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A local limit theorem on the semi-direct product of $\mathbb{R}^{*+}$ and $\mathbb{R}^d$

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ABSTRACT. – Let G be the semi-direct product of $\mathbb{R}^{*+}$ and $\mathbb{R}^d$ and μ a probability measure on G. Let μ^{*n} be the nth power of convolution of μ. Under quite general assumptions on μ, one proves that there exists $\rho \in]0, 1]$ such that the sequence of Radon measures $(\frac{n^{3/2}}{\rho^n}\mu^{*n})_{n\geq 1}$ converges weakly to a non-degenerate measure; furthermore, if μ_2^{*n} is the marginal of μ^{*n} on $\mathbb{R}^d$, the sequence of Radon measures $(\frac{\sqrt{n}}{\rho^n}\mu_2^{*n})_{n\geq 1}$ converges weakly to a non-degenerate measure.

Key words: Random walk, local limit theorem.

RÉSUMÉ. – Soit G le groupe produit semi-direct de $\mathbb{R}^{*+}$ et de $\mathbb{R}^d$ et μ une mesure de probabilité sur G. On note μ^{*n} la $n^{\text{ième}}$ convolée de μ. Sous des hypothèses assez générales sur μ, on établit l'existence d'un réel $\rho \in]0, 1]$ tel que la suite de mesures de Radon $(\frac{n^{3/2}}{\rho^n}\mu^{*n})_{n\geq 1}$ converge vaguement vers une mesure non nulle; de plus, si μ_2^{*n} est la marginale de μ^{*n} sur $\mathbb{R}^d$, la suite $(\frac{\sqrt{n}}{\rho^n}\mu_2^{*n})_{n\geq 1}$ converge vaguement vers une mesure non nulle.

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I. INTRODUCTION

Fix a norm $||.||$ on $\mathbb{R}^d, d \geq 1$, and consider the connected group G of transformations

$$\begin{aligned} g: \quad & \mathbb{R}^d \to \mathbb{R}^d \\ & x \mapsto g.x = ax + b \end{aligned}$$

where $(a, b) \in \mathbb{R}^{*+} \times \mathbb{R}^d$.

Let a (resp. b) be the projection from G on $\mathbb{R}^{*+}$ (resp. on $\mathbb{R}^d$). Consequently, any transformation $g \in G$ is denoted by $(a(g), b(g))$ (or $g = (a, b)$ when there is no ambiguity); for example, $e = (1, 0)$ is the unit element of G.

The group G is also the semi-direct product of $\mathbb{R}^{*+}$ and $\mathbb{R}^d$ with the composition law

$$\forall g = (a, b), \quad \forall g' = (a', b') \in G, \qquad gg' = (aa', ab' + b).$$

Recall that G is a non unimodular solvable group with exponential growth and let m_D be the right Haar measure on $G : m_D(da\ db) = \frac{da\ db}{a}$. Note that if $d = 1$, the group G is the affine group of the real line.

Let μ be a probability measure on G, μ^{*n} its n^{th} power of convolution, $\tilde{\mu}$ the image of μ by the map $g = (a, b) \mapsto \tilde{g} = (\frac{1}{a}, \frac{b}{a})$ and $\overline{\mu}$ the image of μ by the map $g \mapsto g^{-1}$. If λ is a positive measure on $\mathbb{R}^d$, $\mu * \lambda$ denotes the positive measure on $\mathbb{R}^d$ defined by $\mu * \lambda(\varphi) = \int_{G \times \mathbb{R}^d} \varphi(g.x)\ \mu(dg)\ \lambda(dx)$ for any Borel function φ from $\mathbb{R}^d$ into $\mathbb{R}^+$. Finally, δ_x is the Dirac measure at the point x.

In the present paper, we prove under suitable hypotheses that μ satisfies a local limit theorem: there exists a sequence $(\alpha_n)_{n \geq 0}$ of positive real numbers, depending only on the group when μ is centered, such that the sequence $(\alpha_n\ \mu^{*n})_{n \geq 0}$ converges weakly to a non-degenerate measure. This problem has already been tackled by Ph. Bougerol in [5] where he established local limit theorems on some solvable groups with exponential growth, typically the groups NA which occur in the Iwasawa decomposition of a semi-simple group. The affine group of the real line is the simplest example of such a group. In this particular case, Ph. Bougerol proved that, for a class R of centered probability measures μ satisfying some invariance properties, the sequence $(n^{3/2} \mu^{*n})_{n \geq 0}$ converges weakly to a non-degenerate measure on G. His method is roughly the following one : if μ satisfies some invariance properties, it can be lifted on the associated semi-simple group in a measure m_μ (not necessarily bounded) which is bi-invariant under the action of a maximal and compact subgroup. In a second step, using the theory of Guelfand pairs, he showed that the measure m_μ satisfies an analogue of the local limit theorem established in [4]. The aim of the present paper is to obtain such a local limit theorem when the measure μ does not belong to the class R.

This work is also related with the work by N.T. Varopoulos, L. Saloff-Coste and T. Coulhon [19] where there are precise estimates for the heat kernel on a Lie group which is not necessarily unimodular. More recently, N. T. Varopoulos [17] has considered locally compact and nonunimodular groups and has obtained an upperbound for the asymptotic behaviour of the convolution powers μ^{*n} of a probability measure μ which has a continuous density ϕ_μ with respect to the left Haar measure and satisfying some condition at infinity; in [18], he gives a condition on the Lie algebra of an amenable Lie group which characterizes the decay rate at infinity of the heat kernel.

Now, let us introduce some hypotheses on μ

HYPOTHESIS A1. – *There exists $\alpha > 0$ such that*

$$\int_G (\exp(\alpha|\text{Log } a(g)|) + \|b(g)\|^\alpha)\ \mu(dg) < +\infty$$

HYPOTHESIS A2. – $\int_G \text{Log } a(g)\ \mu(dg) = 0.$

HYPOTHESIS A3. – *The probability measure μ has a density ϕ_μ with respect to the Haar measure m_D on G and there exist β and q in $]1, +\infty[$ such that* $\int_0^1 \sqrt[q]{\int_{\mathbb{R}} \phi_\mu^q(a,b)db}\ \frac{da}{a^\beta} < +\infty.$

HYPOTHESIS A3 (bis). – *The image* Log μ_1 *of μ by the application $g = (a,b) \mapsto$* Log a *is aperiodic on $\mathbb{R}$, the support of μ is included in $\mathbb{R}^{*+} \times (\mathbb{R}^+)^d$ and there exists $\gamma > 0$ such that* $\int_G \|b\|^{-\gamma}\mu(da\ db) < +\infty.$

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $g_n = (a_n, b_n), n = 1, 2, \cdots$ be G-valued independent and identically distributed random variables with distribution μ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $\mathcal{F}_n$ the σ-algebra generated by the variables $g_1, g_2, \cdots, g_n$. For any $n \geq 1$, set $G_1^n = g_1 \cdots g_n = (A_1^n, B_1^n)$; a direct computation gives $A_1^n = a_1 a_2 \cdots a_n$ and $B_1^n = \sum_{k=1}^n a_1 a_2 \cdots a_{k-1} b_k$.

THEOREM A. – *Suppose that the probability measure μ satisfies Hypotheses* A1, A2 *and either* A3 *or* A3 (*bis*).

Then, the sequence of finite measures $(n^{3/2}\mu^{*n})_{n \geq 0}$ converges weakly to a non-degenerate Radon measure ν_0 on G.

In other words, for any continuous functions φ *and* ψ *with compact support on* $\mathbb{R}^{*+}$ *and* $\mathbb{R}^d$ *respectively, the sequence*

$$\left(n^{3/2}\mathbb{E}\left[\varphi(a_1\cdots a_n)\psi\left(\sum_{k=1}^{n} a_1\cdots a_{k-1}b_k\right)\right]\right)_{n\geq 1}$$

converges as n *goes to* $+\infty$*; furthermore, one can choose* φ *and* ψ *such that the limit of this sequence is not zero.*

The following theorem deals with the behaviour as n goes to $+\infty$ of the variables B_1^n.

THEOREM B. – *Suppose that the probability measure* μ *satisfies Hypotheses* A1, A2 *and either* A3 *or* A3 (bis). *For any* $n \geq 1$ *denote by* μ_2^{*n} *the image of* μ^{*n} *by the map* $g = (a, b) \mapsto b \in \mathbb{R}^d$.

Then, the sequence of finite measures $(\sqrt{n}\mu_2^{*n})_{n\geq 0}$ *converges weakly to a non-degenerate Radon measure on* $\mathbb{R}^d$.

In other words, for any continuous function ψ *with compact support on* $\mathbb{R}^d$*, the sequence*

$$\left(\sqrt{n}\mathbb{E}\left[\psi\left(\sum_{k=1}^{n} a_1\cdots a_{k-1}b_k\right)\right]\right)_{n\geq 1}$$

converges as n *goes to* $+\infty$*; furthermore, one can choose* ψ *such that the limit of this sequence is not zero.*

Observe that the limit measure in Theorem A should satisfy $\mu * \nu = \nu * \mu = \nu$. Using L. Elie's results [7], we prove under additionnal assumptions that this equation has an unique solution (up to a multiplicative constant) in the space of Radon measure on G and we obtain an explicit form of this solution. Using a ratio-limit theorem due to Y. Guivarc'h [11], the measure ν_0 of theorem A may be identified, up to a multiplicative constant. More precisely, we have the

THEOREM C. – *Suppose that Hypotheses* A1, A2 *and* A3 *hold and assume the additionnal conditions*

C1. *the density* ϕ_μ *of* μ *is continuous with compact support*

C2. $\phi_\mu(e) > 0$

Then, the measure ν_0 *of theorem* A *may be decomposed as follows*

$$\nu_0 = (\delta_1 \otimes \lambda) * \overline{\left(\frac{da}{a} \otimes \lambda_1\right)}$$

where λ (*respectively* λ_1) *is, up to a multiplicative constant, the unique Radon measure on* $\mathbb{R}^d$ *which satisfies the convolution equation* $\mu * \lambda = \lambda$ (resp. $\overline{\mu} * \lambda_1 = \lambda_1$).

Furthermore, for any positive and continuous function φ, $\varphi \not\equiv 0$, *with compact support in* G, *we have* $\nu_0(\varphi) > 0$ *and*

$$\mu^{*n}(\varphi) \sim \frac{\nu_0(\varphi)}{n^{3/2}} \quad as \quad n \to +\infty.$$

When μ is not centered (that is when $\int_G \operatorname{Log} a(g)\, \mu(dg) \neq 0$) we bring back the study to the centered case using the *Laplace transform* of Log μ_1.

THEOREM D. – *Let* μ *be a probability measure on* G *satisfying Conditions*

A′1. *there exists* $\alpha > 0$ *such that for any* $t \in \mathbb{R}$: *the integral* $\int_G (\exp(t|\operatorname{Log}(a(g))|) + \|b(g)\|^\alpha)\mu(dg)$ *is finite.*

A′2. $\int_G \operatorname{Log}\ a(g)\mu(dg) \neq 0$, $\mu\{g \in G : a(g) < 1\} > 0$ *and* $\mu\{g \in G : a(g) > 1\} > 0$.

Then, there exists a unique $t_0 \in \mathbb{R}$ *and* $\rho(\mu) \in]0, 1[$ *such that*

$$\int_G a(g)^{t_0} \mu(dg) = \inf_{t \in \mathbb{R}} \int_G a(g)^t \mu(dg) = \rho(\mu).$$

Moreover, suppose that μ *satisfies either Hypothesis* A3 (bis) *or the following assumption*

A′3. μ *has the density* ϕ_μ *with respect to the Haar measure* m_D *on* G *and there exist* $q \in]1, +\infty[$ *and* $\beta \in]1 - t_0, +\infty[$ *such that* $\int_0^1 \sqrt[q]{\int_{\mathbb{R}} \phi_\mu^q(a, b) db}\, \frac{da}{a^\beta} < +\infty$.

Then, the sequence of finite measures $(\frac{n^{3/2}}{\rho(\mu)^n} \mu^{*n})_{n \geq 1}$ *weakly converges to a non-degenerate Radon measure on* G. *Moreover, if* μ_2^{*n} *is the image of* μ^{*n} *by the map* $g = (a, b) \mapsto b \in \mathbb{R}^d$, *then the sequence of finite measures* $(\frac{\sqrt{n}}{\rho(\mu)^n} \mu_2^{*n})_{n \geq 1}$ *weakly converges to a non-degenerate Radon measure on* $\mathbb{R}^d$.

Let us briefly explain what the *Laplace transform* of Logμ_1 means and connections between Hypotheses A1, A2, A3 and A′1, A′2, A′3. Under Condition A′1, the function $L : t \to \int_G a(g)^t \mu(dg)$ is well defined on $\mathbb{R}$; since it is strictly convex and $\lim_{t \to \pm\infty} L(t) = +\infty$ (this last fact follows by Hypothesis A′2) there exists a unique $t_0 \in \mathbb{R}$ such that

$$\int_G a(g)^{t_0} \mu(dg) = \inf_{t \in \mathbb{R}} \int_G a(g)^t \mu(dg) = \rho(\mu).$$

Equalities $L'(t_0) = 0$, $L(0) = 1$ and $L'(0) = \int_G \operatorname{Log}(a(g))\mu(dg) \neq 0$ imply $\rho(\mu) \in]0, 1[$. Let us thus consider the probability measure

$\mu_{t_0}(dg) = \frac{1}{\rho(\mu)} a(g)^{t_0} \mu(dg)$; one checks that if μ satisfies Hypotheses A$'$1, A$'$2 and either A$'$3 or A3 (bis) then μ_{t_0} satisfies Hypotheses A1, A2 and either A3 or A3 (bis) so that one may apply Theorem A.

There are some close connections between Theorems A and B and the asymptotic behaviour of the probability of non-extinction for branching processes in a random environment. For example, let $(X_n, Y_n)_{n\geq 1}$ be a sequence of $\mathbb{R}^2$-valued independent and identically distributed random variables and set $S_0 = 0$ and $S_n = X_1 + \cdots + X_n, n \geq 1$. Following [1] and [14], the probability of non-extinction for branching processes in a random environment is closely related to the quantities $\mathbb{E}\left[\frac{e^{-aS_n}}{\sum_{k=0}^{n-1} e^{-S_k} Y_{k+1}}\right]$ with $0 \leq a < 1$. As a consequence of Theorems A and B, one obtains the

COROLLARY. – *Suppose that*

(i) $\forall n \geq 1$ $\mathbb{E}[X_n^2] < +\infty$ *and* $\mathbb{E}[X_n] = 0$

(ii) *there exists* $C > 0$ *such that* $\forall n \geq 1$, $\mathbb{P}[Y_n \geq C] = 1$.

Then, the sequence $\left(\sqrt{n}\mathbb{E}\left[\frac{1}{\sum_{k=0}^{n-1} e^{-S_k} Y_{k+1}}\right]\right)_{n\geq 1}$ *converges to a non zero limit.*

Moreover, for any $0 < a < 1$, *the sequence*

$$\left(n^{3/2}\mathbb{E}\left[\frac{e^{-aS_n}}{\sum_{k=0}^{n-1} e^{-S_k} Y_{k+1}}\right]\right)_{n\geq 1}$$

converges to a non zero limit.

The first assertion of this corollary is due to Kozlov [14] and is an easy consequence of Theorem B. The second assertion has been recently proved by Y. Guivarc'h and Q. Liu [12]; it is also a direct consequence of theorem A, the only thing to check being that one may replace the continuous function with compact support $\varphi \otimes \psi$ by the function $(x, y) \mapsto \frac{e^{-ax}}{y}$ defined on $\mathbb{R}^{*+} \times [C, +\infty[$.

Let us now give briefly the ideas of the proofs of Theorems A and B. Set $\mathcal{A} = \{g \in G : a(g) > 1\}$ and consider the transition kernel $P_\mathcal{A}$ associated with the pair $(\mu, \mathcal{A})$ and defined by $P_\mathcal{A}(g, \mathcal{B}) = \int_G 1_{\mathcal{A}^c \cap \mathcal{B}}(gh)\, \mu(dh)$ for any Borel set $\mathcal{B} \subset G$ and any $g \in G$.

In the same way, set $\mathcal{A}' = \{g \in G : a(g) \geq 1\}$ and let $\tilde{P}_{\mathcal{A}'}$ be the operator associated with the pair $(\tilde{\mu}, \mathcal{A}')$. Following Grincevicius's paper, we are led to what we call the Grincevicius-Spitzer identity [10]:

$$\mu^{*n}(\varphi \otimes \psi) = \sum_{k=0}^{n} \int_G \tilde{P}^k_{\mathcal{A}'}(e, dg) \int_G P^{n-k}_{\mathcal{A}}(e, dh) \varphi\left(\frac{a(h)}{a(g)}\right) \psi\left(\frac{b(g) + b(h)}{a(g)}\right)$$

for any continuous functions φ and ψ with compact support in $\mathbb{R}^{*+}$ and $\mathbb{R}^d$ respectively. This formula allows to bring back the study of the asymptotic behaviour of the sequence $(\mu^{*n})_{n\geq 1}$ to the study of powers of operators $P_{\mathcal{A}}$ and $\tilde{P}_{\mathcal{A}'}$. It is the first main idea of this paper.

The second main idea relies on the Grenander's conjecture, proved by Grincevicius in [10] in a weaker form: if $d = 1$ and $\int_G \operatorname{Log} a(g)\mu(dg) = 0$, the asymptotic distribution of $|\operatorname{Log} B_1^n|$ is the same as the asymptotic distribution of $M_n = \max(0, \operatorname{Log} A_1^1, \operatorname{Log} A_1^2 \cdots, \operatorname{Log} A_1^n)$. One may thus expect that the asymptotic behaviour of $(G_1^n)_{n\geq 0}$ is quite similar to the behaviour of $(A_1^n, \exp(M_n))_{n\geq 0}$; we will justify this in section III.

Section II is devoted to the study of the behaviour as n goes to $+\infty$ of the sequence $(\operatorname{Log} A_1^n, M_n)_{n\geq 0}$ and in section III we prove Theorems A, B and C.

II. A PRELIMINARY RESULT

Throughout this section, $X_1, X_2 \cdots$ are independent real valued random variables with distribution p defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(S_n)_{n\geq 0}$ be the associated random walk on $\mathbb{R}$ starting from 0 (that is $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$); the distribution of S_n is the nth power of convolution p^{*n} of the measure p. Set $M_n = \max(0, S_1, \cdots, S_n)$ and denote by $\mathcal{F}_n$ the σ-algebra generated by $X_1, X_2, \cdots, X_n, n \geq 1$. The study of the asymptotic behaviour of the variable M_n is very interesting since many problems in applied probability theory may be reformulated as questions concerning this random variable. Following Spitzer's approach [16], we introduce the two following stopping times T_+ and T'_- with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$:

$$T_+ = \inf\{n \geq 1 : S_n > 0\} \qquad \text{and} \qquad T'_- = \inf\{n \geq 1 : S_n \leq 0\}.$$

Let p_{T_+} (resp. $p_{T'_-}$) be the distribution of the random variable S_{T_+} (resp. $S_{T'_-}$).

In the first part of the present section we give some estimates of the asymptotic behaviour of the sequences $(\mathbb{E}[[T_+ > n];\ \varphi(S_n)])_{n\geq 1}$ and $(\mathbb{E}[[T'_- > n];\ \varphi(S_n)])_{n\geq 1}$ where φ is a bounded Borel function on $\mathbb{R}$, in the second part we use these estimates to study the asymptotic behaviour of $(M_n,\ M_n - S_n)_{n\geq 1}$.

II.a. A local limit theorem for a killed random walk on a half line

We state here a result due to Iglehard [13] concerning the asymptotic behaviour of the sequences $(\mathbb{E}[[T_+ > n]; \varphi(S_n)])_{n\geq 1}$ and $(\mathbb{E}[[T'_- > n]; \varphi(S_n)])_{n\geq 1}$ where φ is a continuous function with compact support on $\mathbb{R}$.

Introducing the operator $P_{]0,+\infty[}$ defined by

$$\forall x \in \mathbb{R} \quad P_{]0,+\infty[}\varphi(x) = 1_{]-\infty,0]}(x) \int_{\mathbb{R}} 1_{]-\infty,0]}(x+y)\varphi(x+y)p(dy),$$

we obtain $\forall n \geq 1 \quad \mathbb{E}[[T_+ > n]; \varphi(S_n)] = P^n_{]0,+\infty[}\varphi(0)$. This section is thus devoted to the asymptotic behaviour as n goes to $+\infty$ of the nth power of the operator $P_{]0,+\infty[}$.

Let us first recall the

DEFINITION II.1. – *Let p be a probability measure on $\mathbb{R}$ and G_p the closed group generated by the support of p. The measure p is aperiodic if there is no closed and proper subgroup H of G_p and no number α such that $p(\alpha + H) = 1$.*

For example, the measure p such that $p(1) = p(3) = 1/2$ is not aperiodic because $G_p = \mathbb{Z}$ but $p(1 + 2\mathbb{Z}) = 1$. Before stating the main result of this section, we recall the following classical.

THEOREM II.2 [6]. – *Suppose that*

(i) *the common distribution p of the variables $X_n, n \geq 1$, is aperiodic;*

(ii) $\sigma^2 = \mathbb{E}[X_1^2] < +\infty$ *and* $\mathbb{E}[X_1] = 0$.

Then $\lim_{n\to+\infty} \sqrt{n}\mathbb{P}[T_+ > n] = \frac{e^\alpha}{\sqrt{\pi}}$ *with* $\alpha = \sum_{n=1}^{+\infty} \frac{\mathbb{P}[S_n \leq 0]-1/2}{n}$.

In the same way, $\lim_{n\to+\infty} \sqrt{n}\mathbb{P}[T'_- > n] = \frac{1}{\sqrt{\pi}}\exp\left(\sum_{n=1}^{+\infty} \frac{\mathbb{P}[S_n > 0]-1/2}{n}\right)$.

Proof. – For the reader's convenience, we sketch a proof, following [16]; we just explain how to obtain the behaviour of $(\mathbb{P}[T_+ > n])_{n\geq 1}$, the one of $(\mathbb{P}[T'_- > n])_{n\geq 1}$ being obtained with obvious modifications. For $s \in [0, 1[$ set $\phi(s) = \sum_{n=0}^{+\infty} s^n \mathbb{P}[T_+ > n]$. By P5(c) in Spitzer's book, page 181 ([16]), we have

$$\forall s \in [0, 1[\qquad \phi(s) = \exp\left(\sum_{n=1}^{+\infty} \frac{s^n}{n} \mathbb{P}[S_n \leq 0]\right).$$

Since the series $\sum_{n=1}^{+\infty} \frac{1}{n}(\mathbb{P}[S_n \leq 0] - \frac{1}{2})$ converges absolutely, it follows that

$$\phi(s) = \frac{e^\alpha}{\sqrt{1-s}}(1 + \epsilon(s))$$

with $\alpha = \sum_{n=1}^{+\infty} \frac{\mathbb{P}[S_n \le 0] - 1/2}{n}$ and $\lim_{s\to 1} \epsilon(s) = 0$. Since the sequence $(\mathbb{P}[T_+ > n])_{n\ge 1}$ decreases, Theorem II.2 follows from a Tauberian theorem for powers series [8]. $\square$

THEOREM II.3. – *Suppose that*

(i) *the distribution p of the variables $X_n, n \ge 1$, is aperiodic*

(ii) $\sigma^2 = \mathbb{E}[X_1^2] < +\infty$ *and* $\mathbb{E}[X_1] = 0$.

Then, for any continuous function φ with compact support on $\mathbb{R}^+$, we have

$$\lim_{n\to+\infty} n^{3/2}\mathbb{E}[[T_+ > n];\ \varphi(-S_n)] = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}^+} \varphi(x)\ \overline{U}_{T'_-} * \lambda_+(dx)$$

where λ_+ denotes the restriction of the Lebesgue measure on $\mathbb{R}^+$ and $\overline{U}_{T'_-}$ the image by the map $x \mapsto -x$ of the σ-finite measure $U_{T'_-} = \sum_{n=0}^{+\infty} (p_{T'_-})^{*n}$.

In the same way, for any continuous function φ with compact support on $\mathbb{R}^+$, we have

$$\lim_{n\to+\infty} n^{3/2}\mathbb{E}[[T'_- > n];\ \varphi(S_n)] = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}^+} \varphi(x)\ U_{T_+} * \lambda_+(dx)$$

where U_{T_+} denotes the σ-finite measure $\sum_{n=0}^{+\infty} (p_{T_+})^{*n}$.

Proof. – For the reader's convenience, we sketch here Iglehard's proof [13]. We just explain how to obtain the asymptotic behaviour of the sequence $(n^{3/2}\ \mathbb{E}[[T_+ > n];\ \varphi(-S_n)])_{n\ge 1}$.

For $a > 0, s \in [0, 1[$ set $\phi_a(s) = \sum_{n=0}^{+\infty} s^n \mathbb{E}[[T_+ > n];\ e^{aS_n}]$. By relations P5($a$) and P5($c$) in Spitzer's book, ([16], page 181) (*see* also [8], chap. XVIII), we have

$$\forall a > 0, \quad \forall s \in [0, 1[, \qquad \phi_a(s) = \sum_{n=0}^{+\infty} \mathbb{E}[s^{T'_-} \exp(aS_{T'_-})]^n$$

and therefore

$$\sum_{n=0}^{+\infty} \mathbb{E}[[T_+ > n];\ e^{aS_n}] = \sum_{n=0}^{+\infty} \mathbb{E}[\exp(aS_{T'_-})]^n$$
$$= \int_{-\infty}^{0} e^{ax}\ U_{T'_-}(dx) = \int_0^{+\infty} e^{-ax}\ \overline{U}_{T'_-}(dx)$$

Note that $-\infty < \mathbb{E}[S_{T'_-}] < 0$ so that the above series converges ([8], [16]). Consequently

$$\forall a > 0 \qquad \int_0^{+\infty} e^{-ax}\ \overline{U}_{T'_-} * \lambda_+(dx) = \int_0^{+\infty} \frac{e^{ax}}{a} \overline{U}_{T'_-}(dx)$$
$$= \sum_{n=0}^{+\infty} \mathbb{E}\left[[T_+ > n]; \frac{e^{aS_n}}{a}\right].$$

Thus, to prove Theorem II.3, it suffices to show that

$$\forall a > 0, \qquad \lim_{n \to +\infty} n^{3/2} \mathbb{E}[[T_+ > n];\ e^{aS_n}] = \frac{1}{\sigma\sqrt{2\pi}} \sum_{n=0}^{+\infty} \mathbb{E}\left[[T_+ > n];\ \frac{e^{aS_n}}{a}\right].$$

Note that $\mathbb{E}[[T_+ > n];\ e^{aS_n}]$ is the nth Taylor coefficient of the function ϕ_a and recall the Spitzer's identity ([16], P5(c), p. 181)

$$\forall s \in [0,1[, \quad \phi_a(s) = e^{A(s)} \qquad \text{with } A(s) = \sum_{n=1}^{+\infty} \frac{s^n}{n} \mathbb{E}\left[[S_n \leq 0];\ e^{aS_n}\right].$$

Let us now state the two following key lemmas whose proofs are given in [13].

LEMMA II.4. – *Let $\sum_{n=0}^{+\infty} d_n s^n = \exp(\sum_{n=1}^{+\infty} b_n s^n)$ for $|s| \leq 1$. If the sequence $(n^{3/2} b_n)_{n \geq 1}$ is bounded, the same holds for $(n^{3/2} d_n)_{n \geq 1}$.*

LEMMA II.5. – *Let $(c_n)_{n \geq 0}$ and $(d_n)_{n \geq 0}$ be two sequences of positive real numbers such that*

(i) $\lim_{n \to +\infty} \sqrt{n} c_n = c > 0$

(ii) $\sum_{n=0}^{+\infty} d_n = d < +\infty$

(iii) *the sequence $(n d_n)_{n \geq 0}$ is bounded.*

If $a_n = \sum_{k=0}^{n-1} c_{n-k} d_k$ then $\lim_{n \to +\infty} \sqrt{n} a_n = cd$.

Differentiating Spitzer's identity with respect to s leads to

$$\sum_{n=1}^{+\infty} n s^{n-1} \mathbb{E}[[T_+ > n];\ e^{aS_n}] = \sum_{n=1}^{+\infty} s^{n-1} \mathbb{E}[[S_n \leq 0];\ e^{aS_n}]\ \phi_a(s)$$

where $|s| < 1$. Set $a_n = n\ \mathbb{E}[[T_+ > n];\ e^{aS_n}], c_n = \mathbb{E}[[S_n \leq 0];\ e^{aS_n}]$ and $\sum_{n=0}^{+\infty} d_n s^n = \phi_a(s)$; we thus have $a_n = \sum_{k=0}^{n-1} d_k c_{n-k}$. By the classical local limit theorem on $\mathbb{R}$, the sequence $(\sqrt{n} c_n)_{n \geq 0}$ converges to $\frac{1}{a\sigma\sqrt{2\pi}}$; by Lemma II.4 it follows that $(n^{3/2} d_n)_{n \geq 1}$ is bounded. We may thus apply Lemma II.5 with $c = \frac{1}{a\sigma\sqrt{2\pi}}$ and $d = \sum_{n=0}^{+\infty} \mathbb{E}[[T_+ > n]; e^{aS_n}]$. The proof of Theorem II.3 is now complete. □

In [15], we give another proof of this theorem quite different from Iglehard's one and based on the following idea : under suitable hypotheses on p the function $z \mapsto \sum_{n=0}^{+\infty} p^{*n}(\varphi) z^n$ may be analytically extended on a certain neighbourhood of the unit complex disc except the pole 1. So the approximation of this function around its singularity may be translated into an approximation of its Taylor coefficients. Unfortunately, this "new" proof requires stronger hypotheses than Theorem II.3 and so it is not as general as Iglehard's one.

II.b. A local limit theorem for the process $(M_n, M_n - S_n)_{n\geq 0}$ on $\mathbb{R}^+ \times \mathbb{R}^+$

Let us first state the following well known theorem concerning the behaviour as n goes to $+\infty$ of the sequence $(\mathbb{E}[\varphi(M_n)])_{n\geq 1}$ where φ is a continuous function with compact support on $\mathbb{R}^+$; in [3] the reader will find a more general statement than the following one.

THEOREM II.6 [3]. – *Suppose that*

(i) *the distribution p of the variables $X_n, n \geq 1$, is aperiodic*

(ii) $\sigma^2 = \mathbb{E}[X_1^2] < +\infty$ *and* $\mathbb{E}[X_1] = 0$.

Then, for any continuous function φ with compact support on $\mathbb{R}^+$, we have

$$\lim_{n\to+\infty} \sqrt{n}\mathbb{E}[\varphi(M_n)] = \frac{e^\alpha}{\sqrt{\pi}} \int_0^{+\infty} \varphi(x) U_{T_+}(dx)$$

with $\alpha = \sum_{n=1}^{+\infty} \frac{\mathbb{P}[S_n \leq 0] - 1/2}{n}$.

Proof. – For the reader's convenience, we present here a simple proof of this theorem. It suffices to show that

$$\forall a > 0 \qquad \lim_{n\to+\infty} \mathbb{E}[e^{-aM_n}] = \frac{e^\alpha}{\sqrt{\pi}} \int_0^{+\infty} e^{-ax} U_{T_+}(dx).$$

The starting point is the following identity due to Spitzer [16] :

$$\forall a > 0, \quad \forall n \geq 1 \qquad \mathbb{E}[e^{-aM_n}] = \sum_{k=0}^{n} \mathbb{E}[[T'_- > k];\, e^{-aS_k}]\, \mathbb{P}[T_+ > n-k].$$

By Theorem II.2, we have $\lim_{n\to+\infty} \sqrt{n}\mathbb{P}[T_+ > n] = \frac{e^\alpha}{\sqrt{\pi}}$ and by Theorem II.3 the sequence $(n^{3/2}\mathbb{E}[[T'_- > n];\, e^{-aS_n}])_{n\geq 0}$ is bounded; furthermore

$$\sum_{n=0}^{+\infty} \mathbb{E}[[T'_- > n]; e^{-aS_n}] = \int_0^{+\infty} e^{-ax} U_{T_+}(dx).$$

Theorem II.4 thus follows from Lemma II.5. □

We now turn to the behaviour of the sequence $(\mathbb{E}[\varphi(M_n, S_n)])_{n\geq 1}$. In [17], N.T. Varopoulos gave an upperbound of the asymptotic behaviour of the sequence $(n^{3/2}\mathbb{P}[M_n \leq a, S_n \geq -b])_{n\geq 1}, a, b \in \mathbb{R}^+$; we obtain here the exact asymptotic behaviour of this sequence and as far as we know this result is new.

THEOREM II.7. – *Suppose that*

(i) *the distribution* p *of the variables* $X_n, n \geq 1$, *is aperiodic*

(ii) $\sigma^2 = \mathbb{E}[X_1^2] < +\infty$ *and* $\mathbb{E}[X_1] = 0$.

Then, for any continuous function φ *with compact support on* $\mathbb{R}^+ \times \mathbb{R}^+$, *we have*

$$\begin{aligned}\lim_{n\to+\infty} & n^{3/2}\mathbb{E}[\varphi(M_n, M_n - S_n)] \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_0^{+\infty}\int_0^{+\infty} \varphi(x,y)\lambda_+ * U_{T_+}(dx)\,\overline{U}_{T'_-}(dy) \\ &\quad + \frac{1}{\sigma\sqrt{2\pi}} \int_0^{+\infty}\int_0^{+\infty} \varphi(x,y)\, U_{T_+}(dx)\,\lambda_+ * \overline{U}_{T'_-}(dy)\end{aligned}$$

where λ_+ *is the restriction of the Lebesgue measure on* $\mathbb{R}^+$, $U_{T_+} = \sum_{n=0}^{+\infty}(p_{T_+})^{*n}$ *and* $\overline{U}_{T_-}$ *is the image by the map* $x \mapsto -x$ *of the potential* $U_{T_-} = \sum_{n=0}^{+\infty}(p_{T_-})^{*n}$.

Proof. – It suffices to show that for any $a, b > 0$ one has

$$\begin{aligned}\lim_{n\to+\infty} & n^{3/2}\mathbb{E}[e^{-aM_n} e^{-b(M_n - S_n)}] \\ &= \int_0^{+\infty}\int_0^{+\infty} \frac{e^{-ax}}{a} e^{-by}\, U_{T_+}(dx)\,\overline{U}_{T'_-}(dy) \\ &\quad + \int_0^{+\infty}\int_0^{+\infty} e^{-ax}\frac{e^{-by}}{b}\, U_{T_+}(dx)\,\overline{U}_{T'_-}(dy)\end{aligned}$$

In his book, F. Spitzer introduces the variable T_n denoting the first time at which $(S_n)_{n\geq 0}$ reaches its maximum M_n during the first n steps. Recall that T_n is not a stopping time with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$; nevertheless, it plays a crucial role in order to obtain the following identity [16]

$$\begin{cases} \forall n \geq 1, \\ \mathbb{E}[e^{-aM_n}\, e^{-b(M_n - S_n)}] = \sum_{k=0}^{n} \mathbb{E}[[T'_- > k];\, e^{-aS_k}]\mathbb{E}[[T_+ > n-k];\, e^{bS_{n-k}}]. \end{cases}$$

Set $\alpha_n = \mathbb{E}[[T'_- > n];\ e^{-aS_n}]$ and $\beta_n = \mathbb{E}[[T_+ > n];\ e^{bS_n}]$. By Theorem II.3 we have

$$\lim_{n\to+\infty} n^{3/2}\alpha_n = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}^+} \frac{e^{-ax}}{a}\, U_{T_+}(dx)$$

and

$$\lim_{n\to+\infty} n^{3/2}\beta_n = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}^+} \frac{e^{-by}}{b}\, \overline{U}_{T'_-}(dy)$$

Furthermore

$$\sum_{n=0}^{+\infty} \alpha_n = \int_0^{+\infty} e^{-ax} U_{T_+}(dx) \quad \text{and} \quad \sum_{n=0}^{+\infty} \beta_n = \int_0^{+\infty} e^{-by} \overline{U}_{T'_-}(dy).$$

Theorem II.5 is thus a consequence of the following lemma

LEMMA II.8. – *Let* $(\alpha_n)_{n\geq 0}$ *and* $(\beta_n)_{n\geq 0}$ *be two sequences of positive real numbers such that* $\lim_{n\to+\infty} n^{3/2}\alpha_n = \alpha$ *and* $\lim_{n\to+\infty} n^{3/2}\beta_n = \beta > 0$. *Then*

(i) *there exists a constant* $C > 0$ *such that, for any* $n \in \mathbb{N}^*$ *and* $0 < i < n - j < n$, *we have*

$$n^{3/2} \sum_{k=i+1}^{n-j} \frac{1}{k^{3/2}(n-k)^{3/2}} \leq C\left(\frac{1}{\sqrt{i}} + \frac{1}{\sqrt{j}}\right).$$

(ii) *one has* $\lim_{n\to+\infty} n^{3/2} \sum_{k=0}^{n} \alpha_k \beta_{n-k} = \alpha B + \beta A$ *where* $A = \sum_{k=0}^{+\infty} \alpha_k$ *and* $B = \sum_{k=0}^{+\infty} \beta_k$.

Proof. – (i) Without loss of generality, one can suppose $i + 1 < [n/2] < n - j$, where $[n/2]$ is the integer part of $n/2$; we have

$$\begin{aligned} n^{3/2} &\sum_{k=i+1}^{n-j} \frac{1}{k^{3/2}(n-k)^{3/2}} \\ &= n^{3/2} \sum_{k=i+1}^{[n/2]} \frac{1}{k^{3/2}(n-k)^{3/2}} + n^{3/2} \sum_{k=[n/2]+1}^{n-j} \frac{1}{k^{3/2}(n-k)^{3/2}} \\ &\leq 2^{3/2} \sum_{k=i+1}^{+\infty} \frac{1}{k^{3/2}} + 2^{3/2} \sum_{k=j}^{+\infty} \frac{1}{k^{3/2}}. \end{aligned}$$

Inequality (i) follows immediately.

(ii) Set $\gamma_n = \sum_{k=0}^{n} \alpha_k \beta_{n-k}$ and fix $1 \leq i < n - j < n$; one has

$$\begin{aligned} |n^{3/2}\gamma_n - \alpha B - \beta A| \leq &\left| n^{3/2} \sum_{k=0}^{i} \alpha_k \beta_{n-k} - \beta \sum_{k=0}^{+\infty} \alpha_k \right| + n^{3/2} \sum_{k=i+1}^{n-j} \alpha_k \beta_{n-k} \\ &+ \left| n^{3/2} \sum_{k=n-j+1}^{n} \alpha_k \beta_{n-k} - \alpha \sum_{k=0}^{+\infty} \beta_k \right| \end{aligned}$$

with

$$\left| n^{3/2} \sum_{k=0}^{i} \alpha_k \beta_{n-k} - \beta \sum_{k=0}^{+\infty} \alpha_k \right| \leq \sum_{k=0}^{i} |n^{3/2}\beta_{n-k} - \beta| \alpha_k + \beta \sum_{k=i+1}^{+\infty} \alpha_k$$

$$\left| n^{3/2} \sum_{k=n-j+1}^{n} \alpha_k \beta_{n-k} - \alpha \sum_{k=0}^{+\infty} \beta_k \right| \leq \sum_{k=0}^{j-1} |n^{3/2}\alpha_{n-k} - \alpha| \beta_k + \alpha \sum_{k=j}^{+\infty} \beta_k$$

and

$$n^{3/2} \sum_{k=i+1}^{n-j} \alpha_k \beta_{n-k} \leq C\left(\frac{1}{\sqrt{i}} + \frac{1}{\sqrt{j}}\right)$$

Fix $\epsilon > 0$ arbitrary small and choose i and j large enough that $\beta \sum_{k=i+1}^{+\infty} \alpha_k < \epsilon/3, \alpha \sum_{k=j}^{+\infty} \beta_k < \epsilon/3$ and $C(\frac{1}{\sqrt{i}} + \frac{1}{\sqrt{j}}) < \epsilon/3$. Letting $n \to +\infty$, one obtains $\limsup_{n \to +\infty} |n^{3/2}\gamma_n - \alpha B - \beta A| \leq \epsilon$. □

III. PROOFS OF THEOREMS A AND B

III.a. Spitzer-Grincevicius factorisation

Let us first recall some notations. Let $g_n = (a_n, b_n), n = 1, 2, \cdots$ be independent and identically distributed random variables with distribution μ. Denote by $\mathcal{F}_n$ the σ-algebra generated by the variables $g_1, g_2, \cdots, g_n, n \geq 1$. For any $n \geq 1$, set $G_1^n = g_1 \cdots g_n = (A_1^n, B_1^n)$; we have $A_1^n = a_1 \cdots a_n$ and $B_1^n = \sum_{k=1}^{n} a_1 \cdots a_{k-1} b_k$. More generally, set $A_n^m = a_n \cdots a_m$ and $B_n^m = \sum_{k=n}^{m} a_n \cdots a_{k-1} b_k$ if $1 \leq n \leq m$ and $A_n^m = 1$, $B_n^m = 0$ otherwise. We also introduce the variables S_n, M_n and T_n defined by $S_n = \operatorname{Log} A_1^n$ and $S_0 = 0$, $M_n = \max(S_0, S_1, \cdots, S_n)$ and $T_n = \inf\{0 \leq k \leq n / S_k = M_n\}$.

In the same way, let $\tilde{\mu}$ be the image of μ by the mapping $g \mapsto (\frac{1}{a(g)}, \frac{b(g)}{a(g)})$; if $\tilde{g}_n = (\tilde{a}_n, \tilde{b}_n)$, $n = 1, 2, \cdots$, are independent and identically distributed random variables with distribution $\tilde{\mu}$ on G, set $\tilde{G}_n^m = \tilde{g}_n \cdots \tilde{g}_m, \tilde{A}_n^m = \tilde{a}_n \cdots \tilde{a}_m, \tilde{B}_n^m = \sum_{k=n}^{m} \tilde{a}_n \cdots \tilde{a}_{k-1} \tilde{b}_k$ for $1 \leq n \leq m$ and $\tilde{G}_n^m = e, \tilde{A}_n^m = 1$ and $\tilde{B}_n^m = 0$ otherwise; set also $\tilde{S}_n = \operatorname{Log} \tilde{A}_1^n$, $\tilde{S}_0 = 0$ and $\tilde{M}_n = \max(\tilde{S}_0, \tilde{S}_1, \cdots, \tilde{S}_n)$. Denote by $\tilde{\mathcal{F}}_n$ the σ-algebra generated by $\tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_n, n \geq 1$.

Fix two positive functions φ and ψ, with compact support, defined respectively on $\mathbb{R}^{*+}$ and $\mathbb{R}^d$. For technical reasons, we suppose that ψ is continuously differentiable on $\mathbb{R}^d$. We are interested in the behaviour of the sequence $(\mathbb{E}[\varphi(A_n)\psi(B_n)])_{n\geq 1}$ as n goes to $+\infty$; following [10], we have

$$\begin{aligned}\mathbb{E}[\varphi(A_1^n)\psi(B_1^n)] &= \sum_{k=0}^{n} \mathbb{E}[[T_n = k]; \varphi(A_1^n)\psi(B_1^n)] \\ &= \sum_{k=0}^{n} \mathbb{E}[[A_1^k > 1] \cap [A_2^k > 1] \cap \cdots \cap [A_k^k > 1] \\ &\quad \cap [A_{k+1}^{k+1} \leq 1] \cap [A_{k+1}^{k+2} \leq 1] \\ &\quad \cdots \cap [A_{k+1}^n \leq 1]; \varphi(A_1^n)\psi(B_1^n)]\end{aligned}$$

The last expectation can be simplified as it is clear that the terms $A_1^k, A_2^k, \cdots, A_k^k$ are independent of the terms $A_{k+1}^{k+1}, A_{k+1}^{k+2} \cdots, A_{k+1}^n$; from the equality $B_1^n = A_1^k(\sum_{j=1}^k \frac{b_j}{A_j^k} + \sum_{j=k+1}^n A_{k+1}^{j-1} b_j)$ and by a duality argument, one obtains

$$\begin{aligned}&\mathbb{E}[\varphi(A_1^n)\psi(B_1^n)] \\ &= \sum_{k=0}^{n} \mathbb{E}\Bigg[[\tilde{A}_1^1 < 1] \cap [\tilde{A}_1^2 < 1] \cap \cdots \cap [\tilde{A}_1^k < 1] \\ &\qquad \cap [A_{k+1}^{k+1} \leq 1] \cap [A_{k+1}^{k+2} \leq 1] \cdots \cap [A_{k+1}^n \leq 1]; \\ &\qquad \varphi\left(\frac{A_{k+1}^n}{\tilde{A}_1^k}\right)\psi\left(\frac{1}{\tilde{A}_1^k}\left(\sum_{j=1}^k \tilde{A}_1^{j-1}\tilde{b}_j + \sum_{j=k+1}^n A_{k+1}^{j-1} b_j\right)\right)\Bigg].\end{aligned}$$

Set $\mathcal{A} = \{g \in G : a(g) > 1\}$ and consider the transition kernel $P_{\mathcal{A}}$ associated with $(\mu, \mathcal{A})$ and defined by $P_{\mathcal{A}}(g, \mathcal{B}) = \int_G 1_{\mathcal{A}^c \cap \mathcal{B}}(gh)\, \mu(dh)$ for any Borel set $\mathcal{B} \subset G$ and any $g \in G$.

Let us give the probabilistic interpretation of $P_{\mathcal{A}}$. Let $T_{\mathcal{A}} = \inf\{n \geq 1 : G_1^n \in \mathcal{A}\}$ be the first entrance time in $\mathcal{A}$ of the random walk $(G_1^n)_{n\geq 0}$; it is a stopping time with respect to $(\mathcal{F}_n)_{n\geq 1}$ and we have

$$\forall n \geq 1 \quad P_{\mathcal{A}}^n(e, B) = \mathbb{P}[[T_{\mathcal{A}} > n] \cap [G_1^n \in B]].$$

In the same way, set $\mathcal{A}' = \{g \in G/a(g) \geq 1\}$, let $\tilde{P}_{\mathcal{A}'}$ be the operator associated with $(\tilde{\mu}, \mathcal{A}')$ and denote by $\tilde{T}_{\mathcal{A}'}$ the first entrance time in $\mathcal{A}'$ of the random walk $(\tilde{G}_1^n)_{n\geq 1}$; $\tilde{T}_{\mathcal{A}'}$ is a stopping time with respect to $(\tilde{\mathcal{F}}_n)_{n\geq 1}$ and we have

$$\forall n \geq 1 \quad \tilde{P}_{\mathcal{A}'}^n(e, B) = \mathbb{P}[[\tilde{T}_{\mathcal{A}'} > n] \cap [\tilde{G}_1^n \in B]].$$

From the previous expression of $\mathbb{E}[\varphi(A_1^n)\psi(B_1^n)]$, we obtain the Spitzer-Grincevicius factorisation:

$$\mathbb{E}[\varphi(A_1^n)\psi(B_1^n)] = \sum_{k=0}^{n} I_{k,n}(\varphi, \psi)$$

where

$$I_{k,n}(\varphi,\psi)=\int_{G\times G}\varphi\left(\frac{a(h)}{a(g)}\right)\psi\left(\frac{b(g)+b(h)}{a(g)}\right)\tilde{P}^k_{\mathcal{A}'}(e,dg)P^{n-k}_{\mathcal{A}}(e,dh).$$

III.b. Proof of Theorem A

The starting point of the proof is the Spitzer-Grincevicius factorisation. First, thanks to the following lemma, we are going to control the sum $\sum_{k=i+1}^{n-j} I_{k,n}(\varphi,\psi)$ for fixed large enough integers i and j.

LEMMA III.1. – *There exists $\lambda_0>0$ such that for any $\lambda\in]0,\lambda_0]$, any $g\in G$ and any $l>0$, we have*

$$\int_G\varphi\left(\frac{a(h)}{a(g)}\right)\psi\left(\frac{b(g)+b(h)}{a(g)}\right)P^l_{\mathcal{A}}(e,dh)\leq\frac{C}{l^{3/2}}\,a(g)^\lambda$$

where C is a positive constant which depends on λ,φ and ψ.

By Theorem II.3, the sequence $\left(k^{3/2}\int_G a(g)^\lambda\tilde{P}^k_{\mathcal{A}'}(e,dg)\right)_{k\geq 0}$ is bounded since

$$\int_G a(g)^\lambda\tilde{P}^k_{\mathcal{A}'}(e,dg)=\mathbb{E}[[\tilde{T}_{\mathcal{A}'}>k];\exp(\lambda\tilde{S}_k)].$$

Hence, using Lemma III.1, we obtain for any $0<k<n$

$$\int_{G\times G}\varphi\left(\frac{a(h)}{a(g)}\right)\psi\left(\frac{b(g)+b(h)}{a(g)}\right)\tilde{P}^k_{\mathcal{A}'}(e,dg)P^{n-k}_{\mathcal{A}}(e,dh)\leq\frac{C_1}{k^{3/2}(n-k)^{3/2}}.$$

Using Lemma II.8 (i), one can thus choose two integers i and j such that $\limsup_{n\to+\infty}n^{3/2}\sum_{k=i+1}^{n-j}I_{k,n}$ is as small as wanted.

Next, we look at the behaviour of the integral

$$\int_G\varphi\left(\frac{a(h)}{a(g)}\right)\psi\left(\frac{b(g)+b(h)}{a(g)}\right)P^l_{\mathcal{A}}(e,dh)$$

as l goes to $+\infty$.

LEMMA III.2. – *For any $g\in G$, the sequence*

$$\left(l^{3/2}\int_G\varphi\left(\frac{a(h)}{a(g)}\right)\psi\left(\frac{b(g)+b(h)}{a(g)}\right)P^l_{\mathcal{A}}(e,dh)\right)_{l\geq 0}$$

converges to a finite limit as l goes to $+\infty$.

In particular $(n^{3/2}I_{0,n}(\varphi,\psi))_{n\geq 1}$ converges in $\mathbb{R}$. On the other hand, for any $i\geq 1$ and any compact set $K\subset\mathbb{R}^{*+}\times\mathbb{R}$, the dominated convergence theorem ensures the existence of a finite limit as n goes to $+\infty$ for $(n^{3/2}\sum_{k=1}^{i} I_{k,n}(\varphi,\psi,K))_{n\geq 0}$ where

$$I_{k,n}(\varphi,\psi,K) = \int_G 1_K(g)\Big(\int_G \varphi\Big(\frac{a(h)}{a(g)}\Big)\psi\Big(\frac{b(g)+b(h)}{a(g)}\Big)P_{\mathcal{A}}^{n-k}(e,dh)\Big) \times \tilde{P}_{\mathcal{A}'}^{k}(e,dg)).$$

One just have to check that the indicator function 1_K does not disturb too much the behaviour of the above integrals. Fix $0<\delta<1$; according to Lemma III.1, we have

$$\begin{aligned}\sum_{k=1}^{i}\int_{\{g\in G:a(g)\leq\delta\}} &\Big(\int_G \varphi\Big(\frac{a(h)}{a(g)}\Big)\psi\Big(\frac{b(g)+b(h)}{a(g)}\Big)P_{\mathcal{A}}^{n-k}(e,dh)\Big) \\ &\times \tilde{P}_{\mathcal{A}'}^{k}(e,dg) \\ &\leq C(\lambda,\varphi,\psi)\sum_{k=1}^{i}\frac{1}{(n-k)^{3/2}}\mathbb{E}[[\tilde{T}_{\mathcal{A}'}>k]\cap[\tilde{S}_k\leq \mathrm{Log}\delta];\ \exp(\lambda\tilde{S}_k)] \\ &\leq C(\lambda,\varphi,\psi)\delta^{\lambda/2}\sum_{k=1}^{i}\frac{1}{(n-k)^{3/2}}E\Big[[\tilde{T}_{\mathcal{A}'}>k];\ \exp\Big(\frac{\lambda}{2}\tilde{S}_k\Big)\Big] \\ &\leq C_1\delta^{\lambda/2}\sum_{k=1}^{i}\frac{1}{(n-k)^{3/2}k^{3/2}}.\end{aligned}$$

On the other hand, by the definition of $\tilde{P}_{\mathcal{A}'}$

$$\begin{aligned}\sum_{k=1}^{i}\int_{\{g\in G:a(g)\geq 1/\delta\}} &\Big(\int_G \varphi\Big(\frac{a(h)}{a(g)}\Big)\psi\Big(\frac{b(g)+b(h)}{a(g)}\Big)P_{\mathcal{A}}^{n-k}(e,dh)\Big) \\ &\times \tilde{P}_{\mathcal{A}'}^{k}(e,dg) = 0.\end{aligned}$$

Now, fix $B>0$; according to Lemma III.1

$$\begin{aligned}\sum_{k=1}^{i}\int_{\{g\in G:\|b(g)\|\geq B\}} &\Big(\int_G \varphi\Big(\frac{a(h)}{a(g)}\Big)\psi\Big(\frac{b(g)+b(h)}{a(g)}\Big)P_{\mathcal{A}}^{n-k}(e,dh)\Big) \\ &\times \tilde{P}_{\mathcal{A}'}^{k}(e,dg) \\ &\leq C(\lambda,\varphi,\psi)\sum_{k=1}^{i}\frac{1}{(n-k)^{3/2}}\mathbb{E}[[\tilde{T}_{\mathcal{A}'}>k]\cap[\|\tilde{B}_k\|\geq B];\ \exp(\lambda\tilde{S}_k)] \\ &\leq \frac{C(\lambda,\varphi,\psi)}{B^{\lambda/2}}\sum_{k=1}^{i}\frac{1}{(n-k)^{3/2}}E[[\tilde{T}_{\mathcal{A}'}>k];\exp(\lambda\tilde{S}_k)\|\tilde{B}_k\|^{\lambda/2}] \\ &\leq \frac{C_1}{B^{\lambda/2}}\sum_{k=1}^{i}\frac{1}{(n-k)^{3/2}k^{3/2}}\end{aligned}$$

where the last inequality is guaranteed by the following

LEMMA III.3. – *There exists* $\epsilon_0 > 0$ *such that for any* $0 < \epsilon < \epsilon_0$

$$\sup_{l \geq 1} \quad l^{3/2}\mathbb{E}[[\tilde{T}_{\mathcal{A}'} > l];\ \exp(2\epsilon \tilde{S}_l)\ \|\tilde{B}_l\|^{\epsilon}] < +\infty.$$

Note that the same upperbounds hold when the sum $\sum_{k=1}^{i}$ is replaced by $\sum_{k=n-j+1}^{n-1}$.

Finally, using the Spitzer-Grincevicius factorisation, we have proved that, for any $\epsilon > 0$, there exist $i, j \in \mathbb{N}$ and a compact set $K \subset G$ such that for any $n > i + j$ one has

$$\begin{aligned}|n^{3/2}\mathbb{E}[\varphi(A_1^n)\psi(B_1^n)] - n^{3/2}\sum_{k=0}^{i} I_{k,n}(\varphi, \psi, K)\\ - n^{3/2}\sum_{k=n-j+1}^{n} I_{k,n}(\varphi, \psi, K)| \leq \epsilon.\end{aligned}$$

On the other hand,

$$\left(n^{3/2}\sum_{k=0}^{i} I_{k,n}(\varphi, \psi, K) + n^{3/2}\sum_{k=n-j+1}^{n} I_{k,n}(\varphi, \psi, K)\right)_{n \geq 0}$$

converges. Hence the sequence of measures $(n^{3/2}\mu^{*n})_{n \geq 1}$ weakly converges to a Radon measure ν_0; the fact that ν_0 is not degenerated follows from the

LEMMA III.4. – *There exist an integer* n_0 *and a compact set* $K_0 \subset G$ *such that*

$$\inf_{n \geq n_0} n^{3/2}\mathbb{P}[G_1^n \in K_0] > 0.$$

The proof of Theorem A is now complete; it just remains to establish Lemmas III.1, III.2, III.3 and III.4.

Proof of Lemma III.1. – First, suppose that Hypotheses A1, A2 and A3 hold.

Fix $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For any $g \in G$ and $l \geq 1$, we have

$$\begin{aligned}
&\int_G \varphi\left(\frac{a(h)}{a(g)}\right)\psi\left(\frac{b(g)+b(h)}{a(g)}\right)P^l_{\mathcal{A}}(e,dh)\\
&= \int_{]0,1]\times\mathbb{R}^d} \mathbb{E}\Bigg[[aA_2^2 \leq 1]\cap\cdots\cap[aA_2^l \leq 1];\ \varphi\left(\frac{aA_2^l}{a(g)}\right)\\
&\qquad\times\psi\left(\frac{b(g)+\sum_{i=2}^l aA_2^{i-1}b_i + b}{a(g)}\right)\Bigg]\phi_\mu(a,b)\frac{dadb}{a}\\
&\leq a(g)^{\frac{1}{p}}\ \|\psi\|_p \int_0^1 \sqrt[q]{\int_{\mathbb{R}^d}\phi_\mu^q(a,b)db}\\
&\quad\times\mathbb{E}\left[[aA_2^2 \leq 1]\cap\cdots\cap[aA_2^l \leq 1];\varphi\left(\frac{aA_2^l}{a(g)}\right)\right]\frac{da}{a}\\
&\leq a(g)^{\frac{1}{p}}\ \|\psi\|_p \int_0^1 \sqrt[q]{\int_{\mathbb{R}^d}\phi_\mu^q(a,b)db}\\
&\quad\times\mathbb{E}\left[\left[\exp(M_{l-1}) \leq \frac{1}{a}\right];\ \varphi\left(\frac{aA_1^{l-1}}{a(g)}\right)\right]\frac{da}{a}\\
&\leq a(g)^{\frac{1}{p}}\ \|\psi\|_p \int_0^1 \sqrt[q]{\int_{\mathbb{R}^d}\phi_\mu^q(a,b)db}\frac{1}{a^{2\epsilon}}\\
&\quad\times\mathbb{E}\left[\exp(-2\epsilon M_{l-1})\varphi\left(\frac{aA_1^{l-1}}{a(g)}\right)\right]\frac{da}{a} \quad \text{for any} \quad \epsilon > 0.
\end{aligned}$$

Since the support of φ is compact in $]0,+\infty[$, there exists $K = K(\epsilon,\varphi) > 0$ such that $\forall a > 0 \quad |\varphi(a)| \leq Ka^\epsilon$; so

$$\begin{aligned}
&\int_G \varphi\left(\frac{a(h)}{a(g)}\right)\psi\left(\frac{b(g)+b(h)}{a(g)}\right)P^l_{\mathcal{A}}(e,dh)\\
&\leq K\ a(g)^{\frac{1}{p}-\epsilon}\ \|\psi\|_p \int_0^1 \sqrt[q]{\int_{\mathbb{R}^d}\phi_\mu^q(a,b)db}\,\frac{da}{a^{1+\epsilon}}\\
&\quad\times\mathbb{E}[[\exp(-\epsilon(M_{l-1}-S_{l-1}))\exp(-\epsilon M_{l-1})].
\end{aligned}$$

Assume $\frac{1}{p} - \epsilon > 0$ and $1 + \epsilon < \beta$; by Theorem II.7 one obtains

$$\int_G \varphi\left(\frac{a(h)}{a(g)}\right)\psi\left(\frac{b(g)+b(h)}{a(g)}\right)P^l_{\mathcal{A}}(e,dh) \leq \frac{C}{l^{3/2}}\ a(g)^{\frac{1}{p}-\epsilon}.$$

Now, replace Hypothesis A3 by Hypothesis A3 (bis)
For any $g \in G$ and $l \geq 1$, we have

$$\varphi\left(\frac{a(h)}{a(g)}\right)\psi\left(\frac{b(g)+b(h)}{a(g)}\right)P_A^l(e,dh)$$
$$= \int_{]0,1]\times\mathbb{R}^d} \mathbb{E}\Bigg[[aA_2^2 \leq 1] \cap \cdots \cap [aA_2^l \leq 1];$$
$$\varphi\left(\frac{aA_2^l}{a(g)}\right)\ \psi\left(\frac{b(g)+\sum_{i=2}^{l} aA_2^{i-1}b_i + b}{a(g)}\right)\Bigg]\mu(da\ db).$$

Since φ and ψ have compact support, for any $\epsilon > 0$ there exists $K = K(\epsilon, \varphi, \psi) > 0$ such that

$$\forall a > 0\ |\varphi(a)| \leq K\ a^\epsilon \qquad \text{and} \qquad \forall b \in (\mathbb{R}^{*+})^d\ |\psi(b)| \leq \frac{K}{\|b\|^{2\epsilon}}.$$

Thus

$$\int_G \varphi\left(\frac{a(h)}{a(g)}\right)\psi\left(\frac{b(g)+b(h)}{a(g)}\right)P_A^l(e,dh)$$
$$\leq K^2 a(g)^\epsilon \int_{]0,1]\times\mathbb{R}^d} \mathbb{E}[[aA_2^2 \leq 1] \cap \cdots \cap [aA_2^l \leq 1]];$$
$$\frac{(A_2^l)^\epsilon}{\|b(g)+\sum_{i=2}^{l} aA_2^{i-1}b_i + b\|^{2\epsilon}} a^\epsilon \mu(da\ db).$$

Hypothesis A3 (bis) implies $\|b(g) + \sum_{i=2}^l aA_2^{i-1}b_i + b\| \geq \|b\|$ $\mathbb{P} - a.s$ so that

$$\int_G \varphi\left(\frac{a(h)}{a(g)}\right)\psi\left(\frac{b(g)+b(h)}{a(g)}\right)P_A^l(e,dh)$$
$$\leq K^2 a(g)^\epsilon \int_{]0,1]\times\mathbb{R}^d} \frac{a^\epsilon}{\|b\|^{2\epsilon}}\mathbb{E}\left[\left[e^{M_{l-1}} \leq \frac{1}{a}\right]; e^{\epsilon S_{l-1}}\right]\mu(da\ db)$$
$$\leq K^2 a(g)^\epsilon \mathbb{E}[e^{-\epsilon(M_{l-1}-S_{l-1})}\ e^{-\epsilon M_{l-1}}]$$
$$\times \int_{\mathbb{R}^{*+}\times(\mathbb{R}^{*+})^d} \frac{1}{a^\epsilon \|b\|^{2\epsilon}}\mu(da\ db).$$

The proof is now complete. $\square$

Proof of Lemma III.2. – Without loss of generality, one may suppose $g = e$. For any $n \in \mathbb{N}^*$, set

$$\nu_n(\varphi, \psi) = n^{3/2}\mathbb{E}[[T_A > n];\ \varphi(A_1^n)\psi(B_1^n)].$$

Fix $i, j \in \mathbb{N}$ such that $1 \leq i < n - j \leq n$ and consider

$$\nu_n(\varphi, \psi, i, j) = n^{3/2}\mathbb{E}[[T_A > n];\ \varphi(A_1^n)\psi(B_1^i + A_1^{n-j}B_{n-j+1}^n)].$$

To obtain the claim, it suffices to prove that

a) $\limsup_{i,j\to+\infty}\ \limsup_{n\to+\infty} |\nu_n(\varphi, \psi) - \nu_n(\varphi, \psi, i, j)| = 0$

b) for any fixed $i, j \in \mathbb{N}$, the sequence $(\nu_n(\varphi, \psi, i, j))_{n\geq 1}$ converges to a finite limit.

Proof of convergence a. – We use the equality $B_1^n = B_1^i + A_1^i B_{i+1}^{n-j} + A_1^{n-j}B_{n-j+1}^n$; since the support of ψ is compact and ψ is continuously differentiable, we have for some $0 < \epsilon < 1$

$$\begin{aligned}
&|\nu_n(\varphi, \psi) - \nu_n(\varphi, \psi, i, j)| \\
&\quad\leq C_1\ n^{3/2}\ \mathbb{E}[[T_A > n];\ \varphi(A_1^n)(A_1^i)^\epsilon \|B_{i+1}^{n-j}\|^\epsilon] \\
&\quad\leq C_1\ n^{3/2} \sum_{k=i+1}^{n-j} \mathbb{E}[[T_A > n];\ \varphi(A_1^n)(A_1^{k-1})^\epsilon \|b_k\|^\epsilon]
\end{aligned}$$

Since the support of φ is compact in $]0, +\infty[$, there exists $K = K(\epsilon, \varphi) > 0$ such that $\forall a > 0\ |\varphi(a)| \leq K a^\epsilon$; thus, for any $i + 1 \leq k \leq n - j$, we have

$$\begin{aligned}
&\mathbb{E}[[T_A > n];\ \varphi(A_1^n)(A_1^{k-1})^\epsilon \|b_k\|^\epsilon] \\
&\quad\leq K\mathbb{E}\Bigg[[T_A > k-1] \cap \left[\max(A_{k+1}^{k+1}, \cdots, A_{k+1}^n) \leq \frac{1}{A_1^{k-1}a_k}\right]; \\
&\qquad\qquad \times (A_1^{k-1})^{2\epsilon} a_k^\epsilon \|b_k\|^\epsilon (A_{k+1}^n)^\epsilon\Bigg] \\
&\quad\leq K\mathbb{E}[[T_A > k-1];\ (A_1^{k-1})^{\epsilon/2} a_k^{-\epsilon/2} \|b_k\|^\epsilon \\
&\qquad\qquad \times \max(A_{k+1}^{k+1}, \cdots, A_{k+1}^n)^{-3\epsilon/2} (A_{k+1}^n)^\epsilon] \\
&\quad\leq K\mathbb{E}[[T_A > k-1];\ (A_1^{k-1})^{\epsilon/2}]\ \mathbb{E}[a_k^{-\epsilon/2}\|b_k\|^\epsilon] \\
&\qquad\qquad \times \mathbb{E}[e^{-\epsilon(M_{n-k} - S_{n-k})} e^{-\frac{\epsilon}{2}M_{n-k}}].
\end{aligned}$$

Consequently, by Theorem II.3, Theorem II.7 and Lemma II.8 (i), there exists $C_2 > 0$ such that

$$|\nu_n(\varphi, \psi) - \nu_n(\varphi, \psi, i, j)| \leq C_2\left(\frac{1}{\sqrt{i}} + \frac{1}{\sqrt{j}}\right)$$

Let i and j go to $+\infty$; we obtain convergence a).

Proof of convergence b. – Fix two integers i and j; we have

$$\nu_n(\varphi,\psi,i,j)=\int_{G^{j+1}} E_n(\varphi,\psi,g,h_1,h_2,\cdots,h_j) \times P^i_{\mathcal{A}}(e,dg)\mu(dh_1)\mu(dh_2)\cdots\mu(dh_j)$$

with

$$\begin{aligned}E_n&(\varphi,\psi,g,h_1,h_2,\cdots,h_j)\\ &=\mathbb{E}\Bigg[\left[\max(A^{i+1}_{i+1},\cdots,A^{n-j}_{i+1})\le\frac{1}{a(g)}\right]\\ &\qquad\cap\left[A^{n-j}_{i+1}\le\min\left(\frac{1}{a(g)},\frac{1}{a(g)a(h_1)},\cdots,\frac{1}{a(g)a(h_1)\cdots a(h_j)}\right)\right];\\ &\qquad\times\varphi(a(g)A^{n-j}_{i+1}a(h_1)\cdots a(h_j))\psi(b(g)+a(g)A^{n-j}_{i+1}b(h_1\cdots h_j)\Bigg]\end{aligned}$$

Using Theorem II-7, one may see that, for any $g,h_1,\cdots,h_j\in G$, the sequence $(n^{3/2}E_n(\varphi,\psi,g,h_1,h_2,\cdots,h_j))_{n\geq1}$ converges to a finite limit. To obtain the convergence b), we have to use Lebesgue dominated convergence theorem and therefore, we have to obtain an appropriate upperbound for $n^{3/2}E_n(\varphi,\psi,g,h_1,h_2,\cdots,h_j)$. Note that

$$\left[\max(A^{i+1}_{i+1},\cdots,A^{n-j}_{i+1})\le\frac{1}{a(g)}\right]\subset\left[\max(1,A^{i+1}_{i+1},\cdots,A^{n-j}_{i+1})\le\frac{1}{a(g)}\right]$$

because $a(g)\le 1$ and

$$\left[A^{n-j}_{i+1}\le\min\left(\frac{1}{a(g)},\frac{1}{a(g)a(h_1)},\cdots,\frac{1}{a(g)a(h_1)\cdots a(h_j)}\right)\right]\subset\left[A^{n-j}_{i+1}\le\frac{1}{a(g)}\right].$$

Since $|\varphi(a)|\le Ka^\epsilon$ for any $a>0$, one thus obtains

$$\begin{aligned}n^{3/2}&E_n(\varphi,\psi,g,h_1,h_2,\cdots,h_j)\\ &\le C\|\psi\|_\infty n^{3/2}\mathbb{E}\Bigg[a(g)^\epsilon(A^{n-j}_{i+1})^\epsilon a(h_1)^\epsilon\cdots a(h_j)^\epsilon\\ &\qquad\times\frac{1}{a(g)^{2\epsilon}\max(1,A^{i+1}_{i+1},\cdots,A^{n-j}_{i+1})^{2\epsilon}}\\ &\qquad\times\frac{1}{(A^{n-j}_{i+1})^{\epsilon/2}a(g)^{\epsilon/2}}\Bigg]\\ &\le C\|\psi\|_\infty a(g)^{-3\epsilon/2}a(h_1)^\epsilon\cdots a(h_j)^\epsilon\, n^{3/2}\\ &\qquad\times\mathbb{E}[(A^{n-j}_{i+1})^{\epsilon/2}\max(1,A^{i+1}_{i+1},\cdots,A^{n-j}_{i+1})^{-2\epsilon}]\\ &\le C_1 a(g)^{-3\epsilon/2}\,a(h_1)^\epsilon\cdots a(h_j)^\epsilon\end{aligned}$$

the last inequality being guaranteed by Theorem II.7. Then, by Hypothesis A2, for ϵ small enough, one may use Lebesgue dominated convergence theorem and convergence $b)$ follows. $\square$

Proof of Lemma III.3. – By a duality argument, it suffices to prove that, for some $\epsilon > 0$

$$\sup_{n\geq 1} n^{3/2}\mathbb{E}[[T_{\mathcal{A}} > n];\ (A_1^n)^{2\epsilon}\|B_1^n\|^{\epsilon}] < +\infty.$$

Using the identity $B_1^n = \sum_{k=1}^n A_1^{k-1}b_k$, we obtain

$$\mathbb{E}[[T_{\mathcal{A}} > n];\ (A_1^n)^{2\epsilon}\|B_1^n\|^{\epsilon}] \leq \sum_{k=1}^{n} \mathbb{E}[[T_{\mathcal{A}} > n];\ (A_1^{k-1})^{3\epsilon}a_k^{2\epsilon}\|b_k\|^{\epsilon}(A_{k+1}^n)^{2\epsilon}].$$

By the definition of $T_{\mathcal{A}}$, we have

$$\begin{aligned}
&\mathbb{E}[[T_{\mathcal{A}} > n];\ (A_1^{k-1})^{3\epsilon}a_k^{2\epsilon}\|b_k\|^{\epsilon}(A_{k+1}^n)^{2\epsilon}] \\
&\leq \mathbb{E}\Bigg[[A_1^1 \leq 1] \cap \cdots [A_1^{k-1} \leq 1] \cap \left[a_k \leq \frac{1}{A_1^{k-1}}\right] \\
&\qquad \cap \left[A_{k+1}^{k+1} \leq \frac{1}{A_1^{k-1}a_k}\right] \cap \cdots \left[A_{k+1}^{n} \leq \frac{1}{A_1^{k-1}a_k}\right]; \\
&\qquad (A_1^{k-1})^{3\epsilon}a_k^{2\epsilon}\|b_k\|^{\epsilon}(A_{k+1}^n)^{2\epsilon}\Bigg] \\
&\leq \int_G a(g)^{3\epsilon} \quad \left[\int_{\{h\in G:a(g)a(h)\leq 1\}} a(h)^{2\epsilon}\|b(h)\|^{\epsilon}K_{k,n}(g,h)\mu(dh)\right] \\
&\quad \times P_{\mathcal{A}}^{k-1}(e,dg)
\end{aligned}$$

with

$$\begin{aligned}
K_{k,n}(g,h) &= \mathbb{E}\Bigg[\left[A_{k+1}^{k+1} \leq \frac{1}{a(g)a(h)}\right] \cap \cdots \\
&\qquad \cap \left[(A_{k+1}^n) \leq \frac{1}{a(g)a(h)}\right];\ (A_{k+1}^n)^{2\epsilon}\Bigg] \\
&= \mathbb{E}\Bigg[\left[A_1^1 \leq \frac{1}{a(g)a(h)}\right] \cap \cdots \\
&\qquad \cap \left[A_1^{n-k} \leq \frac{1}{a(g)a(h)}\right];\ (A_1^{n-k})^{2\epsilon}\Bigg]
\end{aligned}$$

$$\leq \mathbb{E}\left[\left[\max(1, A_1^1, \cdots, A_1^{n-k}) \leq \frac{1}{a(g)a(h)}\right]; \ (A_1^{n-k})^{2\epsilon}\right]$$

since $a(g)a(h) \leq 1$

$$\leq \frac{1}{a(g)^{5\epsilon/2}a(h)^{5\epsilon/2}}$$
$$\times \mathbb{E}\left[\exp\left(-\frac{\epsilon}{2}M_{n-k}\right)\exp(-2\epsilon(M_{n-k} - S_{n-k}))\right]$$
$$\leq \frac{1}{a(g)^{5\epsilon/2}a(h)^{5\epsilon/2}} \frac{C_1}{(n-k)^{3/2}}$$

Hence

$$\mathbb{E}[[T_{\mathcal{A}} > n]; \ (A_1^n)^{2\epsilon} \ \|B_1^n\|^\epsilon]$$
$$\leq \sum_{k=1}^{n} \frac{C_1}{(n-k)^{3/2}} \mathbb{E}\left[\frac{\|b_1\|^\epsilon}{a_1^{\epsilon/2}}\right] \int_G a(g)^{\epsilon/2} P_{\mathcal{A}}^{k-1}(e, dg)$$

One concludes using Hypothesis A2 and the fact that the sequence $(n^{3/2}\sum_{k=1}^{n-1} \frac{1}{k^{3/2}(n-k)^{3/2}})_{n\geq 1}$ is bounded. □

Proof of Lemma III.4. – By Theorem II.3, there exist $n_0 \in \mathbb{N}$, $C_0 > 0$ and $[\alpha, \beta] \subset \mathbb{R}^{*+}$ such that

$$\forall n \geq n_0 \quad n^{3/2}\mathbb{E}[[T_{\mathcal{A}} > n] \cap [\alpha \leq A_1^n \leq \beta]] \geq C_0.$$

On the other hand

$$n^{3/2}\mathbb{E}[[T_{\mathcal{A}} > n] \cap [\alpha \leq A_1^n \leq \beta] \cap [\|B_1^n\| \geq B]]$$
$$\leq \frac{n^{3/2}}{B^\epsilon}\mathbb{E}[[T_{\mathcal{A}} > n] \cap [\alpha \leq A_1^n \leq \beta]; \ \|B_1^n\|^\epsilon].$$

By Lemma III.3, we have $\sup_{n\geq 1} n^{3/2}\mathbb{E}[[T_{\mathcal{A}} > n] \cap [\alpha \leq A_1^n \leq \beta]; \|B_1^n\|^\epsilon] < +\infty$; so, one can choose $B > 0$ such that

$$\forall n \geq n_0 \quad n^{3/2}\mathbb{E}[[T_{\mathcal{A}} > n] \cap [\alpha \leq A_1^n \leq \beta] \cap [\|B_1^n\| \leq B]] \geq \frac{C_0}{2}.$$

The lemma readily follows from the inequality

$$n^{3/2}\mathbb{E}[[\alpha \leq A_1^n \leq \beta] \cap [\|B_1^n\| \leq B]]$$
$$\geq n^{3/2}\mathbb{E}[[T_{\mathcal{A}} > n] \cap [\alpha \leq A_1^n \leq \beta] \cap [\|B_1^n\| \leq B]]. \qquad \square$$

III.c. Proof of Theorem B

We just indicate how to modify the proof in the previous section to obtain Theorem B. For any continuous function ψ with compact support on $\mathbb{R}^d$ we have by the Spitzer-Grincevicius factorisation

$$\mathbb{E}[\psi(B_1^n)] = \sum_{k=0}^{n} J_{k,n}(\psi)$$

with $J_{k,n}(\psi) = \int_G \psi(\frac{b(g)+b(h)}{a(g)})\ P_{\mathcal{A}'}^k(e, dg) P_{\mathcal{A}}^{n-k}(e, dh)$. First, we control the sum $\sum_{k=i+1}^{n-j} J_{k,n}(\psi)$ for fixed large enough integers i and j.

LEMMA III.5. – *There exists $\lambda > 0$ such that for any $\lambda \in]0, \lambda_0]$, any $g \in G$ and any $l > 0$, one has*

$$\int_G \psi\left(\frac{b(g)+b(h)}{a(g)}\right) P_{\mathcal{A}}^l(e, dh) \leq \frac{C}{\sqrt{l}}\ a(g)^\lambda$$

By Theorem II.3, the sequence $(k^{3/2} \int_G a(g)^\lambda \tilde{P}_{\mathcal{A}'}^k(e, dg))_{k \geq 0}$ is bounded since

$$\int_G a(g)^\lambda \tilde{P}_{\mathcal{A}'}^k(e, dg) = \mathbb{E}[[\tilde{T}_{\mathcal{A}'} > k]\ ; \exp(\lambda \tilde{S}_k)].$$

For any $0 < k < n$, we thus have

$$\int_{G \times G} \psi\left(\frac{b(g)+b(h)}{a(g)}\right) \tilde{P}_{\mathcal{A}'}^k(e, dg) P_{\mathcal{A}}^{n-k}(e, dh) \leq \frac{C_1}{k^{3/2}\sqrt{n-k}}.$$

Note that there exists $C > 0$ such that for any n, i, j in $\mathbb{N}^*$, $1 < i < n-j < n$, one has

$$\sqrt{n} \sum_{k=i+1}^{n-j} \frac{1}{k^{3/2}\sqrt{n-k}} \leq C\left(\frac{1}{\sqrt{i}} + \frac{1}{\sqrt{j}}\right);$$

therefore one may choose i and j such that $\limsup_{n \to +\infty} \sqrt{n} \sum_{k=i}^{n-j} J_{k,n}$ is as small as wanted.

Next, we look at the behaviour of the integral $\int_G \psi(\frac{b(g)+b(h)}{a(g)}) P_{\mathcal{A}}^l(e, dh)$ as l goes to $+\infty$.

LEMMA III.6. – *For any $g \in G$, the sequence*

$$\left(\sqrt{l} \int_G \psi\left(\frac{b(g)+b(h)}{a(g)}\right) P_{\mathcal{A}}^l(e, dh)\right)_{l \geq 0}$$

converges to a finite limit as l goes to $+\infty$.

In particular $(\sqrt{n}J_{0,n}(\psi))_{n\geq 1}$ converges in $\mathbb{R}$. Furthermore, for any $i, j \in \mathbb{N}$ and any compact set $K \subset \mathbb{R}^{*+} \times \mathbb{R}$, the dominated convergence theorem ensures the existence of a finite limit as n goes to $+\infty$ for the sequence $(\sqrt{n}\sum_{k=1}^{i} J_{k,n}(\psi, K))_{n\geq 0}$ where

$$J_{k,n}(\psi, K) = \int_G 1_K(g)\left(\int_G \psi\left(\frac{b(g)+b(h)}{a(g)}\right)P_{\mathcal{A}}^{n-k}(e,dh)\right)\tilde{P}_{\mathcal{A}'}^{k}(e,dg).$$

The only thing we have now to check is that the indicator function 1_K does not disturb too much the behaviour of the above integrals. Fix $0<\delta<1$; according to Lemma III.5, we have

$$\begin{aligned}
&\sum_{k=1}^{i}\int_{\{g\in G: a(g)\leq\delta\}}\left(\int_G \psi\left(\frac{b(g)+b(h)}{a(g)}\right)P_{\mathcal{A}}^{n-k}(e,dh)\right)\ \tilde{P}_{\mathcal{A}'}^{k}(e,dg)\\
&\quad\leq C(\lambda,\psi)\sum_{k=1}^{i}\frac{1}{\sqrt{n-k+1}}\mathbb{E}[[\tilde{T}_{\mathcal{A}'}>k]\cap[\tilde{S}_k\leq \mathrm{Log}\delta];\ \exp(\lambda\tilde{S}_k)]\\
&\quad\leq C(\lambda,\psi)\ \delta^{\lambda/2}\sum_{k=1}^{i}\frac{1}{\sqrt{n-k+1}}\ E\left[[\tilde{T}_{\mathcal{A}'}>k];\ \exp\left(\frac{\lambda}{2}\tilde{S}_k\right)\right]\\
&\quad\leq C_1\delta^{\lambda/2}\sum_{k=1}^{i}\frac{1}{\sqrt{n-k+1}k^{3/2}}\\
&\quad\leq C_1\delta^{\lambda/2}\frac{1}{\sqrt{n-i+1}}\sum_{k=1}^{+\infty}\frac{1}{k^{3/2}}
\end{aligned}$$

Note that by definition of $\tilde{P}_{\mathcal{A}'}$ one has

$$\sum_{k=1}^{i}\int_{\{g\in G: a(g)\geq 1/\delta\}}\left(\int_G \psi\left(\frac{b(g)+b(h)}{a(g)}\right)P_{\mathcal{A}}^{n-k}(e,dh)\right)\tilde{P}_{\mathcal{A}'}^{k}(e,dg)=0.$$

On the other hand, fix $B>0$; according to Lemma III.5, we have

$$\begin{aligned}
&\sum_{k=1}^{i}\int_{\{g\in G: \|b(g)\|\geq B\}}\left(\int_G \psi\left(\frac{b(g)+b(h)}{a(g)}\right)P_{\mathcal{A}}^{n-k}(e,dh)\right)\tilde{P}_{\mathcal{A}'}^{k}(e,dg)\\
&\quad\leq C(\lambda,\psi)\sum_{k=1}^{i}\frac{1}{\sqrt{n-k+1}}\mathbb{E}[[\tilde{T}_{\mathcal{A}'}>k]\cap[\|\tilde{B}_k\|\geq B];\ \exp(\lambda\tilde{S}_k)]\\
&\quad\leq \frac{C(\lambda,\psi)}{B^{\lambda/2}}\sum_{k=1}^{i}\frac{1}{\sqrt{n-k+1}}\ E[[\tilde{T}_{\mathcal{A}'}>k];\ \exp(\lambda\tilde{S}_k)\ \|\tilde{B}_k\|^{\lambda/2}]\\
&\quad\leq \frac{C_1}{B^{\lambda/2}}\frac{1}{\sqrt{n-i+1}}\sum_{k=1}^{+\infty}\frac{1}{k^{3/2}},
\end{aligned}$$

where the last inequality is guaranteed by Lemma III.3.

Note the same upperbounds hold when the sum $\sum_{k=1}^{i}$ is replaced by $\sum_{k=n-j+1}^{n-1}$.

Finally, using Spitzer-Grincevicius factorisation, we have proved that, for any $\epsilon > 0$, there exist $i, j \in \mathbb{N}$ and a compact set $K \subset G$ such that for any $n > i + j$ we have

$$|\sqrt{n}\mathbb{E}[\psi(B_1^n)] - \sqrt{n}\sum_{k=0}^{i} J_{k,n}(\psi, K) - \sqrt{n} \sum_{k=n-j+1}^{n} J_{k,n}(\psi, K)|| \leq \epsilon.$$

Since $(\sqrt{n}\sum_{k=1}^{i} J_{k,n}(\psi, K) + \sqrt{n}\sum_{k=n-j+1}^{n} J_{k,n}(\psi, K))_{n\geq 0}$ converges, the sequence $(\sqrt{n}\mathbb{E}[\psi(B_1^n)])_{n\geq 0}$ has a finite limit which is not always zero. It just remains to establish Lemmas III.5 and III.6; they may be easily obtained using Theorems II.2 and II.3 and by obvious modifications in the proofs of Lemmas III.1 and III.2 respectively. □

III.d. Proof of Theorem C : identification of the limit measure ν_0

We are not always able to explicit the form of the limit measure ν_0; nevertheless, if one assumes further hypotheses on μ, it is possible to identify ν_0, up to a multiplicative constant. In this section, we suppose that μ satisfies Hypotheses A1, A2, A3 and also the two following conditions

(C1) *the density ϕ_μ of μ is continuous with compact support.*

(C2) $\phi_\mu(e) > 0$.

Remark. – Note that under these conditions, the semi-group generated by the support S_μ of μ is dense in G. Moreover, there exists $\gamma > 0$ such that $\mu * \mu \geq \gamma\mu$.

To establish Theorem C we first prove that the random walk of distribution μ on G satisfies a ratio-limit theorem and secondly we show that the equation $\mu * \nu = \nu * \mu = \nu$ has a unique solution $\nu_0 \not\equiv 0$ (up to a multiplicative constant) in the class of Radon measures on G. Let $CK^+(G)$ be the space of positive continuous functions with compact support on G; we have

LEMMA III.7. – *Under the hypotheses of Theorem C, we have*

$$\forall\varphi \in CK^+(G),\ \forall g \in G \quad \lim_{n\to+\infty} (\delta_g * \mu^{*n}(\varphi))^{1/n} = 1.$$

In particular $\lim_{n\to+\infty} \frac{\delta_g*\mu^{*(n+1)}(\varphi)}{\delta_g*\mu^{*n}(\varphi)} = 1$ for any $g \in G$ and any function $\varphi \in CK^+(G), \varphi \not\equiv 0$. Since there exists $\gamma > 0$ such that $\mu * \mu \geq \gamma\mu$ we may thus apply the following proposition due to Y. Guivarc'h [11]:

PROPOSITION III.8. – *Suppose that the semi-group generated by the support of* μ *is dense in* G *and that, for any* $\varphi \in CK^+(G)$, *the sequence* $(\frac{\mu^{*(n+1)}(\varphi)}{\mu^{*n}(\varphi)})_{n\geq 1}$ *converges to a constant* c_0 *which does not depend on* φ. *Then, if the equation* $\nu * \mu = \mu * \nu = c_0\nu$ *has a unique solution* $\nu_0 \not\equiv 0$, *up to a multiplicative constant, in the class of Radon measures on* G, *we have*

$$\lim_{n\to+\infty} \frac{\mu^{*n}(\varphi)}{\mu^{*n}(\psi)} = \frac{\nu_0(\varphi)}{\nu_0(\psi)}$$

for any φ *and* $\psi \in CK^+(G)$ *such that* $\nu_0(\psi) > 0$.

We have here $c_0 = 1$; to prove Theorem C, it suffices to establish the following lemma :

LEMMA III.9. – *Under hypotheses of Theorem C, the equation* $\nu * \mu = \mu * \nu = \nu$ *has one and only one (up to a multiplicative constant) solution* $\nu_0 \not\equiv 0$ *in the class of Radon measures on* G. *Moreover, this solution may be decomposed as follows*

$$\nu_0 = (\delta_1 \otimes \lambda) * \overline{\left(\frac{da}{a} \otimes \lambda_1\right)}$$

where λ *(respectively* λ_1*) is, up to a multiplicative constant, the unique Radon measure on* $\mathbb{R}^d$ *which satisfies the convolution equation* $\mu * \lambda = \lambda$ *(resp.* $\overline{\mu} * \lambda_1 = \lambda_1$*).*

By Theorem A one can choose $\psi_0 \in CK^+(G)$ such that $(n^{3/2}\mu^{*n}(\psi_0))_{n\geq 0}$ converges to 1; for any $\varphi \in CK^+(G)$ we thus have

$$\lim_{n\to+\infty} n^{3/2}\mu^{*n}(\varphi) = \frac{\nu_0(\varphi)}{\nu_0(\psi_0)}.$$

This achieves the proof of Theorem C; it remains to establish the Lemmas III.7 and III.9.

Proof of Lemma III.7. – Fix a function $\varphi \in CK^+(G)$ and for any $n \geq 1$ consider the set

$$K_n(\varphi) = \{gh^{-1}/g \in \text{Support } (\varphi) \text{ and } h \in \text{Support } (\mu^{*n})\}.$$

The sets $K_n(\varphi)$, $n \geq 1$, are compact, $K_n(\varphi) \subset K_{n+1}(\varphi)$ and $\bigcup_{n=1}^{+\infty} K_n(\varphi) = G$. Then, there exists n_0 such that the compact set K_0 introduced in Lemma III.4 is included in the interior of $K_{n_0}(\varphi)$.

Consequently, the continuous function $g \mapsto \int_G \varphi(gh)\mu^{*n_0}(dh)$ is strictly positive on K_0 and there exists a constant $C > 0$ such that

$$\forall g \in G \quad \int_G \varphi(gh)\mu^{*n_0}(dh) \geq C\ 1_{K_0}(g).$$

Thus, for any $n \geq 1$, one has $\delta_g * \mu^{*(n_0+n)}(\varphi) \geq C\mu^{*n}(K_0) \geq \frac{C_1}{n^{3/2}}$ with $C_1 > 0$ by Lemma III.4. For any $g \in G$ we thus have $\liminf_{n\to+\infty}(\delta_g * \mu^{*n}(\varphi))^{1/n} \geq 1$. On the other hand $\delta_g * \mu^{*n}(\varphi) \leq \|\varphi\|_\infty$ for any $n \geq 1$ which implies $\limsup_{n\to+\infty}(\delta_g * \mu^{*n}(\varphi))^{1/n} \leq 1$. □

Proof of Lemma III.9. – Let $\mathcal{H}_\mu$ be the set of positive measures ν on G such that $\nu * \mu = \nu$. Recall that $\mathcal{H}_\mu$ is a weakly closed cone with a compact basis and that it is a lattice. By [7] (and more recently by [2] without condition of density) there exists (up to a multiplicative constant) a unique positive measure λ_1 on $\mathbb{R}^d$ such that $\overline{\mu} * \lambda_1 = \lambda_1$; furthermore, λ_1 is a Radon measure on $\mathbb{R}^d$ and the extremal rays of $\mathcal{H}_\mu$ are the positives measures which are proportional, either to the right Haar measure m_D, or to the measures $\delta_{(1,z)} * \overline{(\frac{da}{a} \otimes \lambda_1)}$, $z \in \mathbb{R}^d$. By Choquet's representation theorem, there exist $C_\nu \in \mathbb{R}^+$ and a positive measure m_ν on $\mathbb{R}^d$ such that

$$\nu = C_\nu\ m_D + \int_{\mathbb{R}^d} \delta_{(1,z)} * \overline{\left(\frac{da}{a} \otimes \lambda_1\right)}\ m_\nu(dz)$$

and C_ν and m_ν are unique because $\mathcal{H}_\mu$ is a lattice.

Fix ν in $\mathcal{H}_\mu$; a direct computation leads to

$$\mu * \nu = C_\nu \int_G \operatorname{Log}\ a(g)\ \mu(dg)\ m_D + \int_{\mathbb{R}^d} \delta_{(1,z)} * \overline{\left(\frac{da}{a} \otimes \lambda_1\right)}\ \mu * m_\nu(dz).$$

Then, if one suppose that $\mu * \nu = \nu$, the uniqueness of the Choquet's representation gives

$$C_\nu \int_G \operatorname{Log}\ a(g)\ \mu(dg) = C_\nu \qquad \text{and} \qquad \mu * m_\nu = m_\nu.$$

Since $\int_G a_1\ \overline{\mu}(dg_1) > 1$, one obtains $C_\nu = 0$. On the other hand, by [7], the equation $\mu * m = m$ has a unique solution (up to a multiplicative constant) λ in the set of positive measures on $\mathbb{R}^d$ which leads to the equality $m_\nu = \lambda$.

Finally the solution ν_0 of the equation $\nu * \mu = \mu * \nu = \nu$ is unique (up to a multiplicative constant) in the set of positive measure on $\mathbb{R}^d$, it is a Radon measure and it can be decompose as follows

$$\nu_0 = (\delta_1 \otimes \lambda) * \overline{\left(\frac{da}{a} \otimes \lambda_1\right)} \qquad \square$$

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