ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 33, nº 5 (1997), p. 651-671
<http://www.numdam.org/item?id=AIHPB_1997__33_5_651_0>

Ann. Inst. Henri Poincaré,

Vol. 33, n° 5, 1997, p. 651-671 *Probabilités et Statistiques*

Comparisons between tail probabilities of sums of independent symmetric random variables

by

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ABSTRACT. – We show how estimates for the tail probabilities of sums of independent *identically distributed* random variables can be used to estimate the tail probabilities of sums of *non-identically distributed* independent symmetric random variables which are majorized by a single distribution in the sense of Gut's (1992) weak mean domination. As an application, we prove a weak one-sided extension of a law of large numbers of Chen (1978) to a non-identically distributed case and show how some of Gut's (1992) extensions of Hsu-Robbins type laws of large numbers follow from previously known identically distributed cases. We also extend some theorems of Klesov (1993) to the case of weak mean domination.

One intermediate result of independent interest is that if $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are two collections of independent symmetric random variables such that $P(|X_k| \geq \lambda) \leq P(|Y_k| \geq \lambda)$ for every λ and k, then $P(|Y_1 + \ldots + Y_n| \geq \lambda) \leq 2P(|X_1 + \ldots + X_n| \geq \lambda)$ for all λ.

RÉSUMÉ. – Nous montrons comment utiliser les estimées des probabilités des queues des sommes de variables aléatoires indépendantes et *identiquement distribuées* pour estimer celles des sommes de variables indépendantes, symétriques, mais *non identiquement distribuées*. Nous imposons que ces variables soient faiblement dominées en moyenne, dans

1991 *Mathematics Subject Classification:* 60 E 15, 60 F 05, 60 F 10, 60 F 15.

Key words and phrases. Tail probabilities of sums of independent symmetric random variables, weak mean domination, stochastic domination, regular covering, rates of convergence in the law of large numbers, Hsu-Robbins-Erdős laws of large numbers

le sens de Gut (1992), par une unique distribution. En application, nous adaptons à un cas non equidistribué, un côté de la loi des grands nombres de Chen (1992), et nous montrons comment certaines extensions, dues à Gut (1992), de la loi des grands nombres de type Hsu-Robbins, découlent de résultats précédents obtenus dans le cas equidistribué. Nous étendons aussi certains résultats de Klesov (1993) au cas de la domination faible en moyenne.

Nous obtenons un résultat intermédiaire qui présente un intérêt en lui-même: si $X_1, \ldots, X_n$ et $Y_1, \ldots, Y_n$ sont deux suites des variables aléatoires symétriques indépendantes, telles que $P(|X_k| \geq \lambda) \leq P(|Y_k| \geq \lambda)$ pour tout k, λ, alors $P(|Y_1 + \ldots + Y_n| \geq \lambda) \leq 2P(|X_1 + \ldots + X_n| \geq \lambda)$ pour tout λ.

1. THE MAIN RESULTS

We begin the present section by stating our main comparison inequality for the case of what Gut called "weak mean domination." We then discuss the applications of this inequality to Hsu-Robbins type laws of large numbers. We shall close the section by stating a result on the comparison of tail probabilities of sums of independent symmetric random variables under stochastic domination; this result is of some independent interest and is crucial to the proof of our main inequality. Then, in Section 2 we shall discuss the notion of regular covering, a notion that generalizes C. S. Kahane's [11] randomly sampled Riemann sums and is an important special case of the weak mean domination condition. In Sections 3 and 4 we shall prove the results of Section 1. Finally, in Section 5 we shall give a weak one-sided extension of a law of large numbers of Chen [3]. This will be proved via our main comparison inequality.

1.1. Weak mean domination and the main comparison inequality

Our primary interest is in collections of random variables X_i whose distributions are dominated in the following sense by the distribution of a single random variable X.

DEFINITION (Gut [8]). – Fix $K < \infty$. Then the random variables $X_1, \ldots, X_n$ are K-**weakly mean dominated** by a random variable X if

$$\frac{1}{n}\sum_{k=1}^{n} P(|X_k| \geq \lambda) \leq KP(|X| \geq \lambda), \qquad \forall \lambda > 0.$$

A special case is when for every k we have

$$P(|X_k| \geq \lambda) \leq KP(|X| \geq \lambda), \qquad \forall \lambda > 0.$$

This case was studied by Woyczyński [24], [25] and called **uniform boundedness of tail probabilities**. Another special case of weak mean domination is **regular covering** [18], which we shall discuss in Section 2.

Our main comparison inequality is as follows. Recall that a a random variable X is said to be **symmetric** if X and $-X$ have the same distribution.

THEOREM 1. – *Let $Y_1, \ldots, Y_n$ be independent symmetric random variables which are K-weakly mean dominated by some random variable X. Then there exist constants $C = C(K) < \infty$ and $\alpha = \alpha(K) > 0$ depending only on K such that*

$$P\left(\left|\sum_{k=1}^{n} Y_k\right| \geq \lambda\right) \leq CP\left(\left|\sum_{k=1}^{n} X_k\right| \geq \alpha\lambda\right),$$

for every positive λ, where $X_1, \ldots, X_n$ are independent copies of X.

The proof will be given in Section 4.

Remark 1. – If $K \geq 1$, our proof of Theorem 1 will show that we may take $C = C_0 K$ and $\alpha = \alpha_0/K$, where C_0 and α_0 are absolute constants independent of K. If $K \in \mathbb{Z}^+$ then our proofs show that we may take $C_0 = 16$ and $\alpha_0 = \frac{1}{2}$ in the above expressions for C and α. If $K \geq 1$ is not an integer, then it follows from the above expressions for the integer case that we may take $C_0 = 32$ and $\alpha_0 = \frac{1}{4}$ (just replace K by the smallest integer $\lceil K \rceil$ greater than or equal to K and note that $\lceil K \rceil \leq 2K$ if $K \geq 1$).

OPEN PROBLEM 1. – Is the choice of $\alpha = \alpha_0/K$ in Remark 1 optimal with respect to the order of dependence on K? If not, what then is an optimal choice of α with respect to the order of dependence on K?

OPEN PROBLEM 2. – Can we get any result similar to Theorem 1 for Banach space valued random variables, perhaps with some additional terms dependent on the geometry of the space and may be under some auxiliary conditions on this geometry?

In connection with Problem 2, please note Remark 5, in Section 4, below.

1.2. Applications to Hsu-Robbins type laws of large numbers

We have the following useful corollary of Theorem 1.

COROLLARY 1. – *Fix $K < \infty$ and any random variable X. Let $\{X_{nk}\}_{n \in \mathbb{Z}^+, 1 \leq k \leq k_n}$ be rowwise independent r.v.'s such that $X_{n1}, \ldots, X_{nk_n}$*

are K-weakly mean dominated by X for every fixed n. Let $S_n = X_{n1} + \cdots + X_{nk_n}$ and let T_n be the sum of k_n independent copies of X. Assume that a_n is a numerical sequence such that S_n/a_n tends to zero in probability as $n \to \infty$. Suppose that

$$\sum_{n=1}^{\infty} \tau_n P(|T_n| \geq \varepsilon a_n) < \infty, \qquad \forall \varepsilon > 0.$$

Then,

$$\sum_{n=1}^{\infty} \tau_n P(|S_n| \geq \varepsilon a_n) < \infty, \qquad \forall \varepsilon > 0. \tag{1.1}$$

Remark 2. – Assume $a_n = n^\alpha$, $\alpha > \frac{1}{2}$, $k_n = n$, and $E[|X|^{1/\alpha}] < \infty$. If $\alpha \leq 1$ then additionally assume that

$$\sum_{k=1}^{n} E[X_{nk}] = 0, \qquad \forall n.$$

Then S_n/n^α tends to 0 in probability by standard weak law of large numbers estimates (*see*, e.g., [6, pp. 105-106]).

Our proof of Corollary 1 will use the notion of symmetrization. Given a random variable X, let $X^s = X - \tilde{X}$ where $\tilde{X}$ is an independent copy of X; note that X^s is symmetric. Our symmetrizations will be implicitly chosen in such a way that the symmetrization of a sum of independent random variables will be the sum of the symmetrizations of the random variables, whenever we need this equality.

Proof of Corollary 1. – Let μ_n be a median of S_n. Since $S_n/a_n \to 0$ in probability, it follows that likewise $\mu_n/a_n \to 0$. Standard symmetrization inequalities (*see*, e.g., [15, §17.1.A]) imply that $X_{n1}^s, \ldots, X_{nk_n}^s$ are $2K$-weakly mean dominated by $2X$. Now, since

$$\sum_{n=1}^{\infty} \tau_n P(|2T_n| \geq \varepsilon a_n)$$

converges for every $\varepsilon > 0$, it follows by Theorem 1 that

$$\sum_{n=1}^{\infty} \tau_n P(|S_n^s| \geq \varepsilon a_n)$$

also converges for every $\varepsilon > 0$, since $S_n^s = X_{n1}^s + \ldots + X_{nk_n}^s$. Standard symmetrization inequalities then imply that

$$\sum_{n=1}^{\infty} \tau_n P(|S_n - \mu_n| \geq \varepsilon a_n) \tag{1.2}$$

converges for every $\varepsilon > 0$. Now fix $\varepsilon > 0$. For n sufficiently large we will have $\mu_n/a_n < \varepsilon/2$. But for such n we have

$$P(|S_n| \geq \varepsilon a_n) \leq P(|S_n - \mu_n| \geq \tfrac{1}{2}\varepsilon a_n).$$

From this and the convergence of (1.2), we obtain (1.1), as desired. □

COROLLARY 2. – *Suppose that there is a constant $K < \infty$ and a random variable X such that for every fixed n the variables $X_{n1}, \ldots, X_{nn}$ are independent and K-weakly mean dominated by X. Fix $\alpha > \frac{1}{2}$. Assume that the conditions of Remark 2 are satisfied and that*

$$\sum_{n=1}^{\infty} \tau_n \cdot \min\bigl(1, nP(|X| \geq \varepsilon n^{\alpha})\bigr) < \infty, \qquad \forall \varepsilon > 0. \tag{1.3}$$

Moreover, assume that at least one of the following auxiliary conditions also holds:

(*a*) $\lim_{n\to\infty} n^{-\theta}\tau_n < \infty$ *for some* $\theta > 0$, *and* $E[|X|^r] < \infty$ *for some* $r > \frac{1}{\alpha}$;

(*b*) *there is a slowly varying function L such that*

$$\sum_{n=1}^{\infty} \frac{\tau_n}{(L(n))^{\theta}} < \infty$$

for some $\theta > 0$ *and* $E[|X|^{1/\alpha}(L(|X|^{1/\alpha}))^{\nu}] < \infty$ *for some* $\nu > 0$;

(*c*) *for some* $\theta > 0$ *and some choice of numbers* $T_n \geq \sum_{k=1}^{n} k\tau_k$, *we have* $\sum_{n=m}^{\infty} \tau_n/n^{\theta} = O(T_m/m^{\theta+1})$ *and*

$$\sum_{n=1}^{\infty} T_n P((n-1)^{\alpha} \leq |X| < n^{\alpha}) < \infty.$$

Then,

$$\sum_{n=1}^{\infty} \tau_n P(|S_n| \geq \varepsilon n^{\alpha}) < \infty, \qquad \forall \varepsilon > 0.$$

See [20] for a converse result in the i.i.d. case.

Proof of Corollary 2. – This was in effect shown by Klesov [12] in the independent and identically distributed (i.i.d.) case. Klesov had the slightly stronger assumption that

$$\sum_{n=1}^{\infty} n\tau_n P(|X| \geq \varepsilon n^{\alpha}) < \infty, \qquad \forall \varepsilon > 0,$$

but his proofs can be easily modified to use the weaker (1.3) (*see* Theorem 2 in [20], together with the Remark after Theorem 1 of that paper). The general case then follows from the i.i.d. case together with Remark 2 and Corollary 1. □

OPEN PROBLEM 3. – Find the most general auxiliary condition on X and the τ_n (generalizing the disjunction of (a), (b) and (c), above) under which Corollary 2 holds.

Remark 3. – Klesov's proofs [12] can also be modified to prove our Corollary 2 directly.

Remark 4. – Theorems 2.1 and 5.1 of Gut [8] are special cases of our Corollary 2. To see this, it only suffices to note that if $\tau_n = n^r$ for $r \geq -1$ then condition (c) of Corollary 2 will be satisfied, at least providing the hypotheses of Gut's theorems hold. Moreover, one might recall that Gut [8] had noted that his Theorems 2.1 and 5.1 were generalizations to the weak mean domination case of results already known in the i.i.d. case. In light of Corollary 1 and Remark 2, the weak mean domination versions can thus also be derived from the original i.i.d. results.

Corollary 2 is known as a Hsu-Robbins [10] type law of large numbers. Partial bibliographies on such laws of large numbers may be found in [14] and [17]; *see* also [7].

1.3. A comparison inequality for stochastic domination

We now give a result which will be essential to the proof of Theorem 1 and which is of some independent interest. As usual, we say that a random variable Ξ is **stochastically dominated** by a random variable Υ (possibly defined on a different probability space) if

$$P(\Xi \geq \lambda) \leq P(\Upsilon \geq \lambda),$$

for all $\lambda \in \mathbb{R}$.

THEOREM 2. – *Let $X_1, \ldots, X_n$ be independent symmetric random variables, and suppose $Y_1, \ldots, Y_n$ are also independent symmetric random variables. Assume that for every i we have $|Y_i|$ stochastically dominated by $|X_i|$. Then*

$$P(|Y_1 + \cdots + Y_n| \geq \lambda) \leq 2P(|X_1 + \cdots + X_n| \geq \lambda), \qquad (1.4)$$

for every positive λ.

The proof will be given in Section 3.

Theorem 2 generalizes a lemma of Klesov [12, Lemma 2] which gave the same result in the special case where the X_i are i.i.d. while $Y_i = X_i \cdot 1_{\{|X_i|<a\}}$ for all i. While is easy to see that the constant 2 in (1.4) is optimal, it is not as clear whether it is still optimal in Klesov's special case.

Theorem 2 also bears some resemblance to comparison inequalities of Burkholder [2] for differential subordination of martingales. However, it does not appear that there is any easy logical implication, in either direction, between our result and Burkholder's inequalities.

2. REGULAR COVERING

DEFINITION (Pruss [18]). – The random variables $X_1, \ldots, X_n$ **regularly cover** (the distribution of) a random variable X if

$$\frac{1}{n}\sum_{k=1}^{n} E[g(X_k)] = E[g(X)],$$

for each bounded Borel measurable function g.

This condition is equivalent to asserting that the distribution function of X is the average of the distribution functions of the X_k. It is also equivalent to asserting that the characteristic function of X is the average of the characteristic functions of the X_k. It is clear that if $X_1, \ldots, X_n$ regularly cover X, then $X_1, \ldots, X_n$ are 1-weakly mean dominated by X. Hence, Theorem 1 has some content for the case of regular covering. A result similar to Theorem 1 for the case of regular covering but with better control over the constants will be given as Proposition 1 in Section 4, below.

We have the following generic example.

Example 1. – Let $X_1, \ldots, X_n$ be independent random variables and let A be a random variable independent of them and uniformly distributed on the set $\{1, \ldots, n\}$. Then, $X_1, \ldots, X_n$ are a regular cover of X_A. The easy verification of this is left to the reader (cf. equation (4.9), below).

Example 1 shows that given a set of independent random variables, they always regularly cover *some* random variable. Indeed, this fact is completely clear since we may always choose a random variable whose distribution function is the average of the distribution functions of the original random variables. This construction of a regularly covered random variable will be very important in our work.

Moreover, the above construction, together with Theorem 1 (or the somewhat superior Proposition 1), shows that given any independent symmetric random variables $X_1, \ldots, X_n$, we may estimate the tail probabilities of $X_1 + \ldots + X_n$ by the tail probabilities of $\tilde{X}_1 + \ldots + \tilde{X}_n$, where the latter sum is a sum of independent and *identically distributed* random variables chosen so that the common distribution function of the $\tilde{X}_i$ equals the average of the distribution functions of the X_k.

We now present the following trivial example of regular covering.

Example 2. – Let $X_1, \ldots, X_n$ be identically distributed. Then they regularly cover X_1.

Finally, we present an example which may help to build some intuition as to the meaning of regular covering; it is precisely the following example which has provided the original motivation for the definition of regular covering in [18].

Example 3. – Let f be measurable on $[0, 1]$. For each fixed $n \in \mathbb{Z}^+$, let $x_{n1}, \ldots, x_{nn}$ be independent random variables such that x_{nk} is uniformly distributed over $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ for $1 \leq k \leq n$. Then, for any bounded Borel function g we have

$$\frac{1}{n} \sum_{k=1}^{n} E[g(f(x_{nk}))] = \frac{1}{n} \sum_{k=1}^{n} n \int_{\frac{k-1}{n}}^{\frac{k}{n}} g(f(x))\, dx = \int_0^1 g(f(x))\, dx.$$

Thus $f(x_{n1}), \ldots, f(x_{nn})$ form a regular cover of f, where f is considered a random variable on the probability space $[0, 1]$ equipped with Lebesgue measure. Note that the averaged partial sum

$$R_n f \stackrel{\text{def}}{=} \frac{1}{n} \sum_{k=1}^{\infty} f(x_{nk}),$$

is a randomly sampled Riemann sum. These Riemann sums were introduced by C. S. Kahane [11]. Questions concerning their convergence to the Lebesgue integral of f are addressed in [11] and, more fully, in [18].

3. THE PROOF OF THEOREM 2

The following simple and well-known coupling lemma (*see*, e.g., [23, p. 162]) will be needed.

LEMMA 1. – *Let Ξ and Υ be two positive random variables, possibly defined on different probability spaces, such that Ξ is stochastically dominated by Υ. Then, there exists a probability space (Ω, P) and random variables Ξ^* and Υ^* on Ω such that Ξ^* and Ξ have the same distribution, Υ^* and Υ have the same distribution, and $\Xi^* \leq \Upsilon^*$ with probability* 1.

Given this, we can prove Theorem 2.

Proof of Theorem 2. – By Lemma 1, since $|Y_k|$ is stochastically dominated by $|X_k|$, we may assume that we are given two sets of independent positive random variables $y_1, \ldots, y_n$ and $x_1, \ldots, x_n$ such that $y_k \leq x_k$ almost surely for every k and such that $|Y_k|$ and $|X_k|$ have the same distribution as y_k and x_k, respectively. (Note that y_k will of course *not* be in general independent of x_k.) Let $\varepsilon_1, \ldots, \varepsilon_n$ be i.i.d. Rademacher random variables with $P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = \frac{1}{2}$, and with $\varepsilon_1, \ldots, \varepsilon_n$ independent of $y_1, \ldots, y_n, x_1, \ldots, x_n$. Let $\tilde{X}_k = \varepsilon_k x_k$ and $\tilde{Y}_k = \varepsilon_k y_k$. Then the distributions of $\tilde{X}_k$ and $\tilde{Y}_k$, respectively, are the same as those of X_k and Y_k, respectively. It thus suffices to show that

$$P(|\tilde{Y}_1 + \ldots + \tilde{Y}_n| \geq \lambda) \leq 2P(|\tilde{X}_1 + \ldots + \tilde{X}_n| \geq \lambda), \tag{3.1}$$

for all positive λ. The simple proof of (3.1) given below was kindly communicated to the author by Professor Stephen J. Montgomery-Smith. The technique in this proof is well known (*see* for instance [13, Proposition 1.2.1] or [16, Corollary 5]). The author's original proof was much more complicated. Conditioning on $x_1, \ldots, x_n, y_1, \ldots, y_n$ we may assume that in fact the x_k and y_k are constants, with $0 \leq y_k \leq x_k$ for all k. Let $\alpha_k = y_k/x_k$ for $1 \leq k \leq n$, where $0/0 = 1$. Note that $0 \leq \alpha_k \leq 1$ for all k. Reordering our random variables, we may assume that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. Let $\sigma_n = \alpha_n$, and put $\sigma_k = \alpha_k - \alpha_{k+1}$ for $k < n$. Then,

$$\begin{aligned}\left|\sum_{k=1}^{n} \tilde{Y}_k\right| &= \left|\sum_{k=1}^{n} \alpha_k \tilde{X}_k\right| = \left|\sum_{k=1}^{n}\sum_{i=k}^{n} \sigma_i \tilde{X}_k\right| = \left|\sum_{i=1}^{n} \sigma_i \sum_{k=1}^{i} \tilde{X}_k\right| \\ &\leq \left(\sum_{i=1}^{n} \sigma_i\right) \sup_{1 \leq i \leq n} \left|\sum_{k=1}^{i} \tilde{X}_k\right| = \alpha_1 \sup_{1 \leq i \leq n} \left|\sum_{k=1}^{i} \tilde{X}_k\right| \leq \sup_{1 \leq i \leq n} \left|\sum_{k=1}^{i} \tilde{X}_k\right|,\end{aligned}$$

since $\alpha_1 \leq 1$. Inequality (3.1) then follows from this and from Lévy's inequality. □

4. PROOF OF THE MAIN THEOREM

4.1. Some auxiliary results

For the proof of Theorem 1 we need some auxiliary results. First recall Lévy's inequality. If the X_i are independent and symmetric, then

$$P(\sup_{1 \leq k \leq n} |X_1 + \ldots + X_k| \geq \lambda) \leq 2P(|X_1 + \ldots + X_n| \geq \lambda), \tag{4.1}$$

for all $\lambda \geq 0$.

We will also need to use the very simple result that

$$P(|X_1 + \cdots + X_m| \geq \lambda) \leq \sum_{k=1}^{m} P(|X_k| \geq \lambda/m), \tag{4.2}$$

for every positive λ and for any random variables $X_1, \ldots, X_m$, with no symmetry or independence assumptions being needed.

The following result is essentially due to Montgomery-Smith [16] and will later allow us to assume that X is symmetric in Theorem 1, at the expense of a change of constants.

LEMMA 2. – *Let $X_1, \ldots, X_n$ be i.i.d. random variables. Let $\varepsilon_1, \ldots, \varepsilon_n$ be i.i.d. Rademacher random variables with $P(\varepsilon_1 = 1) = P(\varepsilon_1 = -1) = \frac{1}{2}$ and with the ε_k independent of all the X_i. Then, there is an absolute constant $c \in (0, \infty)$ such that*

$$P(|\varepsilon_1 X_1 + \ldots + \varepsilon_n X_n| \geq \lambda) \leq cP(|X_1 + \ldots + X_n| \geq \lambda/c),$$

for every $\lambda \geq 0$.

Proof. – Let $e_1, \ldots, e_n$ be any real numbers with $|e_k| \leq 1$ for all k. Then, by an inequality of Montgomery-Smith [16, Corollary 5], if the X_k are independent and identically distributed, it follows that

$$P(|e_1 X_1 + \ldots + e_n X_n| \geq \lambda) \leq cP(|X_1 + \ldots + X_n| \geq \lambda/c),$$

for every $\lambda \geq 0$, where $c \in (0, \infty)$ is an absolute constant. Hence,

$$P\big(|\varepsilon_1 X_1 + \ldots + \varepsilon_n X_n| \geq \lambda \bigm| \mathcal{E}\big) \leq cP(|X_1 + \ldots + X_n| \geq \lambda/c),$$

where $\mathcal{E}$ is the σ-field generated by $\varepsilon_1, \ldots, \varepsilon_n$. Taking the unconditional expectation of both sides we obtain the desired inequality. □

4.2. The special case of regular covering

The only other thing we now need for the proof of Theorem 1 is the following result which is of some independent interest.

PROPOSITION 1. – *Suppose that $Y_1, \ldots, Y_n$ are symmetric independent random variables which form a regular cover of a random variable $\tilde{Y}$ (which itself will then automatically be symmetric). Then*

$$P(|Y_1 + \cdots + Y_n| \geq \lambda) \leq 8P(|\tilde{Y}_1 + \cdots + \tilde{Y}_n| \geq \lambda/2),$$

for every positive λ, where $\tilde{Y}_1, \ldots, \tilde{Y}_n$ are independent copies of $\tilde{Y}$.

OPEN PROBLEM 4. – Does Proposition 1 hold without any symmetry assumptions, perhaps with different constants?

Remark 5. – Proposition 1 does work for Banach space valued random variables, since the proof uses nothing deeper than Lévy's inequality which does work in Banach spaces. However, some of our other results (notably, Theorem 2) do not adapt as readily.

OPEN PROBLEM 5. – Determine optimal combinations of constants C and α such that

$$P(|Y_1 + \cdots + Y_n| \geq \lambda) \leq CP(|\tilde{Y}_1 + \cdots + \tilde{Y}_n| \geq \alpha\lambda),$$

in the setting of Proposition 1. Could we for instance choose a $C < \infty$ such that $\alpha = 1$ works?

Remark 6. – In connection with this problem, the anonymous referee has made the following important observation. Let $\varepsilon_1, \ldots, \varepsilon_n$ be i.i.d. Rademacher random variables ($P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$). Let $Y_1 = \varepsilon_1$ and put $Y_2 \equiv Y_3 \equiv \cdots \equiv Y_n \equiv 0$. Then, $Y_1, \ldots, Y_n$ are a regular cover of a variable $\tilde{Y}_1$ such that $P(\tilde{Y}_1 = 0) = 1 - (1/n)$ and $P(\tilde{Y}_1 = 1) = P(\tilde{Y}_1 = -1) = 1/(2n)$. Let $\tilde{Y}_2, \ldots, \tilde{Y}_n$ be independent copies of $\tilde{Y}_1$. Put $T_n = Y_1 + \cdots + Y_n$ and $\tilde{T}_n = \tilde{Y}_1 + \cdots + \tilde{Y}_1$. Let $S_k = \varepsilon_1 + \ldots + \varepsilon_k$, and suppose that K_n is a binomial random variable, independent of the ε_i, and with parameters $(n, 1/n)$. Then, T_n has the same distribution as ε_1, while $\tilde{T}_n$ has the same distribution as S_{K_n}. As $n \to \infty$, the variable K_n converges in distribution to a Poisson variable with parameter n^{-1}. It easily follows that $P(S_{K_n} = 0)$ then converges to

$$p \stackrel{\text{def}}{=} e^{-1} \sum_{j=0}^{\infty} \frac{1}{(2j)!} \cdot \frac{2^{-2j} \cdot (2j)!}{(j!)^2} \approx 0.4658$$

But $P(|T_n| \geq 1) = 1$ and for every $\theta > 0$ we have

$$P(|\tilde{T}_n| \geq \theta) \leq 1 - P(\tilde{T}_n = 0) = 1 - P(S_{K_n} = 0),$$

and hence it follows that for every $\alpha > 0$, the constant C in Problem 6 must satisfy $C \geq (1 - p)^{-1} > 1.87$. The author is most grateful to the anonymous referee for this remark.

4.3. Reduction of Theorem 1 to the regular covering case

Assume Proposition 1 for now. We shall write $\lceil x \rceil$ for the smallest integer greater than or equal to x, and we shall put $\lfloor x \rfloor = -\lceil -x \rceil$. The following proof will reduce the general case of Theorem 1 to the regular covering case of Proposition 1.

Proof of Theorem 1. – Assume that the hypotheses of Theorem 1 are verified. Replacing X_k by $\varepsilon_k X_k$, where the ε_k are as in Lemma 2, we may assume that the X_k are symmetric, at the expense of a change in constants.

We may assume K is a positive integer, replacing K by $\lceil K \rceil$ if necessary. Let $N = Kn$. The variables $Y_1, \ldots, Y_n$ are given. Define the random variables $Y_k \equiv 0$ for $n < k \leq N$. Let $\tilde{X}_1, \ldots, \tilde{X}_N$ be independent identically distributed random variables such that $Y_1, \ldots, Y_N$ are a regular cover of $\tilde{X}_1$, *i.e.*, such that the distribution function of $\tilde{X}_1$ is the average of the distribution functions of $Y_1, \ldots, Y_N$. By Proposition 1 we then have

$$P(|Y_1 + \ldots + Y_N| \geq \lambda) \leq 8P(|\tilde{X}_1 + \ldots + \tilde{X}_N| \geq \lambda/2). \tag{4.3}$$

But of course,

$$Y_1 + \ldots + Y_n = Y_1 + \ldots + Y_N, \tag{4.4}$$

since $Y_k \equiv 0$ for $k > n$.

Now, $X_1, \ldots, X_n$ are independent symmetric identically distributed random variables. Let $X_{n+1}, X_{n+2}, \ldots, X_N$ be independent copies of X_1 such that

$$X_1, \ldots, X_n, X_{n+1}, X_{n+2}, \ldots, X_N$$

are all independent. I claim that $|\tilde{X}_k|$ is stochastically dominated by $|X_k|$ for every $k \in \{1, \ldots, N\}$. By identical distribution, it suffices to check this for $k = 1$. But using regular covering, the vanishing of Y_i for $i > n$ and the assumption of K-weak mean domination, we have

$$\begin{aligned} P(|\tilde{X}_1| \geq \lambda) &= \frac{1}{N} \sum_{i=1}^{N} P(|Y_i| \geq \lambda) = \frac{1}{N} \sum_{i=1}^{n} P(|Y_i| \geq \lambda) \\ &= \frac{n}{N} \cdot \frac{1}{n} \sum_{i=1}^{n} P(|Y_i| \geq \lambda) \\ &\leq \frac{n}{N} \cdot KP(|X_1| \geq \lambda) = P(|X_1| \geq \lambda), \end{aligned}$$

for any $\lambda > 0$, since $N = Kn$. Hence, indeed, $|\tilde{X}_k|$ is stochastically dominated by $|X_k|$ for all k. Applying Theorem 2 we see that

$$P(|\tilde{X}_1 + \ldots + \tilde{X}_N| \geq \lambda) \leq 2P(|X_1 + \ldots + X_N| \geq \lambda), \tag{4.5}$$

for every $\lambda > 0$. But $X_1 + \ldots + X_N$ is actually the sum of K independent copies of $X_1 + \ldots + X_n$, since $N = Kn$. Thus, by (4.2) we have

$$P(|X_1 + \ldots + X_N| \geq \lambda) \leq KP(|X_1 + \ldots + X_n| \geq \lambda/K), \tag{4.6}$$

for all $\lambda > 0$. Combining (4.3)-(4.6) we see that

$$P(|Y_1 + \ldots + Y_n| \geq \lambda) \leq 16KP(|X_1 + \ldots + X_n| \geq (2K)^{-1}\lambda),$$

for all $\lambda > 0$, as desired. $\square$

4.4. Proof of the regular covering case

The proofs of Proposition 1 and of Lemma 3 given below are simpler than the author's original proofs and have yielded better numerical constants. These improvements are due to the referee.

The proof of Proposition 1 depends heavily on the following very simple combinatorial lemma whose proof we include for completeness. Write $|U|$ for the cardinality of a set U.

LEMMA 3. – *If $A_1, \ldots, A_n$ are i.i.d. random variables with values in $\{1, \ldots, n\}$ and each value taken on with equal probability, then*

$$P\big(|\{A_1, \ldots, A_n\}| \geq \lceil n/2 \rceil\big) > \tfrac{1}{2}. \tag{4.7}$$

We shall write $[m] = \{1, \ldots, m\}$ for $m \in \mathbb{Z}_0^+$.

Proof of Lemma 3. – Let $\mathrm{Img}A$ be the random set $\{A_1, \ldots, A_n\}$. Let $\mathcal{S}$ be the collection of all the subsets of $\{1, \ldots, n\}$ which have cardinality $\lfloor n/2 \rfloor$. Then,

$$\begin{aligned} P(|\mathrm{Img}A| < \lceil n/2 \rceil) &\leq \sum_{U \in \mathcal{S}} P(\mathrm{Img}A \subseteq U) = \sum_{U \in \mathcal{S}} \prod_{k=1}^{n} P(A_k \in U) \\ &\leq |\mathcal{S}| \cdot 2^{-n} = \binom{n}{\lfloor n/2 \rfloor} \cdot 2^{-n} \leq \tfrac{1}{2}, \end{aligned}$$

since $P(A_k \in U) = |U|/n \leq \frac{1}{2}$, for all k. Inequality (4.7) follows immediately.

We will now give a proof of Proposition 1, thereby completing the proof of Theorem 1.

Proof of Proposition 1. – Put $n_1 = \lceil n/2 \rceil$. Let $\mathfrak{A} = [n]^n$, so that an element A of $\mathfrak{A}$ is a sequence $(A_1, \ldots, A_n)$ in $[n]$. For any $A \in \mathfrak{A}$, let

Img$A$ be the set $\{A_1, \ldots, A_n\}$, and let $\nu(A)$ be the cardinality of Img$A$. Define $\mathfrak{H}$ to be the set of elements A in $\mathfrak{A}$ with $\nu(A) \geq n_1$.

We now define a certain involution ϕ of $\mathfrak{H}$ as follows. Fix $A \in \mathfrak{H}$. Note that $n - \nu(A) \leq n_1$. Let $k(A) \geq 0$ be the smallest positive integer such that $|\{A_1, \ldots, A_{k(A)}\}| = n - \nu(A)$. Then define ρ_A to be the unique increasing one-to-one map of the set $\{A_1, \ldots, A_{k(A)}\} \subset \mathbb{Z}^+$ onto the set $[n]\backslash$Img$A$. For $i \in [n]$, if $A_i \in \{A_1, \ldots, A_{k(A)}\}$, then let $(\phi(A))_i = \rho_A(A_i)$; otherwise, let $(\phi(A))_i = A_i$. It is easy to see that $\nu(\phi(A)) = \nu(A)$, and hence ϕ maps $\mathfrak{A}$ into itself. In fact, ϕ is an involution. To see this, it suffices to note that if $B = \phi(A)$, then $k(B) = k(A)$ and $\{B_1, \ldots, B_{k(B)}\} = [n]\backslash$Img$A$, so that $\rho_B = \rho_A^{-1}$, and it easily follows that $\phi(B) = A$. Note also that if $A \in \mathfrak{H}$, then (Img$A$) $\cup$ (Img$\phi(A)$) $= [n]$. Extend ϕ to an involution of $\mathfrak{A}$ by defining $\phi(A) = A$ for $A \in \mathfrak{A}\backslash\mathfrak{H}$.

Let A be a random element of $\mathfrak{A}$, where each element of $\mathfrak{A}$ is taken to have equal probability n^{-n}. Lemma 3 then says that $P(A \in \mathfrak{H}) > \frac{1}{2}$. Let $\overline{A} = \phi(A)$. Since we have (Img$A$)$\cup$(Img$\overline{A}$) $= [n]$ for $A \in \mathfrak{H}$, it follows that

$$P((\mathrm{Img}A) \cup (\mathrm{Img}\overline{A}) = [n]) > \tfrac{1}{2}. \tag{4.8}$$

Note also that $\overline{A}$ is a random element of $\mathfrak{A}$ with the same distribution as A since ϕ is an involution of $\mathfrak{A}$ (although evidently A and $\overline{A}$ are not independent).

Let $\{Y_{i,j}\}_{i\in[2n],j\in[n]}$ be an array of independent random variables such that $Y_{i,j}$ has the same distribution as Y_j for every i and j. Assume that the random element $A \in \mathfrak{A}$ is defined on the same probability space as the array $\{Y_{i,j}\}$ and that it is independent of this array. For $i \in [n]$, define the random variables

$$X_i = Y_{i,A_i}$$

and

$$\overline{X}_i = \overline{Y}_{(i+n),\overline{A}_i}.$$

Then, the X_i are independent random variables. Moreover, for any bounded Borel function f we have

$$\begin{aligned} E[f(X_i)] &= \sum_{j=1}^{n} P(A_i = j) E[f(X_i) \mid A_i = j] \\ &= \frac{1}{n} \sum_{j=1}^{n} E[f(Y_j)] = E[f(\tilde{Y})], \end{aligned} \tag{4.9}$$

since Y_j and $Y_{i,j}$ have the same distribution and where we have also used the choice of $\tilde{Y}$ and the definition of regular covering. Hence the X_i are independent copies of $\tilde{Y}$ (cf. Example 1). Likewise, the $\overline{X}_i$ are independent copies of $\tilde{Y}$, since A and $\overline{A}$ have the same distribution.

Put $S_n = X_1 + \ldots + X_n$ and $\overline{S}_n = \overline{X}_1 + \ldots + \overline{X}_n$. Let F be the event that $(\mathrm{Img}A) \cup (\mathrm{Img}\overline{A}) = [n]$. I now claim that

$$P(|Y_1 + \ldots + Y_n| \geq \lambda) \leq 2P(|S_n + \overline{S}_n| \geq \lambda \mid F), \tag{4.10}$$

for all $\lambda \geq 0$. To see this, we condition on A, suppose that we are in F and define $\alpha(i) = A_i$ if $i \leq n$ and $\alpha(i) = \overline{A}_{i-n}$ for $n+1 \leq i \leq 2n$. Since we are in F, it follows that $\{\alpha(1), \ldots, \alpha(2n)\} = [n]$. Remember that we are conditioning on the value of A. Let $U \subseteq [2n]$ be any set with the property that $|U| = n$ and $\{\alpha(i) : i \in U\} = [n]$. Note that

$$S_n + \overline{S}_n = \sum_{i=1}^{2n} Y_{i,\alpha(i)} = \sum_{i \in U} Y_{i,\alpha(i)} + \sum_{i \in [2n] \setminus U} Y_{i,\alpha(i)}.$$

Observe that the two sums here are independent and symmetric, conditionally on A, assuming we are in F. Thus, by Lévy's inequality (4.1),

$$P\left(\left|\sum_{i \in U} Y_{i,\alpha(i)}\right| \geq \lambda \,\middle|\, A\right) \leq 2P(|S_n + \overline{S}_n| \geq \lambda \mid A), \tag{4.11}$$

on F. Moreover,

$$P\left(\left|\sum_{i \in U} Y_{i,\alpha(i)}\right| \geq \lambda \,\middle|\, A\right) = P(|Y_1 + \ldots + Y_n| \geq \lambda), \tag{4.12}$$

on F, because $Y_{i,\alpha(i)}$ has the same distribution as $Y_{\alpha(i)}$, the $Y_{i,j}$ are all independent and are independent of A, while $\{\alpha(i) : i \in U\} = [n]$ and $|U| = n$. Inequality (4.10) follows from (4.11) and (4.12).

From (4.10) and (4.2) we now conclude that

$$\begin{aligned} P(|Y_1 + \ldots + Y_n| \geq \lambda) &\leq 2P(|S_n| \geq \lambda/2 \mid F) + 2P(|\overline{S}_n| \geq \lambda/2 \mid F) \\ &\leq 2P(F)^{-1}[P(|S_n| \geq \lambda/2) + P(|\overline{S}_n| \geq \lambda/2)] \\ &\leq 8P(|S_n| \geq \lambda/2), \end{aligned}$$

since $P(F) > \frac{1}{2}$ by (4.8), and since S_n and $\overline{S}_n$ both have the same distribution. Since S_n is a sum of n independent copies of $\tilde{Y}$, we are done. $\square$

5. A WEAK ONE-SIDED EXTENSION OF A LAW OF LARGE NUMBERS OF CHEN

Let

$$\beta(r,t) = \frac{2t(r-1)}{2r-t}$$

and

$$\kappa(r,t) = \frac{2^{\beta(r,t)/2}\Gamma\left(\frac{1+\beta(r,t)}{2}\right)}{(r-1)\Gamma(\frac{1}{2})}.$$

Chen [3] then proved the following result.

THEOREM A (Chen [3]). – *Fix r and t such that $2 \le t < 2r \le 2t$. Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. random variables such that $E[X_1] = 0$, $E[X_1^2] = 1$ and $E[|X_1|^t] < \infty$. Then,*

$$\lim_{\varepsilon \to 0+} \varepsilon^{\beta(r,t)} \sum_{n=1}^{\infty} n^{r-2} P(|S_n| > \varepsilon n^{r/t}) = \kappa(r,t), \tag{5.1}$$

where $S_n = X_1 + \cdots + X_n$.

This extended an earlier result of Heyde [9] who had proved the same thing in the special case where $r = t = 2$. Heyde's result, in turn, was a significant sharpening of the following result of Hsu and Robbins [10].

THEOREM B (Hsu and Robbins [10]). – *Let $X_1, X_2, \ldots$ be i.i.d. random variables such that*

$$E[X_1] = 0 \qquad \textit{and} \qquad E[X_1^2] < \infty. \tag{5.2}$$

Then

$$\sum_{n=1}^{\infty} P(|S_n| > \varepsilon n) < \infty, \qquad \forall \varepsilon > 0. \tag{5.3}$$

Erdős [4], [5] showed that in fact (5.2) is necessary for (5.3). A result closely related to Heyde's and providing a two sided estimate of the infinite sum in (5.3) valid for all $\varepsilon > 0$ in the i.i.d. case can be found in [19]. As noted before, partial bibliographies on Hsu-Robbins-Erdős laws of large numbers may be found in [14] and [17] (*see* also [7]).

Remark 7. – Note that we cannot hope to get a full result like Chen's theorem, or even Heyde's theorem, in the case of regular covering. To see this, let f be the function on $[0,1]$ which is identically 1 on $[0,\frac{1}{2}]$ and identically -1 on $(\frac{1}{2},1]$. Let the x_{nk} be as in Example 3. Then, $f(x_{n1}),\ldots,f(x_{nn})$ are a regular cover of f, while $E[f^2] = 1$ and $E[f] = 0$.

However, if we put $S_n = f(x_{n1}) + \cdots + f(x_{nn}) = nR_n f$ then we see that for n even we have S_n vanishing with probability 1, while for n odd we have $S_n = f(x_{nk})$ with probability 1 where $k = (n+1)/2$. Then if we put $r = t = 2$, the left hand side of (5.1) will in our case be

$$\lim_{e \to 0+} \varepsilon^2 \sum_{n=1}^{\infty} P(|S_n| > \varepsilon n).$$

If n is even then $P(|S_n| > \varepsilon n) = 0$. If n is odd then $P(|S_n| > \varepsilon n) = 1_{\{1 > \varepsilon n\}}$, so that the left hand side of (5.1) is no bigger than

$$\limsup_{\varepsilon \to 0+} \varepsilon^2 \sum_{n=1}^{\lfloor 1/\varepsilon \rfloor} 1 \le \lim_{\varepsilon \to 0+} \varepsilon = 0,$$

where $\lfloor x \rfloor$ denotes the largest integer not exceeding x, and thus Theorem A cannot hold in this case.

In light of Remark 7, we have little reason to hope for anything more than a one-sided inequality in the general case of regular covering. However, it may be possible to get something more under the auxiliary assumption that each of the random variables $X_{n1}, \ldots, X_{nn}$ which regularly cover X has mean zero, and not just that X has mean zero as in our counter example.

The main result of the present section is as follows and constitutes a partial answer to a question Professor Dominik Szynal asked the author.

THEOREM 3. – *Fix r and t such that $2 \le t < 2r \le 2t$. Fix $K < \infty$. For each n let $X_{n1}, \ldots, X_{nn}$ be independent random variables which are K-weakly mean dominated by X. Assume that $\sum_{k=1}^{n} E[X_{nk}] = 0$, $E[X^2] = 1$ and $E[|X|^t] < \infty$. Then,*

$$\limsup_{\varepsilon \to 0+} \varepsilon^{\beta(r,t)} \sum_{n=1}^{\infty} n^{r-2} P(|S_n| > \varepsilon n^{r/t}) \le C < \infty,$$

where $S_n = X_{n1} + \cdots + X_{nn}$ and C is a constant depending only on K, r and t.

OPEN PROBLEM 6. – Suppose moreover that $X_{n1}, \ldots, X_{nn}$ actually form a regular cover of X for every n. Can we in that case put $C = \kappa(r, t)$? Can we at least do this if $r = t = 2$? Failing that, what is the best value of C for the case of regular covering?

While on the subject of the results of Heyde and Chen, we note that Szynal [22] has shown that the assumption of independence in the Hsu-Robbins [10] theorem (*see* Theorem B, above) can be relaxed to quadruplewise independence, but not to pairwise independence. (However, it is not known whether, under quadruplewise independence, condition (5.2) is necessary for (5.3).) It is not hard to see with Szynal's methods [22] that under the assumption of quadruplewise independence we may obtain analogues of Theorem B even in the case of K-weak mean domination. We have not, however, been able to do this via the methods of the present paper, and, moreover, we have the following question.

PROBLEM 7. – Can we replace the independence of $X_{n1}, \ldots, X_{nn}$ in Theorem 3 by quadruplewise independence?

The answer is not even known in the simplest case where $r = t = 2$ and all the random variables are identically distributed. In the identically distributed cases the proofs of Heyde [9] and Chen [3] use the central limit theorem, but unfortunately the central limit theorem need not hold for quadruplewise independent random variables [21].

Remark 8. – Note that it seems not unlikely that Theorem 3 could also be proved via an estimate of Bikelis [1] as in [18] (*see* also [19]), but given Chen's [3] result and our Theorem 1, it appears to be easier to proceed as we do in the present paper.

Before we give our proof of Theorem 3, we need a simple lemma. Given a statement P, we let $1_{\{P\}}$ equal 1 when P is true and we set $1_{\{P\}}$ equal to 0 when P is false.

LEMMA 4. – *Under the conditions of Theorem* 1, *we have*

$$\varepsilon^{\beta(r,t)} \sum_{n=1}^{\infty} n^{r-2} 1_{\{|\mu(S_n)| > \varepsilon n^{r/t}\}} \leq C_1,$$

for every $\varepsilon > 0$, where C_1 depends only on K, r and t, and where $\mu(S_n)$ is any median of S_n.

Proof of Lemma 4. – We have

$$\begin{aligned} P(|S_n| > \varepsilon n^{r/t}) &\leq \varepsilon^{-2} n^{-2r/t} E[S_n^2] \\ &= \varepsilon^{-2} n^{-2r/t} \mathrm{Var} S_n \\ &= \varepsilon^{-2} n^{-2r/t} \sum_{k=1}^{n} \mathrm{Var} X_{nk} \\ &\leq \varepsilon^{-2} n^{-2r/t} \sum_{k=1}^{n} E[X_{nk}^2] \\ &\leq \varepsilon^{-2} n^{-2r/t} \cdot nKE[X^2] \\ &= \varepsilon^{-2} n^{1-(2r/t)} K. \end{aligned} \tag{5.4}$$

The first equality follows from the fact that $E[S_n] = 0$; the second equality comes from the independence of $X_{n1}, \ldots, X_{nn}$. The last *in*equality in (5.4) followed from the definition of K-weak mean domination and the fact that for any random variable Y we have

$$E[Y^2] = \int_0^\infty 2\lambda P(|Y| \geq \lambda)\, d\lambda.$$

Now, if $P(|S_n| > \varepsilon n^{r/t}) \leq \frac{1}{3}$, then certainly $|\mu(S_n)| \leq \varepsilon n^{r/t}$ (any fraction less than $\frac{1}{2}$ will do in place of $\frac{1}{3}$). Thus by (5.4) we see that $|\mu(S_n)| \leq \varepsilon n^{r/t}$ providing

$$n \geq n_0 \stackrel{\text{def}}{=} \left(\frac{\varepsilon^2}{3K}\right)^{\frac{1}{1-(2r/t)}} = \left(\frac{\varepsilon^2}{3K}\right)^{\frac{-t}{2r-t}}.$$

Thus,

$$\begin{aligned} \sum_{n=1}^\infty n^{r-2} 1_{\{|\mu(S_n)| > \varepsilon n^{r/t}\}} &\leq \sum_{n=1}^{\lfloor n_0 \rfloor} n^{r-2} \leq C(r) n_0^{r-1} \\ &\leq C(r)\left(\frac{\varepsilon^2}{3K}\right)^{\frac{-t(r-1)}{2r-t}} \\ &= C(r)\cdot(3K)^{\beta(r,t)/2}\varepsilon^{-\beta(r,t)}, \end{aligned}$$

where $C(r) < \infty$ depends only on $r > 1$. The desired result follows immediately upon setting $C_1 = C(r)\cdot(3K)^{\beta(r,t)/2}$. $\square$

Proof of Theorem 3. – Note that

$$P(|S_n - \mu_n| > \varepsilon n^{r/t}) \leq 2P(|S_n^s| > \tfrac{1}{2}\varepsilon n^{r/t}),$$

by standard symmetrization inequalities (*see*, e.g., [15, §17.1.A]). On the other hand,

$$P(|S_n| > \varepsilon n^{r/t}) \leq P(|S_n - \mu_n| > \tfrac{1}{2}\varepsilon n^{r/t}) + 1_{\{|\mu_n| > \frac{1}{2}\varepsilon n^{r/t}\}}.$$

Hence in light of Lemma 4, we need only prove that

$$\limsup_{\varepsilon \to 0+} \varepsilon^{\beta(r,t)} \sum_{n=1}^\infty n^{r-2} P(|S_n^s| > \tfrac{1}{4}\varepsilon n^{r/t}) \leq C_2 = C_2(r,t,K) < \infty. \quad (5.5)$$

Let $X' = \varepsilon_1 |X|$, where ε_1 is a Rademacher random variable independent of X, with $P(\varepsilon_1 = 1) = P(\varepsilon_1 = -1) = \frac{1}{2}$. Clearly $|X| = |X'|$ with probability 1, and it follows from standard symmetrization inequalities (*see*, e.g., [15, §17.1.A]) that $X^s_{n1}, \ldots, X^s_{nn}$ are $2K$-weakly mean dominated by $2X'$. Theorem 1 then implies that

$$P(|S^s_n| > \tfrac{1}{4}\varepsilon n^{r/t}) \le C_3 P(|2S'_n| > \tfrac{1}{4}\alpha\varepsilon n^{r/t}), \tag{5.6}$$

where $S'_n = X'_1 + \cdots + X'_n$, for $X'_1, \ldots, X'_n$ independent copies of X', and where C_3 and α depend only on K. Since $E[X'] = 0$ and $E[(X')^2] = E[X^2] = 1$, by using a scaled version of Theorem A we see that

$$\lim_{\varepsilon \to 0+} \varepsilon^{\beta(r,t)} \sum_{n=1}^{\infty} n^{r-2} P(|2S'_n| > \tfrac{1}{4}\alpha\varepsilon n^{r/t}) = \left(\frac{8}{\alpha}\right)^{\beta(r,t)} \kappa(r,t).$$

In light of (5.6), we conclude that (5.5) holds with $C_2 = C_3 \cdot (8\alpha^{-1})^{\beta(r,t)}$, as desired. □

ACKNOWLEDGEMENTS

The research was partially supported by Professor J. J. F. Fournier's NSERC Grant #4822. Most of the research was done at the University of British Columbia but some was done while the author was visiting Maria Curie-Sklodowska University, Lublin, Poland. The author would like to thank Maria Curie-Skłodowska University and Professor Dominik Szynal for their warm hospitality. He would also like to thank Professor Szynal and Professor Stephen J. Montgomery-Smith for a number of interesting discussions, and in particular he would like to thank the latter for the greatly simplified proof of Theorem 2 included in the present version of the paper.

Finally, the author would like to thank the referee for a number of useful suggestions, including a simpler proof (with a better constant) of Proposition 1 which leads to better numerical constants in Theorem 1.

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(Manuscript received April 1st, 1996;
Revised version May 12, 1997.)