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Rates of convergence to the local time of a diffusion

by

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ABSTRACT. – In this paper we consider the approximation of the local time L_t of a 1-dimensional diffusion process X at some level x, say $x = 0$ by normalized sums, say U_t^n, of functions of the values $X_{\frac{i}{n}}$ for $i \leq nt$ as $n \to \infty$. Our main aim is to prove an associated functional central limit theorem fiving a mixed normal limiting value to the sequence of processes $n^{\infty}(U_t^n - L_t)$, for a suitable value of α. © Elsevier, Paris

RÉSUMÉ. – Dans cet article nous considérons l'approximation du temps local L_t d'un processus de diffusion uni-dimensionnel au niveau $x = 0$ par des sommes normalisées U_t^n de fonctions des valeurs $X_{\frac{i}{n}}$ pour $i \leq nt$ quand $n \to \infty$. Notre objectif principal est de montrer un théorème central limite fonctionnel, donnant la convergence de la suite $n^{\infty}(U_t^n - L_t)$ vers une limite qui est un mélange de processus gaussien, pour une valeur convenable de α. © Elsevier, Paris

1. INTRODUCTION AND MAIN RESULTS

1-1) It is well known that one can approximate the local time of a Brownian motion, and more generally of continuous semimartingales, in many ways by some sorts of discretizations: one may either discretize "in space", that is use the random times at which the process hits a grid of mesh $1/n$, say (this includes counting the number of upcrossings from 0 to $1/n$), or one may discretize "in time", that is use the values of the process at times i/n, and in both cases let n go to ∞.

When the basic process is the Brownian motion, space-discretization approximations together with their rates are known. Rates for time-discretization seem to be unknown except in some special cases (see below), and a fortiori no rates are known when the basic process is a diffusion process: finding these rates for time-discretizations, in the form of a central limit theorem, is the main aim of this paper.

Let us introduce our basic assumptions. We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ on which is defined a continuous adapted 1-dimensional process X of the form

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma(X_s)dW_s, \tag{1.1}$$

where W is a standard Brownian motion. The assumptions on b, σ are such:

HYPOTHESIS A. – *σ is a continuously differentiable positive (non vanishing) function on $\mathbb{R}$, such that the equation $dY_t = \sigma(Y_t)dW_t$ with $Y_0 = X_0$ has a (necessarily unique and strong) non-exploding solution. Further, the process b is such that the laws of X and of Y are locally equivalent.* □

Next, we denote by L the local time of the process X at level 0, given by

$$L_t = |X_t| - |X_0| - \int_0^t \mathrm{sign}(X_s)dX_s. \tag{1.2}$$

Next we describe the processes which approximate L. The simplest way is to count how many times $X_{i/n}$ is close enough to 0, that is to consider the processes

$$t \rightsquigarrow \sum_{i=1}^{[nt]} 1_{\{|X_{(i-1)/n}|\leq 1/u_n\}} \tag{1.3}$$

for a sequence u_n of positive numbers going to infinity (here, $[y]$ denotes the integer part of $y \geq 0$; we take $(i-1)/n$ instead of i/n for coherence with further notation). These processes will converge in probability, after normalization, to L. A bit more generally we can consider the processes

$$V(u_n, g)_t^n = \sum_{i=1}^{[nt]} g\left(u_n X_{\frac{i-1}{n}}\right) \tag{1.4}$$

for a function g which "goes to 0 fast enough" at infinity: clearly (1.3) is $V(u_n, g)_t^n$ for $g(x) = 1_{[-1,1]}$. Even more general, and of interest for statistical applications, are the processes

$$U(u_n, h)_t^n = \sum_{i=1}^{[nt]} h\left(u_n X_{\frac{i-1}{n}}, \sqrt{n}\left(X_{\frac{i}{n}} - X_{\frac{i-1}{n}}\right)\right). \tag{1.5}$$

It is known that under suitable assumptions on g and assumptions slightly stronger than (A) on the coefficients b, σ, the processes $\frac{1}{\sqrt{n}} U(\sqrt{n}, g)^n$ converge in probability to cL for some constant c depending on g and on the function σ. Further when $b = 0$ and $\sigma = 1$ (i.e. X is a standard Brownian motion), the normalized differences $n^{1/4}(\frac{1}{\sqrt{n}} U(\sqrt{n}, g)_t^n - cL_t)$ converge in law to $c' W'_{L_t}$ where W' is another Wiener process, independent of X: for example, Azaïs [3] has shown that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} 1_{\{X_{(i-1)/n} X_{i/n} < 0\}} \tag{1.6}$$

converges in probability (and also in $\mathbb{L}^2$) to $\frac{1}{\sigma(0)} \sqrt{\frac{2}{\pi}} L_t$. (1.6) counts the number of "crossings" of the level 0 for the discrete-time process, and it is equal to $\frac{1}{\sqrt{n}} U(\sqrt{n}, h)_t^n$ for $h(x, y) = 1_{\{x(x+y) < 0\}}$. The associated central limit theorem mentioned above has been shown by Borodin ([4],[5]) in a more general context, where $X_{i/n}$ is replaced by $\frac{1}{\sqrt{n}}(Z_1 + .. + Z_i)$ and (Z_i) are centered i.i.d. variables (and in [5] many other related results are exhibited).

Here we will first show that under quite general conditions, the processes $\frac{u_n}{n} U(u_n, h)^n$ converge in probability to the process cL, where c is a number defined below (depending only on h and $\sigma(0)$), as soon as $u_n/n \to 0$ and $u_n \to \infty$. Some related results have been proved by Florens-Zmirou [6].

Next, for the rates of convergence, we restrict to the case where

$$u_n = n^\alpha \qquad \text{for some } \alpha \in (0, 1). \tag{1.7}$$

Then we prove that if $\delta = ((1 - \alpha) \wedge \alpha)/2$, the processes $n^\delta \left(\frac{1}{n^{1-\alpha}} U(n^\alpha, h)^n - cL\right)$ converge in law to a non-trivial limit, provided $\alpha > 1/3$, and we do not know what happens when $\alpha \leq 1/3$. When $\alpha = 1/2$ this essentially reproves the results of Borodin (relative to the Brownian motion), with a different method allowing processes of the form (1.1). Observe that the rate of convergence n^δ is biggest when $\alpha = 1/2$, in which case it is $n^{1/4}$: this is a bit surprising at first glance, but may be interpreted as such: if $\alpha > 1/2$, then $U(n^\alpha, h)^n$ is of "order of magnitude" $n^{1-\alpha}$, that is $n^{1-\alpha}$ is the "number" of nonnegligible terms in (1.5), and it is only natural that the normalizing factor in the associated limit theorem be $n^{(1-\alpha)/2}$. When $\alpha < 1/2$, we still have $n^{1-\alpha}$ nonnegligible terms, but most of them concern values of $X_{(i-1)/n}$ which are too far away from 0 to give an appropriate information about the local time at level 0 (see Remark 2) after Theorem 1-2 for more explanations on this phenomenon).

1-2) Now we proceed to state the main results. Let us begin with some notation. If g is a Borel function on $\mathbb{R}$ and $\gamma \geq 0$ we set

$$\beta_\gamma(g) = \int |x|^\gamma |g(x)| dx \qquad \lambda(g) = \int g(x) dx. \tag{1.8}$$

Let ρ denote the density of the standard normal law $\mathcal{N}(0,1)$. For small enough Borel functions h on $\mathbb{R}^2$, set

$$H_h(x) = \int h(x,y)\rho(y)dy. \tag{1.9}$$

We will assume that h satisfies the following with some $\gamma \geq 0$:

HYPOTHESIS B-γ. – *he function h on $\mathbb{R}^2$ is Borel and satisfies $h(x,v) \leq \hat{h}(x)e^{a|v|}$, where $a \in \mathbb{R}$ and $\hat{h}$ is bounded with $\beta_\gamma(\hat{h}) < \infty$.* □

Finally, associate with any $u > 0$ and any function h on $\mathbb{R}^2$ the function

$$h_u(x,y) = h(ux, uy), \tag{1.10}$$

and consider the following condition on the sequence (u_n):

$$\frac{u_n}{n} \to 0, \qquad u_n \to \infty. \tag{1.11}$$

Recall also that a sequence $(Z^n)_{n\geq 1}$ of processes is said to *converge locally uniformly in time, in probability* to a limiting process Z if for any $t \in \mathbb{R}_+$ the sequence $\sup_{s\leq t} |Z^n_s - Z_s|$ goes to 0 in probability.

THEOREM 1.1. – *Assume (A) and (1.11), and let h satisfy (B-0). The processes $\frac{u_n}{n} U(u_n, h)^n$ converge locally uniformly in time, in probability, to $\frac{1}{\sigma(0)}\lambda(H_{h_{\sigma(0)}})L$.*

For example $h(x,y) = 1_{\{x(x+y)<0\}}$ satisfies (B-γ) for all γ with $a = 1$ and $\hat{h}(x) = e^{-|x|}$, and $\lambda(H_{h_{\sigma(0)}}) = \sqrt{\frac{2}{\pi}}$. So we recover Azaïs' result in a slightly more general situation.

Remark. – When u_n/n does not go to 0, the theorem cannot be valid in general, since the first summand in $V(u_n, g)_t^n$ is $g(u_n x)$ if $X_0 = x$, and this quantity does not go to 0 in general.

When u_n does not go to infinity, the result is also wrong: for example if $u_n = u$ is a constant, by Riemann approximation $\frac{1}{n}V(u, g)_t^n$ converges to $\int_0^t g(X_s)ds$ as soon as g is continuous. □

1-3) Let us turn now to the rates of convergence. For this we need first to recall some facts about *stable convergence*. Let Y_n be a sequence of random variables with values in a Polish space E, all defined on the same probability space $(\Omega, \mathcal{F}, P)$, and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. We say that Y_n *converges $\mathcal{G}$-stably in law* to Y, if Y is an E-valued random variable defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ of the original space and if

$$\lim_n E(Uf(Y_n)) = \tilde{E}(Uf(Y)) \tag{1.12}$$

for every bounded continuous $f : E \to \mathbb{R}$ and all bounded $\mathcal{G}$-measurable random variables U (and then (1.12) holds for all integrable U). This convergence was introduced by Renyi [10] and studied by Aldous and Eagleson [1], see also [8]. It is obviously stronger than the convergence in law, and below it will be applied to càdlàg processes Y^n with E being the space of càdlàg functions endowed with the Skorokhod topology.

For the *conditional Gaussian martingales*, we refer to [9]: denote by $(\mathcal{G}_t)$ the filtration generated by the process X, and $\mathcal{G} = \bigvee \mathcal{G}_t$. A (possibly multidimensional) process Y defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{G}}, (\tilde{\mathcal{G}}_t), \tilde{P})$ of the original filtered space $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ is a $\mathcal{G}$-conditional Gaussian continuous martingale with bracket C if the process C is adapted to the filtration $(\mathcal{G}_t)$ and if Y is a continuous (necessarily Gaussian) martingale with bracket C, for a regular version of the conditional probability knowing $\mathcal{G}$.

Next, we need a whole set of new notation. If f and g are functions on $\mathbb{R}^2$ and $\mathbb{R}$ respectively, we put

$$\bar{H}_{f,g}(x) = \int f(x, y)g(x + y)\rho(y)dy. \tag{1.13}$$

Next, set

$$\hat{g}(x) = \int \rho(y)(|x + y| - |x|)dy,$$

$$\text{which has:} \quad \beta_\alpha(\hat{g}) < \infty \quad \forall\alpha > 0, \quad \lambda(\hat{g}) = 1 \tag{1.14}$$

by a simple calculation.

Next, $(P_t)_{t\geq 0}$ denotes the Brownian semi-group, given by $P_t k(x) = \int k(x+y\sqrt{t})\rho(y)dy$. The following two estimates, where k denotes a Lebesgue-integrable function, are well known (and for the convenience of the reader they will be reproved, along with other related estimates, in Lemma 3-1 below):

$$|P_t k(x) - \frac{\lambda(k)}{\sqrt{2\pi t}} e^{-x^2/2t}| \leq \begin{cases} K\lambda(|k|)/\sqrt{t} \\ Kt^{-3/2}(\beta_1(k) + \beta_2(k)|x|), \end{cases} \tag{1.15}$$

where K denotes a universal constant. Hence if $\beta_1(k) < \infty$ and $\beta_2(k) < \infty$ and $\int k(x)dx = 0$, the series

$$F(k)(x) = \sum_{j\in\mathbb{N}} P_j k(x) \tag{1.16}$$

is absolutely convergent and $|F(k)(x)| \leq K(\beta_1(k)+\beta_2(k)|x|)$. If f has (B-2) the function $k = H_f - \lambda(H_f)\hat{g}$ satisfies the above-mentionned conditions, hence $F(k)$ exists; further, if $\bar{\rho}$ denotes the law of the pair (B_1, ℓ_1) on $\mathbb{R}\times\mathbb{R}_+$, where B is a standard Brownian motion starting at 0, with its local time ℓ at level 0, the following expression is well defined:

$$\begin{aligned}\delta(f) = \sqrt{\frac{2}{\pi}} \int_0^1 \sqrt{\frac{1}{r}-1}\, dr \int \bar{\rho}(dx,dy) y F(H_f - \lambda(H_f)\hat{g})(x\sqrt{1-r}) \\ + \int_0^1 \sqrt{\frac{1}{r}-1}\, dr \int \rho(z)|z|dz \int \bar{\rho}(dx,dy) y f(z\sqrt{r}, x\sqrt{1-r} - z\sqrt{r}).\end{aligned} \tag{1.17}$$

When f and f' have (B-2), we set

$$\begin{aligned}\eta(f,f') = \lambda\Big(H_{ff'} + \bar{H}_{f,F(H_{f'}-\lambda(H_{f'})\hat{g})} + \bar{H}_{f',F(H_f-\lambda(H_f)\hat{g})}\Big) \\ + \frac{8}{3\sqrt{2\pi}}\lambda(H_f)\lambda(H_{f'}) - \lambda(H_f)\delta(f') - \lambda(H_{f'})\delta(f).\end{aligned} \tag{1.18}$$

Using (1.15) again, if $\beta_2(k) < \infty$ and $\lambda(|k|) < \infty$, the integral

$$G(k)(x) = \int_0^\infty \left(P_t k(x) - \frac{\lambda(k)}{\sqrt{2\pi t}} e^{-x^2/2t}\right) dt \tag{1.19}$$

absolutely converges and $|G(k)(x)| \leq K_k(1+|x|)$. So if f and f' have (B-2), we can set

$$\begin{aligned}\eta'(f,f') = \lambda(H_f G(H_{f'}) + H_{f'} G(H_f)) \\ - \lambda(H_f)G(H_{f'})(0) - \lambda(H_{f'})G(H_f)(0).\end{aligned} \tag{1.20}$$

Clearly $\eta(.,.)$ and $\eta'(.,.)$ are bilinear. That $\eta(f,f) \geq 0$ and $\eta'(f,f) \geq 0$ is not obvious from (1.18) and (1.20), but it follows from the fact that these quantities are limits of nonnegative numbers.

Below we state the results for a d-dimensional function $h = (h^i)_{1\leq i\leq d}$ on $\mathbb{R}^2$, so the processes $U(u_n, h)^n$ are d-dimensional, as well as H_h and $\lambda(H_h)$.

THEOREM 1.2. – *Assume (A), let $h = (h^i)_{1\leq i\leq d}$ be a d-dimensional function satisfying (B-r) for some $r > 3$, and set $\delta = ((1-\alpha)\wedge\alpha)/2$. Under either one of the following:*

(i) *the function σ is a constant (so X is a Brownian motion plus a random drift),*

(ii) *the function h is differentiable in the first variable, with a partial derivative satisfying (B-1),*

the processes $n^\delta(\frac{1}{n^{1-\alpha}}U(n^\alpha, h)^n - \lambda(H_{h_{\sigma(0)}})\frac{1}{\sigma(0)}L)$ converge $\mathcal{G}$-stably in law to a process $Y = (Y^i)_{1\leq i\leq d}$, defined on an extension of the space $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$, and which is a $\mathcal{G}$-conditional Gaussian continuous martingale with brackets $\langle Y^i, Y^j\rangle$ as given below, in the following cases:

a) *If $\alpha = 1/2$ (hence $\delta = 1/4$), with*

$$\langle Y^i, Y^j\rangle = \frac{1}{\sigma(0)}\eta(h^i_{\sigma(0)}, h^j_{\sigma(0)})L. \tag{1.21}$$

b) *If $1/2 < \alpha < 1$ (hence $\delta = (1-\alpha)/2$), with*

$$\langle Y^i, Y^j\rangle = \frac{1}{\sigma(0)}\lambda(H_{h^i_{\sigma(0)}h^j_{\sigma(0)}})L. \tag{1.22}$$

c) *If $1/3 < \alpha < 1/2$ (hence $\delta = \alpha/2$), with*

$$\langle Y^i, Y^j\rangle = \frac{1}{\sigma(0)}\eta'(h^i_{\sigma(0)}, h^j_{\sigma(0)})L. \tag{1.23}$$

Remark 1. – There is another, equivalent, way to characterize the limit Y above, when $\langle Y^i, Y^j\rangle = a_{ij}L$. Namely one can construct on an extension of the space a Wiener process $W = (W^i)_{1\leq i\leq d}$ having $E(W^i_t W^j_t) = a_{ij}t$ and independent of X, and we set $Y_t = W_{L_t}$. This formulation is closer to the formulation of Borodin [4] when $\alpha = 1/2$. In this case, the expression (1.18) which is used to compute a_{ij} seems quite different from the corresponding expression in [4], but of course the two agree.

Remark 2. – Suppose that we are in the Brownian case, i.e. $b = 0$ and $\sigma = 1$, and also that $h(x,y) = g(x)$. Denote by L^x the local time at level x, and set

$$A^n_t = \frac{u_n}{n}V(u_n, g)^n_t - \int g(x)L^{x/u_n}_{[nt]/n}dx. \tag{1.24}$$

We have

$$\frac{u_n}{n} V(u_n, g)_t^n - \lambda(g) L_t = A_t^n + \int g(x)(L_{[nt]/n}^{x/u_n} - L_{[nt]/n}) dx + \lambda(g)(L_{[nt]/n} - L_t). \qquad (1.25)$$

If g is twice differentiable, with g, g', g'' satisfying (B-1), one can prove that when $u_n = n^\alpha$ and $\alpha < 1/2$, then $n^{\alpha/2} A_t^n$ goes to 0 in probability, while because L is Hölder in time with any index smaller than $1/2$ we also have $n^{\alpha/2}(L_{[nt]/n} - L_t) \to 0$: then the leading term in Statement (c) of Theorem 1-2 is the second term in the right side of (1.25). When $\alpha > 1/2$ on the contrary, this second term is of order $n^{-\alpha/2}$ (because of the Hölder properties of L_t^x in x) and the leading term in Statement (b) is A_t^n. When $\alpha = 1/2$, both the first and the second terms in (1.25) have an influence on the limit in Statement (a). In a sense, it is more natural to look at the processes (1.24) rather than $\frac{u_n}{n} V(u_n, g)^n - \lambda(g) L$. □

When $f(x, y) = g(x)$ the expression for $\delta(f)$ becomes (see the end of Section 6):

$$\delta(f) = \int g(x) \hat{g}(x) dx + \sqrt{\frac{2}{\pi}} \int_0^1 \sqrt{\frac{1}{r} - 1}\ dr \int \bar{\rho}(dx, dy) y (F(g - \lambda(H_g)\hat{g})(x\sqrt{1-r}). \qquad (1.26)$$

If $H_f = \lambda(H_f)\hat{g}$ and $H_{f'} = \lambda(H_{f'})\hat{g}$, all the terms in (1.18) in which $F(.)$ shows up disappear. In particular if $f(x, y) = \hat{g}(x)$, hence $\lambda(H_f) = 1$, (1.18) becomes

$$\eta(\hat{g}, \hat{g}) = \frac{8}{3\sqrt{2\pi}} - \lambda(\hat{g}^2) = \frac{8}{3\sqrt{\pi}}(\sqrt{2} - 1). \qquad (1.27)$$

The remainder of the paper is organised as follows: in Section 2 we show how to reduce the proof to the case where X is a standard Brownian motion. Section 3 is devoted to some preliminary estimates on the semi-group and on the local time of the Brownian motion. In Section 4 we prove Theorem 1-1. For Theorem 1-2, in Section 5 the problem is reduced to a central limit theorem for a suitable martingale (pretty much as one classically proves a central limit theorem for mixing sequences by reduction to a similar result for martingales), and we consider separately the cases $\alpha = 1/2$, $\alpha > 1/2$ and $\alpha < 1/2$ in Sections 6, 7 and 8.

2. REDUCTION TO THE BROWNIAN CASE

In this section, and throughout a number of steps, we show how our results for the process X having (1.1) can be reduced to the case where X is the standard Brownian motion.

2-1) We will use several times the following method for constructing the limiting processes Y and Z with a bracket of the form cL in Theorem 1-2 (c is a nonnegative symmetric $d \times d$ matrix): according to [9], the process Y itself may be taken as the canonical process on the canonical space $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t))$ of all continuous $\mathbb{R}^d$-valued functions on $\mathbb{R}_+$; then $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t))$ is the product of $(\Omega, \mathcal{G}, (\mathcal{G}_t))$ by $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t))$; finally, the measure on the extension if $\tilde{P}(d\omega, d\hat{\omega}) = P(d\omega)Q(\omega, d\hat{\omega})$, where $Q(d\omega, .)$ is the unique measure under which Y is a Gaussian martingale with deterministic bracket $cL(\omega)$ and $Y_0 = 0$ (observe that Q is a transition probability from $(\Omega, \mathcal{G})$ into $(\hat{\Omega}, \hat{\mathcal{F}})$, and also from $(\Omega, \mathcal{G}_t)$ into $(\hat{\Omega}, \hat{\mathcal{F}}_t)$ for all t, because L is $(\mathcal{G}_t)$-adapted by (1.2)).

2-2) Here we prove that one may assume $\mathcal{F}_t = \mathcal{G}_t$ (recall that $(\mathcal{G}_t)$ is the filtration generated by X).

First, if X satisfies (1.1) and (A) relative to $(\mathcal{F}_t)$, it also satisfies (1.1) and (A) relative to $(\mathcal{G}_t)$, with the same function σ and a drift process b' and a Brownian motion W' adapted to $(\mathcal{G}_t)$: indeed, X is a continuous semimartingale w.r.t. $(\mathcal{F}_t)$, hence w.r.t. $(\mathcal{G}_t)$ as well, with the $(\mathcal{G}_t)$-canonical decomposition $X = X_0 + B_t + M_t$ (where M is the martingale part), and the quadratic variation of M is $\int_0^{\cdot} \sigma(X_s)^2 ds$. Thus $W'_t = \int_0^t (1/\sigma(X_s)) dX_s$ is a $(\mathcal{G}_t)$-Brownian motion. Further, that $B_t = \int_0^t b'_s ds$ for some b'_s follows for example from Girsanov's Theorem and Assumption (A), and clearly the pair (b', σ) satisfies (A) as well.

Next, the local time L as defined by (1.2) is the same for $(\mathcal{F}_t)$ and for $(\mathcal{G}_t)$. The processes $U(u_n, h)^n$ do not depend on the filtration, and Step 2-1 above yields that the limits in Theorem 1-2 depend only on L and on the functions h and σ. Therefore one may always replace $\mathcal{F}_t$ by $\mathcal{G}_t$, or in other words we can and will assume that $\mathcal{F}_t = \mathcal{G}_t$.

2-3) Here we show that we can replace the original space by the "canonical" space.

More precisely let $(\Omega', \mathcal{F}', (\mathcal{F}'_t))$ be the canonical filtered space of all real-valued continuous functions on $\mathbb{R}_+$, endowed with the canonical process X'. Define $\varphi : \Omega \to \Omega'$ by $X = X' \circ \varphi$, so that the law of X under P is $P' = P \circ \varphi^{-1}$. Then standard arguments on changes of space yield that if X satisfies (1.1) and (A) with (b, σ) the process X' (under P') satisfies

(1.1) and (A) with a b' having $b = b' \circ \varphi$ and the same σ, and if L' is the local time of X' under P', then $L' \circ \varphi$ is a version of L. Further the process $U'(u_n, h)^n$ associated with X' by (1.5) has $U(u_n, h)^n = U'(u_n, h)^n \circ \varphi$.

Finally if we define the extension of $(\Omega', \mathcal{F}', (\mathcal{F}_t), P')$ as in Step 2-1 with L', we observe that $Q(\omega, .) = Q'(\varphi(\omega), .)$. Therefore, with obvious notation, we have $\tilde{E}'(U'f(Y)) = \tilde{E}(U' \circ \varphi f(Y))$, while any random variable on $(\Omega, \mathcal{G})$ is a.s. of the form $U' \circ \varphi$: all these facts imply that if our theorems hold for X' under P, they also hold for X.

Henceforth, we may assume in the sequel that $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ is the canonical space, endowed with the canonical process X.

2-4) Here prove the following property: if our results hold for a given pair (σ, b) satisfying (A), they hold for any other such pair (σ, b') with the same function σ.

Indeed, denote by P' the law of the solution of (1.1) with b'. The two measures P and P' are equivalent on each σ-field $\mathcal{F}_t$ (recall that by Step 2-3 we are on the canonical space). By (1.2), the process L is also a version of the local time of X under P'. Further if a sequence A_n of $\mathcal{F}_t$-measurable variables converge in P_x-measure to a limit A (necessarily $\mathcal{F}_t$-measurable as well), we also have $A_n \to A$ in P'_x-measure. So our claim is true for Theorem 1-1.

Now we define the extension for P' as in Step 2-1, except that the measure is now $\tilde{P}'(d\omega, d\hat{\omega}) = P'(d\omega)Q(\omega, d\hat{\omega})$. If $E(Uf(Y^n)) \to \tilde{E}(Uf(Y))$ for all integrable variable U and bounded continuous function f on the space of cádlàg functions, we need to prove the same thing for P' when U is in addition bounded. Since Skorokhod convergence is "local" in time, it is enough to prove it when U is $\mathcal{F}_t$-measurable and f depends only on the function up to time t, for any finite t. But if Z_t denotes the density of P' w.r.t. P on the σ-field $\mathcal{F}_t$, and since $Q(., A)$ is $\mathcal{F}_t$-measurable when $A \in \hat{\mathcal{F}}_t$, we have $E'(Uf(Y^n)) = E(Z_t Uf(Y^n))$, and $\tilde{E}'(Uf(Y)) = \tilde{E}(Z_t Uf(Y))$. Since UZ_t is P_x-integrable when U is bounded, we deduce the result: hence our claim is true also for Theorem 1-2.

2-5) Suppose that our results hold for (1.1), when σ, $1/\sigma$ and σ' are bounded. We prove here, via a well-known localization procedure, that they also hold without the boundedness of σ, $1/\sigma$ and σ'.

In view of 2-4), it is enough to prove this result when $b = 0$. For each $p \geq 1$ choose a continuously differentiable function σ_p which is bounded, as well as $1/\sigma_p$ and σ'_p, and such that $\sigma_p(x) = \sigma(x)$ whenever $|x| \leq p$. Observe that since $b = 0$, Equation (1.1) may be "inverted" to give $W_t = \int_0^t (1/\sigma(X_s))dX_s$. Now let $X(p)$ be the (strong) solution to (1.1), w.r.t. the same W. If $T_p = \inf(t : |X_t| \geq p)$, we clearly have $X_t = X(p)_t$ a.s. for all $t \leq T_p$. Hence all the processes showing up in our results coincide a.s. for X and $X(p)$ on the interval $[0, T_p]$. Since $T_p \uparrow \infty$, the claim is thus obvious.

Hence all what precedes shows that it is enough to prove the results when $\mathcal{F}_t = \mathcal{G}_t$ and σ, $1/\sigma$ and σ' are bounded, and X is the solution to the equation

$$dX_t = \frac{1}{2}(\sigma\sigma')(X_t)dt + \sigma(X_t)dW_t. \tag{2.1}$$

2-6) In our last step, we consider Equation (2.1) with σ, σ' and $1/\sigma$ bounded, and we show how to reduce the two theorems to the case where X is a standard Brownian motion starting at a fixed point x.

Consider an arbitrary twice continuously differentiable function $T : \mathbb{R} \to \mathbb{R}$ such that for some $\varepsilon > 0$:

$$T(0) = 0, \quad T'(0) = 1, \quad \varepsilon \leq T'(x) \leq \frac{1}{\varepsilon}, \quad |T''(x)| \leq \frac{1}{\varepsilon}. \tag{2.2}$$

With any function h on $\mathbb{R}^2$, and with the function T having (2.2) and the sequence (u_n) of positive numbers, we associate the following functions:

$$h_n(x, y) = h\left(u_n T\left(\frac{x}{u_n}\right), \sqrt{n}\left(T\left(\frac{x}{u_n} + \frac{y}{\sqrt{n}}\right) - T\left(\frac{x}{u_n}\right)\right)\right). \tag{2.3}$$

Next, if $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ is the canonical space endowed with the canonical process X, we denote by P_x the unique measure under which X is a standard Brownian motion starting at x. The following auxiliary result will be proved in Section 4:

PROPOSITION 2.1. – *In the above setting, let h satisfy (B-0) and assume (1.11). Let T satisfy (2.2) and associate h_n with T, u_n and h by (2.3). Then*

a) The processes $\frac{u_n}{n} U(u_n, h_n)^n$ tend to $\lambda(H_h)L$ locally uniformly in time in P_x-probability.

b) If h has (B-1), if h is differentiable in the first variable with a partial derivative satisfying (B-1), and if $n/u_n^3 \to 0$, the processes $\sqrt{\frac{u_n}{n}} U(u_n, h_n - h)^n$ tend to 0 locally uniformly in time in P_x-probability.

Suppose also that *Theorem 1-2 holds in the above canonical setting, for each measure P_x*. We will presently see how to deduce the results in the general case.

First there is a version of the local time L which works under each P_x. Let μ be any probability measure on $\mathbb{R}$, and set $P = \int \mu(dx)P_x$ (so under P, X is a standard Brownian motion starting at X_0, and the law of X_0 is μ). Since L is the same under each P_x, it is also a version of the local time under P, and it is obvious that Proposition 2-1 hold also for the measure P. Since in Step 2-1 the transition measure Q depends only on ω through $L_.(\omega)$, if $\tilde{P}_x$ is the extension of P_x, then with the extension of P given by $\tilde{P} = \int \mu(dx)\tilde{P}_x$ it is also obvious that Theorem 1-2 holds for P.

By Step 2-3, Theorem 1-2 and Proposition 2-1 hold also when X is a standard Brownian motion on an arbitrary space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, with an arbitrary (possibly random) initial value, provided $(\mathcal{F}_t)$ is the filtration generated by X.

Let us now consider the case of Equation (2.1) with σ, σ' and $1/\sigma$ bounded. Set

$$S(x) = \int_0^x \frac{1}{\sigma(y)} dy. \tag{2.4}$$

This function is of class C^2, so by Ito's formula, we immediately deduce from (2.1) that the process $X'_t = S(X_t)$ is a standard Brownian motion, starting at the random point $X'_0 = S(X_0)$. From the above, and since $(\mathcal{F}_t)$ is also the filtration generated by X' because S is invertible, Theorem 1-2 and Proposition 2-1 hold for X'. Moreover, we have the

LEMMA 2.2. – *The process* $L' = L/\sigma(0)$ *is a version of the local time of* X' *at level* 0.

Proof. – Let X^+ and X^- (resp. X'^+ and X'^-) be the positive and negative parts of X and X', and let L' be the local time of X'. Not only do we have (1.2), but also

$$X_t^+ = X_0^+ + \frac{1}{2}L_t + \int_0^t 1_{\{X_s>0\}} dX_s, \quad X_t^- = X_0^- + \frac{1}{2}L_t - \int_0^t 1_{\{X_s<0\}} dX_s.$$

Now, $\mathrm{sign(S(x))} = \mathrm{sign(x)}$, hence $X'^+ = S(X^+)$ and $X'^- = -S(-X^-)$, so the property $S'(0) = 1/\sigma(0)$, the fact that L charges only the set where

$X_s = 0$ and Ito's formula yield

$$X_t'^+ = X_0'^+ + \frac{1}{2\sigma(0)} L_t + \int_0^t S'(X_s^+) 1_{\{X_s>0\}} dX_s + \frac{1}{2}\int_0^t S''(X_s^+) 1_{\{X_s>0\}} \sigma(X_s)^2 ds,$$

$$X_t'^- = X_0'^- + \frac{1}{2\sigma(0)} L_t - \int_0^t S'(-X_s^-) 1_{\{X_s<0\}} dX_s - \frac{1}{2}\int_0^t S''(-X_s^-) 1_{\{X_s<0\}} \sigma(X_s)^2 ds.$$

Adding these two equations yields

$$|X_t'| = |X_0'| + \frac{1}{\sigma(0)} L_t + \int_0^t S'(X_s)\mathrm{sign}(\mathrm{X_s})\mathrm{dX_s} + \frac{1}{2}\int_0^t S''(X_s)\mathrm{sign}(\mathrm{X_s})\sigma(\mathrm{X_s})^2 \mathrm{ds}.$$

On the other hand, (1.2) applied to X', together with the fact that $dX_t' = S'(X_t)dX_t + \frac{1}{2}S''(X_t)\sigma(X_t)^2 dt$ (by Ito's formula again) and that $\mathrm{sign}(\mathrm{S}(\mathrm{x})) = \mathrm{sign}(\mathrm{x})$ again yields

$$|X_t'| = |X_0'| + L_t' + \int_0^t S'(X_s)\mathrm{sign}(\mathrm{X_s})\mathrm{dX_s} + \frac{1}{2}\int_0^t \mathrm{S}''(\mathrm{X_s})\mathrm{sign}(\mathrm{X_s})\sigma(\mathrm{X_s})^2 \mathrm{ds}.$$

Comparing the last two equalities gives $L' = L/\sigma(0)$. □

The function S is invertible, and if S^{-1} denotes its inverse we set

$$T(x) = \frac{1}{\sigma(0)} S^{-1}(x). \tag{2.5}$$

This function is twice differentiable and satisfies (2.2). Let h be an $\mathbb{R}^d$-valued function, with which we associate $h' = h_{\sigma(0)}$ by (1.10), while h_n' is associated with h' (and T as above and a given sequence (u_n)) by (2.3). Denote by $U'(u_n, h)^n$ the process defined by (1.5) with X' in place of X. Since $X = \sigma(0)T(X')$ a simple calculation shows:

$$U(u_n, h)^n = U'(u_n, h_n')^n. \tag{2.6}$$

If h satisfies (B-r), so does h'. Thus (2.6) and Lemma 2-2 yield that Theorem 1-1 is exactly Proposition 2-1(a) for X'. It is also clear that (2.6), Lemma 2-2 and Proposition 2-1(b) for X' and Theorem 1-2 for X' yield Theorem 1-2 for X, since if $u_n = n^\alpha$ with $\alpha > 1/3$ implies $n/u_n^3 \to 0$ and since $\sqrt{\frac{u_n}{n}} \geq n^{\delta+\alpha-1}$ (when σ is a constant, we have $T(x) = x$ and thus Proposition 2-1(b) is trivial without any regularity assumption on h because $h_n = h$).

To summarize, at this point we are left to prove Theorem 1-2 and Proposition 2-1, on the canonical space, with the canonical process X and the Wiener measures P_x.

3. SOME ESTIMATES

We give here some estimates on the semigroup (P_t) of the standard Brownian motion. These are more or less known, but simple to prove. Below, K denotes a constant which may change from line to line; when it depends on an additional parameter u, we write it K_u.

LEMMA 3.1. – *If* $t > s > 0$ *and* $\gamma \geq 0$ *we have*

$$|P_t k(x)| \leq K\frac{\lambda(|k|)}{\sqrt{t}}. \tag{3.1}$$

$$|P_t k(x) - \frac{\lambda(k)}{\sqrt{2\pi t}}e^{-x^2/2t}| \leq \frac{K_\gamma}{t}\left(\frac{\beta_1(k)}{1+|x/\sqrt{t}|^\gamma} + \frac{\beta_{1+\gamma}(k)}{1+|x|^\gamma}\right), \tag{3.2}$$

$$|P_t k(x) - \frac{\lambda(k)}{\sqrt{2\pi t}}e^{-x^2/2t}| \leq \frac{K}{t^{3/2}}(\beta_2(k) + \beta_1(k)|x|), \tag{3.3}$$

$$|P_t k(x) - P_t k(y)| \leq K\frac{|x-y|}{t}\lambda(|k|), \tag{3.4}$$

$$|P_t k(x) - P_s k(x)| \leq K\frac{t-s}{s^{3/2}}\lambda(|k|). \tag{3.5}$$

Proof. – The density of the law $\mathcal{N}(0,t)$ is $\rho_t(u) = \frac{1}{\sqrt{t}}\rho(\frac{u}{\sqrt{t}})$. Since $P_t k(x) = \int \rho_t(y)k(x+y)dy$ and ρ is bounded, we have (3.1). Next,

$$i \in \mathbb{N} \;\Rightarrow\; |u/\sqrt{t}|^i \rho_t(u) \leq K_i \rho_{2t}(u) \leq K_i'/\sqrt{t}, \tag{3.6}$$

$$\frac{\partial \rho_t(u)}{\partial u} = -\frac{u}{t}\rho_t(u), \qquad \frac{\partial \rho_t(u)}{\partial t} = \frac{1}{2t}(\frac{u^2}{t} - 1)\rho_t(u). \tag{3.7}$$

Taylor's formula yields, with $\rho_t'(u) = \partial\rho_t(u)/\partial u$:

$$P_t k(x) - \frac{\lambda(k)}{\sqrt{2\pi t}} e^{-x^2/2t} = \int k(u)(\rho_t(x-u) - \rho_t(x))du$$
$$= -\int_0^1 d\theta \int k(u)u\rho_t'(x-\theta u)du. \quad (3.8)$$

If $|x-\theta u| \geq |x|/2$ (3.6) and (3.7) yield $|\rho_t'(x-\theta u)| \leq \frac{K}{t}e^{-x^2/16t}$, while otherwise $|\theta u| > |x|/2$, hence $|u| > |x|/2$ and $|\rho_t'(x-\theta u)| \leq \frac{K}{t} \leq \frac{K_\gamma}{t}\frac{1+|u|^\gamma}{1+|x|^\gamma}$. Hence for all $x, u \in \mathbb{R}$ and $\theta \in [0,1]$ we get

$$|\rho_t'(x-\theta u)| \leq \frac{K_\gamma}{t}\left(e^{-x^2/16t} + \frac{1+|u|^\gamma}{1+|x|^\gamma}\right).$$

Since furthermore $e^{-x^2/16t} \leq K_\gamma \frac{1}{1+|x/\sqrt{t}|^\gamma}$, (3.2) readily follows from (3.8).

Next, (3.6) and (3.7) yield $|\rho_t(x-u) - \rho_t(x)| \leq \frac{K}{t^{3/2}}|u|(|u|+|x|)$, hence (3.3) follows from (3.8).

Next, we have

$$P_t k(x) - P_t k(y) = \int k(u)(\rho_t(x-u) - \rho_t(y-u))du.$$

Again (3.6) and (3.7) yield $|\rho_t(x-u) - \rho_t(y-u)| \leq K|x-y|/t$, hence (3.4).

Finally,

$$P_t k(x) - P_s k(x) = \int k(u)(\rho_t(x-u) - \rho_s(x-u))du.$$

A further application of (3.6) and (3.7) yields $|\rho_t(x-u) - \rho_s(x-u)| \leq K(t-s)/s^{3/2}$, hence (3.5). □

LEMMA 3.2. – *If* $|k(x)| \leq \frac{1}{1+|x/\delta|^\gamma}$ *for some* $\delta \geq 1$ *and* $\gamma > 0$, *we have for all* t:

$$|P_t k(x)| \leq K_\gamma \frac{1+t^{\gamma/2}}{1+|x/\delta|^\gamma}. \quad (3.9)$$

Proof. – Since $|P_t k(x)| \leq \int \frac{1}{1+|(x+y\sqrt{t})/\delta|^\gamma}\rho(y)dy$ and $\frac{1}{1+|(x+y\sqrt{t})/\delta|^\gamma} \leq \frac{K_\gamma(1+|y\sqrt{t}|^\gamma)}{1+|x/\delta|^\gamma}$ by an easy calculation, the result is obvious. □

LEMMA 3.3. – *If* $\gamma(k,x)_t^n = E_x(\sum_{i=2}^{[nt]} k(\sqrt{n}X_{\frac{i-1}{n}}))$, *we have*

$$|\gamma(k,x)_t^n| \leq K\lambda(|k|)\sqrt{nt}. \tag{3.10}$$

If furthermore $\lambda(k) = 0$, *then*

$$\left.\begin{aligned} |\gamma(k,x)_t^n| &\leq K(\beta_2(k)+\beta_1(k)|x|\sqrt{n}) \\ |\gamma(k,x)_t^n| &\leq K\beta_1(k)(1+\log^+(nt)). \end{aligned}\right\} \tag{3.11}$$

Proof. – We have $E_x(k(\sqrt{n}X_{\frac{i-1}{n}})) = P_{i-1}k(x\sqrt{n})$, hence $\gamma(k,x)_t^n \leq \sum_{i=1}^{[nt]-1} |P_i k(x\sqrt{n})|$. The estimates (3.1), (3.2), (3.3), $\sum_{i=1}^{[nt]} \frac{1}{\sqrt{i}} \leq 2\sqrt{nt}$, $\sum_{i=1}^{\infty} \frac{1}{i^{3/2}} < \infty$ and $\sum_{i=1}^{[nt]} \frac{1}{i} \leq 1 + \log^+(nt)$ give the results. □

We end this section with some simple calculations on the local time L. Put

$$G(f,q,x) = E_x(L_1^q f(X_1)), \qquad q \in \mathbb{N}, \tag{3.12}$$

where f satisfies $|f(x)| \leq Ke^{a|x|}$ and $q > 0$. First, according to Revuz and Yor [11], if $L(a)$ denotes the local time of X at level a, under P_0 the processes $(X_t, L(a)_t)$ and $(\frac{1}{c}X_{tc^2}, \frac{1}{c}L(ac)_{tc^2})$ have the same law. Hence

$$\begin{aligned} &n^{q/2}E_{x/\sqrt{n}}(L_{1/n}^q f(\sqrt{n}X_{1/n})) \\ &= n^{q/2}E_0(L(-x/\sqrt{n})_{1/n}^q f(x+\sqrt{n}X_{1/n})) \\ &= E_0(L(-x)_1^q f(x+X_1)) = G(f,q,x). \end{aligned} \tag{3.13}$$

Similarly, for $t > 0$:

$$E_0(L_t^q f(X_t)) = t^{q/2}E_0(L_1^q f(X_1/\sqrt{t})). \tag{3.14}$$

Moreover, on knows (see e.g. [11]) that L_1 under P_0 has the same law than $|X_1|$, hence

$$E_0(L_1) = \sqrt{\frac{2}{\pi}}, \quad E_0(L_1^2) = 1. \tag{3.15}$$

The hitting time T_x of $\{x\}$ for X, under P_0, has the density $r \rightsquigarrow \frac{|x|}{\sqrt{2\pi}r^{3/2}}e^{-x^2/2r}1_{\mathbb{R}_+}(r)$. Then (3.13) and the Markov property yield

$$G(f,q,x) = \begin{cases} E_0(L_1^q f(X_1)) & \text{if } x = 0 \\ \displaystyle\int_0^1 \frac{|x|}{\sqrt{2\pi}r^{3/2}}e^{-\frac{x^2}{2r}}(1-r)^{q/2} & \text{if } x \neq 0. \\ \quad \times E_0(L_1^q f(X_1\sqrt{1-r}))dr & \end{cases} \tag{3.16}$$

4. THEOREM 1-1

4-1) Recall once more that we are on the canonical space, with the canonical process X and the Wiener measure P_x. We first prove a version of Theorem 1-1 in the case $u_n = \sqrt{n}$, and for the processes (1.4). In this setting, this version is indeed more general than Theorem 1-1, and has interest on its own.

THEOREM 4.1. – *a) If g_n is a sequence of functions on $\mathbb{R}$ satisfying for all $x \in \mathbb{R}$, as $n \to \infty$:*

$$\lambda(|g_n|) \to 0, \qquad \frac{g_n(x\sqrt{n})}{\sqrt{n}} \to 0, \tag{4.1}$$

then the processes $\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)^n$ converge locally uniformly in time, in $\mathbb{L}^1(P_x)$, to 0.

b) Let g_n be a sequence of functions on $\mathbb{R}$ satisfying for all $x \in \mathbb{R}$, as $n \to \infty$:

$$\frac{g_n(x\sqrt{n})^2}{n} + \frac{\lambda(g_n^2)}{\sqrt{n}} + \frac{\beta_1(g_n)|g_n(x\sqrt{n})| \log n}{n} + \frac{\beta_1(g_n)\lambda(|g_n|) \log n}{\sqrt{n}} \to 0. \tag{4.2}$$

If $\lambda(g_n) \to \lambda$ we have for all t:

$$\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)_t^n \to^{P_x} \lambda L_t. \tag{4.3}$$

If furthermore $\sup_n \lambda(|g_n|) < \infty$, the processes $\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)^n$ converge locally uniformly in time, in P_x-probability, to λL.

Let us begin with a lemma.

LEMMA 4.2. – *If the functions g_n satisfy (4.2) and $\lambda(g_n) = 0$, we have*

$$E_x(|\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)_t^n|^2) \to 0. \tag{4.4}$$

Proof. – With $\delta(g)_t^n = \sup_{y\in\mathbb{R}, s\in[0,t]} |\gamma(g, y)_s^n|$, we can write (use the Markov property):

$$\begin{aligned}
&E_x(|V(\sqrt{n}, g)_t^n|^2) \\
&= \sum_{i=1}^{[nt]} E_x(g(\sqrt{n}X_{\frac{i-1}{n}})^2) + 2 \sum_{i\leq i<j\leq[nt]} E_x(g(\sqrt{n}X_{\frac{i-1}{n}})g(\sqrt{n}X_{\frac{j-1}{n}})) \\
&= g^2(\sqrt{n}x) + \gamma(g^2, x)_t^n + 2 \sum_{i=1}^{[nt]-1} E_x(g(\sqrt{n}X_{\frac{i-1}{n}})\gamma(g, X_{\frac{i-1}{n}})_{t-(i-1)/n}^n) \\
&\leq g^2(\sqrt{n}x) + \gamma(g^2, x)_t^n + 2\delta(g)_t^n(|g(\sqrt{n}x)| + \gamma(|g|, x)_{t-1/n}^n).
\end{aligned}$$

If $\lambda(g) = 0$, (3.11) yields $\delta(g)_t^n \le K_t\beta_1(g)\log n$ for $n \ge 2$, hence by (3.10):

$$E_x(|V(\sqrt{n},g)_t^n|^2) \\ \le K_t\big(g^2(\sqrt{n}x)+\lambda(g^2)\sqrt{n}+\beta_1(g)(\log n)(|g(\sqrt{n}x)|+\lambda(|g|)\sqrt{n})\big),$$

and (4.4) follows. □

Proof of Theorem 4.1. – We have $E_x(\sup_{s\le t}|V(\sqrt{n},g_n)_s^n|) \le |g_n(x\sqrt{n})| + \gamma(|g_n|,x)_t$, so (a) obviously follows from (3.10).

Let us prove (b). The function $\hat{g}$ of (1.14) is $\hat{g}(x) = E_x(|X_1| - |X_0|)$, and by definition of the local time,

$$E_x(|X_{\frac{i}{n}}| - |X_{\frac{i-1}{n}}||\mathcal{F}_{\frac{i-1}{n}}) = E_x(L_{\frac{i}{n}} - L_{\frac{i-1}{n}}|\mathcal{F}_{\frac{i-1}{n}}) \\ = \frac{1}{\sqrt{n}}\hat{g}(\sqrt{n}X_{\frac{i-1}{n}}), \tag{4.5}$$

so by Lemma (2.14) of [7] we have the following convergence for all x:

$$\frac{1}{\sqrt{n}}V(\sqrt{n},\hat{g})_t^n \ \to^{P_x}\ L_t. \tag{4.6}$$

Now let g_n be a sequence satisfying (4.2) and $\lambda(g_n) \to \lambda$. We set $g'_n = g_n - \lambda(g_n)\hat{g}$. (1.14) implies that the sequence g'_n satisfies (4.2) as well and $\lambda(g'_n) = 0$, hence Lemma 4-2 yields $\frac{1}{\sqrt{n}}V(\sqrt{n},g'_n)_t^n \to^{P_x} 0$. Since $V(\sqrt{n},g_n)^n = V(\sqrt{n},g'_n)^n + \lambda(g_n)V(\sqrt{n},\hat{g})^n$, (4.3) follows from (4.6).

If finally $\sup_n \lambda(|g_n|) < \infty$, up to taking a subsequence we may assume that $\lambda(|g_n|) \to \lambda'$. Then $\lambda(g_n^+) \to b_+ := \frac{\lambda'+\lambda}{2}$ and $\lambda(g_n^-) \to b_- := \frac{\lambda'-\lambda}{2}$. The processes $\frac{1}{\sqrt{n}}V(\sqrt{n},g_n^+)^n$ and $\frac{1}{\sqrt{n}}V(\sqrt{n},g_n^-)^n$ converge in P_x-probability to b_+L and b_-L for all t, and since they are non-decreasing and with a continuous limit this convergence is locally uniform in time. By difference we deduce the second claim in (b). □

4-2) Proof of Theorem 1-1 (under P_x). For further convenience, we first introduce some new notation and some simple properties. For any function h on $\mathbb{R}^2$ having (B-0), the process

$$M(h)^n = U(u_n,h)^n - V(u_n,H_h)^n \tag{4.7}$$

is a P_x-martingale w.r.t. the filtration $\mathcal{F}_t^n = \mathcal{F}_{[nt]/n}$, with predictable bracket given by

$$\langle M(h)^n, M(h)^n\rangle_t = V(u_n,H_{h^2}-(H_h)^2)^n \le V(u_n,H_{h^2})^n. \tag{4.8}$$

Observe also that for any function g on $\mathbb{R}$ we have

$$V(u_n, g)_t^n = V(\sqrt{n}, g_n)_t^n \qquad \text{with} \quad g_n(x) = g(\frac{u_n x}{\sqrt{n}}). \tag{4.9}$$

In (4.9) above, we have $\lambda(|g_n|) = \frac{\sqrt{n}}{u_n}\lambda(|g|)$. Hence (3.10) and (4.9) yield the following useful estimate:

$$E_x\left(\sup_{s\leq t}|\frac{u_n}{n}V(u_n, g)_|^n\right) \leq E_x\left(\frac{u_n}{n}V(u_n, |g|)_t^n\right)$$
$$\leq K\left(\frac{u_n}{n}|g(u_n x)| + \sqrt{t}\lambda(|g|)\right). \tag{4.10}$$

Now we prove three lemmas, which are indeed too strong for proving Theorem 1-1 but will be useful for Proposition 2-1.

LEMMA 4.3. – *Let h_n be a sequence of functions satisfying $|h_n(x,y)| \leq \hat{h}_n(x)e^{a|y|}$ for some $a \in \mathbb{R}_+$, and such that*

$$\sup_n \lambda(\hat{h}_n) \ < \ \infty, \qquad \sup_{n,x} \hat{h}_n(x) \ < \ \infty. \tag{4.11}$$

Then under (1.11) the processes $\frac{u_n}{n}U(u_n, h_n - H_{h_n})^n$ converge locally uniformly in time, in P_x-probability, to 0.

Proof. – Observe that $H_{h_n^2} \leq K\hat{h}_n$, so a combination of (4.7) and (4.8) yields that the martingale $M(h_n)^n$ has a bracket smaller than $KV(u_n, \hat{h}_n)^n$ (use the second part of (4.11)). Therefore $E(\langle\frac{u_n}{n}M(h_n)^n, \frac{u_n}{n}M(h_n)^n\rangle_t) \to 0$ by (4.10) and (4.11) and (1.11), and the result follows from Doob's inequality. □

LEMMA 4.4. – *Let g_n be a sequence of functions satisfying*

$$\sup_n \lambda(|g_n|) \ < \ \infty, \qquad \sup_{n,x} |g_n(x)| \ < \ \infty, \tag{4.12}$$

$$\lim_q \limsup_n \int_{|x|>q} |g_n(x)|dx = 0. \tag{4.13}$$

If further $\lambda(g_n) \to \alpha$ and u_n satisfies (1.11) and $\frac{\log n}{u_n} \to 0$, the processes $\frac{u_n}{n}V(u_n, g_n)^n$ converge locally uniformly in time, in P_x-probability, to αL.

Proof. – a) Assume first that $\sup_n \beta_1(g_n) < \infty$. In view of (4.9), and if $k_n(x) = \frac{u_n}{\sqrt{n}}g_n(\frac{u_n x}{\sqrt{n}})$, we need to prove the convergence, locally uniformly in time in P_x-probability, of the sequence $\frac{1}{\sqrt{n}}V(\sqrt{n}, k_n)^n$ to αL.

We readily check that $|k_n| \leq K u_n/\sqrt{n}$ and $\lambda(|k_n|) \leq K$ and $\beta_1(k_n) \leq K\sqrt{n}/u_n$ and $\lambda(k_n^2) \leq K u_n/\sqrt{n}$. Hence by (1.11) and $\frac{\log n}{u_n} \to 0$ the sequence (k_n) satisfies (4.2), while $\lambda(k_n) = \lambda(g_n) \to \alpha$ by (a): the last statement in Theorem 4-1 gives the result.

b) Let us now consider the general case. Up to taking a subsequence, we can assume that $\int_{\{|x|\leq q\}} g_n(x)ds \to \alpha_q$ for any $q \in \mathbb{N}$, and (4.13) readily gives that $\alpha_q \to \alpha$ as $q \to \infty$.

Set $k_q^n(x) = g_n(x)1_{\{|x|\leq q\}}$. We have $\beta_1(k_q^n) \leq q\lambda(|g_n|) \leq Kq$ by (4.12), so (a) yields that $\frac{u_n}{n}V(u_n, k_q^n)^n$ converges locally uniformly in time in P_x-probability to $\alpha_q L$. Since $\alpha_q \to \alpha$ it thus remain to show that

$$\limsup_n E_x\left(\sup_{s\leq t} \frac{u_n}{n}|V(u_n, g_n - k_q^n)_s^n|\right) \to 0 \tag{4.14}$$

as $q \to \infty$. By (4.10), the expectation above is smaller than $K(\frac{u_n}{n} + \sqrt{t}\lambda(|g_n - k_q^n|))$. Thus the left side of (4.14) is smaller than $K_t \limsup_n \lambda(|g_n - k_q^n|)$ by (1.11), and this quantity goes to 0 as $q \to \infty$ by (4.13): hence (4.14) holds. □

LEMMA 4.5. – *Let g be a bounded and integrable function of $\mathbb{R}$, and let T be a function satisfying (2.2), and set $g_n(x) = g(u_n T(x/u_n))$. If u_n satisfies (1.11) and $u_n^2/n \to 0$ the processes $\frac{u_n}{n}V(u_n, g_n)^n$ converge locally uniformly in time in P_x-probability to $\lambda(g)L$.*

Proof. – By the same argument as in the proof of Theorem 4-1, it suffices to prove the result when $g \geq 0$, in which case it is enough to have the convergence for each time t.

We can write

$$\frac{u_n}{n}V(u_n, g_n)_t^n = \int g_n(x)L_{[nt]/n}^{x/u_n}dx - B_t^n - \alpha_t^n, \tag{4.15}$$

where (by the occupation time formula):

$$\alpha_t^n = \frac{u_n}{n}\int_0^{nt-[nt]} (g_n(u_n X_{\frac{[nt]}{n}+\frac{s}{n}}) - g_n(u_n X_{\frac{[nt]}{n}}))ds,$$

$$B_t^n = \sum_{i=1}^{[nt]} \beta_i^n, \qquad \beta_i^n = \frac{u_n}{n}\int_0^1 (g_n(u_n X_{\frac{i-1}{n}+\frac{s}{n}}) - g_n(u_n X_{\frac{i-1}{n}}))ds.$$

Since g is bounded, $\alpha_t^n \to 0$ follows from (1.11). Next, the variables $L_{[nt]/n}^{x/u_n} - L_t$ tend to 0 and remain smaller (for all x and n) than a fixed finite variable U. Therefore

$$\left|\int g_n(x)(L^{x/u_n}_{[nt]/n} - L_t)dx\right| \leq K \int_{\{|x|\leq q|\}} |L^{x/u_n}_{[nt]/n} - L_t|dx$$
$$+ U \int_{\{|x|>q\}} \left|g\left(u_n T\left(\frac{x}{u_n}\right)\right)\right| dx.$$

The first term in the right side above goes to 0 as $n \to \infty$ for all q, and the second term is smaller than $U \int_{\{|x|>q/\varepsilon\}} |g(x)|dx$, which goes to 0 as $q \to \infty$: thus

$$\int g_n(x) L^{x/u_n}_{[nt]/n} dx - \lambda(g_n)L_t \;\to\; 0.$$

Furthermore $\lambda(g_n) = \int g(x)\bar{T}'(\frac{x}{u_n})dx$, where $\bar{T}$ is the inverse function of T. Since $\bar{T}'(x/u_n)$ tends to 1 and remains bounded by (2.2), it follows that $\lambda(g_n) \to \lambda(g)$. Therefore the first term of the right side of (4.15) goes (for all ω) to $\lambda(g)L_t$.

It remains to prove that $B^n_t \to^{P_x} 0$. Let $C^n_t = \sum_{i=1}^{[nt]} \gamma^n_i$, where $\gamma^n_i = E_x(\beta^n_i | \mathcal{F}_{\frac{i-1}{n}})$. We have

$$E_x(|B^n_t - C^n_t|^2) = E_x\left(\sum_{i=1}^{[nt]} (\beta^n_i - \gamma^n_i)^2\right) \leq \; 2E_x\left(\sum_{i=1}^{[nt]} |\beta^n_i|^2\right), \quad (4.16)$$

and $|\beta^n_i| \leq K u_n^2/n^2$. The above sum is thus smaller than Ktu_n^2/n, which goes to 0 by hypothesis: so it remains to prove that

$$E_x\left(\sum_{i=1}^{[nt]} |\gamma^n_i|\right) \;\to\; 0.$$

Set $v_n = u_n^2/n$. A simple calculation shows that $\gamma^n_i = \frac{u_n}{n} h_n(u_n X_{\frac{i-1}{n}})$, where $h_n = \int_0^1 (P_{sv_n} g_n - g_n)ds$. Since $|g_n| \leq K$, by (4.10) yields that the left side of (4.16) is smaller than $K(\frac{u_n}{n} + \sqrt{t}\lambda(|f_n|))$, and we are left to prove that $\lambda(|f_n|) \to 0$.

We can find for each integer p a function k_p which is Lipschitz with compact support and such that $\lambda(|g - k_p|) \leq 1/p$. The function $k^n_p(x) = k_p(u_n T(x/u_n))$ has $\lambda(|g_n - k^n_p|) \leq 1/p\varepsilon$. Since the action of the kernel P_t is a convolution, we also have $\lambda(|P_{sv_n} g_n - P_{sv_n} k^n_p|) \leq 1/p\varepsilon$. Therefore

$$\lambda(|f_n|) \;\leq\; \frac{2}{p\varepsilon} + \int_0^1 \lambda(|P_{sv_n} k^n_p - k^n_p|)ds.$$

So it remains to prove that for each p, the last term above goes to 0 as $n \to \infty$. By Lebesgue Theorem, it is even enough to prove that for each s, then $\lambda(|P_{sv_n}k_p^n - k_p^n|) \to 0$. But if Y denotes an $\mathcal{N}(0,1)$ random variable, this quantity is less than

$$E\left(\int \left| k_p\left(u_n T\left(\frac{x}{u_n}\right)\right) - k_p\left(u_n T\left(\frac{x + Y\sqrt{sv_n}}{u_n}\right)\right)\right| dx\right).$$

Since $v_n \to 0$ and since k_p is Lipschitz with compact support, and in view of (2.2), this last expression clearly goes to 0 as $n \to \infty$, and we are finished. □

Theorem 1-1 obviously follows from these results: assume (1.11) and let h satisfy (B-0). When $(\log n)/u_n \to 0$, we can apply Lemma 4-3 with $h_n = h$ and Lemma 4-4 with $g_n = H_h$ and (4.7). Otherwise we apply again Lemma 4-3 with $h_n = h$ and Lemma 4-5 with $g = H_h$ and $T(x) = x$ and (4.7) again.

4-3) Proof of Proposition 2-1. This proof will go through three steps.

Step 1. – First, we prove the claim (a). We assume that h satisfies (B-0) with a and $\hat{h}$. Define h_n by (2.3), and set $k_n = H_{h_n}$. In view of (2.2), it is clear that $|h_n(x,y)| \leq \hat{h}_n(x)e^{a'|y|}$, where $a' = a/\varepsilon$ and $\hat{h}_n(x) = \frac{1}{\varepsilon}\hat{h}(u_n T(x/u_n))$.

Obviously (2.2) yields that the sequence $\hat{h}_n$ satisfies (4.11), hence Lemma 4-3 yields that $\frac{u_n}{n}U(u_n, h_n - k_n)^n$ goes to 0 locally uniformly in time in P_x-probability. So it remains to prove that $\frac{u_n}{n}V(u_n, k_n)^n$ tends to $\lambda(H_h)L$ locally uniformly in time in P_x-probability. For this, by using exactly the same argument than in the end of the proof of Lemma 4-4, we can and will assume that $\beta_1(\hat{h}) < \infty$.

Observe that

$$k_n(x) = \int h\left(u_n T\left(\frac{x}{u_n}\right), \sqrt{n}\left(T\left(\frac{x}{u_n} + \frac{y}{\sqrt{n}}\right) - T\left(\frac{x}{u_n}\right)\right)\right)\rho(y)dy.$$

Set $\alpha_n(x,y) = \sqrt{n}\left(\bar{T}(T(\frac{x}{u_n}) + \frac{y}{\sqrt{n}}) - \frac{x}{u_n}\right)$. A change of variable yields

$$k_n(x) = \int h\left(u_n T\left(\frac{x}{u_n}\right), y\right)\delta_n(x,y)dy,$$

where

$$\delta_n(x,y) = \rho(\alpha_n(x,y))\bar{T}'\left(T\left(\frac{x}{u_n}\right) + \frac{y}{\sqrt{n}}\right).$$

Now, (2.2) implies that $|\alpha_n(x,y) - y| \leq K(\frac{y^2}{\sqrt{n}} + |T(\frac{x}{u_n})|)$ and $|\bar{T}'(\frac{y}{\sqrt{n}} + T((\frac{x}{u_n})) - 1| \leq K(\frac{|y|}{\sqrt{n}} + |T(\frac{x}{u_n})|)$ and $\varepsilon \leq \frac{\alpha_n(x,y)}{y} \leq \frac{1}{\varepsilon}$. Thus $\rho(\alpha_n(x,y)) \leq K\rho(y\varepsilon)$, while $|\rho'(y)| \leq K\rho(y/2)$. It follows by Taylor's formula that $|\rho(\alpha_n(x,y)) - \rho(y)| \leq K\rho(y\varepsilon/2)(\frac{y^2}{\sqrt{n}} + |T(\frac{x}{u_n})|)$, and thus $|\delta_n(x,y) - \rho(y)| \leq K\rho(y\varepsilon/2)(\frac{1+y^2}{\sqrt{n}} + |T(\frac{x}{u_n})|)$. Therefore if $f_n(x) = H_h(u_n T(x/u_n))$ we have

$$|k_n(x) - f_n(x)| \leq K\hat{h}\left(u_n T\left(\frac{x}{u_n}\right)\right)\left(\frac{1}{\sqrt{n}} + \left|T\left(\frac{x}{u_n}\right)\right|\right). \quad (4.17)$$

Combining (4.10) and the facts that $\hat{h}$ is bounded and that $\beta_1(\hat{h}) < \infty$, we deduce that $E_x(\frac{u_n}{n}V(u_n, |k_n - f_n|)_t^n) \leq K_t(\frac{1}{u_n} + \frac{1}{\sqrt{n}})$, and thus $\frac{u_n}{n}V(u_n, k_n - f_n)^n$ tends to 0 in P_x-probability locally uniformly in time: therefore it remains to prove that $\frac{u_n}{n}V(u_n, f_n)^n$ tends to $\lambda(H_h)L$ in P_x-probability locally uniformly in time.

Suppose first that $(\log n)/u_n \to 0$. The sequence f_n satisfies (4.12) and (4.13) (for the later, observe that $|f_n| \leq \hat{h}_n$ and that $\int_{|x|>q} \hat{h}_n(x)dx \leq \frac{1}{\varepsilon}\int_{|x|>q\varepsilon} \hat{h}(x)dx$). On the other hand, a change of variable yields

$$\lambda(f_n) = \int H_h(x)\bar{T}'\left(\frac{x}{u_n}\right)dx,$$

which clearly goes to $\lambda(H_h)$ (since $\bar{T}'$ is bounded and goes to 0 at 0): the result then follows from Lemma 4-4.

Next, suppose that $u_n^2/n \to 0$: the result readily follows from Lemma 4-5.

Step 2. – From now on we assume that h satisfies (B-1). In this step we prove that with notation (4.7), the process $M^n = \sqrt{\frac{u_n}{n}}M(h_n - h)^n$ tends to 0 locally uniformly in time in P_x probability. In view of (4.8) and of Doob's inequality, it is enough to prove that $E_x(\frac{u_n}{n}V(u_n, H_{(h_n-h)^2})_t^n) \to 0$.

Set $g_n(x) = H_{(h_n-h)^2}$. As seen at the beginning of Step 1, $|h_n(x,y)| \leq \hat{h}_n(x)e^{a'|y|}$, where $a' = a/\varepsilon$ and $\hat{h}_n(x) = \frac{1}{\varepsilon}\hat{h}(u_n T(x/u_n))$: it readily follows that $g_n(x)$ is bounded uniformly in x, n. Thus in view of (1.11) and (4.10), it remains to show that $\lambda(g_n) \to 0$. We have

$$\lambda(g_n) = \int\left(h\left(u_n T\left(\frac{x}{u_n}\right), \sqrt{n}\left(T\left(\frac{x}{u_n} + \frac{y}{\sqrt{n}}\right) - T\left(\frac{x}{u_n}\right)\right)\right)\right.$$
$$\left. - h(x,y)\right)^2 \rho(y)dxdy. \quad (4.18)$$

Denote below by $\| g \|$ the quantity $\left(\int g(x,y)^2\rho(y)dxdy\right)^{1/2}$. For any $q \in \mathbb{N}$ one may find a continuous function f_q on $\mathbb{R}^2$ with compact support, such that $\| h - g_q \| \leq 1/q$. With the notation of Step 1, a change of variables yields

$$\int\Bigg((h-g_q)^2\Bigg(u_n T\left(\frac{x}{u_n}\right), \sqrt{n}\left(T\left(\frac{x}{u_n}+\frac{y}{\sqrt{n}}\right)-T\left(\frac{x}{u_n}\right)\right)\Bigg)-h(x,y)\Bigg)\rho(y)dxdy$$
$$= \int (h-g_q)^2(x,y)\bar{T}'\left(\frac{x}{u_n}\right)\rho\left(\alpha_n\left(\bar{T}\left(\frac{x}{u_n}\right),y\right)\right)\bar{T}'\left(\frac{x}{u_n}+\frac{y}{\sqrt{n}}\right)dxdy$$
$$\leq K \| h - g_q \|^2$$

for a constant K depending only on ε in (2.2). Now, if α_q^n denotes the right side of (4.18) with g_q instead of h, it then follows that $\sqrt{\lambda(g_n)} \leq \frac{1+\sqrt{K}}{p} + \sqrt{\alpha_p^n}$. Furthermore, since g_q is continuous with compact support, it is immediate (by (2.2) again) that $\alpha_q^n \to 0$ as $n \to \infty$, for all q: thus $\lambda(g_n) \to 0$ readily follows, and Step 2 is complete.

Step 3. – In view of Step 2 and of (4.7), in order to obtain Proposition 2-1(b) it remains to prove that $\sqrt{\frac{u_n}{n}}V(u_n, H_{h_n-h})^n$ tends to 0 locally uniformly in time in P_x-probability when $n/u_n^3 \to 0$ and when h is differentiable in the first variable with a partial derivative satisfying (B-1).

We use again the notation of Step 1. In particular $H_{h_n-h} = k_n - H_h$. Using (4.17), we obtain $\lambda(|k_n - f_n|) \leq K(\frac{1}{\sqrt{n}} + \frac{1}{u_n})$, while by definitions of k_n and f_n and the boundedness of $\hat{h}$ we have $|k_n - f_n| \leq K$. Then (4.10) implies that

$$E_x\left(\sup_{s\leq t}\left|\sqrt{\frac{u_n}{n}}V(u_n, k_n - f_n)_s^n\right|\right) \leq K\left(\sqrt{\frac{u_n}{n}} + \frac{1}{\sqrt{u_n}} + \frac{\sqrt{n}}{u_n^{3/2}}\right).$$

Since we have (1.11) and $n/u_n^3 \to 0$, the above goes to 0, and we are left to prove the convergence of $\sqrt{\frac{u_n}{n}}V(u_n, f_n - H_h)^n = \frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)^n$ to 0, where $g_n(x) = \sqrt{u_n}(f_n - H_h)(xu_n/\sqrt{n})$ (use (4.9)).

We assume that $h' = \partial h/\partial x$ exists and has (B-1). We have

$$g_n(x) = \sqrt{u_n}\left(T\left(\frac{x}{\sqrt{n}}\right)-\frac{x}{\sqrt{n}}\right)\int H_{h'}\left(u_n\left(\frac{x}{\sqrt{n}}+v\left(T\left(\frac{x}{\sqrt{n}}\right)-\frac{x}{\sqrt{n}}\right)\right)\right)dv$$

because $H_{h'}$ is the derivative of H_h. Since $|T(\frac{x}{\sqrt{n}}) - \frac{x}{\sqrt{n}}| \leq K|x|/\sqrt{n}$ by (2.2), it follows that

$$|g_n(x)| \leq \sqrt{u_n}\frac{|x|}{\sqrt{n}}\int |H_{h'}|\left(u_n\left(\frac{x}{\sqrt{n}}+v\left(T\left(\frac{x}{\sqrt{n}}\right)-\frac{x}{\sqrt{n}}\right)\right)\right)dv.$$

Since h', hence $H_{h'}$, satisfy (B-2), we easily deduce that $\lambda(|g_n|) \leq K\sqrt{n}/u_n^{3/2}$, while on the other hand $|g_n| \leq K\sqrt{u_n}$ by construction of g_n: hence the sequence g_n satisfies (4.1), and the result follows from Theorem 4-1.

4-4) Finally we give a result which is simple, but will be used later.

THEOREM 4.6. – *Assume (1.11) and (A), and let $h = (h^i)_{1\leq i\leq d}$ be a d-dimensional function satisfying (B-0). Then the processes $Y^n = \sqrt{\frac{u_n}{n}}(U(u_n,h)^n - V(u_n,H_h)^n)$ converge stably in law to a process Y, defined on an extension of the space, and which is an $\mathcal{F}$-conditional Gaussian continuous martingale with brackets*

$$\langle Y^i, Y^j\rangle = \lambda(H_{h^ih^j} - H_{h^i}H_{h^j})L.$$

Proof. – We have that $Y^n = \sqrt{u_n/n}M(h)^n$ (notation (4.7)) is a martingale w.r.t. the filtration $(\mathcal{F}_{[nt]/n})_{t\geq 0}$. Further, under P_x any martingale (w.r.t. $(\mathcal{F}_t)$) orthogonal to X is constant. Hence by Theorem 3-2 of [9], the result will follow from the next three properties, where for any process Z we put $\Delta_i^n Z = Z_{i/n} - Z_{(i-1)/n}$:

$$\sum_{i=1}^{[nt]} E_x(\Delta_i^n Y^{nl}\Delta_i^n Y^{nj}|\mathcal{F}_{\frac{i-1}{n}}) \to^{P_x} \lambda(H_{h^lh^j} - H_{h^i}H_{h^j})L_t, \tag{4.19}$$

$$\sum_{i=1}^{[nt]} E_x(\Delta_i^n Y^{nj}\Delta_i^n X|\mathcal{F}_{\frac{i-1}{n}}) \to^{P_x} 0, \tag{4.20}$$

$$\sum_{i=1}^{[nt]} E_x(|\Delta_i^n Y^{nj}|^2 1_{\{|\Delta_i^n Y^{nj}|>\varepsilon\}}|\mathcal{F}_{\frac{i-1}{n}}) \to^{P_x} 0 \qquad \forall\varepsilon > 0. \tag{4.21}$$

By polarization, it is enough to prove these when h is 1-dimensional, which we assume in the sequel.

First, by (4.8) the left side of (4.19) is $\frac{u_n}{n}V(u_n, H_{h^2} - (H_h)^2)_t^n$, hence (4.19) obtains by Theorem 1-1.

Second, by a simple computation the left side of (4.20) is $\frac{\sqrt{u_n}}{n}V(u_n,g)_t^n$, where $g(x) = \int h(x,y)y\rho(y)dy$, hence (4.20) obtains by Theorem 1-1 and $u_n \to \infty$.

Third, another computation yields that $|\Delta_i^n Y^n| \leq K\hat{h}(u_n X_{(i-1)/n})\sqrt{u_n/n}e^{a|\sqrt{n}\Delta_i^n X|}$ (where a and $\bar{h}$ are as in (B-0)). Hence $|\Delta_i^n Y^n| > \varepsilon \Rightarrow |\sqrt{n}\Delta_i^n X| > K_\varepsilon \log(n/u_n)$ for some

$K_\varepsilon > 0$. Therefore

$$E_x(|\Delta_i^n Y^{nj}|^2 1_{\{|\Delta_i^n Y^{nj}|>\varepsilon\}}|\mathcal{F}_{\frac{i-1}{n}}) \leq \frac{u_n \gamma_n}{n} \hat{h}(u_n X_{\frac{i-1}{n}}),$$
$$\text{where } \gamma_n = K e^{-K_\varepsilon^2 (\log(n/u_n))/8}$$

(the constant K depends on a and on $\bar{h}$). Therefore the left side of (4.21) is smaller than $(\gamma_n u_n/n)V(u_n, \hat{h})_t^n$: we conclude by Theorem 1-1 once more and by the property $\gamma_n \to 0$.

5. A BASIC MARTINGALE

It remains to prove Theorem 1-2, for the Brownian measures P_x on the canonical space; the d-dimensional function h satisfies (B-r) for some $r > 3$, and $u_n = n^\alpha$ with $1/3 < \alpha < 1$, and $\delta = ((1-\alpha) \wedge \alpha)/2$. For proving the convergence of the sequence $n^\delta(\frac{1}{n^{1-\alpha}} U(n^\alpha, h)^n - \lambda(H_h)L)$ we will use the same method as for Theorem 4-6. However the above processes are not in general martingales, and our first task is to write them as sums of a sequence of martingales to which the previous method applies, and a sequence of processes which go to 0. In this section, we perform this decomposition.

The processes of interest may be decomposed in a sum of four terms:

$$\frac{1}{n^{1-\alpha}} U(n^\alpha, h)^n - \lambda(H_h)L = M^n + N^n + \theta R^n + \theta Q^n, \qquad (5.1)$$

where $M^n = \frac{1}{n^{1-\alpha}} M(h)^n$ (see (4.7)), and $\theta = \lambda(H_h)$, and (recall (1.14) for $\hat{g}$):

$$\left.\begin{aligned} N^n &= \frac{1}{n^{1-\alpha}} V(n^\alpha, H_h)^n - \theta \frac{1}{\sqrt{n}} V(\sqrt{n}, \hat{g})^n, \\ R_t^n &= \frac{1}{\sqrt{n}} V(\sqrt{n}, \hat{g})_t^n - L_{[nt]/n}, \qquad Q_t^n = L_{[nt]/n} - L_t. \end{aligned}\right\} \qquad (5.2)$$

As seen before, the process M^n is a martingale w.r.t. the filtration $(\mathcal{F}_{[nt]/n})_{t\geq 0}$, as well as R^n by (4.5). It is not true for N^n and Q^n, but in this section we prove that $n^\delta Q^n$ is "negligible", while $n^\delta N^n$ is a martingale plus a "negligible" term.

The term Q^n is easy to deal with: the local time L has Hölder paths with index ε for any $\varepsilon < 1/2$ (this is classical result, following for example from Kolmogorov's criterion combined with (3.14)). Hence, since $\delta < 1/2$, we have for all $\omega \in \Omega$:

$$\sup_{s \le t} |n^\delta Q_s^n(\omega)| \ \to \ 0. \tag{5.3}$$

The case of N^n is more complicated. By (4.9), we have $n^\delta N^n = \frac{1}{\sqrt{n}} V(\sqrt{n}, k_n)^n$, where

$$k_n(x) = n^\delta \Big(n^{\alpha - 1/2} H_h(n^{\alpha-1/2}x) - \theta \hat{g}(x) \Big). \tag{5.4}$$

In the rest of the paper, the constants K, K_γ may depend on the function h, via $\hat{h}$ and a. Observe that (B-r) and (1.14) yield

$$\begin{array}{l} |k_n(x)| \ \le \ K n^\delta, \quad \beta_\gamma(k_n) \ \le \ K n^\delta (1 + n^{(\gamma-1)(1/2-\alpha)}) \\ \text{for } \ \gamma \in [0, r], \quad \lambda(k_n) = 0. \end{array} \tag{5.5}$$

We also consider $\beta \in (0,1)$, to be chosen later, and set $w_n = [n^\beta]$ and

$$F_n = \sum_{j=0}^{w_n} P_j k_n, \quad \bar{F}_n = \sum_{j=1}^{w_n} P_j k_n, \quad \hat{F}_n = \sum_{j=1}^{w_n+1} P_j k_n, \quad \breve{F}_n = \sum_{j=0}^{w_n+1} P_j k_n. \tag{5.6}$$

Due to (5.5) and Lemma 3-1 we have for $\gamma \in [0, r-1]$:

$$|\hat{F}_n(x)| + |\bar{F}_n(x)| \ \le \ \begin{cases} K(1 + n^{1/2-\alpha}) n^\delta \log n \\ K_\gamma n^\delta (\log n) \Big(\dfrac{1 + n^{1/2-\alpha}}{1 + |xn^{-\beta/2}|^\gamma} + \dfrac{1 + n^{(\gamma+1)(1/2-\alpha)}}{1 + |x|^\gamma} \Big) \\ K(1 + |x|) n^\delta (1 + n^{1-2\alpha}), \end{cases} \tag{5.7}$$

$$\begin{array}{l} |F_n(x)| + |\breve{F}_n(x)| + |\hat{F}_n(x)| + |\breve{F}_n(x)| \\ \quad \le \ K n^\delta (\log n + n^{1/2-\alpha} \log n + n^{\alpha - 1/2}), \end{array} \tag{5.8}$$

$$|P_{w_n+1} k_n(x)| \ \le \ K_\gamma n^{\delta - \beta} \Big(\frac{1 + n^{1/2-\alpha}}{1 + |xn^{-\beta/2}|^\gamma} + \frac{1 + n^{(\gamma+1)(1/2-\alpha)}}{1 + |x|^\gamma} \Big). \tag{5.9}$$

On the other hand, put

$$\left. \begin{array}{l} \zeta_i^n = \displaystyle\sum_{j=0}^{w_n} \big(E_x(k_n(\sqrt{n} X_{\frac{i+j}{n}} | \mathcal{F}_{\frac{i}{n}}) - E_x(k_n(\sqrt{n} X_{\frac{i+j}{n}} | \mathcal{F}_{\frac{i-1}{n}})\big), \\ W_t^n = \dfrac{1}{\sqrt{n}} \displaystyle\sum_{i=1}^{[nt]} \zeta_i^n, \end{array} \right\} \tag{5.10}$$

$$\left.\begin{aligned} A_t^n &= \frac{1}{\sqrt{n}}(F_n(\sqrt{n}X_0) - F_n(\sqrt{n}X_{[nt]/n})), \\ B^n &= \frac{1}{\sqrt{n}}V(\sqrt{n}, P_{w_n+1}k_n)^n. \end{aligned}\right\} \tag{5.11}$$

Observe that

$$\begin{aligned} \zeta_i^n &= k_n(\sqrt{n}X_{\frac{i}{n}}) + \bar{F}_n(\sqrt{n}X_{\frac{i}{n}}) - \hat{F}_n(\sqrt{n}X_{\frac{i-1}{n}}) \\ &= F_n(\sqrt{n}X_{\frac{i}{n}}) - \hat{F}_n(\sqrt{n}X_{\frac{i-1}{n}}), \end{aligned} \tag{5.12}$$

so a simple computation yields

$$n^\delta N^n = W^n + A^n + B^n. \tag{5.13}$$

First (5.8) and (5.11) yield that $\sup_t |A_t^n|$ is smaller, up to a multiplicative constant, than $(n^{\delta-1/2} + n^{\delta-\alpha} + n^{\delta+\alpha-1})\log n$. Now, the definition of δ implies that all the powers of n in the previous bound are negative, and thus

$$\sup_{t,\omega} |A_t^n(\omega)| \ \to \ 0. \tag{5.14}$$

LEMMA 5.1. – *a) If $\beta \in (1-\alpha, 1)$, then B^n goes to 0 in P_x-probability, uniformly over all finite intervals.*

b) If $\beta \in ((1-2\alpha)^+, 1)$, then $E_x(|B_t^n|^2) \to 0$.

Proof. – Set $g_n = P_{w_n+1}k_n$. Observe that $\lambda(P_t k) = 0$ as soon as $\lambda(k) = 0$, so here (5.5) yields $\lambda(g_n) = 0$. Hence (a) (resp. (b)) follows from Theorem 4-1(a) (resp. Lemma 4-2) if we prove that the sequence g_n satisfies (4.1) when $\beta > 1-\alpha$ (resp. (4.2) when $\beta > (1-2\alpha)^+$).

When $\alpha \geq 1/2$, (5.9) yields

$$|g_n| \ \leq \ Kn^{(1-\alpha)/2-\beta}, \qquad \lambda(|g_n|) \ \leq \ Kn^{(1-\alpha-\beta)/2},$$

$$\beta_1(g_n) \ \leq \ Kn^{(1-\alpha)/2}, \qquad \lambda(|g_n|^2) \ \leq \ Kn^{1-\alpha-3\beta/2},$$

and all the claims are obvious.

When $\alpha < 1/2$, (5.9) yields for all $\varepsilon \in (0, r-3]$:

$$|g_n| \ \leq \ Kn^{1/2-\beta-\alpha/2}, \qquad \lambda(|g_n|) \ \leq \ K_\varepsilon(n^{(1-\alpha-\beta)/2} + n^{\varepsilon-\beta+1-3\alpha/2}),$$

$$\beta_1(g_n) \ \leq \ K_\varepsilon(n^{(1-\alpha)/2} + n^{\varepsilon+3/2-\beta-5\alpha/2}),$$

$$\lambda(|g_n|^2) \ \leq \ K_\varepsilon(n^{1-3\beta/2-\alpha} + n^{\varepsilon+3/2-2\beta-2\alpha}).$$

Then again, all the claims are easy to check, since $\alpha > 1/3$. □

LEMMA 5.2. – *If* $\beta \in ((1-2\alpha)^+, 1)$, *then* $\sup_{s\leq t} |n^\delta N^n_s - W^n_s| \to^{P_x} 0$ *for all* t.

Proof. – Let β be as above, and choose β' in $(1-\alpha, 1)$. With β' we associate the processes W'^n, A'^n, and B'^n. Let $C^n := A^n + B^n = n^\delta N^n - W^n$ and $C'^n = A'^n + B'^n$.

By the previous lemma and (5.14), C'^n tends to 0 in probability, locally uniformly in time, and also C^n_t and C'^n_t tend to 0 in $\mathbb{L}^2(P_x)$. Then the processes $Z^n = C^n - C'^n = W'^n - W^n$ are martingales and satisfy $E_x(|Z^n_t|^2) \to 0$ for all t.

Following Aldous [2], we deduce that in fact Z^n tends in probability to 0, locally uniformly in time, hence the same holds for C^n. □

6. THEOREM 1-2, THE CASE $\alpha = 1/2$

Here we prove Theorem 1-2 when $\alpha = 1/2$, hence $\delta = 1/4$. Let us take $\beta = 1/4$ in the definition (5.10) of W^n. In view of (5.1), (5.3) and Lemma 5-2, we are left to prove that the sequence

$$Y^n = n^{1/4} M^n + W^n + n^{1/4}\theta R^n \tag{6.1}$$

stably converges to the limit Y, as described in Theorem 1-2. Note that Y^n is a locally square-integrable martingale w.r.t. the filtration $(\mathcal{F}_{[nt]/n})_{t\geq 0}$, so exactly as for Theorem 4-6 the result will follow from (4.21), (4.22) and the following (which replaces (4.19)):

$$\sum_{i=1}^{[nt]} E_x(\Delta^n_i Y^{nl} \Delta^n_i Y^{nj} | \mathcal{F}_{\frac{i-1}{n}}) \to^{P_x} \eta(h^l, h^j) L_t, \tag{6.2}$$

By polarization, it is enough to prove these when h is 1-dimensional, which we assume in the sequel. Combining (6.1), (5.2), (5.10), (5.12), (4.7), plus the facts that $k_n = n^{1/4}(H_h - \theta \hat{g})$ and that $\breve{F}_n = \hat{F}_n + k_n$, we obtain

$$\begin{aligned} \Delta^n_i Y^n = & \frac{1}{n^{1/4}} (h(\sqrt{n} X_{\frac{i-1}{n}}, \sqrt{n}\Delta^n_i X) - \theta\sqrt{n}\Delta^n_i L) \\ & + \frac{1}{\sqrt{n}} (F_n(\sqrt{n} X_{\frac{i}{n}}) - \breve{F}_n(\sqrt{n} X_{\frac{i-1}{n}})). \end{aligned} \tag{6.3}$$

Proof of 4.21. – By (6.3), (5.8) and (B-r), $|\Delta^n_i Y^n|^6 \leq K n^{-3/2} (e^{6a|\sqrt{n}\Delta^n_i X|} + (\log n)^6 + n^3 |\Delta^n_i L|^6)$. Since $E_x(n^3 |\Delta^n_i L|^6 | \mathcal{F}_{\frac{i-1}{n}}) =$

$G(1,6,\sqrt{n}X_{\frac{i-1}{n}})$ by (3.13), while (3.16) yields that $G(1,6,.)$ is bounded, we get $E_x(|\Delta_i^n Y^n|^6|\mathcal{F}_{\frac{i-1}{n}})) \leq K(\log n)^6/n^{3/2}$. This yields $E_x(|\Delta_i^n Y^n|^2 1_{|\Delta_i^n Y^n|>\varepsilon}|\mathcal{F}_{\frac{i-1}{n}})) \leq K\varepsilon^{-4}(\log n)^6/n^{3/2}$, and thus (4.21) holds.

Proof of 4.20. – In view of (6.3) and (3.13), the left side of (4.20) is $\frac{1}{\sqrt{n}}V(\sqrt{n},g_n)_t^n + \frac{1}{\sqrt{n}}V(\sqrt{n},g'_n)_t^n + \frac{1}{\sqrt{n}}V(\sqrt{n},g''_n)_t^n$, where

$$g_n(x) = \frac{1}{n^{1/4}}\int y\Big(h(x,y) + \frac{1}{n^{1/4}}k_n(x+y)\Big)\rho(y)dy,$$

$$g'_n(x) = \frac{1}{\sqrt{n}}\int y\bar{F}_n(x+y)\rho(y)dy,$$

$$g''_n(x) = \frac{\theta}{n^{1/4}}(xG(1,1,x) - G(f,1,x)), \quad \text{with } f(x) = x.$$

First, by Cauchy-Schwarz inequality $|g_n(x)| \leq \frac{1}{n^{1/4}}\int |yh(x,y)|\rho(y)dy + \frac{1}{\sqrt{n}}\sqrt{P_1 k_n^2(x)}$. Then (5.5) and (3.2) give $P_1 k_n^2(x) \leq K\sqrt{n}/(1+|x|^{r-1})$, hence

$$|g_n(x)| \leq \frac{K}{n^{1/4}}\Big(\bar{h}(x) + \frac{1}{1+|x|^{(r-1)/2}}\Big).$$

It follows that $|g_n|$ and $\lambda(|g_n|)$ are smaller than $Kn^{-1/4}$, so the sequence g_n satisfies (4.1), and $\frac{1}{\sqrt{n}}V(\sqrt{n},g_n)_t^n \to^{P_x} 0$.

Next, $|g'_n(x)| \leq \frac{1}{\sqrt{n}}(P_1\bar{F}_n^2(x))^{1/2}$. We have $\bar{F}_n(x)^2 \leq Kn^{2\delta}(\log n)^2/(1+|xn^{-1/8}|^{2r-2})$ by (5.7). Then (3.9) yields $(P_1\bar{F}_n^2(x))^{1/2} \leq Kn^{\delta}(\log n)/(1+|xn^{-1/8}|^{r-1})$ and thus

$$|g'_n(x)| \leq K\frac{\log n}{n^{1/4}(1+|xn^{-1/8}|^{r-1})}.$$

Hence $|g_n|$ and $\lambda(|g_n|)$ are smaller than $K(\log n)/n^{1/8}$, so the sequence g_n satisfies (4.1) and $\frac{1}{\sqrt{n}}V(\sqrt{n},g'_n)_t^n \to^{P_x} 0$ by Theorem 4-1.

Finally, consider g''_n. Since $E_0(L_1X_1) = 0$ for symmetry reasons, one has $G(f,1,x) = 0$ by (3.16), so (3.15) and (3.16) give

$$|g''_n(x)| = \begin{cases} 0 & \text{if } x = 0 \\ \dfrac{|\theta|}{\pi n^{1/4}}\displaystyle\int_0^1 \frac{x^2}{r^{3/2}}e^{-x^2/2r}\sqrt{1-r}dr & \text{if } x \neq 0. \end{cases}$$

By the change of variable $r = x^2/s$ we obtain

$$|g''_n(x)| \leq Kn^{-1/4}|x|\int_{x^2}^{\infty}\frac{1}{\sqrt{s}}e^{-s/2}ds \leq Kn^{-1/4}e^{-x^2/4}.$$

Then the sequence g''_n satisfies (4.1) and $\frac{1}{\sqrt{n}}V(\sqrt{n}, g''_n)^n_t \to^{P_x} 0$ by Theorem 4-1 again.

Proof of 6.2. – In this proof, we set $k = H_h - \theta\hat{g} = k_n/n^{1/4}$ and $F'_n = F_n/n^{1/4}$ and $\breve{F}'_n = \breve{F}_n/n^{1/4}$. Observe that $\lambda(k) = 0$, so $F := F(k)$ may be defined by (1.16), and by (5.7) we see that both $F'_n(x)$ and $\breve{F}'_n(x)$ converge to $F(x)$ and stay smaller than $K(1 + |x|)$.

By (6.3) and a simple computation, and if $f_{n,x}(y) = h(x, y-x) + F'_n(y)$, the left side of (6.2) is $\frac{1}{\sqrt{n}}(V(\sqrt{n}, g_n)^n_t + \theta^2 V(\sqrt{n}, g')^n_t - 2\theta V(\sqrt{n}, g''_n)^n_t)$, where (recall (1.13), (3.12) and (3.13)):

$$g_n(x) = H_{h^2}(x) + P_1 F'^2_n(x) - \breve{F}'_n(x)^2 + 2\bar{H}_{h,F'_n}(x),$$

$$g'(x) = G(1, 2, x), \qquad g''_n(x) = G(f_{n,x}, 1, x).$$

First we observe that by (3.15) and (3.16),

$$g'(x) = \begin{cases} 1 & \text{if } x = 0 \\ \displaystyle\int_0^1 \frac{|x|}{\sqrt{2\pi}r^{3/2}} e^{-\frac{x^2}{2r}}(1-r)dr & \text{if } x \neq 0. \end{cases}$$

Hence $\lambda(g') = \sqrt{\frac{2}{\pi}}\int_0^1(\frac{1}{\sqrt{r}} - \sqrt{r})dr = \frac{8}{3\sqrt{2\pi}}$, and we obtain that $\frac{1}{\sqrt{n}}V(\sqrt{n}, g')^n_t$ converges in P_x-probability to $\frac{8}{3\sqrt{3\pi}}L_t$ by Theorem 1-1. So in order to prove (6.2), and in view of Theorem 4-1, it suffices to show that the two sequences g_n and g''_n satisfy (4.2) and

$$\lambda(g_n) \to \lambda(H_{h^2} + 2\bar{H}_{h,F}), \qquad \lambda(g''_n) \to \delta(h). \tag{6.4}$$

The sequence g_n: First we have

$$\begin{aligned}\lambda(g_n) &= \lambda(H_{h^2} + F'^2_n - \breve{F}'^2_n + 2\bar{H}_{h,F'_n}) \\ &= \lambda(H_{h^2} - (P_{w_n+1}k)(2F'_n + P_{w_n+1}k) + 2\bar{H}_{h,F'_n}).\end{aligned}$$

Since $|P_{w_n+1}k(x)| \leq K/n^{1/4}(1+|x/n^{1/8}|^{s-1})$ by (5.9) and $|F'_n| \leq K\log n$ by (5.8), we see that $\lambda((P_{w_n+1}k)(2F'_n + P_{w_n+1}k)) \to 0$. Since $F'_n(x) \to F(x)$ and $|F'_n(x)| \leq K(1 + |x|)$, we see that $\bar{H}_{h,F'_n}(x) \to \bar{H}_{h,F}(x)$ and that $|\bar{H}_{h,F'_n}(x)| \leq K\bar{h}(x)(1 + |x|)$; so $\lambda(\bar{H}_{h,F'_n}) \to \lambda(\bar{H}_{h,F})$, hence the first property (6.4).

If $\bar{F}'_n = \bar{F}_n/n^{1/4}$, we have $P_1 F'^2_n \leq 2P_1k^2 + 2P_1\bar{F}'_n$ and $\bar{H}_{h,F'_n} = \bar{H}_{h,k} + \bar{H}_{h,\bar{F}'_n}$. (B-$r$) implies $\beta_s(k^2) < \infty$ for all $s \in [0, r]$, so (3.2) yields $P_1k^2(x) \leq K(e^{-x^2/2} + \frac{1}{1+|x|^r}) \leq \frac{K}{1+|x|^r}$, while clearly $|\bar{H}_{h,k}| \leq \bar{h}\sqrt{P_1k^2} \leq K\bar{h}$ by Cauchy-Schwarz inequality. On the other hand we have seen in the proof of (4.20) that $P_1\bar{F}_n^2(x) \leq K\sqrt{n}(\log n)^2/(1+|xn^{-1/8}|^{2r-2})$, and by (5.7) the same majoration holds for $\breve{F}_n^2(x)$, while $|\bar{H}_{h,F'_n}| \leq K\bar{h}\sqrt{P_1F_n'^2}$ by Cauchy-Schwarz inequality again. Putting all these together yields

$$|g_n(x)| \;\leq\; K\left(\bar{h} + + \frac{1}{1+|x|^r} + \frac{(\log n)^2}{1+|x/n^{1/8}|^{2r-2}}\right).$$

Then $|g_n| \leq K(\log n)^2$ and $\lambda(|g_n|) \leq K(\log n)^2 n^{1/8}$ and $\lambda(g_n^2) \leq K(\log n)^4 n^{1/4}$ and $\beta_1(g_n) \leq K(\log n)^2 n^{1/4}$. It readily follows that the sequence g_n satisfies (4.2).

The sequence g_n'': (3.16) yields for $x \neq 0$:

$$g_n''(x) = \begin{cases} E_0(L_1(h(0,X_1) + F_n'(X_1))) & \text{if } x = 0 \\ \displaystyle\int_0^1 \frac{|x|}{\sqrt{2\pi}r^{3/2}} e^{-\frac{x^2}{2r}}\sqrt{1-r} & \text{if } x \neq 0 \\ \quad \times E_0(L_1(h(x, X_1\sqrt{1-r} - x)) + F_n'(X_1\sqrt{1-r})))dr & \end{cases} \tag{6.5}$$

The change of variable $x = y\sqrt{r}$ gives

$$\lambda(g_n'') = \int_0^1 \sqrt{\frac{1}{r} - 1}\, dr$$
$$\int |y| E_0(L_1(h(y\sqrt{r}, X_1\sqrt{1-r} - y\sqrt{r}) + F_n'(X_1\sqrt{1-r})))\rho(y)dy.$$

Then $F_n'(x) \to F(x)$ and $|F_n'(x)| \leq K(1+|x|)$ readily give the last property (6.4) (recall (1.17), and observe that $\bar{\rho}$ is the law of (X_1, L_1) under P_0). Further, (H-r) implies that the expectation in the expression (6.5) for $x \neq 0$ is smaller than $Ke^{a|x|}$. Hence

$$|g_n''(x)| \;\leq\; K|x|e^{a|x|}\int_{x^2}^{\infty} \frac{1}{\sqrt{y}} e^{-y/2} dy \;\leq\; Ke^{a|x| - x^2/2}.$$

It is then obvious that the sequence g_n'' satisfies (4.2).

Proof of 1.26. – Finally, let us verify that (1.26) holds when $h(x,y) = g(x)$. First we have $H_h = g$. Second,

$$\int_0^1 \left(\frac{1}{\sqrt{r}} - \sqrt{r}\right) dr \int |y| g(y\sqrt{r}) E_0(L_1)\rho(y)dy = \int g(x)G(1,1,x)dx$$

by (3.16) and the change of variable $x = y\sqrt{r}$, while $G(1,1,x) = \hat{g}(x)$.

7. THEOREM 1-2, THE CASE $\alpha > 1/2$

Here we prove Theorem 1-2 when $\alpha \in (1/2, 1)$, hence $\delta = (1-\alpha)/2$. We will choose β such that $0 < \beta < (1-\alpha) \wedge (2\alpha - 1)$: this choice is possible, and yields $\beta < 1/3$ and $\beta < \alpha$.

First, observe that $n^{1/4}(\frac{1}{\sqrt{n}} V(\sqrt{n}, \hat{g})^n - L) = n^{1/4}(R^n + Q^n)$ (this is (5.1) when $h(x, y) = \hat{g}(x)$), and by Theorem 1-2 applied with $\alpha = 1/2$ and $h(x, y) = \hat{g}(x)$ these processes converge stably in law. In view of (5.3), it follows that $n^{1/4} R^n$ stably converge in law as well. Coming back to the case $\alpha > 1/2$, hence $\delta < 1/4$, we deduce that $n^\delta R^n$ converges in law to 0. This and (5.3) and Lemma 5-2 imply that for Theorem 1-2 with $\alpha > 1/2$ we are left to prove that the sequence

$$Y^n = n^\delta M^n + W^n \tag{7.1}$$

stably converges to the limit as described in the theorem. Again Y^n is a locally square-integrable martingale w.r.t. the filtration $(\mathcal{F}_{[nt]/n})_{t \geq 0}$, so exactly as for Theorem 4-6 the result will follow from the properties (4.20), (4.21) and

$$\sum_{i=1}^{[nt]} E_x(\Delta_i^n Y^{nl} \Delta_i^n Y^{nj} | \mathcal{F}_{\frac{i-1}{n}}) \ \to^{P_x} \ \lambda(H_{h^l h^j}) L_t. \tag{7.2}$$

By polarization, it is enough to prove these when h is 1-dimensional, which we assume in the sequel. By (4.7), (5.10), (5.12) and (7.1), we have

$$\begin{aligned} \Delta_i^n Y^n = & \frac{1}{n^\delta}(h(n^\alpha X_{\frac{i-1}{n}}, \sqrt{n}\Delta_i^n X) - H_h(n^\alpha X_{\frac{i-1}{n}})) \\ & + \frac{1}{\sqrt{n}}(F_n(\sqrt{n} X_{\frac{i}{n}}) - \hat{F}_n(\sqrt{n} X_{\frac{i-1}{n}})). \end{aligned} \tag{7.3}$$

Proof of 4.21. – (B-r), (5.8) and (7.3) yield $|\Delta_i^n Y^n| \leq K(n^{-\delta} e^{a|\sqrt{n}\Delta_i^n X|} + n^{\delta - 1/2} \log n)$, hence $E(|\Delta_i^n Y^n|^q | \mathcal{F}_{\frac{i-1}{n}}) \leq K_q(n^{-q\delta} + n^{q(\delta - 1/2)} \log n)$. This quantity is smaller than $1/n^2$ if q is large enough, hence (4.21) follows from Tchebicheff's inequality.

Proof of 4.20. – By (7.3), the left side of (4.20) is $\frac{1}{\sqrt{n}} V(\sqrt{n}, g_n)_t^n$, where

$$g_n(x) = \int y \left(\frac{1}{n^{(1-\alpha)/2}} h(x n^{\alpha - 1/2}, y) + \frac{1}{\sqrt{n}} F_n(x + y) \right) \rho(y) dy.$$

In view of Theorem 4-1(a), it suffices to prove that the sequence g_n satisfies (4.1). Since $\alpha > 1/2$, it follows from (5.7) and (3.9) that

$P_1\bar{F}_n^2(x) \leq Kn^{1-\alpha}(\log n)^2/(1+|xn^{-\beta/2}|^{2r-2})$. On the other hand, (5.5) and (1.9) yield $P_1k_n^2(x) \leq Kn^{1-\alpha}(e^{-x^2/2}+1/(1+|x|^{r-1}))$ hence

$$P_1F_n^2(x) \leq Kn^{1-\alpha}(\log n)^2\left(\frac{1}{1+|xn^{-\beta/2}|^{2r-2}}+\frac{1}{1+|x|^{r-1}}\right). \quad (7.4)$$

By Cauchy-Schwarz inequality $|g_n(x)| \leq \frac{1}{n^{(1-\alpha)/2}}\bar{h}(xn^{\alpha-1/2}) + \frac{1}{\sqrt{n}}\sqrt{P_1F_n^2(x)}$. Thus (7.4) yields

$$|g_n(x)| \leq K\left(\frac{1}{n^{(1-\alpha)/2}}\bar{h}(xn^{\alpha-1/2})+\frac{\log n}{n^{\alpha/2}(1+|x/n^{\beta/2}|^{r-1})}\right.$$
$$\left.+\frac{1}{n^{\alpha/2}(1+|x|^{r/2-1/2}}\right).$$

Hence $|g_n| \leq Kn^{-(1-\alpha)/2}$ and $\lambda(|g_n|) \leq (\log n)/n^{(\alpha-\beta)/2}$: since $\beta<\alpha$, we obtain the desired result.

Proof of 7.2. – In view of (7.3), the left side of (7.2) is $n^{\alpha-1}V(n^\alpha, H_{h^2}-(H_h)^2)_t^n + \frac{1}{\sqrt{n}}V(\sqrt{n},g_n)_t^n + \frac{1}{\sqrt{n}}V(\sqrt{n},\ell_n)_t^n$, where

$$g_n = \frac{1}{\sqrt{n}}(P_1F_n^2 - \hat{F}_n^2),$$

$$\ell_n(x) = \frac{2}{n^\delta}\int (h(n^{\alpha-1/2}x,y) - H_h(n^{\alpha-1/2}x))F_n(x+y)\rho(y)dy.$$

First, Theorem 1-1 gives that $n^{\alpha-1}V(n^\alpha,H_{h^2}-(H_h)^2)_t^n \to \lambda(H_{h^2}-(H_h)^2)L_t$ in P_x-probability. Thus it is enough to prove that

$$\frac{1}{\sqrt{n}}V(\sqrt{n},g_n)_t^n \to^{P_x} \lambda(H_h^2)L_t, \qquad \frac{1}{\sqrt{n}}V(\sqrt{n},\ell_n)_t^n \to^{P_x} 0. \quad (7.5)$$

Let us study first g_n: we will prove that this sequence satisfies (4.2) and $\lambda(g_n) \to \lambda(H_h^2)$, thus obtaining the first condition in (7.5) by Theorem 4-1. By (5.7) and (7.4) we have

$$|g_n(x)| \leq K_\gamma n^{1/2-\alpha}(\log n)^2\left(\frac{1}{1+|xn^{-\beta/2}|^{2r-2}}+\frac{1}{1+|x|^{r-1}}\right).$$

Hence $|g_n| \leq K(\log n)^2/n^{\alpha-1/2}$, and $\lambda(|g_n|) \leq K(\log n)^2/n^{\alpha-1/2-\beta/2}$, and $\beta_1(g_n) \leq K(\log n)^2/n^{\alpha-1/2-\beta}$ and $\lambda(|g_n|^2) \leq K(\log n)^4/n^{2\alpha-1-\beta/2}$: in view of $\beta<\alpha$ and $\beta<2\alpha-1$, we readily observe that the sequence g_n satisfies (4.2).

Next, we easily obtain $\lambda(g_n) = \lambda(\frac{1}{\sqrt{n}}(F_n^2 - \hat{F}_n^2))$. But we may write $\frac{1}{\sqrt{n}}(F_n^2 - \hat{F}_n^2) = v_n + y_n + z_n$, where

$$v_n(x) = n^{\alpha-1/2} H_h(n^{\alpha-1/2}x)^2,$$

$$y_n(x) = n^{1/2-\alpha}\theta^2 \hat{g}(x)^2 - 2n^{1/2-\alpha}\hat{g}(x) n^{\alpha-1/2} H_h(n^{\alpha-1/2}x),$$

$$z_n = \frac{1}{\sqrt{n}}(k_n(\hat{F}_n + \bar{F}_n) - (P_{w_n+1}k_n)(F_n + \hat{F}_n)).$$

Then $|y_n(x)| \leq Kn^{1/2-\alpha}(\hat{g}(x) + n^{\alpha-1/2}\bar{h}(n^{\alpha-1/2}x))$, and it follows that $\lambda(y_n) \to 0$. Moreover, (5.5), (5.7), (5.8) and (5.9) give $|z_n(x)| \leq Kn^{1/2-\alpha}(\log n)/(1 + |xn^{-\beta/2}|^{r-1})$, hence $\lambda(|z_n|) \leq Kn^{-(\alpha-1/2-\beta/2)} \log n \to 0$. Finally, we trivially have $\lambda(v_n) = \lambda(H_h^2)$: hence the first part of (7.5) holds.

Let us turn now to the sequence ℓ_n. We have $|\ell_n(x)| \leq Kn^{-\delta}\bar{h}(n^{\alpha-1/2}x)\sqrt{P_1 F_n^2(x)}$. By (7.4) we obtain that $|\ell_n(x)| \leq K(\log n)\bar{h}(n^{\alpha-1/2}x)$: thus $|\ell_n| \leq K \log n$ and $\lambda(|\ell_n|) \leq Kn^{1/2-\alpha} \log n \to 0$, and Theorem 4-1(a) yields the second condition in (7.5).

8. THEOREM 1-2, THE CASE $\alpha < 1/2$

Here we prove Theorem 1-2 when $\alpha \in (1/3, 1/2)$, hence $\delta = \alpha/2$. We will choose β such that

$$1-2\alpha < \beta < \frac{1}{3} \qquad (\Rightarrow \quad \beta < 1-\alpha, \quad \beta < 2\alpha, \quad 3\beta < 1+2\alpha). \qquad (8.1)$$

This choice is possible, since $1/3 < \alpha < 1/2$.

As seen in Section 7, $n^{1/4}R^n$ stably converge in law. Since $\delta < 1/4$, we deduce that $n^\delta R^n$ converges in law to 0. On the other hand, $n^{(1-\alpha)/2}M^n$ converges in law by Theorem 4-6 and since $\delta < (1-\alpha)/2$ it follows that the sequence $n^\delta M^n$ tends in law to 0. These two facts, plus (5.3) and Lemma 5-2, imply that for Theorem 1-2 with $\alpha < 1/2$ we are left to prove that the sequence of martingales $Y^n = W^n$ stably converges to the limit as described in the theorem. Hence once again the result will follow from the properties (4.20), (4.21) and

$$\sum_{i=1}^{[nt]} E_x(\Delta_i^n Y^{nl} \Delta_i^n Y^{nj} | \mathcal{F}_{\frac{i-1}{n}}) \to^{P_x} \eta'(h^l, h^j) L_t. \qquad (8.2)$$

By polarization, it is enough to prove these when h is 1-dimensional, which we assume in the sequel. By (5.10) and (5.12), we have

$$\Delta_i^n Y^n = \frac{1}{\sqrt{n}}(F_n(\sqrt{n}X_{\frac{i}{n}}) - \hat{F}_n(\sqrt{n}X_{\frac{i-1}{n}})). \tag{8.3}$$

Proof of 4.21. – (5.8) yields $|\Delta_i^n Y^n| \leq K(\log n)/n^{\alpha/2}$, and thus (4.21) obviously holds.

Proof of 4.20. – In view of (8.3), the left side of (4.20) is $\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)_t^n + \frac{1}{\sqrt{n}}V(\sqrt{n}, g'_n)_t^n$, where

$$g_n(x) = \frac{1}{\sqrt{n}}\int yk_n(x+y)\rho(y)dy, \qquad g'_n(x) = \frac{1}{\sqrt{n}}\int y\bar{F}_n(x+y)\rho(y)dy.$$

First, (1.9) and (5.5) yield $P_1 k_n^2(x) \leq K_\gamma n^\alpha(e^{-x^2/2} + \frac{n^{\gamma(1/2-\alpha)}}{1+|x|^\gamma})$ for $\gamma \in [1, r-1]$, hence

$$|g_n(x)| \leq K_\gamma \frac{n^{\alpha/2+\gamma(1/2-\alpha)-1/2}}{1+|x|^\gamma} \qquad \text{for } \gamma \in [1/2, (r-1)/2].$$

Then clearly $|g_n| \leq Kn^{-1/4}$ and $\lambda(|g_n|) \leq K_\varepsilon n^{\varepsilon-\alpha/2}$ for $\varepsilon > 0$ small enough: the sequence g_n satisfies (4.1) and $\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)_t^n \to^{P_x} 0$ by Theorem 4-1.

Next, observe that $\lambda(k_n) = 0$, hence $\lambda(F_n) = 0$ and $\lambda(g'_n) = 0$. So in view of Theorem 4-1, it suffices to prove that the sequence g'_n satisfies (4.2). It follows from (5.7) and (3.9) that for $\gamma \in [0, r-1]$:

$$\sqrt{P_1\bar{F}_n^2(x)} \leq K_\gamma n^\delta(\log n)\left(\frac{n^{1/2-\alpha}}{1+|xn^{-\beta/2}|^\gamma} + \frac{n^{(\gamma+1)(1/2-\alpha)}}{1+|x|^\gamma}\right).$$

By Cauchy-Schwarz inequality $|g'_n(x)| \leq \frac{1}{\sqrt{n}}(P_1\bar{F}_n^2(x))^{1/2}$. Thus for $\varepsilon > 0$ small enough,

$$|g'_n(x)| \leq K\frac{\log n}{n^{\alpha/2}}, \qquad \lambda(|g'_n|) \leq K_\varepsilon(\log n)(n^{\beta/2-\alpha/2} + n^{3\alpha/2-1/2+\varepsilon}),$$

$$\beta_1(g'_n) \leq K_\varepsilon(\log n)(n^{\beta-\alpha/2} + n^{5\alpha/2-1+\varepsilon}),$$

$$\lambda(|g'_n|^2) \leq K_\varepsilon(\log n)^2(n^{\beta/2+1/2-\alpha} + n^{5\alpha/2-1+\varepsilon}).$$

In view of (8.1) it is then obvious that (4.2) is met by g'_n.

Proof of 8.2. – By (8.3), the left side of (8.2) is $\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)_t^n$, where $g_n = \frac{1}{\sqrt{n}}(P_1 F_n^2 - \hat{F}_n^2)$. In fact $\hat{F}_n = F_n - k_n + P_{w_n+1}k_n$, so we can write $g_n = y_n + z_n + z'_n$, with

$$y_n = \frac{2}{\sqrt{n}} k_n F_n, \qquad z_n = -\frac{1}{\sqrt{n}}(P_{w_n+1}k_n)(\bar{F}_n + \breve{F}_n),$$

$$z'_n = \frac{1}{\sqrt{n}}(P_1 F_n^2 - F_n^2).$$

Then (8.3) will follows from the next three properties:

$$\frac{1}{\sqrt{n}}V(\sqrt{n}, y_n)_t^n \ \to \ \eta'(h,h)L_t, \quad \frac{1}{\sqrt{n}}V(\sqrt{n}, z_n)_t^n \ \to \ 0,$$

$$\frac{1}{\sqrt{n}}V(\sqrt{n}, z'_n)_t^n \ \to \ 0. \tag{8.4}$$

2) By (5.8) we have $|z_n| \leq Kn^{-\alpha/2}(\log n)|P_{w_n+1}k_n|$, while (5.9) gives $|P_{w_n+1}k_n| \leq Kn^{1/2-\beta-\alpha/2}$ and $\lambda(|P_{w_n+1}k_n|) \leq Kn^{(1-\beta-\alpha)/2}$ because $\beta > 1 - 2\alpha$. It readily follows that the sequence z_n satisfies (4.1), and the second property in (8.4) is satisfied.

3) In order to prove the last property in (8.4) it is enough by Theorem 4-1 to show that the sequence z'_n satisfies (4.2), since obviously $\lambda(z'_n) = 0$. By (3.4) and (5.5) we have $|P_j k_n(x+y) - P_j k_n(x)| \leq K|y|n^{\alpha/2}/j$. Then $|F_n(x+y) - F_n(x)| \leq Kn^{\alpha/2}(1+|y|)\log n$, and

$$\begin{aligned} |z'_n(x)| &\leq Kn^{\alpha/2-1/2}(\log n)\int \rho(y)dy(1+|y|)(|F_n(x+y)| + |F_n(x)|) \\ &\leq Kn^{\alpha/2-1/2}(\log n)\left(|k_n(x)| + \int \rho(y)(1+|y|)|k_n(x+y)|dy\right) \\ &\quad + n^{\alpha-1/2}(\log n)^2\left(\frac{n^{1/2-\alpha}}{1+|xn^{-\beta/2}|^\gamma} + \frac{n^{(\gamma+1)(1/2-\alpha)}}{1+|x|^\gamma}\right), \end{aligned}$$

where the last equality follows from (5.7) and (3.9). It follows clearly (using again $\beta > 1 - 2\alpha$, and (5.5)) that

$$|z'_n| \ \leq \ K(\log n)^2, \qquad \lambda(|z'_n|) \ \leq \ K(\log n)^2 n^{\beta/2},$$

$$\beta_1(z'_n) \ \leq \ K(\log n)^2 n^\beta, \qquad \lambda(z'^2_n) \ \leq \ K(\log n)^4 n^{\beta/2}.$$

Hence the sequence z'_n satisfies (4.2) as soon as $\beta < 1/3$, which is met by (8.1): so we have the last property in (8.4).

4) In view of Theorem 4-1, the first property in (8.4) will hold if we prove that the sequence y_n satisfies (4.2) and $\lambda(y_n) \to \eta'(h,h)$. By (5.8), we have $|y_n| \leq n^{-\alpha/2}(\log n)|k_n|$, therefore (5.5) yields $|y_n| \leq K \log n$, and $\lambda(|y_n|) \leq K \log n$, and $\beta_1(y_n) \leq Kn1/2 - \alpha \log n$ and $\lambda(y_n^2) \leq K(\log n)^2$, which clearly imply that (4.2) is satisfied.

5) To simplify the notation we write $v_n = n^{\alpha - 1/2}$. Set $\ell_n(x) = H_h(x) - \theta \hat{g}(x/v_n)/v_n$, so $k_n(x) = v_n n^\delta \ell_n(v_n x)$. A simple calculation shows $P_j k_n(x) = v_n n^\delta P_{jv_n^2} \ell_n(v_n x)$, and thus

$$F_n(x) = v_n n^\delta \sum_{j=0}^{w_n} P_{jv_n^2} \ell_n(v_n x). \tag{8.5}$$

Now, $\beta_i(|\ell_n|) \leq K$ for $i = 0, 1, 2$ (use the fact that $v_n \leq 1$). So we deduce from (3.1), from (3.3) and $\lambda(\ell_n) = 0$, and from (3.5), that

$$\begin{aligned} |\frac{1}{v_n} \int_0^{v_n^2} (P_s \ell_n) ds| \ &\leq \ K, \\ |\frac{1}{v_n} \int_{(w_n+1)v_n^2}^{\infty} (P_s \ell_n) ds| \ &\leq \ K \frac{1 + |x|}{v_n^2 \sqrt{w_n}}, \\ |v_n P_{jv_n^2} \ell_n - \frac{1}{v_n} \int_{jv_n^2}^{(j+1)v_n^2} (P_s \ell_n) ds| \ &\leq \ \frac{K}{j^{3/2}}. \end{aligned}$$

Putting all these together with (8.5) yields

$$|F_n(x) - \frac{n^\delta}{v_n} \int_0^\infty P_s \ell_n(v_n x) ds| \ \leq \ K n^\delta \left(1 + \frac{1}{v_n^2 \sqrt{w_n}} + \frac{|x|}{v_n \sqrt{w_n}} \right).$$

Hence $y'_n(x) = 2n^{-\delta} k_n(x) \int_0^\infty P_s \ell_n(v_n x) ds$ has $|y_n(x) - y'_n(x)| \leq K|k_n(x)|(n^{(\alpha-1)/2} + n^{1-\beta-3\alpha)/2} + |x| n^{-(\alpha+\beta)/2})$. Since $\lambda(|k_n|) \leq K n^\delta$ and $\beta_1(k_n) \leq K n^{\delta - \alpha + 1/2}$, we obtain $\lambda(|y_n - y'_n|) \leq K(n^{\alpha - 1/2} + n^{(1-\beta-2\alpha)/2}) \to 0$ because $\beta + 2\alpha > 1$ and $\alpha < 1/2$. Further if $y''_n = 2\ell_n \int_0^\infty P_s \ell_n ds$ we have $\lambda(y'_n) = \lambda(y''_n)$. Hence it remains to prove that $\lambda(y''_n) \to \eta'(h,h)$.

6) Now we study $P_s \ell_n = P_s H_h - \theta P_s \hat{g}_n$, where $\hat{g}_n(x) = \hat{g}(x/v_n)/v_n$. In fact we compare $\int_0^\infty P_s \ell_n(x) ds$ with $G(x)$, where $G = G(H_h)$.

If we set $\gamma_s(x) = P_s H_h(x) - \frac{\theta}{\sqrt{2\pi s}} e^{-x^2/2s}$ we have $G(x) = \int_0^\infty \gamma_s(x) ds$. Set also $\gamma_s^n(x) = \frac{1}{\sqrt{2\pi s}} e^{-x^2/2s} - P_s \hat{g}_n(x)$, so that

$$\int_0^\infty P_s \ell_n(x) ds - G(x) = \theta \int_0^\infty \gamma_s^n(x) ds. \tag{8.6}$$

Now, $\beta_i(\hat{g}_n) = v_n^i \beta_i(\hat{g})$, so (3.1) and (3.3) and $v_n \leq 1$ yield

$$|\gamma_s(x)| + |\gamma_s^n(x)| \leq \frac{K}{\sqrt{s}}, \tag{8.7}$$

$$|\gamma_s(x)| \leq K \frac{1+|x|}{s^{3/2}}, \qquad |\gamma_s^n(x)| \leq K v_n \frac{1+|x|}{s^{3/2}}. \tag{8.8}$$

Dividing the integral in (8.6) in two pieces, from 0 to ε and from ε to ∞, and using (8.7) for the first piece and (8.8) for the second piece, we get

$$\left| \int_0^\infty P_s \ell_n(x) ds - G(x) \right| \leq K \left(\sqrt{\varepsilon} + \frac{v_n}{\sqrt{\varepsilon}}(1+|x|) \right).$$

If $f_n = 2\ell_n G$, and taking $\varepsilon = v_n$ above, we thus get $|y_n''(x) - f_n(x)| \leq K|\ell_n(x)|\sqrt{v_n}(1+|x|)$. Thus $\lambda(|y_n'' - f_n|) \leq K\sqrt{v_n} \to 0$, and it remains to prove that $\lambda(f_n) \to \eta'(h,h)$.

7) Apply (3.4) and the majorations $|\frac{1}{\sqrt{2\pi s}}(e^{-x^2/2s} - 1)| \leq K|x|/s$ (easily deduced from (3.6) and (3.7)) to obtain $|\gamma_s(x) - \gamma_0(x)| \leq K|x|/s$. Thus if we cut the integral $G(v_n x) - G(0) = \int_0^\infty (\gamma_s(v_n x) - \gamma_s(0)) ds$ in three pieces, from 0 to ε (use (8.7)), from ε to A (use what precedes) and from A to ∞ (use (8.8)), we get if $0 < \varepsilon < 1 < A < \infty$:

$$|G(v_n x) - G(0)| \leq K \left(\sqrt{\varepsilon} + v_n|x|(\log \frac{A}{\varepsilon}) + \frac{1+v_n|x|}{\sqrt{A}} \right).$$

Taking $\varepsilon = v_n$ and $A = 1/v_n^2$ gives

$$|G(v_n x) - G(0)| \leq K\sqrt{v_n}(1+|x|). \tag{8.9}$$

Now we can write

$$\begin{aligned} \lambda(f_n) &= 2\int H_h(x)G(x)dx - 2\theta \int \hat{g}\left(\frac{x}{v_n}\right) G(x) \frac{1}{v_n} dx \\ &= 2\lambda(H_h G) - 2\theta \int \hat{g}(x) G(v_n x) dx. \end{aligned}$$

Recalling that $\eta'(h,h) = 2\lambda(H_h G) - 2\theta G(0)$ and that $\lambda(\hat{g}) = 1$, it follows that

$$\begin{aligned} |la(f_n) - \eta'(h,h)| &\leq K \int \hat{g}(x)|G(v_n x) - G(0)| dx \\ &\leq K\sqrt{v_n}(1+\beta_1(\hat{g})) \to 0 \end{aligned}$$

by (8.9), and we are finished.

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