v}, \partial^2_{(\alpha_0,\alpha)} \Psi(\alpha_0, \alpha)\overline{v}\rangle = \sum_{n=1}^{+\infty} \sum_{x \in \mathbb{Z}^d} \langle \overline{v}, \partial^2_{(\alpha_0,\alpha)}\Big(e^{\langle \alpha, x\rangle + \alpha_0 n}\Big)\overline{v}\rangle (1+\theta_\delta)^{n-1} e^{-(n-1)\gamma} p^{(n)}(0, x).$$

Thus for $(\alpha_0, \alpha) \in \Omega \cap \mathbb{R}^{d+1}$ the relations (55) and (56) give

$$\text{(57)} \qquad |\langle \overline{v}, \partial^2_{(\alpha_0,\alpha)}(\widetilde{\mathcal{J}}(\delta, \alpha, e^{\alpha_0}))\overline{v}\rangle| \le \theta_\delta \langle \overline{v}, \partial^2_{(\alpha_0,\alpha)} \Psi(\alpha_0, \alpha)\overline{v}\rangle.$$

But for $(\alpha_0, \alpha) \in \Omega$

$$\langle \overline{v}, \partial^2_{(\alpha_0,\alpha)} \Psi(\alpha_0, \alpha)\overline{v}\rangle = \frac{1}{(1 - (1+\theta_\delta)e^{-\gamma}R(\alpha)e^{\alpha_0})^2} \langle \overline{v}, \partial^2_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0})\overline{v}\rangle + \frac{2(1+\theta_\delta)e^{-\gamma}}{(1 - (1+\theta_\delta)e^{-\gamma}R(\alpha)e^{\alpha_0})^3} \langle \overline{v}, \partial_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0})\rangle^2,$$

where for $(\alpha_0, \alpha) \in \mathbb{R}^{d+1}$ lemma 8.1 gives

$$\langle \overline{v}, \partial_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0})\rangle^2 \le R(\alpha)e^{\alpha_0} \langle \overline{v}, \partial_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0})\rangle^2.$$

Hence for $(\alpha_0, \alpha) \in \Omega \cap \mathbb{R}^{d+1}$, from (57) it follows

$$\text{(58)} \quad |\langle \overline{v}, \partial^2_{(\alpha_0,\alpha)}(\widetilde{\mathcal{J}}(\delta, \alpha, e^{\alpha_0}))\overline{v}\rangle| \le \theta_\delta \frac{2r+1}{(1-r)^3} \langle \overline{v}, \partial^2_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0})\overline{v}\rangle,$$

with $r = (1+\theta_\delta)e^{-\gamma}R(\alpha)e^{\alpha_0}$. Moreover, since the function $\frac{2r+1}{(1-r)^3}$ is increasing on the interval $0 < r < 1$, then the relation (58) holds also for

$$(1+\theta_\delta)e^{-\gamma}R(\alpha)e^{\alpha_0} \le r < 1.$$

From this we get

$$\begin{aligned}(59)\qquad &\langle \overline{v}, \partial^2_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0} + \widetilde{\mathcal{J}}(\delta,\alpha,e^{\alpha_0}))\overline{v}\rangle \\ &\geq \left(1 - \theta_\delta \frac{2r+1}{(1-r)^3}\right)\langle \overline{v}, \partial^2_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0})\overline{v}\rangle\end{aligned}$$

for all $(\alpha_0,\alpha) \in \mathbb{R}^{d+1}$, such that $(1+\theta_\delta)e^{-\gamma}R(\alpha)e^{\alpha_0} \leq r < 1$. But $\theta_\delta = C_0 e^{\gamma}|\delta|$ and because of Lemma 8.1 the function $R(\alpha)e^{\alpha_0}$ is strictly convex everywhere in $\mathbb{R}^{d+1}$. Hence for $\overline{v} \neq 0$ and

$$(60)\qquad |\delta| < \delta_2(r) = \frac{(1-r)^3}{C_0 e^{\gamma}(1+2r)},$$

the inequality (59) gives

$$\langle \overline{v}, \partial^2_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0} + \widetilde{\mathcal{J}}(\delta,\alpha,e^{\alpha_0}))\overline{v}\rangle > 0$$

for all $(\alpha_0,\alpha) \in \mathbb{R}^{d+1}$, such that $(1+\theta_\delta)e^{-\gamma}R(\alpha)e^{\alpha_0} \leq r < 1$. Finally, since $\overline{v} \neq 0$ is arbitrary, Lemma 8.2 follows. □

LEMMA 8.3. – *Let* $0 < r < 1$ *and*

$$\delta_2(r) = \frac{(1-r)^3}{C_0 e^{-\gamma}(1+2r)},$$

then for $\delta \in \mathbb{R}$ *such that* $|\delta| < \delta_2(r)$

$$\frac{\partial}{\partial \alpha_0}(R(\alpha)e^{\alpha_0} + \widetilde{\mathcal{J}}(\delta,\alpha,e^{\alpha_0})) > 0;$$

everywhere in the domain

$$V_r = \{(\alpha_0,\alpha) \in \mathbb{R}^{d+1} : e^{\alpha_0}(1+\theta_\delta)e^{-\gamma}R(\alpha) < r\}.$$

Proof of Lemma 8.3. – The proof of this lemma is similar to that of Lemma 8.2. Using the same arguments as in the proof of Lemma 8.2 one can easily show that for $(\alpha_0,\alpha) \in \mathbb{R}^{d+1}$ such that $e^{\alpha_0}(1+\theta_\delta)e^{-\gamma}R(\alpha) < 1$ the following relation holds

$$\left|\frac{\partial}{\partial \alpha_0}\widetilde{\mathcal{J}}(\delta,\alpha,e^{\alpha_0})\right| \leq \theta_\delta \frac{\partial}{\partial \alpha_0}\Psi(\alpha,\alpha_0),$$

where

$$\Psi(\alpha_0,\alpha) = \frac{R(\alpha)e^{\alpha_0}}{1-(1+\theta_\delta)e^{-\gamma}R(\alpha)e^{\alpha_0}}.$$

But

$$\frac{\partial}{\partial \alpha_0}\Psi(\alpha,\alpha_0) = \frac{d}{dz}\frac{z}{1-(1+\theta_\delta)e^{-\gamma}z}\Big|_{z=R(\alpha)e^{\alpha_0}} \times R(\alpha)e^{\alpha_0}$$
$$= \frac{R(\alpha)e^{\alpha_0}}{(1-(1+\theta_\delta)e^{-\gamma}R(\alpha)e^{\alpha_0})^2}.$$

Hence for $(1+\theta_\delta)e^{-\gamma}R(\alpha)e^{\alpha_0} \le r < 1$

$$\left|\frac{\partial}{\partial \alpha_0}\tilde{\mathcal{J}}(\delta,\alpha,e^{\alpha_0})\right| \le \theta_\delta \frac{1}{(1-r)^2}R(\alpha)e^{\alpha_0} \le \theta_\delta \frac{2r+1}{(1-r)^3}R(\alpha)e^{\alpha_0},$$

and consequently

$$\frac{\partial}{\partial \alpha_0}(R(\alpha)e^{\alpha_0} + \tilde{\mathcal{J}}(\delta,\alpha,e^{\alpha_0})) > 0$$

if

$$\theta_\delta \frac{2r+1}{(1-r)^3} = C_0 e^{\gamma}\delta\frac{2r+1}{(1-r)^3} < 1.$$

Lemma 8.3 is therefore proved. □

Proof of Proposition 8.1. – We have to show that for $\delta \in \mathbb{R}$ small enough

(61) $$\langle v, \partial_\alpha^2 H_\delta(\alpha)v\rangle > 0$$

for all $v \in \mathbb{R}^d\backslash\{0\}$ and $\alpha \in \mathbb{R}^d$, where $\partial_\alpha^2 H_\delta(\alpha)$ is the matrix of the second derivatives of the function $H_\delta(\alpha)$, and

$$\langle v, \partial_\alpha^2 H_\delta(\alpha)v\rangle = \sum_{i,j=1}^{d}\frac{\partial^2}{\partial\alpha_i\partial\alpha_j}H_\delta(\alpha)v_i v_j.$$

Consider $e^{-\gamma} < r < 1$, and let $\delta_1(r)$ be the constant of Proposition 6.1. Denote

$$\delta(r) = \min\{\delta_2(r), \delta_1(r)\},$$

where $\delta_2(r)$ is the constant of Lemma 8.2, and suppose that $\delta \in \mathbb{R}$ and $|\delta| < \delta(r)$. Then for any $\alpha \in \mathbb{R}^d$ Proposition 6.2 yields $H_\delta(\alpha) = -\log z_0(\alpha,\delta)$ where $z_0(\alpha,\delta)$ is a unique simple solution of the equation

$$R(\alpha)z + \tilde{\mathcal{J}}(\delta,\alpha,z) = 1.$$

in the disk $Q_\alpha^r = \{z \in \mathbb{C} : |z| < \frac{r}{(1+\theta_\delta)e^{-\gamma}R(\alpha)}\}$. Hence for any $v = (v_1,\ldots,v_d) \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}^d$

(62) $$\langle v, \partial_\alpha^2(R(\alpha)e^{-H_\delta(\alpha)} + \tilde{\mathcal{J}}(\delta,\alpha,e^{-H_\delta(\alpha)}))v\rangle = 0,$$

that is

$$\langle \widetilde{v}(\alpha), \partial^2_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0} + \widetilde{\mathcal{J}}(\delta,\alpha,e^{\alpha_0}))\widetilde{v}(\alpha)\rangle\Big|_{\alpha_0=-H_\delta(\alpha)} - \langle v, \partial^2_\alpha H_\delta(\alpha)v\rangle \times \frac{\partial}{\partial\alpha_0}(R(\alpha)e^{\alpha_0} + \widetilde{\mathcal{J}}(\delta,\alpha,e^{\alpha_0}))\Big|_{\alpha_0=-H_\delta(\alpha)} = 0, \tag{63}$$

where

$$\widetilde{v}(\alpha) = (v_0(\alpha), v_1, \ldots, v_d),$$

with

$$v_0(\alpha) = -\sum_{i=1}^{d} v_i \frac{\partial}{\partial\alpha_i}(H_\delta(\alpha)).$$

Furthermore, since $z_0(\alpha,\delta) = e^{-H_\delta(\alpha)} \in Q^r_\alpha$, then obviously $(\alpha_0,\alpha) \in V_r$, and therefore Lemma 8.2 and Lemma 8.3 yield

$$\frac{\partial}{\partial\alpha_0}(R(\alpha)e^{\alpha_0} + \widetilde{\mathcal{J}}(\delta,\alpha,e^{\alpha_0}))\Big|_{\alpha_0=-H_\delta(\alpha)} > 0, \tag{64}$$

and

$$\langle \widetilde{v}(\alpha), \partial^2_{(\alpha_0,\alpha)}(R(\alpha)e^{\alpha_0} + \widetilde{\mathcal{J}}(\delta,\alpha,e^{\alpha_0}))\widetilde{v}(\alpha)\rangle\Big|_{\alpha_0=-H_\delta(\alpha)} > 0, \tag{65}$$

if $\widetilde{v}(\alpha) \neq 0$. But $\widetilde{v}(\alpha) = 0$ if and only if $v = 0$. Thus for $v \neq 0$ from (63), (64) and (65) it follows

$$\langle v, \partial^2_\alpha(H_\delta(\alpha))v\rangle > 0.$$

Finally, since $v \in \mathbb{R}^d\backslash\{0\}$ and $\alpha \in \mathbb{R}^r$ are arbitrary, then the function $H_\delta(\alpha)$ is strictly convex everywhere in $\mathbb{R}^d$. Proposition 8.1 is therefore verified with $\delta_2 = \sup_{e^{-\gamma}<r<1} \min\{\delta_1(r), \delta_2(r)\}$. □

9. PROOF OF THEOREM 1

We use now the results of the previous sections to prove Theorem 1. Indeed, because of Proposition 6.2 for all $\alpha \in \mathbb{R}^d$ and $\delta \in \mathbb{R}$ such that $|\delta| < \delta_1$ there exists a limit

$$\lim_{t\to+\infty} \frac{1}{t} \log E \exp\langle\alpha, X_t - X_0\rangle = H_\delta(\alpha).$$

Using Gärtner-Ellis theorem we conclude therefore that the sequence of measures $(\mu_{\delta,t})_{t\in\mathbb{Z}_+}$ satisfies the large deviation principle with a good rate function L_δ which is the Fenchel-Legendre transform of the function $H_\delta(\alpha)$.

Moreover, because of Proposition 8.1 for $\delta \in \mathbb{R}, |\delta| < \delta_2$ the function $H_\delta(\alpha)$ is strictly convex with respect to α everywhere in $\mathbb{R}^d$. We conclude therefore (see for example [15]) that

- the Fenchel-Legendre transform $L_\delta(v)$ of the function $H_\delta(\alpha)$ is strictly convex with respect to v everywhere in $\mathbb{R}^d$,
- for any $v \in \mathbb{R}^d$ there exists a unique $\alpha(\delta, v) \in \mathbb{R}^d$ such that

$$L_\delta(v) = \langle \alpha(\delta, v), v \rangle - H_\delta(\alpha(\delta, v)),$$

- $\alpha(\delta, v)$ is a unique solution of the system

$$\frac{\partial}{\partial \alpha_i} H_\delta(\alpha) = v_i, \; i = 1, \ldots, d, \tag{66}$$

and

$$\det \left(\frac{\partial^2}{\partial \alpha_i \partial \alpha_j} H_\delta(\alpha) \right)_{i,j=1}^{d} \neq 0$$

for any $\alpha \in \mathbb{R}^d$.

Using now Corollary 7.1 and implicit function theorem applied to the system (66) it follows that the function $\alpha(\delta, v)$ is analytic with respect to (δ, v) for all $v \in \mathbb{R}^d$ and $|\delta| < \delta_2$. Therefore the function $L_\delta(v)$ is also analytic with respect to (δ, v) for all $v \in \mathbb{R}^d$ and $|\delta| < \delta_2$. Theorem 1 is proved.

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