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P. J. FITZSIMMONS

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The quasi-sure ratio ergodic theorem

by

P. J. FITZSIMMONS

Department of Mathematics, University of California,
San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0112 USA.
e-mail: pfitzsim@ucsd.edu

ABSTRACT. – Let $(P_t)_{t\geq 0}$ be the transition semigroup of a right Markov process, and let m be a conservative (P_t)-invariant measure. Let f and g be elements of $L^1(m)$ with $g > 0$. We show that, with the exception of an m-polar set of starting points x, the ratio $\int_0^t P_s f(x)\,ds / \int_0^t P_s g(x)\,ds$ converges as $t \to +\infty$, and we identify the limit as a ratio of conditional expectations with respect to the appropriate invariant σ-algebra. This improves upon earlier work of M. Fukushima and M.G. Shur, in which the exceptional set was shown to be m-semipolar. The proof is based on Neveu's presentation of the Chacon-Ornstein filling scheme, adapted to continuous time. The method yields, as a by-product, a local limit theorem for the ratio of the "characteristics" of two continuous additive functionals, extending a result of G. Mokobodzki. © Elsevier, Paris

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RÉSUMÉ. – Soit P_t, $t \geq 0$, le semi-groupe de transition d'un processus de Markov droit avec espace d'états E, et soit m une mesure conservative (donc (P_t)-invariante) sur E. Soit f et $g > 0$ des fonctions m-intégrable sur E. Nous montrons que, en dehors d'un ensemble m-polaire des points de départ x, le rapport $\int_0^t P_s f(x)\,ds / \int_0^t P_s g(x)\,ds$ converge comme $t \to +\infty$, et nous identifier la limite comme le rapport des espérances conditionnel (par rapport à une tribu invariante approprié). Ceci s'améliore sur les premiers travaux de M. Fukushima et M.G. Shur, dans lequel l'ensemble exceptionnel s'est avéré m-semi-polaire. La démonstration est basée sur la présentation

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de Neveu de le schéma de remplissage de Chacon-Ornstein, adaptée au temps continu. La méthode donne, comme sous-produit, un théorème de limite locale pour le rapport des « caractéristiques » de deux functionals additifs continus, étendant un résultat de G. Mokobodzki.

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1. INTRODUCTION

Let $X = (X_t, P^x)$ be a right Markov process with state space E and transition semigroup $(P_t)_{t\geq 0}$. Let m be a *conservative* excessive measure of X; in particular, m is σ-finite and *invariant*: $mP_t = m$ for all $t > 0$. Recall that a Borel set B is *m-polar* provided $P^m(T_B < \infty) = 0$, where $T_B := \inf\{t > 0 : X_t \in B\}$ is the hitting time of B by X. We shall prove the following *quasi-sure* form of the celebrated Chacon-Ornstein ratio ergodic theorem [9]. Since m is σ-finite there is a bounded Borel function $q > 0$ on E such that $m(q) = 1$; let μ denote the probability measure $q \cdot m$.

THEOREM 1.1. – *If f and $g \geq 0$ are Borel functions in $L^1(m)$, then there is an m-polar Borel set $B \subset E$ such that*

$$\lim_{t\to\infty} \frac{\int_0^t P_s f(x)\,ds}{\int_0^t P_s g(x)\,ds} = \frac{\mu(f/q|\mathcal{I})(x)}{\mu(g/q|\mathcal{I})(x)} \tag{1.2}$$
$$\forall x \in \left\{y \in E : \int_0^\infty P_s g(y)\,ds > 0\right\} \setminus B.$$

(The invariant σ-algebra $\mathcal{I}$ is defined in section 3.)

As we shall see, if $f \in L^1(m)$ then there is an m-polar set B_f such that (i) $P^x(T_{B_f} < \infty) = 0$ for all $x \in E \setminus B_f$ and (ii) $x \mapsto \int_0^t P_s f(x)\,ds$ is real-valued and finely continuous on $E \setminus B_f$ for all $t > 0$.

The original ratio ergodic theorem of Chacon and Ornstein is a discrete-time result to the effect that $\sum_{k=0}^n P^k f / \sum_{k=0}^n P^k g$ converges m-a.e on $\{\sum_{k=0}^\infty P^k g > 0\}$, whenever P is a positivity preserving linear contraction operating in $L^1(m)$. It is a straightforward matter to apply the Chacon-Ornstein theorem in the present context to deduce the weaker form of (1.2) in which the exceptional set B is m-null; *cf.* [5; Thm. II.1]. Refinements of this basic result, wherein it is shown that the exceptional set B is at worst m-semipolar (*i.e.*, $P^m(X_t \in B$ for uncountably many $t > 0) = 0$) have been discovered by Fukushima [20] and Shur [39, 40] under duality hypotheses, and under an absolute continuity hypothesis [40; Thm. 3, p. 705]; see also [12]. (Actually, Fukushima shows the exceptional set to be m-polar under the assumption that semipolar sets are m-polar, an assumption which is satisfied if (P_t) is symmetric with respect to m, or nearly so: [17; (4.13)].) In these works the ratio ergodic theorem is deduced from a continuous-time version of Brunel's maximal inequality [8, 2], which also allows for an explicit description of the limit as in (1.2). See [10] for a description of the limit in the original Chacon-Ornstein theorem, and [27] for a proof of the identification via Brunel's inequality.

Neveu [34] has shown how an amplified form of the filling scheme of Chacon and Ornstein leads to a rapid proof of the ratio ergodic theorem in discrete time. Inspired by this work, and that of Rost [37] and Meyer [28] on the filling-scheme in continuous time, we show how Neveu's approach can be adapted to continuous time to prove Theorem (1.1). It is a testament to Neveu's insight that this approach leads to an m-polar exceptional set with relative ease. Moreover, there is no need (as in [20, 39]) to assume that the functions f and g appearing in Theorem (1.1) are bounded; the relaxation of this hypothesis was achieved in [40] only by means of fairly delicate arguments. As a bonus, we deduce from (1.1) a sharp form of Brunel's inequality (Theorem (4.11)), improving (in our specific context) on the analogous results found in [3, 14, 16, 20, 39, 40].

After developing the basic properties of the continuous-time filling scheme in section 2, we briefly discuss the invariant σ-algebra in section 3. Section 4 is devoted to the proof of Theorem (1.1). Further applications of the filling scheme appear in section 5 (where we present a generalization of Mokobodzki's local limit theorem) and in section 6 (where we present the "abelian" form of the ratio ergodic theorem).

In the remainder of this section we describe the context in which (1.1) will be proved and we establish some basic notation. If $(F, \mathcal{F}, \mu)$ is a measure space, then $b\mathcal{F}$ (resp. $p\mathcal{F}$) denotes the class of bounded real-valued (resp. $[0, \infty]$-valued) $\mathcal{F}$-measurable functions on F. For $f \in p\mathcal{F}$ we use $\mu(f)$ to denote the integral $\int_F f\, d\mu$; similarly, if $D \in \mathcal{F}$ then $\mu(f; D)$ denotes $\int_D f\, d\mu$. We write $\mathcal{F}^*$ for the universal completion of $\mathcal{F}$; that is, $\mathcal{F}^* = \cap_\nu \mathcal{F}^\nu$, where $\mathcal{F}^\nu$ is the ν-completion of $\mathcal{F}$ and the intersection runs over all finite measures on $(F, \mathcal{F})$. If $(E, \mathcal{E})$ is a second measurable space and $K = K(x, dy)$ is a kernel from $(F, \mathcal{F})$ to $(E, \mathcal{E})$ (*i.e.*, $F \ni x \mapsto K(x, A)$ is $\mathcal{F}$-measurable for each $A \in \mathcal{E}$ and $K(x, \cdot)$ is a measure on $(E, \mathcal{E})$ for each $x \in F$), then μK denotes the measure $A \mapsto \int_F \mu(dx) K(x, A)$ and Kf the $\mathcal{F}$-measurable function $x \mapsto \int_E K(x, dy) f(y)$.

Throughout the paper, $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ will be the canonical realization of a Borel right Markov process with Lusin state space $(E, \mathcal{E})$; see [23] or [38]. Briefly, this means that (i) E is homeomorphic to a Borel subset of some compact metric space and $\mathcal{E}$ is the Borel σ-algebra on E, (ii) under P^x (the law of X started at $x \in E$) the E-valued process $t \mapsto X_t$ is a right-continuous strong Markov process, (iii) the transition operators $P_t f(x) := P^x(f(X_t))$ satisfy $P_t(b\mathcal{E}) \subset b\mathcal{E}$.

Since our concern is with "recurrent" processes, we assume throughout that X is honest, in the sense that $P_t 1 \equiv 1$ for all $t \geq 0$.

The sample space Ω is the set of all right-continuous E-valued paths $\omega : [0, \infty[\to E$, and $X_t(\omega) = \omega(t)$ is the coordinate process. The shift operators θ_t, $t \geq 0$, are defined on Ω by $X_s(\theta_t \omega) = X_{s+t}(\omega)$.

The resolvent operators associated with (P_t) will be denoted $U^\alpha := \int_0^\infty e^{-\alpha t} P_t \, dt$, $\alpha > 0$, and we write U instead of U^0 for the potential kernel $\int_0^\infty P_t \, dt$. In addition we introduce the notation U_t for the partial potential operator $\int_0^t P_s \, ds$.

Upon occasion we shall need to consider sets more general than Borel sets. Recall that, for $\alpha \geq 0$, a function $f \in p\mathcal{E}^*$ is *α-excessive* provided $t \mapsto e^{-\alpha t} P_t f(x)$ is decreasing and right-continuous for each $x \in E$. We write $\mathcal{E}^e$ for the σ-algebra on E generated by the α-excessive functions ($\alpha > 0$), and note [38; (8.6)] that $P_t(b\mathcal{E}^e) \subset b\mathcal{E}^e$ and $U_t(b\mathcal{E}^e) \subset b\mathcal{E}^e$. Since the process $t \mapsto f(X_t)$ is optional whenever f is α-excessive, the hitting time $T_B := \inf\{t > 0 : X_t \in B\}$ of any $B \in \mathcal{E}^e$ is a stopping time.

A measure m on $(E, \mathcal{E})$ is *excessive* for X provided it is σ-finite and $mP_t \leq m$ for all $t > 0$. It is known that since X is a right process, we have $\lim_{t \to 0+} mP_t(f) = m(f)$ for all $f \in p\mathcal{E}$, for any excessive measure m.

Let m be an excessive measure. Then (P_t) respects m-classes in the sense that if $f = 0$, m-a.e., then $P_t f = 0$, m-a.e. for all $t > 0$. By the triangle inequality, the restriction of $(P_t)_{t \geq 0}$ to $b\mathcal{E} \cap L^1(m)$ extends uniquely to a contraction semigroup of linear operators in $L^1(m)$. The "uniqueness of charges" property [24; (2.12)] enjoyed by any right process ensures that this semigroup is strongly continuous.

2. THE FILLING SCHEME IN CONTINUOUS TIME

In this section we develop the continuous-time form of Neveu's presentation [34] of the filling scheme. Our point of departure is Meyer's observation [27] that the filling scheme (in discrete or continuous time) is equivalent to a construction involving space-time réduites. In this way we are able to avoid the time-discretization arguments that appear to be the source of the m-semipolar exceptional set in earlier forms of (1.1). The reader familiar with [27] will note we use the backward space-time process, while Meyer used the forward space-time process. This is quite natural since our construction is "dual" to his in the obvious sense.

Let $\overline{X}$ denote the backward space-time process associated with X:

$$\overline{X}_t(\omega, r) := \begin{cases} (X_t(\omega), r-t) & 0 \le t < r, \\ \Delta & t \ge r, \end{cases}$$

where Δ is a cemetery state adjoined to $E\times]0,\infty[$ as an isolated point. (By convention, functions and measures defined on $E\times]0,\infty[$ are taken to vanish at Δ.) We realize $\overline{X}$ on the product space $\Omega\times]0,\infty[$ endowed with the product measures $\overline{P}^{x,r} := P^x \otimes \epsilon_r$, where ϵ_r denotes the unit point mass at r. It is well known [38; pp. 86–88] that $\overline{X}$ is a transient Borel right Markov process with transition semigroup

$$\overline{P}_t(x, r; \overline{h}) := \overline{P}^{x,r}(\overline{h}(\overline{X}_t)) = P_t h_{r-t}(x) 1_{\{t<r\}}, \qquad \overline{h} \in p[\mathcal{E} \otimes \mathcal{B}_{]0,\infty[}],$$

where (here and in the sequel) we use the notational convention $h_u(x) := \overline{h}(x, u)$. The potential kernel of $\overline{X}$ is given by

$$\overline{U}(x, r; \overline{h}) = \int_0^r P_t h_{r-t}(x)\, dt.$$

Notice that if $f \in p\mathcal{E}^*$, then

$$\overline{U}(x, r; f \otimes 1) = U_r f(x).$$

Let us now fix $f, g \in p\mathcal{E}$ such that $U_t(f+g)(x) < +\infty$ for all $t \ge 0$ and all $x \in E$. Let $\overline{G}$ denote the least $\overline{X}$-supermedian majorant of the (finite) space-time potential $\overline{U}((g-f)\otimes 1)$; see [11; XII.2]. Then $\overline{G}$ is $\overline{X}$-excessive and strongly dominated by the finite potential $\overline{U}(g \otimes 1)$, in the sense that the function $\overline{U}(g\otimes 1) - \overline{G}$ is $\overline{X}$-excessive. Consequently, there exists $\overline{g} : E\times]0,\infty[\to [0,\infty[$, measurable over the universal completion of $\mathcal{E} \otimes \mathcal{B}_{]0,\infty[}$, such that

$$\overline{G} = \overline{U}\overline{g}.$$

The asserted strong domination means that we can take $\overline{g}$ to be dominated by $g \otimes 1$. Moreover we can (and do) assume that $\overline{g}$ is supported in the set $\{\overline{G} = \overline{U}((g-f)\otimes 1)\}$. See [29] or [11; Thm. 31, p. 17].

Define

$$\overline{F} := \overline{G} - \overline{U}((g-f)\otimes 1), \qquad \overline{H} := \overline{U}(g\otimes 1) - \overline{G},$$

and notice that both $\overline{F}$ and $\overline{H}$ are non-negative functions. Then for all $t > 0$ and $x \in E$,

$$U_t f(x) = F_t(x) + H_t(x) \tag{2.1}$$

$$U_t g(x) = \int_0^t P_s g_{t-s}(x)\, ds + H_t(x) \tag{2.2}$$

$$0 \le g_t(x) \le g(x) \cdot 1_{\{F_t=0\}}(x). \tag{2.3}$$

The ordered pair $(\overline{F}, \overline{G})$ embodies the continuous-time filling scheme associated with the ordered pair (f, g)—compare (2.1)-(2.2) to [34; (5), p. 369]. In the sequel we shall refer to $(\overline{F}, \overline{G})$ as the filling scheme for (f, g).

Each of the functions $\overline{U}(f\otimes 1)$, $\overline{U}(g\otimes 1)$, $\overline{G}$, $\overline{H}$ is excessive for $\overline{X}$; in particular, these functions are $\overline{X}$-finely continuous. Consequently, $\overline{F}$ is also $\overline{X}$-finely continuous. In the following proposition we record several additional properties of $\overline{F}$.

PROPOSITION 2.4. – (a) $t \mapsto \overline{F}(x,t)$ *is monotone increasing and continuous for each fixed* $x \in E$.

(b) *Define*

$$Z := \{x \in E : F_t(x) = 0, \forall t > 0\} = \cap_{0<r\in\mathbf{Q}}\{x : F_r(x) = 0\}, \tag{2.5}$$

and let $D_Z := \inf\{t \ge 0 : X_t \in Z\}$ denote the entry time of Z. Then

$$F_t(x) \le P^x \int_0^{D_Z \wedge t} f(X_s)\, ds \qquad \forall t > 0, x \in E. \tag{2.6}$$

In particular, Z is finely closed.

Proof. – (a) By the theory of optimal stopping ([13] or [15; Chap. II]), we have

$$\overline{F}(x,r) = \sup_T \overline{P}^{x,r}(\overline{A}_T) = P^x \int_0^{T(r)} [f(X_s) - g(X_s)]\, ds, \tag{2.7}$$

where T ranges over the stopping times of $\overline{X}$, $\overline{A}_t(\omega, r) := \int_0^{t\wedge r}(f-g)(X_s(\omega))\,ds$, and

$$T(r) := \inf\{s \geq 0 : F_{r-s}(X_s) = 0\} \wedge r.$$

Since (2.7) follows easily from facts about $\overline{F}$ already in evidence, we provide a quick proof. The process

$$M_t(\omega, r) := F_{r-t}(X_t(\omega))1_{\{t<r\}} + \int_0^{t\wedge r} (f-g+g_{r-s})(X_s(\omega))\,ds, \qquad t \geq 0,$$

is a uniformly integrable right-continuous $\overline{P}^{x,r}$-martingale with initial value $F_r(x) = \overline{F}(x, r)$. Thus, for any $\overline{X}$ stopping time T,

$$\overline{F}(x,r) = \overline{P}^{x,r}\left(\overline{F}(\overline{X}_T)1_{\{T<r\}} + \overline{A}_T + \int_0^{T\wedge r} \overline{g}(\overline{X}_s)\,ds\right) \geq \overline{P}^{x,r}(\overline{A}_T). \tag{2.8}$$

Moreover, the inequality in (2.8) is an equality for the $\overline{X}$-stopping time $T(\omega, r) := T(r)(\omega)$ because of (2.3). This proves (2.7).

For $t > 0$ set $\mathcal{F}_t^\circ := \sigma\{X_s : 0 \leq s < t\}$, and for $r > 0$ let $\mathcal{T}_r$ denote the class of $(\mathcal{F}_{t+}^\circ)$ stopping times S such that $\{S < t\} \in \mathcal{F}_r^\circ$ whenever $t > r$. If T is a stopping time of $\overline{X}$, then for fixed (x, r) there exists $S \in \mathcal{T}_r$ such that $\overline{P}^{x,r}(\overline{S} \neq T) = 0$, where $\overline{S}(\omega, r) := S(\omega)$. Thus, by (2.7),

$$\overline{F}(x,r) = \sup_{S\in\mathcal{T}_r} P^x(A_{S\wedge r}),$$

where $A_t(\omega) := \int_0^t (f-g)(X_u(\omega))\,du$. But $\mathcal{T}_r \subset \mathcal{T}_s$ for any $s > r$, so

$$\overline{F}(x,r) = \sup_{S\in\mathcal{T}_r} P^x(A_{S\wedge r}) \leq \sup_{S\in\mathcal{T}_s} P^x(A_{S\wedge s}) = \overline{F}(s,x),$$

which establishes the monotonicity of $F(\cdot, x)$. Next we note that

$$F_s - F_r \leq P_r F_{s-r} \leq P_r U_{s-r} f \tag{2.9}$$

for all $0 \leq r \leq s$, where $F_0 := 0$; *cf.* [37; Prop. 4], [28; Prop. 11]. The second inequality in (2.9) follows immediately from the definitions. To see the first we use (2.7) and the obvious relation

$$T(s) = r + T(s-r)\circ\theta_r \qquad \text{on } \{T(s) \geq r\}.$$

Therefore,

$$\begin{aligned}F_s(x) &= P^x(A_{T(s)}; T(s) < r) + P^x(A_r + A_{T(s-r)} \circ \theta_r; T(s) \geq r) \\ &= P^x(A_{T(s)\wedge r}) + P_r F_{s-r}(x) \leq F_r(x) + P_r F_{s-r}(x),\end{aligned}$$

as claimed. Now $P_r U_{s-r} f(x) = \int_r^s P_v f(x)\, dv \to 0$ as $s - r \to 0$, by dominated convergence, so (2.9) ensures the continuity of $t \mapsto F_t(x)$ on $[0, \infty[$.

(b) Notice that $T(t) := \inf\{s \geq 0 : F_{t-s}(X_s) = 0\} \wedge t \leq D_Z \wedge t$. Therefore (2.6) follows from the second equality in (2.7). From (2.6) and the inequality $D_Z \leq T_Z$ we deduce that if $x \in Z^r := \{x : P^x(T_Z = 0) = 1\}$ (the regular points for Z) then $x \in Z$. That is, Z is finely closed. □

3. INVARIANT SETS AND FUNCTIONS

We now fix an excessive measure m of X. Recall that a set $B \in \mathcal{E}^e$ is m-polar provided $P^m(T_B < \infty) = 0$. A statement $S(x)$ holding for each x outside an m-polar subset of E is said to hold *m-quasi-everywhere* (m-q.e.).

In this section and the next we assume that m is *conservative* ($m \in$ Con) in the sense that the only potential νU dominated by m is the zero potential.

A function $h \in b\mathcal{E}^e \cup p\mathcal{E}^e$ is *m-invariant* provided $P_t h = h$ for all $t > 0$, m-q.e. It is easy to check that if $h_0 \in b\mathcal{E}^e \cup p\mathcal{E}^e$ satisfies $P_t h_0 = h_0$, m-a.e. for each $t > 0$, then $h := U^1 h_0$ is an m-invariant function equal to h_0, m-a.e.

In the following lemma we record some well-known consequences of the hypothesis $m \in$ Con. For proofs see [7] and [24; pp. 9–15].

LEMMA 3.1.

(a) *If h is excessive, then $P^x(h(X_t) = h(x), \forall t > 0) = 1$ for m-q.e. $x \in E$.*

(b) *If $B \in \mathcal{E}^e$, then $\varphi_B \in \{0, 1\}$, m-q.e., where $\varphi_B(x) := P^x(T_B < \infty)$.*

(c) *If $B \in \mathcal{E}^e$, then $P^m(0 < L_B < \infty) = 0$, where $L_B := \sup\{t > 0 : X_t \in B\}$.*

(d) *If $f \in p\mathcal{E}^*$, then $Uf \in \{0, +\infty\}$, m-q.e.*

In particular, (3.1)(a) implies that any excessive function of X is m-invariant.

Let $\mathcal{H}$ denote the class of all bounded m-invariant functions, and define $\mathcal{I} := \{B \in \mathcal{E}^e : 1_B \in \mathcal{H}\}$. Let $\mathcal{I}'$ denote the class of elements of $\mathcal{E}^e$ of the form

$$\{Uf = +\infty\} \triangle N$$

where $f \in bp\mathcal{E}^* \cap L^1(m)$, $N \in \mathcal{E}^e$ is m-polar, and $\triangle$ denotes symmetric difference.

LEMMA 3.2. – *(a)* $\mathcal{I}' = \mathcal{I}$. *(b)* $\mathcal{I} = \sigma(\mathcal{H})$ *and* $\mathcal{H} = b\mathcal{I}$.

Proof. – (a) If $B \in \mathcal{I}$, then $U1_B = (+\infty) \cdot 1_B$, m-q.e., so $B \in \mathcal{I}'$. Conversely, suppose $B = \{Uf = +\infty\}\triangle N \in \mathcal{I}'$, and define $F := \{Uf = +\infty\}$. Then 1_F is a supermedian function; *i.e.*, $P_t 1_F \leq 1_F$ for all $t > 0$. Moreover, (3.1)(a) implies that, for m-q.e. x, if $Uf(x) = +\infty$, then $Uf(X_s) = +\infty$ for all $s > 0$, P^x-a.s.; for such x we must have $P_t 1_F(x) = 1$ for all $t > 0$. It follows that $1_F \in \mathcal{H}$, and then that $B \in \mathcal{I}$.

(b) Property (3.1)(a) implies that $\mathcal{H}$ is a vector lattice contained in $b\mathcal{E}^e$, and that $\mathcal{H}$ contains the constant functions. It is also clear that $\mathcal{H}$ is closed under bounded monotone convergence. Thus $\mathcal{I}$ is a σ-algebra. Moreover, if $h \in \mathcal{H}$ and $b \in \mathbf{R}$, then $h_n := [n(h-b)^+] \wedge 1 \in \mathcal{H}$ for all $n \in \mathbf{N}$, and so $1_{\{h>b\}} =\uparrow \lim_n h_n \in \mathcal{H}$ as well. It follows that $\sigma(\mathcal{H}) \subset \mathcal{I}$. The reverse inclusion $\mathcal{I} \subset \sigma(\mathcal{H})$ is trivial. Of course, $\mathcal{I} = \sigma(\mathcal{H})$ implies $\mathcal{H} \subset b\mathcal{I}$; the reverse inclusion follows by a routine application of the monotone class theorem. □

From [24; (4.17)] we know that, because $m \in \mathrm{Con}$, if $B \in \mathcal{E}^e$ then the balayage of m on B, $R_B m$, is a conservative excessive measure, and is related to m by

$$R_B m = \varphi_B \cdot m.$$

LEMMA 3.3. – *Given* $A \in \mathcal{E}^e$, *we have (writing* $\subset_m$ *for inclusion modulo* m*-polars)*

(a) $A \subset_m \{\varphi_A = 1\}$, *and*

(b) $A \subset_m B \in \mathcal{I} \Rightarrow \{\varphi_A = 1\} \subset_m B$.

Proof. – (a) Suppose that $A \setminus \{\varphi_A = 1\}$ is not m-polar. Then there is a stopping time T with

$$P^m(0 < T < \infty) > 0, \text{ and}$$
$$X_T \in A \cap \{\varphi_A = 0\},\ P^m\text{-a.s. on } \{T < \infty\},$$

since $\{\varphi_A < 1\} =_m \{\varphi_A = 0\}$ by (3.1)(b). Thus, by the strong Markov property,

$$\begin{aligned}0 < P^m(0 < T < \infty) &\leq P^m(T < \infty, T_A < \infty, \varphi_A(X_T) = 0)\\ &= P^m(T < \infty, T_A < \infty, T_A \circ \theta_T = \infty) \leq P^m(0 < L_A < \infty),\end{aligned}$$

resulting in a contradiction because of (3.1)(c).

(b) Note that $A \subset_m B$ implies that $\varphi_A \leq \varphi_B$, m-q.e., and $B\triangle\{\varphi_B = 1\}$ is m-polar because $B \in \mathcal{I}$. □

4. PROOF OF THEOREM (1.1)

Before embarking on the proof of Theorem (1.1) we prepare the way with a lemma. Let $\mathcal{L}^1(m)$ denote the class of m-measurable real-valued functions such that $\int_E |f|\, dm < +\infty$. A set $A \in \mathcal{E}^e$ is *absorbing* provided $P^x(T_{E\setminus A} < \infty) = 0$ for all $x \in A$. The restriction of X to an absorbing set $A \in \mathcal{E}^e$ is a right process with state space A, and a Borel right process provided $A \in \mathcal{E}$. Following [22] we say that $N \in \mathcal{E}^e$ is *m-inessential* provided N is m-polar and $E \setminus N$ is absorbing. From [22; (6.12)] we know that if $N_0 \in \mathcal{E}^e$ is m-polar, then there exists an m-inessential Borel set $N \supset N_0$.

The following result is valid for arbitrary excessive measures.

LEMMA 4.1. – *Let m be an excessive measure and let f be an element of $\mathcal{L}^1(m)$. Then there exists $f_0 \in p\mathcal{E}$ with $m(f_0 \neq f) = 0$ and an m-inessential set $B_f \in \mathcal{E}$ such that (i) $\{x : U_t f_0(x) \neq U_t f(x)$ for some $t > 0\} \subset B_f$, (ii) $U_t f_0(x) < +\infty$ for all $t > 0$ and $x \in E \setminus B_f$, (iii) $E \setminus B_f \ni x \mapsto U_t f_0(x)$ is finely continuous for each $t > 0$, (iv) $t \mapsto U_t f_0(x)$ is continuous for each $x \in E \setminus B_f$.*

Proof. – Fix m and f as in the statement of the lemma, and choose functions f_0 and f_1 in $p\mathcal{E}$ such that $f_0 \leq f \leq f_1$ and $m(f_0 < f_1) = 0$. Clearly $f_1 \in \mathcal{L}^1(m)$, so $U^1 f_1 \in \mathcal{L}^1(m)$ as well. It follows that $U^1 f_1$ is finite m-a.e., hence m-q.e. (by the proof of [6; II(3.5)]). Let $B_1 \in \mathcal{E}$ be an m-inessential set off which $U^1 f_1$ is finite. Now the 1-excessive function $U^1(f_1 - f_0)$ vanishes m-a.e. and so the set $B_2 := \{U^1(f_1 - f_0) \neq 0\} \in \mathcal{E}$ is m-inessential. Since $U_t(f_1 - f_0) \leq e^t U^1(f_1 - f_0)$, we have $U_t f(x) = U_t f_0(x) < +\infty$ for all $t > 0$ whenever $x \in E \setminus (B_1 \cup B_2)$. The set $B_f := B_1 \cup B_2$ is an m-inessential Borel set and it is easy to verify properties (i), (ii), and (iv) in the statement of the lemma. As for property (iii), note that if $t > 0$ and $x \in E \setminus B_f$, then $s \mapsto U_t f_0(X_s)$ is the P^x-optional projection of the (P^x-a.s.) right-continuous process $s \mapsto \int_s^{s+t} f_0(X_u)\, du$. Thus $s \mapsto U_t f_0(X_s)$ is right continuous, P^x-a.s., so $U_t f_0$ is finely continuous on $E \setminus B_f$ by [6; II(4.8)]. □

Proof of Theorem 1.1. – It is enough to show that if $f, g \in p\mathcal{E} \cap L^1(m)$, then

$$(4.2)\qquad \limsup_{t\to\infty} \frac{U_t f}{U_t g} \leq \frac{\mu(f/q|\mathcal{I})}{\mu(g/q|\mathcal{I})}, \qquad m\text{-q.e. on } \{Ug = +\infty\}.$$

where $\mu = q \cdot m$ and q is a strictly positive bounded Borel function with $\mu(q) = 1$. (The substitution of $\{Ug = +\infty\}$ for $\{Ug > 0\}$ is justified by

(3.1)(d).) Indeed, (4.2) and its counterpart when f and g exchange roles establishes (1.2) m-q.e. on $\{Uf = Ug = +\infty\}$, while (1.2) is trivial on $\{Uf < +\infty = Ug\}$.

In what follows we shall write $\tilde{f}$ and $\tilde{g}$ for the conditional expectations $\mu(f/q|\mathcal{I})$ and $\mu(g/q|\mathcal{I})$, respectively.

In view of Lemma (4.1), at the cost of working with the restriction of X to $E \setminus (B_f \cup B_g)$, we can (and do) assume in the sequel that $U_t f$ and $U_t g$ enjoy the smoothness properties listed in the statement of the lemma on all of E.

To prove (4.2) it suffices to show that for any real $b > 0$

$$\text{(4.3)} \quad \limsup_{t\to\infty} U_t f / U_t g \leq b \qquad m\text{-q.e. on } \{\tilde{f} < b \cdot \tilde{g}\} \cap \{Ug = +\infty\}.$$

This assertion is an immediate consequence of the following two lemmas. Fix $b > 0$ and let $(\overline{F}, \overline{G})$ denote the filling scheme for (f, bg). Recall from section 2 the finely closed set $Z := \{x \in E : F_\infty(x) = 0\}$, where $F_\infty(x) :=\uparrow \lim_{t\to\infty} F_t(x)$.

LEMMA 4.4. – *We have* $\limsup_{t\to\infty} U_t f / U_t g \leq b$, *$m$-q.e. on the set* $\{\tilde{f} < b \cdot \tilde{g}\} \cap \{Ug = +\infty\} \cap \{\varphi_Z = 1\}$.

Proof. – Define $B := \{\tilde{f} < b \cdot \tilde{g}\} \cap \{Ug = +\infty\} \cap \{\varphi_Z = 1\} \in \mathcal{I}$. By (2.1) and (2.2),

$$\frac{U_t f(x)}{U_t g(x)} \leq \frac{F_t(x)}{U_t g(x)} + b,$$

provided $U_t g(x) > 0$. Thus the assertion will follow once we show that $F_\infty < +\infty$, m-q.e. on B. To this end define $v(x) := P^x \int_0^{D_Z} f(X_t)\, dt$, so that $F_\infty \leq v$ (by (2.6)), and $v(x) = P^x \int_0^{T_Z} f(X_t)\, dt$ if $x \notin Z$. We are going to show that $\{v = +\infty\}$ is m^*-polar, where m^* is the conservative excessive measure defined by $m^* := R_B m = \varphi_B \cdot m$. It suffices to show that $P^{m^*}(T_K < \infty) = 0$ for every compact $K \subset \{v = +\infty\}$. Fix such a compact set K.

The finely open set $E \setminus Z$ can be viewed as the state space of the subprocess (X, T_Z) (X killed at time T_Z), for which the restriction of m^* to $E \setminus Z$ is a dissipative excessive measure. Since $f \in L^1(m)$, it follows that v (which is the potential of f relative to (X, T_Z)) is finite m^*-a.e. Therefore we can choose a strictly positive function $\ell \in b\mathcal{E}$ such that $m^*(\ell \cdot v) < +\infty$. Then by the strong Markov property and the terminal time property of T_Z ($u < T_Z$ implies $T_Z = u + T_Z \circ \theta_u$),

$$\begin{aligned}
+\infty > m^*(\ell \cdot v) &= P^{m^*}\left(\ell(X_0)\int_0^{T_Z} f(X_t)\,dt\right) \\
&\geq P^{m^*}\left(\ell(X_0)\int_{T_K}^{T_K+T_Z\circ\theta_{T_K}} f(X_t)\,dt;\, T_K < T_Z\right) \\
&= P^{m^*}(\ell(X_0)v(X_{T_K});\, T_K < T_Z),
\end{aligned}$$

which leads to a contradiction unless $P^{m^*}(T_K < T_Z) = 0$ because $X_{T_K} \in K \subset \{v = +\infty\}$ a.s. on $\{T_K < \infty\}$.

Since $P^{m^*}(T_K = T_Z) = 0$, it only remains to show that $P^{m^*}(T_Z < T_K < \infty) = 0$. To this end let Γ denote the set of strictly positive left endpoints of the excursions of X from Z. By [19; (6.6), (6.10)]

$$P^{m^*}\sum_{s\in\Gamma} e^{-s}\left[\int_0^{T_Z} f(X_t)\,dt\right]\circ\theta_s \leq m^*(f) < +\infty. \tag{4.5}$$

If $T_Z(\omega) < T_K(\omega) < \infty$, then there is a unique $s \in \Gamma(\omega)$ such that $s \leq T_K(\omega) < s + T_Z(\theta_s\omega)$, and for this s we have $[\int_0^{T_Z} f(X_t)](\theta_s\omega) \geq [\int_0^{T_Z} f(X_t)\,dt](\theta_{T_K}\omega)$, by the terminal time property of T_Z. Consequently,

$$\begin{aligned}
&P^{m^*}\left(\sum_{s\in\Gamma} e^{-s}\left[\int_0^{T_Z} f(X_t)\,dt\right]\circ\theta_s;\, T_Z < T_K < \infty\right) \\
&\qquad \geq P^{m^*}\left(e^{-T_K}\left[\int_0^{T_Z} f(X_t)\,dt\right]\circ\theta_{T_K};\, T_Z < T_K < \infty\right) \\
&\qquad = P^{m^*}(e^{-T_K}v(X_{T_K});\, T_Z < T_K < \infty),
\end{aligned}$$

which contradicts (4.5) unless $P^{m^*}(T_Z < T_K < \infty) = 0$. □

LEMMA 4.6. – *The set* $\{\tilde{f} < b \cdot \tilde{g}\} \cap \{\varphi_Z < 1\}$ *is* m*-polar.*

Proof. – Recall from (3.1)(b) that $\varphi_Z = 0$, m-q.e. on $\{\varphi_Z < 1\}$. Let $C := \{\tilde{f} < b \cdot \tilde{g}\} \cap \{\varphi_Z = 0\} \in \mathcal{I}$ and define a conservative excessive measure $m_C := 1_C \cdot m$. We are going to show that if m_C is not the zero measure, then $m_C(Z) > 0$. This will prove the assertion because $\varphi_Z = 1$, m-a.e. on Z.

So let us assume that the measure m_C is non-zero. Then

$$m_C(bg - f) = \mu((bg/q - f/q) \cdot 1_C) = \mu((b\tilde{g} - \tilde{f}) \cdot 1_C) > 0. \tag{4.7}$$

Also, since $m_C \in$ Con is invariant,

$$\begin{aligned} t \cdot m_C(bg - f) &= m_C U_t(bg - f) = m_C(G_t - F_t) \leq m_C(G_t) \\ &= m_C \int_0^t P_s g_{t-s}\, ds = m_C \int_0^t g_u\, du \\ &\leq \int_0^t m_C(g; F_u = 0)\, du, \end{aligned}$$

the final equality following from (2.3). Consequently

$$\liminf_{t\to\infty} t^{-1} \int_0^t m_C(g; F_u = 0)\, du \geq m_C(bg - f) > 0. \tag{4.8}$$

But $\{F_u = 0\} \downarrow \cap_{t>0}\{F_t = 0\} = Z$ as $u \to \infty$. So (4.8) implies

$$m_C(g; Z) > 0,$$

as desired. □

Remark 4.9. – Given $k \in bp\mathcal{E}^*$, define an operator U^k on $p\mathcal{E}^*$ by the formula

$$U^k f(x) := P^x \int_0^\infty \exp\left(- \int_0^t k(X_s)\, ds \right) f(X_t)\, dt, \qquad x \in E.$$

Following Neveu [32], let us say that a function $f \in p\mathcal{E}$ is *m-special* if $U^k f$ is a bounded function for each $k \in bp\mathcal{E}^*$ with $m(k) > 0$. Notice that if f is m-special then $U_t f(x) < +\infty$ for each $t > 0$ and $x \in E$. One can also show that m-special functions are m-integrable. If f and g are both m-special and if $g > 0$, then the argument used above to prove Theorem (1.1) shows that

$$\lim_{t\to\infty} \frac{U_t f(x)}{U_t g(x)} = \frac{m(f)}{m(g)}, \qquad \forall x \in E. \tag{4.10}$$

The lack of an exceptional set in (4.10) may seem surprising until we recall that the existence of a single strictly positive m-special function guarantees that X is Harris recurrent (in the sense of [5]; see [32; Sect. 7]), in which case m is the unique (up to constant multiple) σ-finite invariant measure for X. (This explains why the invariant σ-algebra $\mathcal{I}$ does not appear on the right side of (4.10).) Port and Stone [36; Thm. 5.3] have established a ratio ergodic theorem without exceptional set in the context of Lévy processes in locally compact abelian groups. Analogous results for Harris recurrent

Markov chains on general state spaces can be found in the work of Orey [35], Neveu [33] and Métivier [26]. □

As a corollary of Theorem (1.1) we have the following sharp continuous-time form of Brunel's maximal inequality. As noted in section 1, Fukushima [20] and Shur [39, 40] based their proofs of the "semipolar" form of the ratio ergodic theorem on weaker forms of this inequality. We leave it to the reader to verify that, conversely, Theorem (1.1) can be deduced from Theorem (4.11).

THEOREM 4.11. – *Fix* $h \in L^1(m)$ *and let* $B \in \mathcal{E}^e$ *be a subset of* $\{x \in E : \limsup_{t\to\infty} U_t h(x) > 0\}$. *Then* $R_B m(h) \geq 0$.

Proof. – Write $h = f - g$ where f and g are positive elements of $L^1(m)$. If $g = 0$, m-a.e. then there is nothing to prove. Otherwise, $C := \{Ug = +\infty\} \in \mathcal{I}$ is not m-polar. By Theorem (1.1), $\tilde{f}/\tilde{g} = \lim_{t\to\infty} U_t f / U_t g \geq 1$, m-q.e. on $B \cap C$. Consequently, $\tilde{f} - \tilde{g} \geq 0$ on $\{\varphi_B = 1\} \cap C$. Thus

$$R_B m(h) = m((f - g)\varphi_B) = \mu((\tilde{f} - \tilde{g}); \varphi_B = 1) \geq 0,$$

since $\tilde{g} = 0$, m-q.e. on $E \setminus C$. □

We end this section with another corollary of Theorem (1.1)—the ratio ergodic theorem for general additive functionals. An additive functional (AF) of X is an increasing, adapted, right-continuous process $(A_t)_{t\geq 0}$ such that $A_t < +\infty$ and $A_{t+s} = A_t + A_s \circ \theta_t$ for all $s, t \geq 0$, P^x-a.e. for m-q.e. $x \in E$. (It would be more precise to term such a process an additive functional admitting an m-polar exceptional set; *cf.* [21; p. 181] or [18; (3.1)].)

The characteristic measure of the AF A (relative to m) is the measure on $(E, \mathcal{E})$ defined by the formula

$$\nu_A(C) := \uparrow \lim_{t\to 0} t^{-1} P^m \int_0^t 1_C(X_s)\, dA_s, \qquad C \in \mathcal{E}. \tag{4.12}$$

(Actually, this definition makes sense for any excessive measure m.) We say that A is *integrable* provided $\nu_A(E) < +\infty$. If A is integrable then, since $m \in$ Con is invariant, the expectation appearing on the right side of (4.12) is equal to $t \cdot P^m \int_0^1 1_C(X_s)\, dA_s$ and

$$\nu_A(f) = P^m \int_0^\infty \phi(t) f(X_t)\, dA_t \qquad \forall f \in p\mathcal{E}, \tag{4.13}$$

where $\phi : [0, +\infty[\to [0, +\infty[$ is any Borel function with $\int_0^\infty \phi(t)\, dt = 1$. Just as in the special case of an AF of the form $\int_0^t f(X_s)\, ds$ ($f \in pL^1(m)$), if A is an integrable AF then $P^x(A_t) < +\infty$ for all $t \geq 0$, for m-q.e. $x \in E$.

The "semipolar exceptional set" form of the following ratio ergodic theorem was proved by Direev [12] under duality hypotheses. See also [4; Thm. 3.1] for the "polar" exceptional set form of the result, under hypotheses amounting to Harris recurrence. Let the function $q > 0$ and the measure $\mu = q \cdot m$ be as in the statement of Theorem (1.1). We are still assuming that the excessive measure m is conservative.

PROPOSITION 4.14. – *Let A and B be integrable AFs of X, and define $f(x) := P^x(A_1)$, $g(x) := P^x(B_1)$. Then*

$$\lim_{t\to\infty} \frac{P^x(A_t)}{P^x(B_t)} = \frac{\mu(f/q|\mathcal{I})(x)}{\mu(g/q|\mathcal{I})(x)}$$

for m-q.e. x in the set $\{y \in E : \sup_t P^y(B_t) > 0\}$.

Proof. – Apply Theorem (1.1) to the pair (f, g), taking note of the obvious inequalities

$$P^x(A_{t+1}) - f(x) \le U_t(f) \le P^x(A_{t+1}),$$
$$P^x(B_{t+1}) - g(x) \le U_t(g) \le P^x(B_{t+1}). \quad \square$$

5. LOCAL LIMIT THEOREM

As a further application of the filling scheme we present an extension of Mokobodzki's local limit theorem [29; Cor. 19]; *cf.* [38; (66.9)]. Our argument is based on a local form of E. Hopf's maximal inequality, appearing as Lemma (5.2) below.

Let $A = (A_t)_{t\ge 0}$ and $B = (B_t)_{t\ge 0}$ be continuous additive functionals (CAFs) of X in the sense of [6; IV(1.15)]. We assume, for simplicity that the "characteristics" $\alpha_t(x) := P^x(A_t)$ and $\beta_t(x) := P^x(B_t)$ are finite for all $t \ge 0$ and all $x \in E$. Then $\overline{\alpha}(x,t) := \alpha_t(x)$ and $\overline{\beta}(x,t) := \beta_t(x)$ are (finite, regular) space-time potentials, and the filling-scheme construction of section 2 can be carried out as follows. Let $\overline{G}$ denote the least space-time supermedian majorant of $\overline{\beta} - \overline{\alpha}$, and define non-negative functions $\overline{F} := \overline{G} - \overline{\beta} + \overline{\alpha}$ and $\overline{H} := \overline{\beta} - \overline{G}$. The potential $\overline{G}$ is strongly dominated by $\overline{\beta}$, so by Motoo's theorem ([31], [38; (66.2)]) and the result of Mokobodzki already cited in section 2, there exists $\overline{g} : E\times]0,\infty[\to]0,+\infty[$, measurable over the universal completion of $\mathcal{E} \otimes \mathcal{B}_{]0,\infty[}$, such that $0 \le \overline{g} \le 1_{\{\overline{F}=0\}}$ and

$$\overline{G}(x,r) = \overline{P}^{x,r} \int_0^t \overline{g}(\overline{X}_s)\, d\overline{B}_s, \qquad \forall r > 0, x \in E, \tag{5.1}$$

where $\overline{B}_s(\omega, r) := B_{s\wedge r}(\omega)$ is a CAF of $\overline{X}$. The obvious analogue of Proposition (2.4) is valid in the present context; in particular, $t \mapsto F_t(x)$ is increasing and continuous on $[0, \infty[$.

LEMMA 5.2. – *Fix an excessive measure m of X, and assume that the characteristic measures ν_A and ν_B (as defined by (4.12)) have finite total mass. Let $(\overline{F}, \overline{G})$ be the filling scheme associated with $\overline{\alpha}$ and $\overline{\beta}$ as above, and define $D := \cap_{t>0}\{F_t > 0\}$. Then $\nu_A \geq \nu_B$ on D.*

Proof. – It follows easily from the definitions that

(5.3)
$$\begin{aligned} 0 \leq F_{t+s}(x) &= \alpha_s(x) - \beta_s(x) + P_s F_t(x) + P^x \int_0^s g_{t+s-u}(X_u)\, dB_u \\ &\leq \alpha_s(x) - \beta_s(x) + P_s F_t(x) + P^x \int_0^s 1_{\{F_t=0\}}(X_u)\, dB_u. \end{aligned}$$

Sending t to 0 in (5.3) we find that if $w \in bp\mathcal{E}^e$ and $s > 0$, then

$$s^{-1} \int_E w \cdot \alpha_s\, dm \geq s^{-1} \int_E w(x) P^x \left(\int_0^s 1_D(X_u)\, dB_u \right) m(dx). \tag{5.4}$$

Let us now take w in (5.4) of the form $U^\alpha f$, ($f : E \to [0, +\infty[$ bounded and continuous). Since X is a right process, $t \mapsto e^{-\alpha t} w(X_t)$ is a positive bounded right-continuous supermartingale. Thus the left limit $w(X)_{t-} := \lim_{s\uparrow t} w(X_s)$ exists for all $t > 0$, almost surely. The time-set $\{t > 0 : w(X_t) \neq w(X)_{t-}\}$ is at most countable (almost surely), hence not charged by the random measures dA_t, dB_t. Therefore, by the *proof* of (8.7) in [22], we have

$$\lim_{s\to 0} s^{-1} \int_E w\alpha_s\, dm = \nu_A(w)$$

and

$$\lim_{s\to 0} s^{-1} \int_E w(x) P^x \left(\int_0^s 1_D(X_u)\, dB_u \right) m(dx) = \nu_B(w 1_D).$$

Using these facts to pass to the limit as $s \to 0$ in (5.4), we obtain

$$\nu_A(U^\alpha f) \geq \nu_B(1_D U^\alpha f) \tag{5.5}$$

for every bounded continuous $f \geq 0$. But for such f, $\alpha U^\alpha f$ converges boundedly and pointwise to f as $\alpha \to \infty$. Therefore (5.5) implies

$$\nu_A(f) \geq \nu_B(1_D f) \tag{5.6}$$

for every bounded continuous function f. Since $\mathcal{E}$ is the Borel σ-algebra of a metrizable Lusin measurable space, we conclude from (5.6) that $\nu_A \geq 1_D \cdot \nu_B$ on all of $\mathcal{E}$. □

Remark 5.7. – A minor modification of the above argument shows that for any $t > 0$,

$$\nu_A(F_t > 0) \geq \nu_B(F_t > 0) \tag{5.8}$$

an inequality which is closely related to a result of Mokobodzki [30; Thm. 9]. We leave it to the reader to show that (5.8) implies the (generalized) Hopf maximal inequality: for each $t > 0$,

$$\nu_A(\Lambda_t) \geq \nu_B(\Lambda_t) \quad \text{where } \Lambda_t := \left\{ x : \sup_{0<s\leq t} P^x(A_s - B_s) > 0 \right\}. \tag{5.9}$$

THEOREM 5.10. – *Let m be an excessive measure, and assume that the characteristic measures ν_A and ν_B (computed relative to m as in (4.12)) have finite total mass. Let $\nu_A = \varphi \cdot \nu_B + \nu_A^{(s)}$ be the Lebesgue-Radon-Nikodym decomposition of ν_A with respect to ν_B. Then*

$$\lim_{t\to 0} \frac{P^x(A_t)}{P^x(B_t)} = \varphi(x), \qquad \textit{for } \nu_B\textit{-a.e. } x \in E. \tag{5.11}$$

Proof. – By a standard reduction, we can assume that $A_t \leq B_t$ for all $t \geq 0$, P^m-a.s., in which case $\nu_A \leq \nu_B$ and $\varphi := d\nu_A/d\nu_B \leq 1$. Fix $b > 0$ and let $(\overline{F}, \overline{G})$ be the filling scheme associated with $\alpha_t(x) := P^x(A_t)$ and $b\beta_t(x) := P^x(b \cdot B_t)$, as discussed above. By Lemma (5.2),

$$\begin{aligned} b\nu_B(\{\varphi < b\} \cap D) &= \nu_{b\cdot B}(\{\varphi < b\} \cap D) \leq \nu_A(\{\varphi < b\} \cap D) \\ &= \int_{\{\varphi < b\}\cap D} \varphi \, d\nu_B, \end{aligned}$$

which is absurd unless $\nu_B(\{\varphi < b\} \cap D) = 0$. Thus, for ν_B-a.e. $x \in \{\varphi < b\}$ there exists $t(x) > 0$ such that $F_t(x) = 0$ for $0 < t < t(x)$, in which case

$$\frac{P^x(A_t)}{P^x(B_t)} = b\frac{F_t(x) + H_t(x)}{G_t(x) + H_t(x)} \leq b, \qquad \forall\, 0 < t < t(x).$$

It follows that $\limsup_{t\to 0} P^x(A_t)/P^x(B_t) \le b$ for ν_B-a.e. $x \in \{\varphi < b\}$. Varying b, we conclude that $\limsup_{t\to 0} P^x(A_t)/P^x(B_t) \le \varphi$ for ν_B-a.e. $x \in E$. The same argument applied to the CAF $B_t - A_t$ shows that $\limsup_{t\to 0}[P^x(B_t - A_t)]/P^x(B_t) \le 1 - \varphi(x)$ for ν_B-a.e. x, so $\liminf_{t\to 0} P^x(A_t)/P^x(B_t) \ge \varphi(x)$ for ν_B-a.e. x. □

We now apply Theorem (5.10) in space-time to obtain the promised extension of Mokobodzki's local limit theorem. See also Airault and Föllmer [1; (5.31)] for a similar result obtained under much more restrictive conditions. (In addition to a harmless transience hypothesis, they assume the existence of a nice Martin boundary providing an integral representation for excessive functions.) For the statement of the theorem let A and B be CAFs, and recall that a set $D \in \mathcal{E}^*$ is of B-potential zero provided $P^x \int_0^\infty 1_D(X_t)\, dB_t = 0$ for all $x \in E$. By the theorem of Motoo cited earlier in this section, there is a function $\varphi \in p\mathcal{E}^*$, uniquely determined modulo sets of B-potential zero, such that

$$A_t = \int_0^t \varphi(X_s)\, dB_s + C_t, \qquad \forall t \ge 0, \tag{5.12}$$

P^x-a.s. for all $x \in E$, where C is a CAF that is singular with respect to B in the sense that there exists $D \in \mathcal{E}^*$ of B-potential zero such that D^c is of C-potential zero.

THEOREM 5.13. – *Let A and B be CAFs with finite characteristics $P^x(A_t)$ and $P^x(B_t)$ respectively. Let the "Motoo density" φ be as in* (5.12). *Then for all x outside a set of B-potential zero,*

$$\lim_{t\to 0} \frac{P^x(A_t)}{P^x(B_t)} = \varphi(x). \tag{5.14}$$

Proof. – Define CAFs $\overline{A}$ and $\overline{B}$ of the space-time process $\overline{X}$ by setting $\overline{A}_t(\omega, r) := A_{t\wedge r}(\omega)$ and $\overline{B}_t(\omega, r) := B_{t\wedge r}(\omega)$. Notice that $\overline{P}^{x,r}(\overline{A}_t) = P^x(A_{t\wedge r})$, and similarly for $\overline{B}$. Given $(x, r) \in E\times]0, \infty[$, the measure $\overline{m} := \overline{U}(x, r; \cdot)$ is an excessive measure of $\overline{X}$. Since $\overline{m}$ is a potential, the characteristic measure $\nu_{\overline{A}}$ of $\overline{A}$ (computed with respect to $\overline{m}$) is given by the formula

$$\nu_{\overline{A}}(\Phi) = \overline{P}^{x,r} \int_0^\infty \Phi(\overline{X}_s)\, d\overline{A}_s = P^x \int_0^r \Phi(X_s, s)\, dA_s.$$

(See [24; (8.11)].) Notice that $\nu_{\overline{A}}(E\times]0, \infty[) = P^x(A_r) < \infty$. Similar remarks apply to $\nu_{\overline{B}}$. In particular, the function $\varphi \otimes 1$ is a version of the Lebesgue-Radon-Nikodym derivative $d\overline{\nu}_{\overline{A}}/d\overline{\nu}_{\overline{B}}$. By Theorem (5.10) (applied to $\overline{X}$, the CAFs $\overline{A}$ and $\overline{B}$, and the $\overline{X}$-excessive measure $\overline{m}$),

$$(5.15)\qquad \lim_{t\to 0}\frac{P^y(A_t)}{P^y(B_t)} = \lim_{t\to 0}\frac{\overline{P}^{y,s}(\overline{A}_t)}{\overline{P}^{y,s}(\overline{B}_t)} = \varphi(y)$$

for $\overline{\nu}_{\overline{B}}$-a.e. $(y,s) \in E \times]0,\infty[$. That is, if $D := \{y \in E : P^y(A_t)/P^y(B_t) \not\to \varphi(y)$ as $t \to 0\}$, then

$$(5.16)\qquad P^x \int_0^r 1_D(X_t)\, dB_t = \overline{P}^{x,r} \int_0^\infty 1_{D\times]0,\infty[}(\overline{X}_t)\, d\overline{B}_t = 0.$$

Since $(x,r) \in E\times]0,\infty[$ was arbitrary, we conclude from (5.16) that D is of B-potential 0, which proves the assertion. □

6. RESOLVENT RATIO LIMIT THEOREMS

All of the results presented in previous sections have their "abelian" analogues. We illustrate the possibilities with one example, the quasi-sure form of a ratio-ergodic theorem of Edwards [14; Thm. 8]. Edwards was concerned with the resolvent associated with a strongly continuous semigroup of positive L^1 contractions. A more general result has been obtained by Feyel [16; Cor. 23] for general resolvent families consisting of positive L^1 contractions, with no assumption of strong continuity. Neither of these authors restricted attention to resolvents that contract the L^∞ norm, as we do here.

As before, $q > 0$ is a Borel function such that $m(q) = 1$, and $\mu := q \cdot m$.

THEOREM 6.1. – *Let m be a conservative excessive measure and let f and $g \geq 0$ be elements of $L^1(m)$. Then*

$$(6.2)\qquad \lim_{\lambda\to 0+}\frac{U^\lambda f}{U^\lambda g} = \frac{\mu(f/q|\mathcal{I})}{\mu(g/q|\mathcal{I})}$$

m-q.e. on $\{x : Ug(x) > 0\}$.

Sketch of proof. – We assume without loss of generality that $f \geq 0$ and that $U^\lambda(f+g)(x) < +\infty$ for all $\lambda > 0$ and all $x \in E$. Fix $\lambda > 0$, let G^λ denote the least λ-supermedian majorant of $U^\lambda(g-f)$, and define $F^\lambda := G^\lambda - U^\lambda(g-f)$ and $H^\lambda := U^\lambda g - G^\lambda$. Then $F^\lambda \wedge H^\lambda \geq 0$ and $G^\lambda = U^\lambda g^\lambda$ for some function $g^\lambda \in p\mathcal{E}^*$ satisfying the estimate

$g^\lambda \leq g \cdot 1_{\{F^\lambda = 0\}}$. The function F^λ is the value function of a discounted optimal stopping problem: For $\lambda > 0$,

$$F^\lambda(x) = \sup_T P^x \int_0^T e^{-\lambda s}[f(X_s) - g(X_s)]\, ds \tag{6.3}$$
$$= P^x \int_0^{T(\lambda)} e^{-\lambda s}[f(X_s) - g(X_s)]\, ds,$$

where T ranges over the finite stopping times of X and $T(\lambda) := \inf\{t \geq 0 : F^\lambda(X_t) = 0\}$. The first equality in (6.3) implies that $\lambda \mapsto F^\lambda(x)$ is decreasing and continuous, and the second yields the estimate

$$F^\lambda(x) \leq P^x \int_0^{D_Z} f(X_s)\, ds, \qquad \forall \lambda > 0, x \in E.$$

where $Z := \cap_{\lambda > 0}\{x : F^\lambda(x) = 0\}$. With these facts in hand, one can follow the line of reasoning used in section 4 to complete the proof. □

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