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The asymptotic distribution of the bootstrap sample mean of an infinitesimal array *

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ABSTRACT. – Bootstrapping from a sample of rare events leads in a natural way to the consideration of triangular arrays for the study of the behaviour of bootstrap procedures. In this paper we consider the different limit laws which the bootstrap mean can produce when the bootstrap sample is obtained from a triangular array of (row-wise) independent and identically distributed random variables. Our framework requires the original array to be infinitesimal and to verify the General Central Limit Theorem, and includes the consideration of different resampling sizes as well as different norming constants. Small resampling sizes are object of special attention and we point out the difficulties that appear in this general framework. © Elsevier, Paris

RÉSUMÉ. – Le bootstrap d'un échantillon d'événements rares amène de façon naturelle à la considération de tableaux triangulaires pour étudier le comportement des procédures de bootstrap. Dans ce travail on considère les différentes lois limites que la moyenne bootstrap peut produire quand l'échantillon bootstrap est pris à partir d'un tableau triangulaire de variables aléatoires indépendantes et de même loi sur chaque ligne.

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Notre cadre exige que le tableau soit infinitésimal et vérifie le théorème central limite généralisé, et il inclut la considération de différentes tailles d'échantillonnage ainsi que de différentes constantes de normalisation. Les petites tailles d'échantillonnage font l'objet d'une étude particulière, et nous mettons en évidence les difficultés qui apparaissent dans ce cadre général.

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1. INTRODUCTION

The study of the validity of the bootstrap methodology has been the object of much work in recent years. In particular, the prominent role of the mean in probability and statistics has motivated considerable attention on the bootstrap of the mean. Broadly speaking, this work adressed the solution of the following question:

Question 1

How well does the bootstrap work if we assume that the distribution of a sum can be approximated by some law for a wide range of sample sizes n and that the distribution of the terms in the sum do not depend on n?

In these studies (mainly beginning with Bickel and Freedman [7]) the usual framework to study the bootstrap has been that of a suitably normalized sequence of independent identically distributed random variables (i.i.d.r.v.'s), including later the consideration of domains of attraction of stable laws (Athreya [4, 5, 6], Hall [11], Arcones and Giné [2, 3], Swanepoel [16], Knight [13] and Deheuvels, Mason and Shorack [9]).

In this paper we are interested in the following more general question:

Question 2

Assumed that we know, for a fixed sample size n, that the distribution of a sum can be approximated by some law, what can we say about the bootstrap?

This question cannot be solved in the preceding framework, the adequate one (which we will consider in this paper) being that infinitesimal arrays, $\{X_j^n, j = 1, ..., k_n, n \in \mathcal{N}\}$, $k_n \to \infty$, constituted by i.i.d.r.v.'s in each row, which satisfy the general Central Limit Theorem (CLT) with a general infinitely divisible limit law $N(0,\sigma^2) * c_\tau\text{Pois } \mu$. That is, if $S_n := \sum_{j=1}^{k_n} X_j^n$ then for some sequence $\{a_n\}_n$ of real numbers

$$S_n - a_n \to_w N(0, \sigma^2) * c_\tau\text{Pois } \mu. \tag{1}$$

Now, for every $n \in \mathcal{N}$, let $\{Y_j^n, j = 1, ..., m_n\}$, $m_n \to \infty$, be the bootstrap variables, *i.e.*, conditionally independent r.v.'s given $\{X_j^n, j = 1, ..., k_n\}$ with the same conditional distribution defined by

$$P^*[Y_1^n = X_j^n] = k_n^{-1}, j = 1, ..., k_n,$$

and let us consider the sequence of partial sums $S_n^* := \sum_{j=1}^{m_n} Y_j^n$. We are interested in the study of the conditional asymptotic law of a suitable normalization, $r_n^{-1}(S_n^* - A_n)$, of S_n^* given $\{X_j^n, j = 1, ..., k_n, n \in \mathcal{N}\}$.

Among the different types of convergence considered in related work we will use the so-called *convergence in law in law* introduced by Athreya [6], which allows the consideration of random limit distributions (this implies that the limit distribution that we obtain may depend on the values taken by the r.v.'s in the parent array).

The main results in this paper can be summarized as follows (A_n are centering r.v.'s usually related to S_n).

– *If* $\lim_n m_n/k_n = \infty$ *and* $r_n = (m_n/k_n)^{1/2}$ *then (Theorem 6) the limit law of* $r_n^{-1}(S_n^* - A_n)$ *is normal with variance* X^2, *where* X^2 *is a positive random variable with law given by*

$$\mathcal{L}(X^2) = \delta_{\{\sigma^2 + \int_0^{\tau^2} x\mu^*(dx)\}} * c_{\tau^2} \mathit{Pois}\ \mu^*, \text{ where } \mu^* = \mu \circ \sqrt{\cdot}.$$

– *If* $\lim_n m_n/k_n = c \in (0, \infty)$ *and* $r_n = (m_n/k_n)^{1/2}$ *then (Theorem 11)* $r_n^{-1}(S_n^* - A_n)$ *has a random limit law with characteristic function*

$$\Phi(t) = \exp\left\{-(t^2\sigma^2/2) + c\int(e^{itxc^{-1/2}} - 1 - itc^{-1/2}\alpha_{\sqrt{c}.\tau}(x))N(dx)\right\},$$

where $\alpha_\tau(x) = xI_{[-\tau,\tau]}(x)$, *and* N *is a random measure such that* $N(A)$ *and* $N(B)$ *are independent Poisson random variables with parameters* $\mu(A)$ *and* $\mu(B)$ *if* A, B *are Borel measurable disjoint subsets of* $(-\delta, \delta)^c$, $\delta > 0$.

– *If* $\lim_n m_n/k_n = 0$ *then we have that*

– *It is not always possible to find a normalizing sequence giving a non-degenerate limit law (Example 14 and Remark 14.1).*

– *Even when a normalizing sequence giving a non-degenerate limit law may exist (see Theorem 13), it is not obvious how the asymptotic laws of* S_n *and* S_n^* *are related (Example 15).*

Through standard arguments, it is straightforward to show that the convergence in law in law to a non-random (*i.e.* fixed) distribution is equivalent to the convergence in law in probability. Therefore most known results concerning the bootstrap CLT can be derived from our Theorems 6, 11 and 13. Moreover, our results for large resampling sizes solve an open question stated in Arcones and Giné [2] (see the comments preceding Theorem 2.5 there).

The case "small resampling sizes" (*i.e.* the case when $m_n/k_n \to 0$) deserves special comments. First notice that the involved appearance of the hypotheses in Theorem 13 is mainly due to the difficulties in reconciling the behaviour in the bootstrap of the normal and Poisson parts (see conditions (8) and (9) there).

Moreover Theorem 13 contains the available first order results concerning stable laws (Remark 13.1) like those in Arcones and Giné [2], Athreya [6] and Bickel, Goetze and Van Zwet [8].

Finally, Theorem 13 shows a main difference between the answers to the present questions in our framework and in the i.i.d. setting which consists in that, in the i.i.d. case, the only difficulties arise with heavy-tailed distributions and can be solved without additional assumptions by using small resampling sizes (alternatively other resampling methods have been proposed with the same aim, as those developed in Politis and Romano [15] and Bickel, Goetze and Van Zwet [8]).

However, according to Theorem 13, the bootstrap does not necessarily work without additional hypotheses on the triangular arrays even with small resampling sizes. This is caused by the fact (see Example 15) that the mean intensity of rare events can be modified by varying the sample sizes and renorming the r.v.'s. This possibility leads us to consider a Lévy measure (see condition (8) in Theorem 13) different from that controlling the rare events in the original array. The role of this condition is related to the fact that bootstrap subsampling is in some way equivalent to classical bootstrap from a subset of the sample. A work in preparation will report the singularity of that situation for limiting stable laws.

Therefore our results warn about the dangers of the indiscriminate use of classical bootstrap with small resampling sizes; they are related to the results in Hall and Jing Bing [12]. In that paper it is shown, through Edgeworth expansions, that bootstrap subsampling performs poorly in that the error between the subsample bootstrap approximation and the true distribution can be larger than that of an asymptotic approximation.

2. NOTATION AND PRELIMINARY RESULTS

As previously stated, we are concerned with an infinitesimal array $\{X_j^n, j = 1, ..., k_n, n \in \mathcal{N}\}$, $k_n \to \infty$, constituted by i.i.d.r.v.'s in each row, and satisfying (1), where $N(0,\sigma^2) * c_\tau \text{Pois } \mu$ is the convolution of a normal law and a generalized Poisson law. Here, see e.g. Chapter 2 in Araujo and Giné [1], μ is a Lévy measure and $\tau > 0$ is a constant related to a certain truncation procedure, such that $\mu\{-\tau,\tau\} = 0$. In the sequel such an array will be called an *impartial array* to stress the additional fact that every summand has exactly the same influence on the whole sum.

Convergence in distribution, in law or weak convergence will be terms indistinctly used throughout the paper, and, as already employed will be denoted by $\to_w$. The limit in distribution will be denoted as $w - \lim$.

We will often denote by P^* and E^* respectively the conditional probability and expectation given $\{X_j^n, j = 1,...,k_n, n \in \mathcal{N}\}$. Writing the laws of r.v.'s we use $\mathcal{L}(Z)$ for the distribution of the r.v. Z and $\mathcal{L}^*(Z)$ for the conditional distribution of Z given $\{X_j^n, j = 1,...,k_n, n \in \mathcal{N}\}$. By $\delta_{\{x\}}$ we denote Dirac's measure on x.

To simplify the exposition, our approach to consider convergence in law in law of $S_n^* := \sum_{j=1}^{m_n} Y_j^n$ does not involve the convergence of the finite dimensional distributions of the stochastic process (as x varies) $P^*\left[\frac{S_n^*-A_n}{r_n} \leq x\right]$. Actually we consider the *convergence in law in law* as the convergence in law of the random characteristic functions, $\phi_n^*(t)$, of $\frac{S_n^*-A_n}{r_n}$, for each $t \in \Re$, instead of consideryng, as in Athreya [6], the convergence of the joint law of $(\phi_n^*(t_1),...,\phi_n^*(t_k))$ for every k and every $t_1,...,t_k \in \Re$. Still our arguments could be easily modified to cope with that requirement.

For the r.v.'s already considered and $\delta, \tau > 0$ given, we will use throughout the paper the following notation concerning truncated r.v.'s.

$$\begin{aligned}
(Z)_\tau &:= Z I_{\{|Z|\leq\tau\}}, & (Z)^\tau &:= Z I_{\{|Z|>\tau\}}.\\
X_{j,\delta}^n &:= X_j^n I_{\{|X_j^n|\leq\delta\}}, & X_j^{n,\delta} &:= X_j^n I_{\{|X_j^n|>\delta\}},\ j = 1,...,k_n;\\
Y_{j,\delta}^n &:= Y_j^n I_{\{|Y_j^n|\leq\delta\}}, & Y_j^{n,\delta} &:= Y_j^n I_{\{|Y_j^n|>\delta\}},\ j = 1,...,m_n;\\
R_{n,\delta} &:= \sum_{j=1}^{k_n} X_{j,\delta}^n, & T_{n,\delta} &:= \sum_{j=1}^{k_n} X_j^{n,\delta};\\
R_{n,\delta}^* &:= \sum_{j=1}^{m_n} Y_{j,\delta}^n, & T_{n,\delta}^* &:= \sum_{j=1}^{m_n} Y_j^{n,\delta};\\
X_{j,\tau}^{n,\delta} &:= X_j^n I_{\delta<\{|X_j^n|\leq\tau\}}, \quad j = 1,...,k_n; & &\\
\bar{X}_{n,\delta} &:= \frac{1}{k_n}\sum_{j=1}^{k_n} X_{j,\delta}^n. & &
\end{aligned}$$

We end this section by stating two results related to the general CLT.. The first result is stated for the sake of completeness and will be often used in the sequel. It is included in (a) and (d) of Corollary 4.8, pag. 63, in Araujo and Giné [1].

PROPOSITION 1. – *Let* $\{X_j^n, j = 1, ..., k_n, n \in \mathcal{N}\}$ *be an impartial array and let* $\{a_n\}_n$ *be a sequence of real numbers. Then:*

(a) $\mathcal{L}(S_n - a_n) \to_w N(a, \sigma^2)$ *if and only if the following three conditions are satisfied:*

(a.i) $\lim_n (ER_{n,\tau} - a_n) = a$,

(a.ii) $\lim_n k_n E(X_{1,\tau}^n - EX_{1,\tau}^n)^2 = \sigma^2$ *for some* $\tau > 0$, *and*

(a.iii) for every $\epsilon > 0$, $\lim_n k_n P\{|X_1^n| > \epsilon\} = 0$.

(b) Let μ *be a Lévy measure and let* $\tau > 0$ *be such that* $\mu\{-\tau, \tau\} = 0$. *Then* $\mathcal{L}(S_n - a_n) \to_w c_\tau$ *Pois* μ *if and only if the following three conditions are satisfied:*

(b.i) $k_n \mathcal{L}(X_1^n)|\{|x| > \delta\} \to_w \mu|\{|x| > \delta\}$ *for every* $\delta > 0$ *such that* $\mu\{-\delta, \delta\} = 0$,

(b.ii) $\lim_{\delta \to 0^+} \limsup_n k_n E(X_{1,\delta}^n - EX_{1,\delta}^n)^2 = 0$, *and*

(b.iii) $\lim_n (ER_{n,\tau} - a_n) = 0$.

In the spirit of the so-called découpage de Lévy, we obtain from the arguments in Chapter 2 of Araujo and Giné [1] the following version of the general CLT.

PROPOSITION 2. – *Let* $\{X_j^n, j = 1, ..., k_n, n \in \mathcal{N}\}$ *be an impartial array. Let* μ *be a Lévy measure and let* $\tau > 0$ *be such that* $\mu\{-\tau, \tau\} = 0$. *In order that* $\mathcal{L}(S_n - ER_{n,\tau}) \to_w N(0, \sigma^2) * c_\tau$*Pois* μ, *it is necessary and sufficient that there exists a sequence* $\{\delta_n\}$, $\delta_n \downarrow 0$, *such that*

(i) $\mathcal{L}(R_{n,\delta_n} - ER_{n,\delta_n}) \to_w N(0, \sigma^2)$,

(ii) $\mathcal{L}[T_{n,\delta_n} - E(R_{n,\tau} - R_{n,\delta_n})] \to_w c_\tau$*Pois* μ, *and that*

(iii) *the laws* $\mathcal{L}(S_n)$ *and* $\mathcal{L}(R_{n,\delta_n}) * \mathcal{L}(T_{n,\delta_n})$ *are asymptotically equivalent.*

Proof. – Since the reverse implication is obvious, it suffices to show the existence of such a sequence, $\{\delta_n\}_n$, from the convergence of $\mathcal{L}(S_n - ER_{n,\tau})$.

Let us consider the functions

$$f(n, \alpha) := k_n E[X_{1,\alpha}^n - EX_{1,\alpha}^n]^2, \text{ and } g(n, \alpha) := k_n E|X_{1,\alpha}^n| P[|X_1^n| > \alpha].$$

From the general CLT setting it follows that

$$\lim_{\alpha \to 0+} \begin{bmatrix} \overline{\lim} \\ \underline{\lim} \end{bmatrix}_{n\to\infty} f(n,\alpha) = \sigma^2; \text{ and that } \lim_{n\to\infty} g(n,\alpha) = 0.$$

Therefore, by standard arguments it is possible to construct a sequence $\{\delta_n\}_n$ satisfying

$$\lim_n \delta_n = \lim_n P[|X_1^n| > \delta_n] = \lim_n g(n,\delta_n) = 0 \text{ and } \lim_n f(n,\delta_n) = \sigma^2, \tag{2}$$

while the corresponding associated sequences satisfy (i), (ii) and (iii). □

Remark 2.1. – Notice that the sequence $\{\delta_n\}_n$ obtained in the previous proposition also satisfies that for every $\tau > 0$,

$$|k_n E X_{1,\delta_n}^n E X_{1,\tau}^{n,\delta_n}| \leq \tau g(n,\delta_n) \to 0,$$

so that the random variables R_{n,δ_n} and $(R_{n,\tau} - R_{n,\delta_n})$ are asymptotically uncorrelated. □

3. RESULTS

In the following proposition we provide some general results to be employed later.

PROPOSITION 3. – *Let $\{X_j^n, j = 1, ..., k_n, n \in \mathcal{N}\}$ be an impartial array, and let $\{A_n\}_n$ be a sequence of random variables. Then*

a) if $\{r_n\}_n$ is a non-random sequence of positive numbers such that $\lim_n r_n = \infty$, and X^2 is a positive random variable, then

$$\mathcal{L}^*\left(r_n^{-1}(S_n^* - A_n)\right) \to_w N(0, X^2)$$

in law if the following three conditions are satisfied

(a.i) $r_n^{-1}\left(E^* R_{n,\tau r_n}^* - A_n\right) \to_p 0$,

(a.ii) $r_n^{-2} m_n E^* \left[Y_{1,\tau r_n}^n - E^* Y_{1,\tau r_n}^n\right]^2 \to_w X^2$, for some $\tau > 0$, and

(a.iii) for every $\epsilon > 0$, $m_n P^*[|Y_1^n| > \epsilon r_n] \to_p 0$.

b) given the Lévy measure ν, the sequence $\{\mathcal{L}^*(S_n^* - A_n)\}_n$ w-converges to c_τPois ν in probability if the following three conditions are satisfied

(b.i) $m_n \mathcal{L}^*(Y_1^n)|\{|x| > \delta\} \to_p \nu|\{|x| > \delta\}$, for every $\delta > 0$ such that $\nu\{-\delta, \delta\} = 0$,

(b.ii) $\limsup_n E\left[m_n E^*(Y^n_{1,\delta} - E^*Y^n_{1,\delta})^2\right] \leq \int_{-\delta}^{\delta} x^2 d\nu(x) + g(\delta)$, *where* $\lim_{\delta\to 0+} g(\delta) = 0$, *and*

(b.iii) $\lim_n \left(E^* R^*_{n,\tau} - A_n\right) = 0$ *in probability.*

Proof. – It is easy to get an a.s. Skorohod construction leading to a.s. statements (a.i), (a.ii), (a.iii) (resp. (b.i), (b.ii), (b.iii)), so that verifying the hypotheses in Proposition 1 with probability one. Therefore the original sequence will verify a) (resp. b)). □

We divide our study in three parts, whose treatments are radically different, depending on the relative asymptotic rate between $\{m_n\}_n$ and $\{k_n\}_n$.

3.1. Large resampling sizes

First let us consider the case when $\lim_n m_n/k_n = \infty$.

We start with a proposition, of some independent interest, where we obtain the asymptotic behaviour of the sums of the squares of an impartial array. Note that its proof works not only for impartial arrays but also for general infinitesimal ones.

PROPOSITION 4. – *Let* $\{X^n_j, j = 1, ..., k_n, n \in \mathcal{N}\}$ *be an impartial array. Let* μ *be a Lévy measure and let* $\tau > 0$ *such that* $\mu\{-\tau, \tau\} = 0$. *If* $\mathcal{L}(S_n - ER_{n,\tau}) \to_w N(0, \sigma^2) * c_\tau Pois\ \mu$, *then*

$$\mathcal{L}\left(\sum_{j=1}^{k_n} (X^n_j)^2 - k_n E(X^n_{1,\tau})^2\right) \to_w c_{\tau^2} Pois\ \mu^*$$

where μ^* *is the Lévy measure on the positive real line defined by* $\mu^* = \mu \circ \sqrt{\ }$ *(i.e.:* $\mu^*(0, a) = \mu[(-a^{1/2}, a^{1/2}) - \{0\}], a > 0$*).*

Proof. – First note that for every r.v. Z such that $|Z| \leq \tau$, μ-a.s., if $\mathcal{L}(Z) = Q$, we have

$$\begin{aligned}
\mathrm{Var}(Z^2) &= \int \left[E\{x^2 - Z^2\}\right]^2 dQ(x) = \int \left[E|(x+Z)(x-Z)|\right]^2 dQ(x) \\
&\leq (2\tau)^2 \int \left[|x - EZ| + E|Z - EZ|\right]^2 dQ(x) \\
&= 4\tau^2 \int \left[|x - EZ|^2 + \{E|Z - EZ|\}^2 \right.\\
&\quad \left. + 2|x - EZ|E|Z - EZ|\right] dQ(x) \\
&\leq 16\tau^2 E(Z - EZ)^2 = 16\tau^2 \mathrm{Var} Z.
\end{aligned}$$

Now the result is obtained by introducing a sequence $\{\delta_n\}_n$ constructed as in Proposition 2, and applying Proposition 1. □

In the following corollary we obtain the asymptotic behaviour of the sums of the squares of the variables in impartial arrays, in the line of Raikov's theorem (see e.g. Gnedenko-Kolmogorov [10], Theorem 5, p. 143).

COROLLARY 5. – *Let $\{X_j^n, j = 1, ..., k_n, n \in \mathcal{N}\}$ be an impartial array. Let μ be a Lévy measure, μ^* be the Lévy measure defined in Theorem 4, and let $\tau > 0$ such that $\mu\{-\tau, \tau\} = 0$.*

a) If $\mathcal{L}(S_n - ER_{n,\tau}) \rightarrow_w N(0, \sigma^2)$, then $\sum_{j=1}^{k_n}(X_j^n - EX_{j,\tau}^n)^2 \rightarrow_p \sigma^2$.

b) If $\mathcal{L}(S_n - ER_{n,\tau}) \rightarrow_w c_\tau Pois\ \mu$, then

$$\mathcal{L}\left(\sum_{j=1}^{k_n}(X_j^n - EX_{j,\tau}^n)^2\right) \rightarrow_w \delta_{\{\int_0^{\tau^2} x d\mu^*(x)\}} * c_{\tau^2} Pois\ \mu^*.$$

c) If $\mathcal{L}(S_n - ER_{n,\tau}) \rightarrow_w N(0, \sigma^2) * c_\tau Pois\ \mu$, then

$$\mathcal{L}\left(\sum_{j=1}^{k_n}(X_j^n - EX_{j,\tau}^n)^2\right) \rightarrow_w \delta_{\{\sigma^2 + \int_0^{\tau^2} x d\mu^*(x)\}} * c_{\tau^2} Pois\ \mu^*.$$

Proof. – Part a) is just Raikov's theorem.

To obtain part b) note that infinitesimality implies $EX_{1,\tau}^n \rightarrow 0$, so that

$$EX_{1,\tau}^n \sum_{j=1}^{k_n}(X_j^n - EX_{j,\tau}^n) \rightarrow_p 0.$$

On the other hand (see e.g. Problems 13 and 14 in Chapter 2 in Araujo and Giné [1]) the sequence $(R_{n,\tau} - ER_{n,\tau})^2$ is uniformly integrable, hence

$$k_n E[X_{1,\tau}^n - EX_{1,\tau}^n]^2 \rightarrow \int_{-\tau}^{\tau} x^2 \mu(dx),$$

and b) follows from Proposition 4. To prove part c) let $\{\delta_n\}_n$ be a sequence obtained as in Proposition 2, and let us consider the decomposition

$$\sum_{j=1}^{k_n}(X_j^n - EX_{j,\tau}^n)^2 = \sum_{j=1}^{k_n}(X_{j,\delta_n}^n - EX_{j,\delta_n}^n)^2 + \sum_{j=1}^{k_n}(X_j^{n,\delta_n} - EX_{j,\tau}^{n,\delta_n})^2$$
$$+ 2\sum_{j=1}^{k_n}(X_{j,\delta_n}^n - EX_{j,\delta_n}^n)(X_j^{n,\delta_n} - EX_{j,\tau}^{n,\delta_n}).$$

Now the result is deduced from infinitesimality and the hypothesis (recall also Remark 2.1). □

In the following theorem we obtain the conditional asymptotic behaviour of the bootstrap mean when $\lim_n m_n/k_n = \infty$.

THEOREM 6. – *Let us assume that the impartial array* $\{X_j^n, j = 1, ..., k_n, n \in \mathcal{N}\}$ *satisfies* $\mathcal{L}(S_n - a_n) \to_w N(0,\sigma^2) * c_\tau Pois\ \mu$, *where* μ *is a Lévy measure such that* $\mu\{-\tau,\tau\} = 0$. *If* $\lim_n m_n/k_n = \infty$ *and we define* $r_n = (m_n/k_n)^{1/2}$, *then*

$$r_n^{-1}\sum_{j=1}^{m_n}(Y_j^n - \bar{X}_{n,r_n}) \to_w N(0,X^2) \text{ in distribution,}$$

where X^2 *is a positive random variable with law given by*

$$\mathcal{L}(X^2) = \delta_{\{\sigma^2+\int_0^{\tau^2} x\mu^*(dx)\}} * c_{\tau^2} Pois\ \mu^*, \text{ where } \mu^* = \mu \circ \sqrt{\cdot}.$$

Proof. – Let $\delta > 0$. It is obvious that conditions (a.i), (a.ii) and (a.iii) in Proposition 3 are satisfied with $A_n = \bar{X}_{n,\delta r_n}$ if conditions 1 and 2 below hold

1. $\sum_{j=1}^{k_n}(X_{j,\delta r_n}^n)^2 - \frac{1}{k_n}\left(\sum_{j=1}^{k_n} X_{j,\delta r_n}^n\right)^2 \to_w X^2$.
2. $\frac{m_n}{k_n}\sum_{j=1}^{k_n} I_{\{|X_j^n|>\epsilon r_n\}} \to_p 0$, for every $\epsilon > 0$.

But since $r_n \to \infty$, if $\epsilon > 0$,

$$\lim_n P\left[\bigcup_{j=1}^{k_n}\{|X_j^n| > \epsilon r_n\}\right] \leq \lim_n k_n P[|X_1^n| > \epsilon r_n] \leq \lim_n k_n P[|X_1^n| > \alpha] = \mu[-\alpha,\alpha]^c$$

for every $\alpha > 0$ such that $\mu\{-\alpha,\alpha\} = 0$. Therefore condition 2 is satisfied because μ is a Lévy measure and

$$\lim_n P\left[\frac{m_n}{k_n}\sum_{j=1}^{k_n} I_{\{|X_j^n|>\epsilon r_n\}} \neq 0\right] \leq \lim_n P\left[\bigcup_{j=1}^{k_n}\{|X_j^n| > \epsilon r_n\}\right] = 0. \quad (3)$$

To show condition 1 under our hypotheses, note that (3) also implies (take $\epsilon = \delta$) the equivalence between the w-convergence of the sequences $\left\{\mathcal{L}\left(\sum_j X_{j,\delta r_n}^n\right)\right\}_n$ and $\left\{\mathcal{L}\left(\sum_j X_j^n\right)\right\}_n$ and that of $\left\{\mathcal{L}\left(\sum_j (X_{j,\delta r_n}^n)^2\right)\right\}_n$ and $\left\{\mathcal{L}\left(\sum_j (X_j^n)^2\right)\right\}_n$.

Therefore, the convergence in law of $\{\mathcal{L}(S_n - ER_{n,\tau})\}_n$ and the fact that $\lim_n k_n = \infty$ imply that $\left\{k_n^{-1}\left[\left(\sum_{j=1}^{k_n} X_j^n\right) - ER_{n,\tau}\right]^2\right\}_n$ converges in probability to zero, hence from the convergence obtained in Corollary 5 c) and the fact that $EX_{1,\tau}^n \sum_{j=1}^{k_n}(X_j^n - EX_{j,\tau}^n) \to_p 0$, we obtain that

$$\begin{aligned}
w - \lim_n \mathcal{L}&\left(\sum_{j=1}^{k_n}(X_{j,\delta r_n}^n)^2 - \frac{1}{k_n}\left(\sum_{j=1}^{k_n} X_{j,\delta r_n}^n\right)^2\right) \\
&= w - \lim_n \mathcal{L}\left(\sum_{j=1}^{k_n}(X_j^n)^2 - \frac{1}{k_n}\left(\sum_{j=1}^{k_n} X_j^n\right)^2\right) \\
&= w - \lim_n \mathcal{L}\left(\sum_{j=1}^{k_n}(X_j^n)^2 - \frac{1}{k_n}\left[\left(\sum_{j=1}^{k_n}(X_j^n - E(X_{j,\tau}^n))\right)^2\right.\right. \\
&\qquad \left.\left. +2k_n EX_{1,\tau}^n \sum_{j=1}^{k_n}(X_j^n - EX_{j,\tau}^n) + k_n^2(EX_{1,\tau}^n)^2\right]\right) \\
&= w - \lim_n \mathcal{L}\left(\sum_{j=1}^{k_n}(X_j^n)^2 - k_n(EX_{1,\tau}^n)^2\right) \\
&= w - \lim_n \mathcal{L}\left(\sum_{j=1}^{k_n}(X_j^n - EX_{j,\tau}^n)^2\right) \\
&= \delta_{\{\sigma^2 + \int_{-\tau}^{\tau} x^2\mu(dx)\}} * c_{\tau^2}\text{Pois } \mu^*,
\end{aligned}$$

so that condition 1 is satisfied with

$$\mathcal{L}(X^2) = \delta_{\{\sigma^2 + \int_0^{\tau^2} x\mu^*(dx)\}} * c_{\tau^2}\text{Pois } \mu^*.$$

Finally note that (3) implies that

$$\frac{m_n}{k_n}(\bar{X}_{n,\delta r_n} - \bar{X}_{n,r_n}) \to_p 0. \quad \square$$

3.2. Moderate resampling sizes

Our next objective is to analyze the case where m_n and k_n are of the same order. We first study separately the normal limit law case and the Poisson one and then we will merge both results.

The Normal case has been exhaustively studied by several authors when the infinitesimal array is a suitably normalized sequence of i.i.d.r.v.'s. However to get Theorem 13 we are interested not only in general infinitesimal arrays, but in infinitesimal arrays with a random number of elements. On the other hand the proof of Theorem 2.2 in Arcones and Giné [2] (see also Theorem 1 in Mammen [14]) remains essentially valid, proving the next result.

PROPOSITION 7. – *Let $\{c_n\}$ and $\{w_n\}$ be two sequences of integer-valued random variables such that $\lim_n c_n = \lim_n w_n = \infty$ and that $\lim_n \frac{c_n}{w_n} = c \in (0,\infty)$ in probability. For every $n \in \mathcal{N}$, let $\{X_j^n, j = 1, ..., c_n\}$ be random variables which are independent and identically distributed given c_n, and such that the conditional laws of $S_n - ES_n$ given c_n, $n \in \mathcal{N}$, w-converge to $N(0,\sigma^2)$ almost surely, where $\sigma^2 \in \Re$ is not random.*

Let $\{Y_j^n, j = 1, ..., w_n\}$ be a bootstrap sample of size w_n taken from $\{X_j^n, j = 1, ..., c_n\}$. If we define $\bar{X}_n := c_n^{-1}\sum_{j=1}^{c_n} X_j^n$ and $r_n := (w_n/c_n)^{1/2}$ then we have

$$\mathcal{L}^*\left[r_n^{-1}\sum_{j=1}^{w_n}(Y_j^n - \bar{X}_n)\right] \to_w N(0,\sigma^2),$$

in probability, where, in this case, $\mathcal{L}^*(Z)$ denotes the conditional distribution of the random variable Z given $\{X_j^n, j = 1, ..., c_n, n \in \mathcal{N}\}$.

Proof. – Every subsequence of $\{(c_n, w_n)\}_n$ contains a new subsequence $\{(c_{n'}, w_{n'})\}_{n'}$ such that

$$\lim_{n'} c_{n'} = \lim_{n'} w_{n'} = \infty \text{ and that } \lim_{n'} \frac{c_{n'}}{w_{n'}} = c$$

a.s. If we apply the argument in the proof of Theorem 2.2 in Arcones and Giné [2] to the second subsequence, we have that every subsequence of $\left\{\mathcal{L}^*\left(r_n^{-1}\sum_{j=1}^{w_n}(Y_j^n - \bar{X}_n)\right)\right\}_n$ contains a further subsequence such that the conditional laws of $r_{n'}^{-1}\sum_{j=1}^{w_{n'}}(Y_j^{n'} - \bar{X}_{n'})$ given $\{X_j^{n'}, j = 1, ..., c_{n'}\}$ w-converge to $N(0,\sigma^2)$ in probability. Then the original sequence also satisfies this property. □

Next we analyze the case when $\mathcal{L}(S_n - a_n) \to_w c_\tau \text{Pois}\ \mu$. First we study the situation when the Lévy measure μ is finite. In the first proposition we obtain an equivalent condition to the convergence in law of the sequence we are interested in.

PROPOSITION 8. – *Let* $\{X_j^n, j = 1, ..., k_n, n \in \mathcal{N}\}$ *be an impartial array such that* $\mathcal{L}(S_n - a_n) \rightarrow_w c_\tau Pois\ \mu$, *where* μ *is a Lévy measure such that* $\mu\{-\tau, \tau\} = 0$.

If $\mu(\Re - \{0\}) < \infty$, $\lim_n \frac{m_n}{k_n} = c \in (0, \infty)$ *and* ϕ_n^* *denotes the characteristic function of the conditional distribution of* S_n^* *given* $\{X_j^n, j = 1, ..., k_n\}$, *then*

$$w - \lim_n \phi_n^*(t) = w - \lim_n \exp\left(\frac{m_n}{k_n}\sum_{j=1}^{k_n}\left[e^{itX_j^n} - 1\right]\right), \textit{ for every } t \in \Re.$$

Proof. – Since $\mathcal{L}^*(Y_1^n) = k_n^{-1}\sum_{j=1}^{k_n}\delta_{\{X_j^n\}}$, we have

$$\sup_A \left|\mathcal{L}^*\left(\sum_{j=1}^{m_n} Y_j^n\right)(A) - \text{ Pois }\left[\frac{m_n}{k_n}\sum_{j=1}^{k_n}\delta_{X_j^n}\right](A)\right| \leq \sum_{j=1}^{m_n}\left(\frac{\#\{i : X_j^n \neq 0\}}{k_n}\right)^2$$

which converges in probability to 0 because $\lim_n k_n P[X_1^n \neq 0] = \mu[\Re - \{0\}]$, thus the (unconditional) expectation of the last term converges to zero.

Now, if $\{Z_j^n, j = 1, ..., k_n\}$ is a family of r.v.'s which are conditionally independent given $\{X_1^n, ..., X_{k_n}^n\}$ with conditional distributions $\mathcal{L}^*(Z_j^n) = \text{Pois}\left[\frac{m_n}{k_n}\delta_{\{X_j^n\}}\right]$, $j = 1, ..., k_n$; then $\mathcal{L}^*\left[\sum_{j=1}^{k_n} Z_j^n\right] = \text{Pois}\left[\frac{m_n}{k_n}\delta_{\{X_1^n\}}\right] * \ldots * \text{Pois}\left[\frac{m_n}{k_n}\delta_{\{X_{k_n}^n\}}\right] = \text{Pois}\left[\frac{m_n}{k_n}\sum_{j=1}^{k_n}\delta_{\{X_j^n\}}\right]$, therefore the w-limit of $\{\mathcal{L}^*(S_n^*)\}_n$ coincides with that of $\left\{\mathcal{L}^*\left(\sum_{j=1}^{k_n} Z_j^n\right)\right\}_n$, whose characteristic function is

$$E^*[\exp(itZ_j^n)] = \left(\text{Pois}\left[\frac{m_n}{k_n}\delta_{\{X_j^n\}}\right]\right)^{\widehat{}}(t) = \exp\left[\frac{m_n}{k_n}e^{itX_j^n} - \frac{m_n}{k_n}\right]. \quad \square$$

PROPOSITION 9. – *Let* $\{X_j^n, j = 1, ..., k_n, n \in \mathcal{N}\}$ *be an impartial array. Let* μ *be a Lévy measure such that* $\mu[\Re - \{0\}] < \infty$ *and let* $\zeta, \delta, \tau > 0$ *be such that* $\mu\{-\zeta, \zeta, -\delta, \delta, -\tau, \tau\} = 0$. *Assume that* $\mathcal{L}(S_n - a_n) \rightarrow_w c_\zeta$ *Pois* μ *and that* $\lim_n \frac{m_n}{k_n} = c \in (0, \infty)$. *Let* $\alpha_\tau(x) := xI_{[-\tau,\tau]}(x)$. *Then*

$$\mathcal{L}\left(E^*\left[\exp\left\{it\left(T_{n,\delta}^* - \frac{m_n}{k_n}\sum_{j=1}^{k_n} X_{j,\tau}^{n,\delta}\right)\right\}\right]\right) \rightarrow_w \mathcal{L}\left[\exp\left(c\int\left[e^{itx} - 1 - it\alpha_\tau(x)\right]dN^\delta(x)\right)\right]$$

where N^δ is a random measure with the property that for any Borel measurable disjoint sets, A, B, $N^\delta(A)$ and $N^\delta(B)$ are independent random variables with Poisson laws and parameters $\mu(A \cap [-\delta,\delta]^c)$ and $\mu(B \cap [-\delta,\delta]^c)$.

Proof. – To simplify the notation, given $t \in \Re$ let us define

$$f_t(x) := e^{itx} - 1 - it\alpha_\tau(x).$$

As a first step we prove the proposition when there exists a denumerable set $\{h_m, m \in \mathcal{N}\} \subset [-\delta,\delta]^c$, without finite accumulation points, such that $P[X_1^n \in \{0\} \cup \{h_1, h_2, ...\}] = 1$, $n \in \mathcal{N}$.

It is enough to show that

$$\mathcal{L}\left[\sum_{j=1}^{k_n} f_t(X_j^n)\right] \to_w \mathcal{L}\left[\int f_t(x)dN^\delta(x)\right], \tag{4}$$

but, under the assumed conditions, if we denote $N_m^n = \#\{j : X_j^n = h_m, j = 1, ..., k_n\}$, $m \in \mathcal{N}$ then

$$\sum_{j=1}^{k_n} f_t(X_j^n) = \sum_{m=1}^{\infty} N_m^n f_t(h_m),$$

and, as it is well known, $(N_1^n, N_2^n, ..., N_r^n) \to_w (N_1, N_2, ..., N_r)$ for every r, where $\{N_m\}_m$ are independent r.v.'s $\text{Pois}[\mu(h_m)\delta_1]$, $m \in \mathcal{N}$. Therefore, for every $r \in \mathcal{N}$,

$$\sum_{m=1}^{r} N_m^n f_t(h_m) \to_w \sum_{m=1}^{r} N_m f_t(h_m).$$

On the other hand,

$$\begin{aligned} P\left[\sum_{m=1}^{\infty} N_m^n f_t(h_m) \neq \sum_{m=1}^{r} N_m^n f_t(h_m)\right] &\leq P\left[\cup_{j=1}^{k_n}\{X_j^n \in \{h_{r+1}, h_{r+2}, ...\}\}\right] \\ &\leq k_n P[X_1^n \in \{h_{r+1}, h_{r+2}, ...\}] \\ &\to \sum_{m=r+1}^{\infty} \mu(h_m) < \infty, \end{aligned}$$

so that it is straightforward to show the weak convergence of $\sum_{m=1}^{\infty} N_m^n f_t(h_m)$ to the same weak limit as

$$w - \lim_{r \to \infty} \sum_{m=1}^{r} N_m f_t(h_m) = \sum_{m=1}^{\infty} N_m f_t(h_m) = \int f_t(x) dN^{\delta}(x).$$

To cover the general case let $\eta > 0$ and consider the denumerable partition of $\Re$, $\{A_k, k \in Z\}$, given by

$$A_k = \begin{cases} [-\delta, \delta], & \text{if } k = 0, \\ (\delta + (k-1)\eta, \delta + k\eta], & \text{if } k > 0, \\ [\delta + k\eta, \delta + (k+1)\eta), & \text{if } k < 0. \end{cases}$$

Let $a_0 = 0$, a_k, $k \neq 0$, be any value in A_k, and let us define $\tilde{X}_j^n := \sum_{k \in Z} a_k I_{A_k}(X_j^n)$, $j = 1, ..., k_n$ and $\tilde{Y}_j^{n,\delta} := \sum_{k \in Z} a_k I_{A_k}(Y_j^{n,\delta}) = \sum_{k \in Z} a_k I_{A_k}(Y_j^n)$, $j = 1, \ldots, m_n$. As a matter of fact these r.v.'s depend also on η but we suppress this dependence in the notation in order to simplify it. Thus $\{\tilde{Y}_1^{n,\delta}, \cdots, \tilde{Y}_{m_n}^{n,\delta}\}$ can be considered as a bootstrap sample obtained from $\{\tilde{X}_1^n, \cdots, \tilde{X}_{k_n}^n\}$ which, in turns, are r.v.'s which satisfy the hypothesis in the first part of this proof. Therefore if we define the random measure

$$N^{\eta,\delta}(A) := \sum_{k: a_k \in A} N_k^{\eta,\delta},$$

where $N_k^{\eta,\delta}$ are independent r.v.'s $\text{Pois}[\mu(A_k)\delta_1]$, then the sequence

$$\left\{ \mathcal{L}^* \left(\sum_{j=1}^{m_n} \tilde{Y}_j^{n,\delta} - \frac{m_n}{k_n} \sum_{j=1}^{k_n} \alpha_\tau(\tilde{X}_j^{n,\delta}) \right) \right\}_n$$

w-converges to the random law whose characteristic function is

$$\phi_{\eta,\delta}(t) := \exp \left[c \int f_t(x) dN^{\eta,\delta}(x) \right].$$

On the other hand, by construction, $|Y_j^{n,\delta} - \tilde{Y}_j^{n,\delta}| \leq \eta$, $\tilde{Y}_j^{n,\delta} = Y_j^{n,\delta}$, if $Y_j^{n,\delta} = 0$, and $|\alpha_\tau(X_j^{n,\delta}) - \alpha_\tau(\tilde{X}_j^{n,\delta})| \leq \eta$, if η is chosen (without loss of generality) satisfying $\eta = \frac{\tau - \delta}{k}$ for some $k \in \mathcal{N}$ in the case that

$\delta < \tau$. Therefore,

$$
\begin{aligned}
E\bigg| E^* \exp &\bigg\{ it\bigg(\sum_{j=1}^{m_n} Y_j^{n,\delta} - \frac{m_n}{k_n}\sum_{j=1}^{k_n} \alpha_\tau(X_j^{n,\delta})\bigg)\bigg\} \\
&- E^* \exp\bigg\{ it\bigg(\sum_{j=1}^{m_n} \tilde{Y}_j^{n,\delta} - \frac{m_n}{k_n}\sum_{j=1}^{k_n} \alpha_\tau(\tilde{X}_j^{n,\delta})\bigg)\bigg\}\bigg| \\
&\leq |t| E\bigg[E^* \bigg| \sum_{j=1}^{m_n} Y_j^{n,\delta} - \sum_{j=1}^{m_n} \tilde{Y}_j^{n,\delta}\bigg| \\
&\quad + \frac{m_n}{k_n}\bigg| \sum_{j=1}^{k_n}\bigg(\alpha_\tau(X_j^{n,\delta}) - \alpha_\tau(\tilde{X}_j^{n,\delta})\bigg)\bigg|\bigg] \\
&\leq \eta|t|\frac{m_n}{k_n} E\bigg[\sum_{j=1}^{k_n} I_{\{|X_j^n|\geq\delta\}} + \sum_{j=1}^{k_n} I_{\{\delta\leq|X_j^n|<\eta\}}\bigg] \\
&\leq 2\eta|t|\frac{m_n}{k_n} k_n P\bigg[|X_1^n| \geq \delta\bigg] \to_{n\to\infty} 2c\eta|t|\mu(-\delta,\delta)^c.
\end{aligned}
$$

Thus we obtain that the conditional characteristic functions of $T_{n,\delta}^* - \frac{m_n}{k_n}\sum_{j=1}^{k_n}\alpha_\tau(X_j^{n,\delta})$ converge in distribution to $w - \lim_{\eta\to 0+}\mathcal{L}(\phi_{\eta,\delta})$ if this limit exists, which follows from standard techniques. □

The classical truncation technique allows us the following final extension.

PROPOSITION 10. – *Let $\{X_j^n, j = 1, ..., k_n, n \in \mathcal{N}\}$ be an impartial array, let μ be a Lévy measure and let $\zeta, \tau > 0$ be such that $\mu\{-\zeta, \zeta, -\tau, \tau\} = 0$. Assume that $\mathcal{L}(S_n - a_n) \to_w c_\zeta$ Pois μ and that $\lim_n \frac{m_n}{k_n} = c \in (0, \infty)$.*

If ϕ_n^* denotes for the characteristic function of the conditional distribution of $S_n^* - m_n\bar{X}_{n,\tau}$ given $\{X_j^n, j = 1, ..., k_n\}$ then

$$
\mathcal{L}[\phi_n^*(t)] \to_w \mathcal{L}\bigg\{ \exp\bigg[c\int \big(e^{itx} - 1 - it\alpha_\tau(x)\big)N(dx)\bigg]\bigg\}, \textit{for every } t \in \Re
$$

where $\alpha_\tau(x) = xI_{[-\tau,\tau]}(x)$, and N is a random measure such that if A, B are two Borel measurable disjoint subsets of $(-\delta,\delta)^c$, $\delta > 0$, then $N(A)$ and $N(B)$ are independent Poisson random variables with parameters $\mu(A)$ and $\mu(B)$respectively.

Proof. – Let $0 < \delta < \tau$ and note that by (b.ii) in Proposition 3:

$$
\begin{aligned}
E\Bigg| E^* \exp\left\{ it\left(S_n^* - \frac{m_n}{k_n}\sum_{j=1}^{k_n} \alpha_\tau(X_j^n) \right) \right\} \\
- E^* \exp\left\{ it\left(T_{n,\delta}^* - \frac{m_n}{k_n}\sum_{j=1}^{k_n} \alpha_\tau(X_j^{n,\delta}) \right) \right\}\Bigg| \\
\leq |t| E\big[E^*|S_{n,\delta}^* - m_n \bar{X}_{n,\delta}|\big] \\
\leq |t|\big(E\big[E^*(S_{n,\delta}^* - m_n \bar{X}_{n,\delta})^2\big]\big)^{\frac{1}{2}} \\
= |t|\left(m_n \frac{k_n - 1}{k_n} E(X_{1,\delta}^n - EX_{1,\delta}^n)^2 \right)^{\frac{1}{2}}.
\end{aligned}
$$

Thus if we define

$$
\psi(t) := \int_{\Re - \{0\}} \big(e^{itx} - 1 - it\alpha_\tau(x)\big) dN(x)
$$

$$
\psi_\delta(t) := \int_{(-\delta,\delta)^c} \big(e^{itx} - 1 - it\alpha_\tau(x)\big) dN(x),
$$

it suffices to prove that $\lim_{t \to 0} \exp[c\psi(t)] = 1$ and that

$$
w - \lim_{\delta \to 0+} \psi_\delta(t) = \psi(t). \tag{5}
$$

Now, taking into account that

$$
\lim_{x \to 0} \frac{e^{itx} - 1 - it\alpha_\tau(x)}{x^2} = -\frac{t^2}{2}.
$$

it is possible to find a nondecreasing function $H(t) \geq 0$ such that

$$
|e^{itx} - 1 - it\alpha_\tau(x)| \leq H(t)x^2,
$$

Then by a standard uniform integrability argument, (5) will be proved if we show that

$$
w - \lim_{\delta \to 0+} \int_{(-\delta,\delta)^c} x^2 dN(x) = \int_{\Re - \{0\}} x^2 dN(x).
$$

For this we only need to show that the family $\{\int_{(-\delta,\delta)^c} x^2 dN(x)\}_{\delta > 0}$ is a Cauchy family in probability when δ approaches zero. This will be a consequence of

$$
E\int_{(-\delta,-\delta')\cup(\delta',\delta)} x^2 dN(x) = \int_{(-\delta,-\delta')\cup(\delta',\delta)} x^2 d\mu(x), \tag{6}
$$

because by the definition of a Lévy measure, $x^2 I_{(-1,1)}$ is μ-integrable and, if $0 < \delta' < \delta$, then

$$E\left|\int_{(-\delta,\delta)^c} x^2 N(dx) - \int_{(-\delta',\delta')^c} x^2 N(dx)\right| = E\int_{(-\delta,-\delta')\cup(\delta',\delta)} x^2 N(dx).$$

Moreover (6) arises by standard arguments from the obvious fact that if A is a Borel set such that $A \subset (-\delta, -\delta') \cup (\delta', \delta)$, then

$$E\int I_A(x)dN(x) = E[N(A)] = \mu(A) = \int I_A(x)d\mu(x).$$

Finally, since $H(t)$ is non-deacreasing, by applying the dominated convergence theorem, we obtain that $\lim_{t\to 0}\psi(t) = 0$. $\square$

We end this subsection with a theorem which merges all the results obtained until now related to the conditional asymptotic distribution of S_n^* for moderate resampling sizes, and covers the general case when the sums of the original r.v.'s in the impartial array w-converge to a general infinitely divisible distribution.

THEOREM 11. – *Let $\{m_n\}_n$ be a sequence of natural numbers such that $\lim_n \frac{m_n}{k_n} = c \in (0,\infty)$. Let $\{X_j^n, j = 1,\dots,k_n, n \in \mathcal{N}\}$ be an impartial array. Let μ be a Lévy measure and let $\zeta, \tau > 0$ be such that $\mu\{-\zeta, \zeta, -\sqrt{c}\tau, \sqrt{c}\tau\} = 0$. Assume that*

$$\mathcal{L}\left(\sum_{j=1}^{k_n} X_j^n - a_n\right) \to_w N(0,\sigma^2) * c_\zeta \text{ Pois } \mu,$$

where $\sigma \geq 0$.

Given $n \in \mathcal{N}$, let $\{Y_j^n, j = 1,\dots,m_n\}$ be a bootstrap sample taken from $\{X_j^n, j = 1,\dots,k_n\}$. If we define $S_n^* := \sum_{j=1}^{m_n} Y_j^n$ and $r_n := (m_n/k_n)^{1/2}$, then

$$\mathcal{L}^*[r_n^{-1}(S_n^* - m_n\bar{X}_{n,\tau r_n})] \to_w N(0,\sigma^2) * Q$$

in law, where Q is a random distribution whose (random) characteristic function is given by

$$\Phi(t) = \exp\left\{c\int (e^{itxc^{-1/2}} - 1 - itc^{-1/2}\alpha_{\sqrt{c}.\tau}(x))N(dx)\right\},$$

where α_τ and N are as described in Proposition 10.

Proof. – The r.v.'s S_n^* and $R_{n,\alpha}^* + T_{n,\alpha}^*$ coincide for every $\alpha > 0$, thus the result will follow from the w-convergence of the conditional characteristic functions of the r.v.'s $r_n^{-1}\big(R^*_{n,\delta_n} + T^*_{n,\delta_n} - m_n\bar{X}_{n,\tau r_n}\big)$, $n \in \mathcal{N}$, for some sequence $\{\delta_n\}_n$. Therefore let $\{\delta_n\}_n$ be a sequence obtained as in Proposition 2, and let us analyze the asymptotic behaviour of the conditional characteristic function of $S^*_{n,1} + S^*_{n,2}$, which we denote by ϕ^*_n, where

$$\begin{aligned} S^*_{n,1} &:= r_n^{-1}\big(R^*_{n,\delta_n} - m_n\bar{X}_{n,\delta_n}\big), \text{ and} \\ S^*_{n,2} &:= r_n^{-1}\big[T^*_{n,\delta_n} - m_n(\bar{X}_{n,\tau r_n} - \bar{X}_{n,\delta_n})\big]. \end{aligned}$$

Let $w_n := \#\{j : |Y^n_j| \le \delta_n\}$ and $\{\tilde{X}^n_j, j = 1, ..., c_n\} = \{X^n_j : |X^n_j| \le \delta_n\}$. Therefore $c_n := \#\{j : |X^n_j| \le \delta_n\}$ and the r.v.'s $S^*_{n,1}$ and $S^*_{n,2}$ are conditionally independent given $\{w_n,\ X^n_j,\ j = 1, ..., k_n\}$. Then

$$\begin{aligned} \phi^*_n(t) &= E^*\Big\{E^*\Big[e^{itS^*_{n,1}+itS^*_{n,2}}\Big/w_n\Big]\Big\} \\ &= E^*\Big\{E^*\Big[e^{itS^*_{n,1}}\Big/w_n\Big]E^*\Big[e^{itS^*_{n,2}}\Big/w_n\Big]\Big\}. \end{aligned}$$

Now, if $\{\tilde{Y}^n_j, j = 1, ..., w_n\}$ is a bootstrap sample of size w_n taken from $\{\tilde{X}^n_j, j = 1, ..., c_n\}$ and $\tilde{S}^*_n := r_n^{-1}\Big(\sum_{j=1}^{w_n} \tilde{Y}^n_j - \frac{w_n}{c_n}\sum_{j=1}^{c_n} \tilde{X}^n_j\Big)$, then the conditional distribution of $S^*_{n,1}$ given w_n and $\{X^n_j, j = 1, ..., k_n\}$ coincides with that of $\tilde{S}^*_n$ given $\{\tilde{X}^n_j, j = 1, ..., c_n\}$. Thus we will have that $E^*\big[e^{itS^*_{n,1}}/w_n\big] \to_p e^{-t^2\sigma^2/2}$ as soon as the conditional expectation of $e^{it\tilde{S}^*_n}$ given $\{\tilde{X}^n_j, j = 1, ..., c_n\}$, $\tilde{E}^*\Big[e^{it\tilde{S}^*_n}\Big]$, satisfies

$$\tilde{E}^*\Big[e^{it\tilde{S}^*_n}\Big] \to_p e^{-t^2\sigma^2/2}. \tag{7}$$

To prove (7) first note that $\lim_n \frac{c_n}{k_n} = \lim_n \frac{w_n}{m_n} = 1$ in probability because, given $\epsilon > 0$,

$$\begin{aligned} P\left[\left|\frac{w_n}{m_n} - 1\right| > \epsilon\right] &= P\left[\left|\frac{\#\{j : |Y^n_j| > \delta_n\}}{m_n}\right| > \epsilon\right] \\ &\le \frac{1}{\epsilon m_n} E\big[E^*\#\{j : |Y^n_j| > \delta_n\}\big] \\ &= \frac{1}{\epsilon k_n} E\big[\#\{j : |X^n_j| > \delta_n\}\big] = \epsilon^{-1}P[|X^n_1| > \delta_n], \end{aligned}$$

which converges to zero by (2). The proof for $\lim_n \frac{c_n}{k_n} = 1$ in probability is similar. From this we obtain, in particular, that $\lim_n c_n = \lim_n w_n = \infty$ and that $\lim_n \frac{c_n}{w_n} = c$ in probability.

Moreover, from a) in Proposition 1 and (2) it is trivial that $\{\tilde{X}^n_j,\ j = 1, ..., c_n\}$ is an infinitesimal array such that $\{\mathcal{L}(\sum_{j=1}^{k_n} \tilde{X}^n_j - k_n E\tilde{X}^n_1)\}_n$

w-converges to $N(0,\sigma^2)$. Therefore, from Proposition 7 we obtain (7). Now taking into account that the terms $E^*\left[e^{itS^*_{n,1}}/w_n\right]$, $E^*\left[e^{itS^*_{n,2}}/w_n\right]$ and $e^{-t^2\sigma^2/2}$ are bounded by 1, we have that

$$\lim_n \phi^*_n(t) = e^{-t^2\sigma^2/2}\lim_n E^*\left[e^{itS^*_{n,2}}\right].$$

Thus the theorem follows from Proposition 10 taking into account that the choice made for the sequence $\{\delta_n\}_n$ implies that $\mathcal{L}[T_{n,\delta_n} - E(S_{n,\tau} - R_{n,\delta_n})] \rightarrow_w c_\tau$ Pois μ. $\square$

3.3. Small resampling sizes

Finally let us consider the case $\frac{m_n}{k_n} \rightarrow 0$. The behaviour of $\mathcal{L}^*(S^*_n)$ without scaling constants is easily obtained in the following proposition.

PROPOSITION 12. – *Let us assume that $\{X^n_j, j=1,\ldots,k_n, n\in\mathcal{N}\}$ is an impartial array such that $\mathcal{L}(S_n - a_n) \rightarrow_w N(0,\sigma^2) * c_\tau Pois\ \mu$ and let $\frac{m_n}{k_n} \rightarrow 0$. Then $\mathcal{L}^*(S^*_n - \frac{m_n}{k_n}a_n) \rightarrow_w \delta_{\{0\}}$ in probability.*

Proof. – From the necessary conditions for the general CLT, for small δ

$$E\left[m_n E^*(Y^n_{1,\delta} - E^*Y^n_{1,\delta})^2\right] = m_n\frac{k_n - 1}{k_n}E(X^n_{1,\delta} - EX^n_{1,\delta})^2 \longrightarrow 0$$

and for every $\delta > 0$

$$E[m_n\mathcal{L}^*(Y^n_1)[-\delta,\delta]^c] = m_nE[P^*(|Y^n_1| > \delta)] = m_nP[|X^n_1| > \delta] \longrightarrow 0$$

Therefore $\frac{m_n}{k_n}(R_{n,\tau} - a_n) \rightarrow_P 0$ from b) in Proposition 1, taking into account that $E^*R^*_{n,\tau} = \frac{m_n}{k_n}R_{n,\tau}$ and that $\mathcal{L}(R_{n,\tau} - a_n) \rightarrow_w N(0,\sigma^2) * c_\tau$Pois $[\mu|\{|x| \leq \tau\}]$. $\square$

In Theorem 2.5 in Arcones and Giné [2] it is proved that if the triangular array we are considering is in fact a suitably normalized sequence of i.i.d.r.v.'s in the domain of attraction of a stable law, then it is possible to obtain a nondegerate limit in the previous proposition if we choose an appropriate sequence of normalizing constants for the bootstrap sequence. This suggests the possibility that this result also holds in our framework.

Next we provide sufficient conditions for this property to hold. Then we also include some examples to show that this is not always the case.

THEOREM 13. – *Let $\{X^n_j, j=1,\ldots,k_n, n\in\mathcal{N}\}$ be an impartial array. Let μ be a Lévy measure and let $\zeta > 0$ be such that $\mu\{-\zeta,\zeta\} = 0$. Assume that*

$$\mathcal{L}\left(\sum_{j=1}^{k_n} X^n_j - a_n\right) \rightarrow_w N(0,\sigma^2) * c_\zeta\ Pois\ \mu,$$

where $\sigma \geq 0$, that $\lim_n \frac{m_n}{k_n} = 0$ and also assume that there exists a sequence $\{r_n\}_n$ of real numbers, a sequence $\{h_n\}_n$ of natural numbers such that $\lim_n h_n = \infty$, and a Lévy measure ν satisfying

$$m_n \mathcal{L}(r_n^{-1} X_1^n)|\{|x| > \delta\} \to_w \nu|\{|x| > \delta\}, \tag{8}$$

for every δ such that $\nu\{-\delta, \delta\} = 0$, and that

$$\sup_{\eta > 0} \left| m_n P[r_n^{-1}|X_1^n| > \eta] - k_{h_n} P[|X_1^{h_n}| > \eta] \right| \leq t_n \tag{9}$$

where $\{t_n\}_n$ is a bounded sequence. Then, if $\tau > 0$ verifies $\nu\{-\tau, \tau\} = 0$,

$$\mathcal{L}^*\left(r_n^{-1}(S_n^* - m_n \bar{X}_{n,\tau r_n})\right) \to_w N(0, \sigma^2) * c_\tau Pois\, \nu \text{ in probability.}$$

Proof. – We employ the same scheme as in Theorem 11. Let $\{\delta_n\}_n$ be a sequence obtained as in Proposition 2 and, by (9), satisfying

$$\lim_n m_n \delta_n^2 P[|r_n^{-1} X_1^n| \geq \delta_n] = 0. \tag{10}$$

If we define

$$S_{n,1}^* := r_n^{-1}\left(R_{n,\delta_n r_n}^* - m_n \bar{X}_{n,\delta_n r_n}\right), \text{ and}$$
$$S_{n,2}^* := r_n^{-1}\left[T_{n,\delta_n r_n}^* - m_n(\bar{X}_{n,\tau r_n} - \bar{X}_{n,\delta_n r_n})\right],$$

it will be enough to show that

$$E^*\left[e^{itS_{n,1}^*}\right] \to_w e^{-t^2\sigma^2/2} \tag{11}$$

and that

$$E^*\left[e^{itS_{n,2}^*}\right] \phi_{\tau,\nu}(t) \tag{12}$$

where $\phi_{\tau,\nu}$ is the characteristic function of the law c_τ Pois ν.

Statement (9) implies that $\{r_n^{-1} X_j^n, j = 1, ..., k_n, n \in \mathcal{N}\}$ is an impartial array. Thus, according to the argument in Proposition 3, to prove (11) it is enough to show that

$$m_n E^*\left[(r_n^{-1} Y_1^n)_{\delta_n} - E^*(r_n^{-1} Y_1^n)_{\delta_n}\right]^2 \to_w \sigma^2 \tag{13}$$

and that

$$m_n P^*[|(r_n^{-1} Y_1^n)_{\delta_n}| > \epsilon] \to_w 0.$$

Since $\delta_n \to 0$, the last relation is trivial. With respect to (13) we have:

$$E\Big(m_n E^*\big[(r_n^{-1}Y_1^n)_{\delta_n} - E^*(r_n^{-1}Y_1^n)_{\delta_n}\big]^2\Big)$$
$$= m_n \frac{k_n - 1}{k_n} E\big[(r_n^{-1}X_1^n)_{\delta_n} - E(r_n^{-1}X_1^n)_{\delta_n}\big]^2$$

and

$$\begin{aligned}
&E\Big(m_n E^*\big[(r_n^{-1}Y_1^n)_{\delta_n} - E^*(r_n^{-1}Y_1^n)_{\delta_n}\big]^2\Big)\\
&= m_n \frac{k_n - 1}{k_n} E\big[(r_n^{-1}X_1^n)_{\delta_n} - E(r_n^{-1}X_1^n)_{\delta_n}\big]^2\\
&\le m_n E\big[(r_n^{-1}X_1^n)_{\delta_n} - EX_{1,\delta_n}^{h_n}\big]^2\\
&= m_n E\Big[(r_n^{-1}X_1^n - EX_{1,\delta_n}^{h_n})^2 I_{\{|r_n^{-1}X_1^n|<\delta_n\}}\Big]\\
&\quad + m_n [EX_{1,\delta_n}^{h_n}]^2 P[|r_n^{-1}X_1^n| \ge \delta_n]\\
&\le \int_{-\delta_n}^{\delta_n} (x - EX_{1,\delta_n}^{h_n})^2 d(k_{h_n} P_{X_1^{h_n}})(x)\\
&\quad + (2\delta_n)^2 t_n + m_n \delta_n^2 P[|r_n^{-1}X_1^n| \ge \delta_n]\\
&\le k_{h_n} E\big[X_{1,\delta_n}^{h_n} - EX_{1,\delta_n}^{h_n}\big]^2 + (2\delta_n)^2 t_n + m_n \delta_n^2 P[|r_n^{-1}X_1^n| \ge \delta_n]
\end{aligned}$$

where inequality (14) comes from (9). Now, if we apply the same reasoning to the term $k_{h_n} E\big[X_{1,\delta_n}^{h_n} - EX_{1,\delta_n}^{h_n}\big]^2$ we obtain that

$$\begin{aligned}
&k_{h_n} E\big[X_{1,\delta_n}^{h_n} - EX_{1,\delta_n}^{h_n}\big]^2\\
&\quad \le m_n E\big[(r_n^{-1}X_1^n)_{\delta_n} - E(r_n^{-1}X_1^n)_{\delta_n}\big]^2 + (2\delta_n)^2 t_n + k_{h_n} \delta_n^2 P[|X_1^{h_n}| \ge \delta_n]\\
&\quad \le k_{h_n} E\big[X_{1,\delta_n}^{h_n} - EX_{1,\delta_n}^{h_n}\big]^2 + 9\delta_n^2 t_n + 2m_n \delta_n^2 P\big[|r_n^{-1}X_1^n| \ge \delta_n\big].
\end{aligned}$$

By (10) we have that the third term in the last expression converges to zero and we obtain that

$$\begin{aligned}
\lim_n E\Big(m_n E^*\big[(r_n^{-1}Y_1^n)_{\delta_n} - E^*(r_n^{-1}Y_1^n)_{\delta_n}\big]^2\Big) &= \lim_n m_n \text{Var}\,[(r_n^{-1}X_1^n)_{\delta_n}]\\
&= \lim_n \text{Var}(X_{1,\delta_n}^{h_n}) = \sigma^2
\end{aligned}$$

by construction of the sequence $\{\delta_n\}_n$.

Moreover, from the expression for the variance of the sample variance and the inequality in the proof of Proposition 4, we have that

$$\begin{aligned}
&\text{Var}\Big(m_n E^*\big[(r_n^{-1}Y_1^n)_{\delta_n} - E^*(r_n^{-1}Y_1^n)_{\delta_n}\big]^2\Big)\\
&= m_n^2 \left(k_n^{-1} \text{Var}\big[(r_n^{-1}X_1^n)_{\delta_n} - E(r_n^{-1}X_1^n)_{\delta_n}\big]^2 + \frac{s_n}{k_n^2}\right)\\
&\le \frac{m_n^2}{k_n} 64\delta_n^2 \text{Var}\,[(r_n^{-1}X_1^n)_{\delta_n}] + \frac{s_n m_n^2}{k_n^2}
\end{aligned}$$

where $\{s_n\}_n$ is a bounded sequence. Thus both terms converge to zero proving (13) and, in consequence, we obtain (11).

In the same way, to show (12) we only need to check that if $\nu\{-\tau,\tau\} = 0$, then

$$m_n \mathcal{L}^*\left[(r_n^{-1}Y_1^n)^{\delta_n}\right] | \{|x| > \tau\} \to_w \nu | \{|x| > \tau\} \tag{15}$$

and that

$$\limsup_n E\Big[m_n E^*\left[\left((r_n^{-1}Y_1^n)^{\delta_n}\right)_\tau - E^*\left((r_n^{-1}Y_1^n)^{\delta_n}\right)_\tau\right]^2\Big] \le g(\tau), \tag{16}$$

where $\lim_{\tau\to 0} g(\tau) = 0$.

To show (15) note that

$$\begin{aligned}
&E\big(m_n\mathcal{L}^*\left[(r_n^{-1}Y_1^n)^{\delta_n}\right](\tau,\infty) - m_n\mathcal{L}\left[(r_n^{-1}X_1^n)^{\delta_n}\right](\tau,\infty)\big)^2 \\
&\quad = m_n^2 E\left(n^{-1}\sum_{j=1}^{k_n} I_{\{r_n^{-1}X_j^n > \tau\}} - P\left[r_n^{-1}X_1^n > \tau\right]\right)^2 \\
&\quad = \frac{m_n^2}{k_n} P[r_n^{-1}X_1^n > \tau](1 - P[r_n^{-1}X_1^n > \tau]) \to 0
\end{aligned}$$

by (8), taking into account that the first identity holds from a certain index onward because $\delta_n \to 0$. Similarly we have that

$$\lim_n E\big[m_n\mathcal{L}^*\left[(r_n^{-1}Y_1^n)^{\delta_n}\right](-\infty,-\tau) - m_n\mathcal{L}\left[(r_n^{-1}X_1^n)^{\delta_n}\right](-\infty,-\tau)\big]^2 = 0$$

and (15) is satisfied because, trivially,

$$w - \lim_n m_n\mathcal{L}\left[(r_n^{-1}X_1^n)^{\delta_n}\right] | \{|x| > \tau\} = w - \lim_n m_n\mathcal{L}\left(r_n^{-1}X_1^n\right) | \{|x| > \tau\}.$$

With respect to (16), from the same sort of computations leading to (14), we obtain

$$\begin{aligned}
&m_n E\left[\left((r_n^{-1}X_1^n)^{\delta_n}\right)_\tau - E\left((r_n^{-1}X_1^n)^{\delta_n}\right)_\tau\right]^2 \\
&\le \int_{(-\tau,-\delta_n)\cup(\delta_n,\tau)} \left(x - E[(X_1^{h_n})^{\delta_n}]_\tau\right)^2 d\Big(k_{h_n} P_{X_1^{h_n}}(x)\Big) \\
&\quad + 4\tau^2 t_n + m_n\tau^2 P[|r_n^{-1}X_1^n| \ge \tau].
\end{aligned}$$

As a result, we obtain (16) with

$$g(\tau) = \int_{-\tau}^{\tau} x^2 d\mu(x) + 4\tau^2 H + \tau^2\nu(-\tau,\tau)^c$$

where H is an upper bound for $\{t_n\}_n$, and the result is proved. □

Remark 13.1. – The previous result covers all cases in which the triangular array under consideration is a suitably normalized sequence of r.v.'s: Let us assume that there exists a sequence $\{X_n\}_n$ of i.i.d.r.v.'s and a sequence of positive numbers $\{a_n\}_n$ such that $X_j^n = X_j/a_n, j = 1, ..., n$. Then the hypotheses in Theorem 13 are trivially satisfied if we take $r_n = a_n/a_{m_n}$, $h_n = m_n$ and $\mu = \nu$, and, in consequence, we have that the bootstrap always works, with small resampling sizes, for sequences of normalized partial sums. In particular, Theorem 2.5 in Araujo and Giné [1] is included in the previous proposition. □

With the next example we show that it is not always possible to find a normalizing sequence to get a nondegenerate limit if $\lim_n \frac{m_n}{k_n} = 0$.

Example 14. – Let us assume that the triangular array we are considering satisfies

$$P[X_1^n = x] = \begin{cases} n^{-1}, & \text{if } x = 1, \\ 1 - \frac{1}{n}, & \text{if } x = 0. \end{cases}$$

It is well-known that if $k_n = n$ then $\mathcal{L}(S_n) \to_w$ Pois $\delta_{\{1\}}$, and that there exists a (non random) sequence $\{t_n\}_n$ of real numbers such that $\lim_n t_n n^{-1} = 0$ and that

$$P\big[\#\{j : X_j^n \neq 0\} \geq t_n \text{ i.o. }\big] = 0.$$

Therefore we can assume without loss of generality that $\#\{j : X_j^n \neq 0\} \leq t_n$ for every $n \in \mathcal{N}$. Now let us assume that $\lim_n m_n t_n n^{-1} = 0$. Then, trivially, we have that

$$\lim_n P^*[S_n^* = 0] = \lim_n \left(1 - \frac{\#\{j : X_j^n \neq 0\}}{n}\right)^{m_n} \geq \lim_n \left(1 - \frac{t_n}{n}\right)^{m_n} = 1;$$

so that no re-scaling of S_n^* can give a non-degenerate limit. □

Remark 14.1. – It could be argued that the asymptotic distribution of $\mathcal{L}^*(S_n^*)$ we obtained in the previous example is based on the fact that zero does not belong to the support of the Lévy measure associated to the problem. However this is not the reason because the same result could have been obtained for the following triangular array:

Let $\{h_n\}_n$ be a sequence of integers such that $h_n \to \infty$ and let us assume that

$$P[X_1^n = x] = \begin{cases} n^{-1}, & \text{if } x = i^{-1}, i = 1, 2, ..., h_n; \\ \frac{n-h_n}{n}, & \text{if } x = 0. \end{cases}$$

In this case $\lim_n nP[X_1^n = x] = 1$, if $x = i^{-1}, i = 1, 2, ...$ but the argument used in the previous example works to prove that

$\mathcal{L}^*(S_n^* - E^* R_{n,\tau}^*)$ is also degenerate for every sequence $\{h_n\}_n$ which converges to infinity slowly enough. $\square$

A question remains. Is the new Lévy measure ν in Theorem 13 really needed? We mean that, as in the case of the domain of attraction of an stable law, it could happen that the Lévy measure ν in that theorem coincides with the original Lévy measure μ. We include an example to show that this is not the general situation.

Example 15. – Given $n \in \mathcal{N}$ let us assume that $k_n = n$ and that

$$P[X_1^n = x] = \frac{1}{n}, \text{ where } x \in \{1, 2^{-1}, ..., n^{-1}\}.$$

In this case $\mathcal{L}(S_n - ES_{n,\tau}) \to_w c_\tau$ Pois μ where $\mu(A) = \#(A \cap \{1, 2^{-1}, ...\})$. Let $\delta > 0$. Then

$$nP[X_1^n > \delta] = \#\{i \leq n : i^{-1} > \delta\} = \left| \begin{array}{ll} n, & \text{if } \delta < n^{-1} \\ [\delta^{-1}]^*, & \text{if } \delta \geq n^{-1} \end{array} \right.$$

where $[x]^*$ denotes the greatest integer strictly lower than the real number x. Thus $[x]^*$ coincides with the integer part of x unless x is integer; in which case $[x]^* = x - 1$. On the other hand if $r_n > 0$ and $\delta \in (0, 1)$ then

$$m_n P[r_n X_1^n > \delta] = m_n \frac{1}{n} \#\{i \leq n : r_n i^{-1} > \delta\}$$

$$= \left| \begin{array}{ll} m_n, & \text{if } \delta < r_n n^{-1} \\ \frac{m_n}{n} \left[\frac{r_n}{\delta}\right]^* & \text{if } \delta \geq r_n n^{-1}. \end{array} \right.$$

Therefore, if $r_n n^{-1} \leq \delta < 1$ we have that

$$\frac{m_n}{n}\left(\frac{r_n}{\delta} - 1\right) \leq m_n P[r_n X_1^n > \delta] < \frac{m_n}{n}\frac{r_n}{\delta}$$

and, if we assume that $\lim_n \frac{m_n}{n} = 0$ and we take $r_n = \frac{n}{m_n}$ we obtain that $\lim_n m_n P[r_n X_1^n > \delta] = \delta^{-1}$. However, if we take $h_n = m_n$, then trivially

$$\sup_{\delta > 0} |m_n P[r_n X_1^n > \delta] - nP[X_1^n > \delta]| \leq \sup\left(\frac{m_n}{n}, 1\right).$$

and all the hypotheses in Theorem 13 are verified but in this case $\nu \neq \mu$. $\square$

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