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ABSTRACT. – We study k-fold self-intersection local times $\alpha(x_2, \ldots, x_k; t)$, and k-fold renormalized self-intersection local times $\gamma(x_2, \ldots, x_k; t)$ for Levy processes. Our main result says that the k-fold renormalized self-intersection local time $\gamma(x_2, \ldots, x_k; t)$ for the symmetric stable process of order β in $\mathbf{R}^2$ is jointly continuous almost surely if and only if $(2k-1)(2-\beta) < 2$.

Key words: Intersections, renormalization.

1. INTRODUCTION

The object of this paper is to establish the almost sure joint continuity of intersection local time and renormalized intersection local time for the multiple intersections of a large class of Lévy process in $\mathbf{R}^m$ including the symmetric stable processes in the plane.

Intersection local times were originally envisioned as a means of "measuring" the amount of self-intersections of a stochastic process $X_t \in \mathbf{R}^m$. Formally, the k-fold intersection local time is

$$\alpha_k(t) = \int \cdots \int_{\{0 \le t_1 \le \cdots \le t_k \le t\}} \prod_{j=2}^{k} \delta(X_{t_j} - X_{t_{j-1}})\, dt_1 \ldots dt_k$$

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where $\delta(x)$ denotes the δ-function.

More precisely, we can set

$$(1.1)\quad \alpha_{k,\varepsilon}(t)=\int\cdots\int_{\{0\le t_1\le\cdots\le t_k\le t\}}\prod_{j=2}^{k} f_\varepsilon(X_{t_j}-X_{t_{j-1}})\,dt_1\ldots dt_k$$

where f_ε is an approximate δ-function, and try to take the $\varepsilon\to 0$ limit. In general, this limit will not exist! This gives rise to the problem of renormalization: the attempt to subtract from $\alpha_{k,\varepsilon}(t)$ terms involving lower order intersections, $\alpha_{j,\varepsilon}(t)$ for $j<k$, in such a way that a finite $\varepsilon\to 0$ limit results. This was originally done for double intersections of Brownian motion by Varadhan [17], and gave rise to a large literature summarized in Dynkin [3]. Such renormalized intersection local times have turned out to be the right tool for the solution of certain "classical" problems: the asymptotic expansion of the area of the Wiener and stable sausage in the plane, and the fluctuations of the range of stable random walks, *see* Le Gall [9], [8], Le Gall-Rosen [11] and Rosen [14]. For a clear account of progress concerning Brownian intersection local times up to 1990 *see* Le Gall's lecture notes [10]. Dynkin introduced the idea of studying $\alpha_{k,\varepsilon}(t)$ for $t=\zeta$ an independent mean 1 exponential time, and demonstrated how one could exploit the resulting simplifications. Dynkin's renormalization, introduced for Brownian motion in the plane, is

$$\gamma_k(\zeta)=\lim_{\varepsilon\to 0}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}g_\varepsilon^j(0)\alpha_{k-j,\varepsilon}(\zeta)$$

with $g_\varepsilon(x)=\int f_\varepsilon(y-x)g(y)\,dy$, where $g(x)=\int_0^\infty e^{-t}p_t(x)\,dt$. Here, we use the convention $\alpha_{1,\varepsilon}(t)=t$.

In [5] we suggested studying

$$(1.2)\quad \alpha_{k,\varepsilon}(x_2,x_3,\ldots,x_k;t)$$
$$=\int\cdots\int_{\{0\le t_1\le\cdots\le t_k\le t\}}\prod_{j=2}^{k} f_\varepsilon(X_{t_j}-X_{t_{j-1}}-x_j)\,dt_1\ldots dt_k.$$

$\alpha_k(x_2,x_3,\ldots,x_k;t)=\lim_{\varepsilon\to 0}\alpha_{k,\varepsilon}(x_2,x_3,\ldots,x_k;t)$ can often be realized as an occupation density which "measures" near-intersections. The occupation density formula states that

$$(1.3)\int\Phi(x_2,\ldots,x_k)\alpha_k(x_2,\ldots,x_k;t)\,dx_2\ldots dx_k$$
$$=\int\cdots\int_{\{0\le t_1\le\cdots\le t_k\le t\}}\Phi(X_{t_2}-X_{t_1},\ldots,X_{t_k}-X_{t_{k-1}})\,dt_1\ldots dt_k$$

for all bounded Borel measurable functions Φ on $(\mathbf{R}^m)^{k-1}$. We formally recover $\alpha_k(t)$ if we take $(x_2, x_3, \ldots, x_k) = (0, 0, \ldots, 0)$. However, in the cases we shall consider, $\alpha_k(x_2, x_3, \ldots, x_k; t)$ will exist only for $x_i \neq 0, \forall i$.

Following the work of Le Gall [7] for the case $k = 2$, we recast the renormalization problem as an attempt to subtract from the intersection local time $\alpha_k(x_2, x_3, \ldots, x_k; t)$ terms involving lower order intersection local times so that the resulting function has a continuous extension to all $(x_2, x_3, \ldots, x_k; t)$. This continuous extension will allow us to exhibit the exact form of the asymptotics of $\alpha_k(x_2, x_3, \ldots, x_k; t)$ as the $x_i \to 0$.

In [16], [15] we showed how to construct a jointly continuous renormalized triple intersection local time for Brownian motion and stable processes in the plane. In a recent paper Bass and Khoshnevisan [1] have constructed a jointly continuous renormalized intersection local time for k-multiple intersections of planar Brownian motion. Werner [18] then showed that for $t = \zeta$ an independent mean 1 exponential time, the renormalization of Bass and Khoshnevisan agrees with that of Dynkin when $(x_2, x_3, \ldots, x_k) = (0, 0, \ldots, 0)$ and provides a natural generalization for arbitrary x_i.

These recent results of Bass, Khoshnevisan and Werner have motivated the present paper. We define the renormalized k-fold intersection local time for $x = (x_2, \ldots, x_k) \in (\mathbf{R}^m - \{0\})^{k-1}$ by

$$\gamma_k(x; t) = \sum_{A \subseteq \{2, \ldots, k\}} (-1)^{|A|} \Big(\prod_{j \in A} g(x_j)\Big) \alpha_{k-|A|}(x_{A^c}; t) \tag{1.4}$$

where for any $B = \{i_1 < \cdots < i_{|B|}\} \subseteq \{2, \ldots, k\}$

$$x_B = (x_{i_1}, x_{i_2}, \ldots, x_{i_{|B|}}). \tag{1.5}$$

Here, we use the convention $\alpha_1(t) = t$. We now state our main result.

THEOREM 1. – *Let X_t denote the symmetric stable process of order β in $\mathbf{R}^2$. If $(2k - 1)(2 - \beta) < 2$, then, restricted to $(x, t) \in (\mathbf{R}^2 - \{0\})^{k-1} \times R_+$*

$$\alpha_k(x; t) = \lim_{\varepsilon \to 0} \alpha_{k,\varepsilon}(x; t) \tag{1.6}$$

exists and is jointly continuous a.s., and $\gamma_k(x; t)$, defined for $(x, t) \in (\mathbf{R}^2 - \{0\})^{k-1} \times R_+$, has an extension to $(\mathbf{R}^2)^{k-1} \times R_+$ which is jointly continuous a.s.

Note that this result is best possible in the sense that we know from [13] that our renormalization will not converge if $(2k - 1)(2 - \beta) \geq 2$. We also note that the symmetric stable process of order β in $\mathbf{R}^2$ will have k-fold intersections a.s. if and only if $k(2-\beta) < 2$. For this and other details concerning k-multiple points of Levy processes *see* [4], [6] and references therein.

Our renormalization is similar in form to the renormalization used by Bass and Khoshnevisan for Brownian motion in $\mathbf{R}^2$, and our proof gives an alternate derivation of their a.s. joint continuity result. However, our methods do not yield the a.s. joint Holder continuity which they obtain.

Simple combinatorics show that

$$\alpha_k(x;t) = \sum_{A\subseteq\{2,\ldots,k\}} \Big(\prod_{j\in A} g(x_j)\Big)\gamma_{k-|A|}(x_{A^c};t).$$

Since the γ_j are continuous, we can read off the asymptotics of $\alpha_k(x;t)$ as the $x_i \to 0$.

Let us define the approximate renormalized k-fold intersection local time for $x = (x_2, \ldots, x_k) \in (\mathbf{R}^m)^{k-1}$ by

$$\gamma_{k,\varepsilon}(x;t) = \sum_{A\subseteq\{2,\ldots,k\}} (-1)^{|A|}\Big(\prod_{j\in A} g_\varepsilon(x_j)\Big)\alpha_{k-|A|,\varepsilon}(x_{A^c};t). \tag{1.7}$$

THEOREM 2. – *Let X_t denote the symmetric stable process of order β in $\mathbf{R}^2$. If $(2k-1)(2-\beta) < 2$, then $\gamma_{k,\varepsilon}(x;t)$ converges to $\gamma_k(x;t)$ locally uniformly on $(\mathbf{R}^2)^{k-1} \times R_+$ as $\varepsilon \to 0$ with probability* 1.

In particular we have

$$\gamma_k(0;t) = \lim_{\varepsilon\to 0} \gamma_{k,\varepsilon}(0;t) = \lim_{x\to 0} \gamma_k(x;t) \quad a.s. \tag{1.8}$$

Remark 1. – A function $Z_\varepsilon(x)$ indexed by $\varepsilon \in (0,1]$ and x in a topological space $\mathcal{S}$ will be said to converge locally uniformly on $\mathcal{S}$ as $\varepsilon \to 0$ if for any compact $K \in \mathcal{S}$, $Z_\varepsilon(x)$ converges uniformly in $x \in K$ as $\varepsilon \to 0$.

Our paper is organized as follows. After laying the groundwork in sections 2 and 3, we establish general criteria in section 4 for the existence and almost sure continuity of $\alpha_k(x_2, x_3, \ldots, x_k; \zeta)$ when $x_i \neq 0, \forall i$, where ζ is an independent mean-1 exponential random variable. Section 5 gives a general criteria for existence of an almost sure continuous extension to $(\mathbf{R}^m)^{k-1}$ of the renormalized k-fold intersection local time, again at an independent exponential time. In section 6 we obtain a.s. joint continuity. In section 7 we show that our theorems cover the symmetric stable processes of order β in the plane when $(2k-1)(2-\beta) < 2$.

2. INTERSECTION LOCAL TIMES: MOMENTS

Let X_t denote a Lévy process in $\mathbf{R}^m$ with transition densites $p_t(x)$. In this section we introduce approximate intersection local times for X_t and compute the expectations of their moments.

Let f denote a smooth positive function supported on the unit ball of $\mathbf{R}^m$ with $\int f(x)\,dx = 1$, and for any $\varepsilon > 0$ let

$$f_\varepsilon(y) = \frac{1}{\varepsilon^m} f\left(\frac{y}{\varepsilon}\right)$$

and $f_{\varepsilon,x}(y) = f_\varepsilon(y - x)$. We define the approximate intersection local time of order k as

$$\begin{aligned}(2.1)\quad & \alpha_{k,\varepsilon}(x_2, x_3, \ldots, x_k; t) \\ &= \int \cdots \int_{\{0 \le t_1 \le \cdots \le t_k \le t\}} \prod_{j=2}^{k} f_{\varepsilon,x_j}(X_{t_j} - X_{t_{j-1}})\, dt_1 \ldots dt_k.\end{aligned}$$

We often abbreviate this as $\alpha_{k,\varepsilon}(x;t)$ where $x = (x_2, x_3, \ldots, x_k) \in (\mathbf{R}^m)^{k-1}$. Let $g(x) = \int_0^\infty e^{-t} p_t(x)\,dt$ denote the Green's function for X_t, and let ζ denote a mean-1 exponential random variable independent of X_t. The following expectation follows easily from the Markov property for X_t.

LEMMA 1.

$$\begin{aligned}(2.2)\qquad & E\left(\prod_{i=1}^{n} \alpha_{k_i,\varepsilon_i}(x^i;\zeta)\right) \\ &= \sum_{v\in\mathcal{V}} \int \prod_{i=1}^{n} \prod_{j=2}^{k_i} f_{\varepsilon_i,x_j^i}(y_j^i - y_{j-1}^i) \\ &\quad \times \prod_{p=1}^{k} g(w_{v(p)} - w_{v(p-1)})\, dw_1 \ldots dw_k\end{aligned}$$

where $x^i = (x_2^i, x_3^i, \ldots, x_{k_i}^i)$, $k = \sum_{i=1}^n k_i$, $(w_1, \ldots, w_k) = (y^1, \ldots, y^n) \in (\mathbf{R}^m)^k$ *and* $\mathcal{V}$ *is the set of bijections* v *of* $\{1, 2, \ldots, k\}$ *such that whenever* $w_{v(p)} = y_j^i, w_{v(\tilde p)} = y_{\tilde j}^i$ *we have* $p > \tilde p \iff j > \tilde j$.

A change of variables leads to the following more useful formula.

LEMMA 2.

$$(2.3)\ E\left(\prod_{i=1}^{n}\alpha_{k_i,\varepsilon_i}(x^i;\zeta)\right)$$

$$=\sum_{s\in\mathcal{S}}\int\prod_{i=1}^{n}\prod_{j=2}^{k_i}f(y_j^i)\prod_{p=1}^{k}g\left(z_{s(p)}+\sum_{j=2}^{c(p)}(\varepsilon_{s(p)}y_j^{s(p)}+x_j^{s(p)})\right.$$

$$\left.-\left(z_{s(p-1)}+\sum_{j=2}^{c(p-1)}(\varepsilon_{s(p-1)}y_j^{s(p-1)}+x_j^{s(p-1)})\right)\right)dy_j^i\,dz_1\dots dz_n$$

where $x^i=(x_2^i,x_3^i,\dots,x_{k_i}^i)$, $k=\sum_{i=1}^{n}k_i$, $\mathcal{S}$ *is the set of mappings s:* $\{1,2,\dots,k\}\mapsto\{1,\dots,n\}$ *such that* $|s^{-1}(i)|=k_i,\forall 1\le i\le n$, *and* $c(p)=|\{u\le p\mid s(u)=s(p)\}|$.

3. THE BASIC LEMMA

A finite chain C is a finite linearly ordered set. If $i\in C$ we use $i-1,i+1$ to denote the immediate predecessor and successor of i (when they exist). If i is the first element in C, we set $i-1=0$. We use f to denote the final element of C. As usual, $|C|$ will denote the cardinality of the finite set C.

Let us define a k-fold n-colored chain as a pair $\{C,r\}$ consisting of a finite chain C and a function r from C to $\{1,2,\dots,n\}$ satisfying the following two properties

1. for each $1\le j\le n$, $r^{-1}(j)$ is non-empty with at most k elements.
2. $r(i)\ne r(i-1)$ for all $i,i-1\in C$.

We think of r as a coloring of C. $r(i)$ indicates the color of the element $i\in C$. Property 1 says that each of n possible colors appears at least once–but no more than k times, while property 2 says that adjacent elements in C must have different colors.

The following simple lemma is the key to our results.

LEMMA 3 (The Basic Lemma). – *Let* $h\ge 0$ *be a spherically symmetric function on* $\mathbf{R}^m$ *with*

$$(3.1)\qquad \|h\|_j<\infty,\qquad \forall j\le 2k-1$$

and such that $h(x)$ *is monotonically decreasing in* $|x|$.

Then if $\{C,r\}$ *is any k-fold n-colored chain we have*

$$(3.2)\qquad \sup_{\bar a\in(\mathbf{R}^m)^{|C|}}\int\prod_{i\in C}h(x_{r(i)}-x_{r(i-1)}+a_i)\,dx_1\,dx_2\cdots dx_n<\infty$$

where we set $x_{r(0)} = 0$, *and* $\bar{a} = (a_1, \ldots, a_{|C|}) \in (\mathbf{R}^m)^{|C|}$.

We first prove a variant of the Basic Lemma:

LEMMA 4. – *Let* $h \geq 0$ *be a spherically symmetric function on* $\mathbf{R}^m$ *with*

$$(3.3) \qquad \|h\|_j < \infty, \qquad \forall j \leq 2k-1$$

and such that for some $d < \infty$

$$(3.4) \qquad \inf\{h(x), h(y)\} \leq dh(x+y), \qquad \forall x, y \in \mathbf{R}^m.$$

Then if $\{C, r\}$ *is any k-fold n-colored chain we have*

$$(3.5) \qquad \sup_{\bar{a} \in (\mathbf{R}^m)^{|C|}} \int \prod_{i \in C} h(x_{r(i)} - x_{r(i-1)} + a_i)\, dx_1\, dx_2 \cdots dx_n < \infty$$

where we set $x_{r(0)} = 0$, *and* $\bar{a} = (a_1, \ldots, a_{|C|}) \in (\mathbf{R}^m)^{|C|}$.

Proof of Lemma 5. – Fix k. We will argue by induction on n. $n = 1$ is trivial and if $n = 2$ our integral must be of the form

$$\int h(x) \prod_{i=1}^{j} h(x - y + b_i)\, dx\, dy \leq \|h\|_1 \|h\|_j^j$$

where $j = r^{-1}(1) + r^{-1}(2) - 1 \leq 2k - 1$, so that by (3.3) our lemma holds when $n = 2$.

We now give the inductive step. Let $\{C, r\}$ be a k-fold n-colored chain as in our lemma. By relabeling the colors we may assume that r maps the last element in C to n. Then the integral in our lemma will have at most $2k - 1$ factors of h containing the variable x_n. To motivate our proceedure, note that by (3.3) we could integrate out these factors. However, there would be no guarantee that the remaning integral can be associated with a k-fold n-1 colored chain. In particular, after integrating out all factors involving e.g. the variables $x_n, x_{n-1}, \ldots, x_{j+1}$ we may find that there no longer remain any h factors involving x_j, so that the dx_j integral will be infinite.

To handle this we proceed in increasing order through the elements of C. We first give an overview of our procedure, and then provide the details. Set $C_0 = C$. At the step corresponding to element i we will replace C_{i-1} by a subset $C_i \subseteq C_{i-1}$ such that when r is restricted to C_i, $\{C_i, r\}$ will form a new k-fold n-colored chain, ($n - 1$ colored if $i = f$). We will bound our integral (3.5) in terms of a new integral

$$(3.6) \qquad \int \prod_{j \in C_i} h(x_{r(j)} - x_{r(j-1)} + a_j^i) F_i\, dx_1\, dx_2 \cdots dx_n$$

where the constants $a_j^i \in \mathbf{R}^m$ will be specified and F_i will involve a product of h functions containing x_n. Note that in (3.6), (and throughout this proof), the expression $j-1$ refers to the immediate predecessor of j in C_i. After the final step, C_f will no longer contain any elements j with $r(j)=n$, so that in (3.6) the variable x_n will only appear in F_f. F_f will involve a non-empty product of no more than $2k-1$ h functions containing x_n, hence the dx_n integral will be finite and eliminate F_f from (3.6). The remaining integral will be finite by the induction hypothesis.

We now describe our procedure in detail. Set $a_j^0 = a_j$ and $F_0 = 1$. Assume that either i is the first element of C or that we have already completed the steps associated with all elements preceeding i. We describe the step associated with the element i.

I) Assume first that i is not the final element in our chain.

A) If $r(i) \neq n$ we do nothing, *i.e.* set $C_i = C_{i-1}, a_j^i = a_j^{i-1}, F_i = F_{i-1}$. This completes the step associated with i in this case.

B) If $r(i) = n$, and $r(i-1) = r(i+1)$, set $C_i = C_{i-1} - \{i, i+1\}, a_j^i = a_j^{i-1}$ and $F_i = F_{i-1} h(x_{r(i)} - x_{r(i-1)} + a_i^{i-1}) h(x_{r(i+1)} - x_{r(i)} + a_{i+1}^{i-1})$. Thus we have removed the elements $i, i+1$ from our chain, and moved two factors involving $x_{r(i)} = x_n$ to F_i. Our new $\{C_i, r\}$ is a k-fold n-colored chain: the only point to note is that if e.g. $r(i-1) = r(i+1) = v$ (where of course $n \neq v$), although we have removed one element, $i+1$, from $r^{-1}(v)$, the latter will remain non-empty since it contains $i-1$, while $r^{-1}(n) \neq \emptyset$ since it contains the last element of C. We note also that (3.6) corresponding to C_{i-1} contained the factor

$$h(x_{r(i+2)} - x_{r(i+1)} + a_{i+2}^{i-1}).$$

If we set $p = i+2$, then in C_i, $p-1 = i-1$, and since $r(i-1) = r(i+1)$ we can write the above factor as

$$h(x_{r(p)} - x_{r(p-1)} + a_p^i).$$

This completes the step associated with i in the present case, and since C_i no longer contains $i+1$, our next step will be associated with the element $p = i+2$, the immediate successor of $i-1$ in C_i.

C) If $r(i) = n$, and $r(i-1) \neq r(i+1)$, we set $C_i = C_{i-1} - \{i\}$. This is easily checked to give a new k-fold n-colored chain. We now use the bound which comes from (3.4):

$$h(x_{r(i)} - x_{r(i-1)} + a_i^{i-1}) h(x_{r(i+1)} - x_{r(i)} + a_{i+1}^{i-1}) \leq d\tilde{q}_i q_i \tag{3.7}$$

where $\tilde{q}_i = h(x_{r(i+1)} - x_{r(i-1)} + a_{i+1}^{i-1} + a_i^{i-1}), q_i = h(x_{r(i)} - x_{r(i-1)} + a_i^{i-1}) + h(x_{r(i+1)} - x_{r(i)} + a_{i+1}^{i-1})$. Set $F_i = F_{i-1} q_i$, while $\tilde{q}_i$ will be the factor in (3.6) corresponding to the element $p = i + 1$, with $i - 1$ the immediate predecesor of p in C_i. Thus, we set $a_p^i = a_{i+1}^{i-1} + a_i^{i-1}$, while $a_j^i = a_j^{i-1}$, $\forall j \neq p$. This completes the step associated with the element i in this case.

II) Finally, if $i = f$ is the last element in our chain, by assumption $r(i) = n$. Set $C_i = C_{i-1} - \{i\}$, $F_i = F_{i-1}h(x_{r(i)} - x_{r(i-1)} + a_i^{i-1})$, and $a_j^i = a_j^{i-1}$. It is easy to see that $\{C_f, r\}$ is a k-fold n-1 colored chain, which no longer contains any element j with $r(j) = n$, while F_f contains at least 1 but no more than $2k - 1$ h factors, all of which involve x_n. As described above, this completes the proof of our lemma. □

Proof of Lemma 4. – By the triangle inequality

$$|x + y|/2 \leq \max\{|x|, |y|\}$$

for all $x, y \in \mathbf{R}^m$, so that by monotonicity

$$\inf\{h(x), h(y)\} \leq h\left(\frac{x+y}{2}\right), \qquad \forall x, y \in \mathbf{R}^m. \tag{3.8}$$

We now mimic the proof of Lemma 5. Whenever we applied (3.4) in that proof we now use (3.8). We then get (3.6) except that one h factor, the factor arising from (3.8), has its argument $x_{r(p)} - x_{r(p-1)} + a_p^i$ replaced by $\frac{1}{2}(x_{r(p)} - x_{r(p-1)} + a_p^i)$. By monotonicity we bound (3.6) by halving the arguments appearing in all other factors. We then have (3.6) with h replaced by $\tilde{h}(x) = h(x/2)$, and this allows us to continue following the proof of Lemma 5 to prove our present lemma. □

4. INTERSECTION LOCAL TIMES: EXISTENCE AND CONTINUITY AT EXPONENTIAL TIMES

We first consider the intersection local time α_k at an independent exponential time.

THEOREM 3. – *Assume that we can find a positive, spherically symmetric function $h(x)$ on $\mathbf{R}^m$, monotonically decreasing in $|x|$, with*

$$||h||_j < \infty, \qquad \forall j \leq 2k - 1 \tag{4.1}$$

and such that for some $\delta > 0$ we have

$$g(x) \leq h(x)$$

$$|g(x+a)-g(x)| \leq |a|^{\delta}(h(x)+h(x+a))$$

for all $a, x \in \mathbf{R}^m$.

Then, restricted to $x \in (\mathbf{R}^m - \{0\})^{k-1}$

$$\alpha_k(x;\zeta) = \lim_{\varepsilon\to 0} \alpha_{k,\varepsilon}(x;\zeta) \tag{4.2}$$

exists and is continuous a.s. and in all L^p.

Furthermore, the occupation density formula holds:

(4.3)

$$\int \Phi(x_2,\dots,x_k)\alpha_k(x_2,\dots,x_k;\zeta)\,dx_2\dots dx_k$$
$$= \int\cdots\int_{\{0\leq t_1\leq\cdots\leq t_k\leq\zeta\}} \Phi(X_{t_2}-X_{t_1},\dots,X_{t_k}-X_{t_{k-1}})\,dt_1\dots dt_k$$

for all bounded Borel measurable functions Φ *on* $(\mathbf{R}^m)^{k-1}$.

Remark 2. – The occupation density formula (4.3) shows that $\alpha_k(x;\zeta)$ is independent of the particular f used to define $\alpha_{k,\varepsilon}(x;\zeta)$.

Proof of Theorem 3. – We will show that for n even and $\gamma > 0$

$$E(\{\alpha_{k,\varepsilon}(x;\zeta)-\alpha_{k,\varepsilon'}(x';\zeta)\}^n) \leq c_{n,\gamma}|(\varepsilon,x)-(\varepsilon',x')|^{\delta n/2} \tag{4.4}$$

for all $0 < \varepsilon,\varepsilon' \leq \gamma/2$ and all $x, x' \in (\mathbf{R}^m_\gamma)^{k-1}$ where $\mathbf{R}^m_\gamma = \{x \in \mathbf{R}^m \,|\, |x| \geq \gamma\}$. The multidimensional version of Kolmogorov's lemma then gives us that for any $\delta' < \delta$ and any $M < \infty$ we have

$$|\alpha_{k,\varepsilon}(x;\zeta)-\alpha_{k,\varepsilon'}(x';\zeta)| \leq c_{n,\gamma}(\omega)|(\varepsilon,x)-(\varepsilon',x')|^{\delta'/2} \tag{4.5}$$

for all rational $0 < \varepsilon,\varepsilon' \leq \gamma/2$ and all rational $x, x' \in (\mathbf{R}^m_\gamma)^{k-1}$, $|x|, |x'| \leq M$. Since $\alpha_{k,\varepsilon}(x;\zeta)$ is clearly continuous as long as $\varepsilon > 0$, this will establish (4.2).

To establish (4.4) we first handle the variation in ε. From Lemma 1 we have

$$\begin{aligned} (4.6)\quad & E(\{\alpha_{k,\varepsilon}(x;\zeta)-\alpha_{k,\varepsilon'}(x;\zeta)\}^n) \\ &= \sum_{v\in\mathcal{V}} \int \prod_{i=1}^{n}\left(\prod_{j=2}^{k} f_{\varepsilon,x_j}(y^i_j-y^i_{j-1}) - \prod_{j=2}^{k} f_{\varepsilon',x_j}(y^i_j-y^i_{j-1})\right) \\ &\quad\times \prod_{p=1}^{nk} g(w_{v(p)}-w_{v(p-1)})\,dw_1\dots dw_{nk} \end{aligned}$$

We expand this as a sum of many terms using

$$\prod_{j=2}^{k} a_j - \prod_{j=2}^{k} b_j = \sum_{j=2}^{k}\left(\prod_{p=1}^{j-1} a_p\right)(a_j - b_j)\prod_{q=j+1}^{k} b_q \tag{4.7}$$

so that in each term there is one difference of the form

$$f_{\varepsilon,x_j}(y_j^i - y_{j-1}^i) - f_{\varepsilon',x_j}(y_j^i - y_{j-1}^i)$$

for each $1 \leq i \leq n$, where in the last display $j = j(i)$. We then change variables as in lemma 2, and once more employ (4.7) to obtain a sum of many terms, each of which contains for each $1 \leq i \leq n$ a difference involving one g factor. Since each g factor in (2.3) involves at most two i's, whenever our procedure gives two differences involving the same g factor we write one of the differences as two terms. The upshot is that the integral in (4.6) can be written as a sum of many terms of the form appearing in (2.3) except that at least $n/2$ of the g factors have been replaced by factors of the form

$$\begin{aligned}(4.8)\quad \Delta_{\varepsilon,\varepsilon',j}\,g\Bigg(& z_{s(p)} + \sum_{j=2}^{c(p)}(\tilde{\varepsilon}_{s(p)}y_j^{s(p)} + x_j^{s(p)}) \\ &- \left(z_{s(p-1)} + \sum_{j=2}^{c(p-1)}(\tilde{\varepsilon}_{s(p-1)}y_j^{s(p-1)} + x_j^{s(p-1)})\right)\Bigg)\end{aligned}$$

where $\tilde{\varepsilon}$ can be variously $\varepsilon, \varepsilon'$ and the notation $\Delta_{\varepsilon,\varepsilon',j}$ denotes a difference between two g factors of the above form in which one of the ε's, multiplying $y_j^{s(p)}$ or $y_j^{s(p-1)}$ has been replaced by ε'.

For a fixed $s \in \mathcal{S}$ we will say that p is a "good p" if $s(p) \neq s(p-1)$, while p is a "bad p" if $s(p) = s(p-1)$. Assume first that (4.8) involves a bad p. Then, setting $s(p) = i$, (4.8) can be written simply as

$$g(\varepsilon y_j^i + x_j^i) - g(\varepsilon' y_j^i + x_j^i),$$

and we have the bound

$$\begin{aligned}(4.9)\qquad &|g(\varepsilon y_j^i + x_j^i) - g(\varepsilon' y_j^i + x_j^i)| \\ &\leq 2h(\gamma/2)|\varepsilon - \varepsilon'|^\delta\end{aligned}$$

for $\varepsilon, \varepsilon' \leq \gamma/2$. (Recall that each $|x_j^i| \geq \gamma$ and each y_j^i is integrated with respect to the density f which is supported in the unit ball in $\mathbf{R}^m$ so that we can assume $|y_j^i| \leq 1$). Similarly, if one of the g factors not involving a subtraction contains a bad p, we use the bound

$$|g(\varepsilon y_j^i + x_j^i)| \leq h(\gamma/2).$$

We now turn to the g factors containing good p's. If such a g doesn't involve a subtraction, we simply bound it by h of the same argument. If g is of the form considered in (4.8) we use the bound

$$(4.10) \qquad |g(z + \varepsilon y) - g(z + \varepsilon' y)| \leq (h(z + \varepsilon y) + h(z + \varepsilon' y))|\varepsilon - \varepsilon'|^\delta.$$

Finally, applying the Basic Lemma to the dz integral now establishes (4.4) for the variation in ε, and the variation in x is handled similarly. We write out $E(\{\alpha_{k,\varepsilon}(x;\zeta) - \alpha_{k,\varepsilon}(x';\zeta)\}^n)$ as we did for the ε variation in (4.6), leading to the following analogue of (4.8)

$$(4.11) \ \Delta_{x,x',j} g\left(z_{s(p)} + \sum_{j=2}^{c(p)} (\varepsilon_{s(p)} y_j^{s(p)} + \tilde{x}_j^{s(p)}) - \left(z_{s(p-1)} + \sum_{j=2}^{c(p-1)} (\varepsilon_{s(p-1)} y_j^{s(p-1)} + \tilde{x}_j^{s(p-1)}) \right) \right)$$

where $\tilde{x}$ can be variously x, x'. We follow along the lines of the proof for the ε variation, replacing (4.9) by

$$(4.12) \qquad \begin{aligned} &|g(\varepsilon y_j^i + x_j^i) - g(\varepsilon y_j^i + x_j'^i)| \\ &\leq 2h(\gamma/2)|x - x'|^\delta \end{aligned}$$

and (4.10) by

$$(4.13) \qquad |g(z + x) - g(z + x')| \leq (h(z + x) + h(z + x'))|x - x'|^\delta.$$

This completes the proof of (4.2).

To prove the occupation density formula (4.3) we note that

$$(4.14) \qquad \begin{aligned} &\int \Phi(x_2, \ldots, x_k)\alpha_{k,\varepsilon}(x_2, \ldots, x_k; \zeta)\, dx_2 \ldots dx_k \\ &= \int \cdots \int_{\{0 \leq t_1 \leq \cdots \leq t_k \leq \zeta\}} \\ &\Phi * F_\varepsilon(X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}})\, dt_1 \ldots dt_k \end{aligned}$$

where $F_\varepsilon(x_2, \dots, x_k) = \prod_{j=2}^k f_\varepsilon(x_j)$. Hence, by what we have established above, we can take the $\varepsilon \to 0$ limit in (4.14) to yield (4.3) whenever Φ is a bounded continuous function supported on $(\mathbf{R}_\gamma^m)^{k-1}$ where $\mathbf{R}_\gamma^m = \{x \in \mathbf{R}^m \| x| \geq \gamma\}$. The monotone convergence theorem then allows us to obtain (4.3) for all bounded Borel measurable Φ. This completes the proof of theorem 2. □

5. RENORMALIZED INTERSECTION LOCAL TIMES: CONTINUITY AT EXPONENTIAL TIMES

By Theorem 2 and Lemma 2 we have

$$(5.1)\ E\left(\prod_{i=1}^n \alpha_{k_i}(x^i;\zeta)\right) = \sum_{s\in\mathcal{S}} \int \prod_{p=1}^k g\left(z_{s(p)} + \sum_{j=2}^{c(p)} x_j^{s(p)} - \left(z_{s(p-1)} + \sum_{j=2}^{c(p-1)} x_j^{s(p-1)}\right)\right) dz_1 \dots dz_n.$$

We note in particular that if p is a bad integer, *i.e.* $s(p) = s(p-1)$, the g factor in the above product has the form

$$(5.2)\qquad g(x_{c(p)}^{s(p)})$$

Recall that we have defined the renormalized intersection local time of order k for $x = (x_2, \dots, x_k) \in (\mathbf{R}^m - \{0\})^{k-1}$ by

$$(5.3)\qquad \gamma_k(x;t) = \sum_{A\subseteq\{2,\dots,k\}} (-1)^{|A|}\Big(\prod_{j\in A} g(x_j)\Big)\alpha_{k-|A|}(x_{A^c};t)$$

where for any $B = \{i_1 < \dots < i_{|B|}\} \subseteq \{2, \dots, k\}$

$$(5.4)\qquad x_B = (x_{i_1}, x_{i_2}, \dots, x_{i_{|B|}}).$$

We use the convention $\alpha_1(t) = t$.

Let us now analyze the changes which occur in (5.1) when we replace the factor $\alpha_{k_r}(x^r;\zeta)$ by $(\prod_{j\in A} g(x_j^r))\alpha_{k_r-|A|}(x_{A^c}^r;\zeta)$. Keeping in mind (5.2) we see that now s runs over those $s \in \mathcal{S}$ such that

$s(p) = r, c(p) \in A \Rightarrow s(p-1) = r$, *i.e.* such p's are bad, and in the integrand on the right hand side of (5.1), aside from the factor $\prod_{j \in A} g(x_j^r)$, all other occurences of x_i^r, $i \in A$ are deleted.

If $h(x)$ is any function of the variable x we use the notation

$$\mathcal{D}_x h = h(x) - h(0)$$

for the difference between the value of h at x and it's value at $x = 0$. If $s \in \mathcal{S}$ we set $B_s = \{p | s(p) = s(p-1)\}$. The upshot is that we have

LEMMA 5.

$$(5.5) E\left(\prod_{i=1}^{n} \gamma_{k_i}(x^i;\zeta)\right)$$
$$= \sum_{s \in \mathcal{S}} \left(\prod_{p \in B_s} g(x_{c(p)}^{s(p)})\right) \int \left(\prod_{p \in B_s} \mathcal{D}_{x_{c(p)}^{s(p)}}\right)$$
$$\cdot \prod_{p \in B_s^c} g\left(z_{s(p)} + \sum_{j=2}^{c(p)} x_j^{s(p)} - \left(z_{s(p-1)} + \sum_{j=2}^{c(p-1)} x_j^{s(p-1)}\right)\right) dz_1 \dots dz_n.$$

where $x^i = (x_2^i, x_3^i, \dots, x_{k_i}^i)$, $k = \sum_{i=1}^n k_i$, $\mathcal{S}$ *is the set of mappings* s: $\{1, 2, \dots, k\} \mapsto \{1, \dots, n\}$ *such that* $|s^{-1}(i)| = k_i, \forall 1 \le i \le n$, *and* $c(p) = |\{u \le p \mid s(u) = s(p)\}|$.

We can now state our continuity theorem for renormalized intersection local time at an exponential time. In section 6 we will show that the conditions of our theorem are satisfied by the symmetric stable processes in $\mathbf{R}^2$. We recall the standard notation for difference operators

$$\Delta_a g(x) = g(x+a) - g(x).$$

THEOREM 4. – *Assume that*

$$(5.6) \qquad g(y) \le c g(x), \qquad \forall |x| \le |y|$$

and that for some $\delta > 0$, *and all* $M > 0$

$$(5.7) \qquad |\Delta_a g(x)| \le c_M |a|^\delta g(x)/|x|^\delta$$

for all $|a| \le |x|/4 \le M$.

Assume further that we can find a positive, spherically symmetric function $h(x)$ *on* $\mathbf{R}^m$, *monotonically decreasing in* $|x|$, *with*

$$(5.8) \qquad \|h\|_j < \infty, \qquad \forall j \le 2k-1$$

and such that we have

$$g(x) \leq h(x)$$

for all $x \in \mathbf{R}^m$, *and with the notation* $h_u(x) = (h(x) \vee 1)^{u-1} h(x)$ *we have*

$$\prod_{j=1}^{u} g(a_j) \left| \left(\prod_{i=1}^{u+d} \Delta_{a_i} \right) g(x) \right| \leq \left(\prod_{i=1}^{u+d} |a_i|^{\delta} \right) \Big(\sum_{A \subseteq \{1,\ldots,u+d\}} h_{u+1}(x + \sum_{i \in A} a_i) \Big) \tag{5.9}$$

for $d = 0, 1, 2$ *and all integers* $u \geq 0$, *and all* $x, a_1, a_2, \ldots, a_{u+d} \in \mathbf{R}^m$.

Then $\gamma_k(x;\zeta)$, *defined for* $x = (x_2, \ldots, x_k) \in (\mathbf{R}^m - \{0\})^{k-1}$, *has an extension to* $(\mathbf{R}^m)^{k-1}$ *which is continuous a.s. and in all* L^p.

Proof of Theorem 4. – We will show that for all even n

$$E(\{\gamma_k(y;\zeta) - \gamma_k(w;\zeta)\}^n) \leq c_n |y - w|^{\delta n} \tag{5.10}$$

with c_n independent of $y, w \in (\mathbf{R}^m - \{0\})^{k-1}$ with $|y|, |w| \leq M$.

We begin by showing how to obtain a bound

$$E\left(\prod_{i=1}^{n} \gamma_k(x^i;\zeta) \right) \leq c \tag{5.11}$$

independent of $x^i \in (\mathbf{R}^m - \{0\})^{k-1}$ with $|x^i| \leq M$. We use Lemma 6, and fix for the moment one $s \in \mathcal{S}$. We have $|s^{-1}(j)| = k$ for each $1 \leq j \leq n$. Set $C = B_s^c$ and r to be s restricted to C. Then $\{C, r\}$ forms an k-fold n-colored chain. We partition B_s into the subsets $B_{s,j} = \{p \in B_s | s(p) = j\}$. We have $|B_{s,j}| = k - |r^{-1}(j)|$. We can expand the summand in (5.5) corresponding to s as a sum of many terms, in which each $\mathcal{D}_{x^{s(p)}_{c(p)}}$, $p \in B_s$ is applied to exactly one of the g factors to its right. Necessarily, this will be a g factor containing $z_{s(p)}$. For a given term, let $B_{s,r(i),i}$ denote the set of $p \in B_{s,r(i)}$ such that $s(p) = r(i)$ and $\mathcal{D}_{x^{s(p)}_{c(p)}}$ is applied to the factor $g(z_{r(i)} - z_{r(i-1)} + e_i)$. Similarly, we let $\tilde{B}_{s,r(i-1),i-1}$ denote the set of $p \in B_{s,r(i-1)}$ such that $s(p) = r(i-1)$ and $\mathcal{D}_{x^{s(p)}_{c(p)}}$ is applied to the factor $g(z_{r(i)} - z_{r(i-1)} + e_i)$. Any such term can be written in the form

$$\int \prod_{i \in C} \Big(\prod_{p \in B_{s,r(i),i}} g(x^{r(i)}_{c(p)}) \Delta_{x^{r(i)}_{c(p)}} \Big) \Big(\prod_{p \in \tilde{B}_{s,r(i-1),i-1}} g(x^{r(i-1)}_{c(p)}) \Delta_{x^{r(i-1)}_{c(p)}} \Big) g(z_{r(i)} - z_{r(i-1)} + e_i) \, dz_1 \ldots dz_n. \tag{5.12}$$

(We take $\prod_D$ to be the identity operator whenever $D = \emptyset$.) Clearly, for each j, the sets $\{B_{s,j,i}, \tilde{B}_{s,j,i},\ i \in C\}$ form a partition of $B_{s,j}$. Therefore if we set $b(i) = |B_{s,r(i),i}|$, $\tilde{b}(i) = |\tilde{B}_{s,r(i),i}|$ we see that

$$\sum_{i \in r^{-1}(j)} (1 + b(i) + \tilde{b}(i)) = k$$

for all $1 \leq j \leq n$.

Using (5.9) with $d = 0$ we see that (5.12) can be bounded by a sum of terms of the form

$$(5.13)\quad \Big(\prod_{p \in B_s} |x_{c(p)}^{s(p)}|^{\delta}\Big) \int \prod_{i \in C} h_{1+b(i)+\tilde{b}(i-1)}(z_{r(i)} - z_{r(i-1)} + e_i)\, dz_1\, dz_2 \cdots dz_n$$

The following variant of the Basic Lemma will now establish (5.11).

LEMMA 6. – *Let $h \geq 0$ be a spherically symmetric function on $\mathbf{R}^m$ with*

$$(5.14)\quad \|h\|_j < \infty, \qquad \forall j \leq 2k - 1$$

and such that $h(x)$ is monotonically decreasing in $|x|$.

Then if $\{C, r\}$ is any k-fold n-colored chain, and $b, \tilde{b}$ are integer valued functions on C such that $\tilde{b}(f) = 0$ and

$$(5.15)\quad \sum_{i \in r^{-1}(j)} (1 + b(i) + \tilde{b}(i)) \leq k \qquad \forall 1 \leq j \leq n$$

then

$$(5.16)\quad \sup_{\bar{a} \in (\mathbf{R}^m)^{|C|}} \int \prod_{i \in C} h_{1+b(i)+\tilde{b}(i-1)}(x_{r(i)} - x_{r(i-1)} + a_i)\, dx_1\, dx_2 \cdots dx_n < \infty$$

where we set $x_{r(0)} = 0$, and $\bar{a} = (a_1, \ldots, a_{|C|}) \in (\mathbf{R}^m)^{|C|}$.

Proof of Lemma 6. – Set

$$(5.17)\quad D(j) = \sum_{i \in r^{-1}(j)} (1 + \tilde{b}(i-1) + b(i)) + (1 + \tilde{b}(i) + b(i+1))1_{i \neq f}.$$

$D(j)$ counts the number of factors in (5.16) containing the variable x_j, and thus gives a measure of the divergence in x_j. Using (5.15) and the disjointness of the sets $r^{-1}(j)$ we see that

$$(5.18)\quad \sum_{j=1}^{n} D(j) = \sum_{i\in C}\{(1+\tilde{b}(i-1)+b(i)) + (1+\tilde{b}(i)+b(i+1))1_{i\neq f}\}$$
$$= \sum_{i\in C} 2(1+b(i)+\tilde{b}(i)) - (1+b(1))$$
$$= 2\sum_{j=1}^{n}\sum_{i\in r^{-1}(j)} (1+b(i)+\tilde{b}(i)) - (1+b(1))$$
$$\leq 2nk-1$$

where we use $b(1)$ to denote the value of b on the initial element in C.

This shows that for some j, $D(j) \leq 2k-1$. By relabeling the colors we can assume that $D(n) \leq 2k-1$.

We now proceed along the lines of the proof of Lemma 5. We use $\tilde{b}_i, b_i$ to denote functions at stage i satisfying (5.15) with $\tilde{b}_i(f) = 0$. We shall only describe the modifications. In case B), we set $F_i = F_{i-1}h_{1+b_{i-1}(i)+\tilde{b}_{i-1}(i-1)}(x_{r(i)}-x_{r(i-1)}+a_i^{i-1})h_{1+b_{i-1}(i+1)+\tilde{b}_{i-1}(i)}(x_{r(i+1)}-x_{r(i)}+a_{i+1}^{i-1})$, and $\tilde{b}_i(i-1)=\tilde{b}_{i-1}(i+1)$. In case C), we use (3.7) only on the h factors, while all $\tilde{h}(x)=h(x)\vee 1$ factors are thrown into F_i. Thus in the notation of the proof of Lemma 5, $F_i = F_{i-1}q_i\tilde{h}^{b_{i-1}(i)+\tilde{b}_{i-1}(i-1)}(x_{r(i)}-x_{r(i-1)}+a_i^{i-1})\tilde{h}^{\tilde{b}_{i-1}(i)+b_{i-1}(i+1)}(x_{r(i+1)}-x_{r(i)}+a_{i+1}^{i-1})$. We then set $\tilde{b}_i(i-1)=b_i(i+1)=0$. Finally, as in case C), if $i=f$ and $r(f)=n$, we set $F_i = F_{i-1}h_{1+b_{i-1}(i)+\tilde{b}_{i-1}(i-1)}(x_{r(i)}-x_{r(i-1)}+a_i^{i-1})$, and $\tilde{b}_i(i-1)=0$.

It is easy to check that F_f contains at least one h factor, but no more than $D(n)\leq 2k-1$ factors altogether, hence the dx_n integral is finite and will eliminate the factor F_f. The remaining integral will then be of the form (5.16) with $b=b_f, \tilde{b}=\tilde{b}_f$ satisfying (5.15) and $\tilde{b}(f)=0$ and our proof will be completed by induction. □

Proof of Theorem 4 (continued). – With these results, we now turn to the bound (5.10). For ease of exposition we use y^i, w^i to denote the y, w's in the i'th factor; in the end we will set $y^i = y, w^i = w$. We may assume that y, w differ only in the v'th coordinate, and we set $a = y_v - w_v$. We use Lemma 6 to expand (5.10) as a sum of many terms similar in form to (5.5), where now x^i is variously y^i or w^i, and each term is preceeded by

$$(5.19)\qquad \prod_{i=1}^{n}\mathcal{D}_{y_v^i,w_v^i}$$

where, if h is a function of the variable r or s, then $\mathcal{D}_{r,s}h = h(r) - h(s)$. We reorganize such a term in the form of (5.12), again preceeded by (5.19). Expanding as before, we can associate each $\mathcal{D}_{y_v^j, w_v^j}$ with one of the factors

$$\Big(\prod_{p\in B_{s,r(i),i}} g(x_{c(p)}^{r(i)})\Delta_{x_{c(p)}^{r(i)}}\Big)\Big(\prod_{p\in \tilde{B}_{s,r(i-1),i-1}} g(x_{c(p)}^{r(i-1)})\Delta_{x_{c(p)}^{r(i-1)}}\Big)$$
$$g(z_{r(i)} - z_{r(i-1)} + e_i)$$

of (5.12).

If $(j,v) \neq (s(p), c(p))$ for any $p \in B_{s,r(i),i}, \tilde{B}_{s,r(i-1),i-1}$, the effect of $\mathcal{D}_{y_v^j, w_v^j}$ is simply to add another Δ_a to one of the $g(z_{r(i)} - z_{r(i-1)} + e_i)$ factors in (5.12) and using (5.9) with $d = 1, 2$ we see that (5.13) will now have an extra factor of $|a|^\delta$.

Assume now that $(j,v) = (s(p), c(p))$ for some $p \in B_{s,r(i),i}$, $\tilde{B}_{s,r(i-1),i-1}$, so that either $x_{c(p)}^{s(p)} = y_v^j$ or $x_{c(p)}^{s(p)} = w_v^j$. If we have that both $|a| \geq |y_v^j|/4$ and $|a| \geq |w_v^j|/4$, (5.13) directly provides us with a factor $|x_{c(p)}^{s(p)}|^\delta \leq c|a|^\delta$.

If on the other hand, say $|a| \leq |y_v^j|/4$ so that $|a| \leq |w_v^j|/2$, we make use of the identity

$$g(y_v^j)\Delta_{y_v^j} - g(w_v^j)\Delta_{w_v^j} = (g(y_v^j) - g(w_v^j))\Delta_{y_v^j} + g(w_v^j)(\Delta_{y_v^j} - \Delta_{w_v^j}) \tag{5.20}$$

For the first term in (5.20) we use (5.7) and (5.13), while for the second term we use (5.6) and (5.13) noting that

$$|g(w_v^j)(\Delta_{y_v^j} - \Delta_{w_v^j})g(x)| \leq cg(a)|\Delta_a g(x + w_v^j)| \tag{5.21}$$

Thus, each factor in (5.19) contributes a factor $|a|^\delta$. This completes the proof of (5.10), hence of our theorem. □

Recall the approximate k-fold renormalized intersection local time

$$\gamma_{k,\varepsilon}(x;t) = \sum_{A\subseteq\{2,\ldots,k\}} (-1)^{|A|}\Big(\prod_{j\in A} g_\varepsilon(x_j)\Big)\alpha_{k-|A|,\varepsilon}(x_{A^c};t) \tag{5.22}$$

with

$$g_\varepsilon(x) = \int f_\varepsilon(y-x)g(y)\,dy = \int f(y)g(\varepsilon y + x)\,dy. \tag{5.23}$$

THEOREM 5. – *Under the assumptions of Theorem 4,*

$$\lim_{\varepsilon\to 0} \gamma_{k,\varepsilon}(x,\zeta) = \gamma_k(x,\zeta)$$

a.s. and in all L^p, and this convergence is locally uniform on $(\mathbf{R}^m)^{k-1}$.

Proof of Theorem 5. – We will show that for all even n

$$E(\{\gamma_{k,\varepsilon}(x;\zeta)-\gamma_{k,\varepsilon'}(x';\zeta)\}^n)\le c'_n|(\varepsilon,x)-(\varepsilon',x')|^{\delta n} \tag{5.24}$$

with c_n independent of $0<\varepsilon,\varepsilon'\le 1$ and $|x|,|x'|\le M$.

Comparing Lemma 2 and (5.1) we see that $E(\prod_{i=1}^n \gamma_{k_i,\varepsilon_i}(x^i;\zeta))$ can be expressed as in lemma 6 except that each occurence of x^i_j is replaced by $\varepsilon_i y^i_j + x^i_j$ and the resulting expression is integrated with respect to $dF(y)=\prod_{i=1}^n\prod_{j=2}^{k_i} f(y^i_j)\,dy^i_j$.

The proof of theorem 4 now shows that the left hand side of (5.24) is bounded by

$$c_n\int\prod_{i=1}^n|(\varepsilon y^i+x)-(\varepsilon' y^i+x')|^\delta\,dF(y)$$

where $y^i=(y^i_2,y^i_3,\dots,y^i_k)$ and c_n is as in (5.10). (5.24) immediately follows.

From this it follows that $\gamma_{k,\varepsilon}(x,\zeta)$ converges locally uniformly on $(\mathbf{R}^m)^{k-1}$ as $\varepsilon\to 0$. The limit must be a continuous function of $x\in(\mathbf{R}^m)^{k-1}$. Since we know from Theorem 2 that for $x\in(\mathbf{R}^m-\{0\})^{k-1}$ this limit is $\gamma_k(x,\zeta)$, and from Theorem 4 that $\gamma_k(x,\zeta)$ has a unique continuous extension to $(\mathbf{R}^m)^{k-1}$, our present theorem follows. □

6. JOINT CONTINUITY

Recall the approximate k'th order renormalized intersection local time.

$$\gamma_{k,\varepsilon}(x;t)=\sum_{A\subseteq\{2,\dots,k\}}(-1)^{|A|}\Big(\prod_{j\in A}g_\varepsilon(x_j)\Big)\alpha_{k-|A|,\varepsilon}(x_{A^c};t). \tag{6.1}$$

THEOREM 6. – *Under the assumptions of theorem 4,* $\gamma_{\varepsilon,k}(x;t)$ *converges locally uniformly on* $(\mathbf{R}^m)^{k-1}\times\mathbf{R}_+$ *as* $\varepsilon\to 0$, *with probability* 1. *Hence*

$$\gamma_k(x;t)\stackrel{\text{def}}{=}\lim_{\varepsilon\to 0}\gamma_{\varepsilon,k}(x;t) \tag{6.2}$$

is almost surely continuous in $(x,t)\in(\mathbf{R}^m)^{k-1}\times\mathbf{R}_+$.

Proof. – Let Y_t denote our Levy process X_t killed at an independent mean-1 exponential time ζ. From now on $\gamma_{\varepsilon,k}(x;t)$ will be defined for the process Y_t in place of X_t. By Fubini's theorem it suffices to show that $\gamma_{\varepsilon,k}(x;t)$ converges locally uniformly on $(\mathbf{R}^m)^{k-1} \times [0,\zeta)$ as $\varepsilon \to 0$ with probability 1.

If $\mathcal{S}$ is a subset of Euclidean space we will say that $\{Z_\varepsilon(x);\ (\varepsilon,x) \in (0,1] \times \mathcal{S}\}$ converges rationally locally uniformly on $\mathcal{S}$ as $\varepsilon \to 0$ if for any compact $K \in \mathcal{S}$, $Z_\varepsilon(x)$ converges uniformly in $x \in K$ as $\varepsilon \to 0$ when restricted to dyadic rational x, ε. We note that since $\gamma_{\varepsilon,k}(x;t)$ for $\varepsilon > 0$ is continuous in ε, x, t, saying that $\gamma_{\varepsilon,k}(x;t)$ converges locally uniformly or converges rationally locally uniformly on $(\mathbf{R}^m)^{k-1} \times [0,\zeta)$ as $\varepsilon \to 0$ are equivalent.

We know from Theorem 5 that $\gamma_{\varepsilon,k}(x;\infty)$ converges locally uniformly on $(\mathbf{R}^m)^{k-1}$ as $\varepsilon \to 0$ with probability 1. Using martingale techniques we will see that the right continuous martingale

$$\Gamma_{k,\varepsilon}(x;t) \stackrel{def}{=} E\{\gamma_{k,\varepsilon}(x;\infty) \,|\, \mathcal{F}_t\}$$

converges rationally locally uniformly on $(\mathbf{R}^m)^{k-1} \times \mathbf{R}_+$ as $\varepsilon \to 0$ with probability 1. $\Gamma_{k,\varepsilon}(x;t)$ is not the same as $\gamma_{\varepsilon,k}(x;t)$, but we will see that they differ by terms of "lower order", and we will be able to complete our proof by induction. Given all the tools we have developped so far in this paper, the proof of joint continuity is conceptually fairly straightforward, but in order to treat the "lower order" terms systematically we need to introduce some notation. This we now proceed to do.

We first define the approximate k'th order generalized intersection local time

$$(6.3)\quad \alpha_{k,\varepsilon}(x_2,x_3,\ldots,x_k;\phi;t) = \int_{\{0\le t_1\le\cdots\le t_k\le t\}} \prod_{j=2}^{k} f_{\varepsilon,x_j}(Y_{t_j} - Y_{t_{j-1}})\phi(Y_{t_k})\,dt_1\ldots dt_k$$

and set

$$\alpha_{k,\varepsilon}(x_2,x_3,\ldots,x_k;\phi) = \alpha_{k,\varepsilon}(x_2,x_3,\ldots,x_k;\phi;\infty).$$

$\alpha_{k,\varepsilon}(x_2,x_3,\ldots,x_k;\phi)$ is the approximate k'th order generalized total intersection local time. For ease of notation in later formulas, we also set

$$\alpha_{0,\varepsilon}(\phi;t) = \int \phi(z)\,dz$$

although $\alpha_{0,\varepsilon}(\phi;t)$ is actually independent of ε, t.

Observe that

(6.4)

$$\begin{aligned}
&E\{\alpha_{k,\varepsilon}(x_2, x_3, \ldots, x_k; \phi) \mid \mathcal{F}_t\} \\
&= \alpha_{k,\varepsilon}(x_2, x_3, \ldots, x_k; \phi; t) \\
&\quad + \sum_{i=0}^{k-1} E\Bigg(\int_{\{0 \le t_1 \le \cdots \le t_i \le t \le t_{i+1} \cdots \le t_k\}} \\
&\qquad \prod_{j=2}^{k} f_{\varepsilon,x_j}(Y_{t_j} - Y_{t_{j-1}})\phi(Y_{t_k})\, dt_1 \ldots dt_k \mid \mathcal{F}_t\Bigg) \\
&= \alpha_{k,\varepsilon}(x_2, x_3, \ldots, x_k; \phi; t) \\
&\quad + \sum_{i=0}^{k-1} \int_{\{0 \le t_1 \cdots \le \le t_i \le t \le t_{i+1} \cdots \le t_k\}} \prod_{j=2}^{i} f_{\varepsilon,x_j}(Y_{t_j} - Y_{t_{j-1}}) \\
&E\left(\prod_{j=i+1}^{k} f_{\varepsilon,x_j}(Y_{t_j} - Y_{t_{j-1}})\phi(Y_{t_k}) \mid \mathcal{F}_t\right) dt_1 \ldots dt_k \\
&= \alpha_{k,\varepsilon}(x_2, x_3, \ldots, x_k; \phi; t) \\
&\quad + \sum_{i=0}^{k-1} \int_{\{0 \le t_1 \le \cdots \le t_i \le t\}} \prod_{j=2}^{i} f_{\varepsilon,x_j}(Y_{t_j} - Y_{t_{j-1}}) \\
&\quad \times \lambda_{k-i,\varepsilon}[\phi; x_{i+1}, \ldots, x_k; Y_t](Y_{t_i})\, dt_1 \ldots dt_i \\
&= \alpha_{k,\varepsilon}(x_2, x_3, \ldots, x_k; \phi; t) \\
&\quad + \sum_{i=0}^{k-1} \alpha_{i,\varepsilon}(x_2, x_3, \ldots, x_i; \lambda_{k-i,\varepsilon}[\phi; x_{i+1}, \ldots, x_k; Y_t]; t)
\end{aligned}$$

where

(6.5)

$$\begin{aligned}
&\lambda_{k-i,\varepsilon}[\phi; x_{i+1}, \ldots, x_k; u](v) \\
&\qquad = \int f_{\varepsilon,x_{i+1}}(z_{i+1} + u - v) \prod_{j=i+2}^{k} f_{\varepsilon,x_j}(z_j) \\
&\quad \phi(z_{i+1} + \cdots + z_k + u) \prod_{j=i+1}^{k} g(z_j)\, dz_{i+1} \ldots dz_k \\
&\qquad = \int g(z_{i+1} + v - u) \prod_{j=i+2}^{k} g(z_j) \\
&\quad \phi(z_{i+1} + \cdots + z_k + v) \prod_{j=i+1}^{k} f_{\varepsilon,x_j}(z_j)\, dz_{i+1} \ldots dz_k.
\end{aligned}$$

Setting

$$(6.6)\quad \lambda_{k-i}[\phi; z_{i+1}, \dots, z_k; u](v) = g(z_{i+1} + v - u) \prod_{j=i+2}^{k} g(z_j)\phi(z_{i+1} + \dots + z_k + v)$$

we have

$$(6.7)\quad \lambda_{k-i,\varepsilon}[\phi; x_{i+1}, \dots, x_k; u](v) = \int \lambda_{k-i}[\phi; z_{i+1}, \dots, z_k; u](v) \prod_{j=i+1}^{k} f_{\varepsilon, x_j}(z_j)\, dz_{i+1} \dots dz_k.$$

We next define the approximate k'th order generalized renormalized intersection local time

$$(6.8)\quad \gamma_{k,\varepsilon}(x; \phi; t) = \sum_{A \subseteq \{2,\dots,k\}} (-1)^{|A|} (\prod_{j \in A} g_\varepsilon(x_j)) \alpha_{k-|A|,\varepsilon}(x_{A^c}; \phi; t)$$

and set

$$\gamma_{k,\varepsilon}(x_2, x_3, \dots, x_k; \phi) = \gamma_{k,\varepsilon}(x_2, x_3, \dots, x_k; \phi; \infty).$$

$\gamma_{k,\varepsilon}(x_2, x_3, \dots, x_k; \phi)$ is the approximate k'th order generalized total renormalized intersection local time. As before, for ease of notation in later formulas, we also set

$$\gamma_{0,\varepsilon}(\phi; t) = \int \phi(z)\, dz$$

although $\alpha_{0,\varepsilon}(\phi; t)$ is actually independent of ε, t. Using (6.4) we find that

$$(6.9)\quad E\{\gamma_{k,\varepsilon}(x_2, x_3, \dots, x_k; \phi) \,|\, \mathcal{F}_t\} = \gamma_{k,\varepsilon}(x_2, x_3, \dots, x_k; \phi; t) + \sum_{i=0}^{k-1} \gamma_{i,\varepsilon}(x_2, x_3, \dots, x_i; \Lambda_{k-i,\varepsilon}[\phi; x_{i+1}, \dots, x_k; Y_t]; t)$$

where

$$(6.10)\quad \Lambda_{k-i,\varepsilon}[\phi; x_{i+1}, \dots, x_k; u](v) = \int \Lambda_{k-i}[\phi; z_{i+1}, \dots, z_k; u](v) \prod_{j=i+1}^{k} f_{\varepsilon, x_j}(z_j)\, dz_{i+1} \dots dz_k$$

and

$$(6.11)\quad \Lambda_{k-i}[\phi; z_{i+1}, \ldots, z_k; u](v) = g(z_{i+1} + v - u) \prod_{j=i+2}^{k} g(z_j)\mathcal{D}_{z_j}\phi(z_{i+1} + \cdots + z_k + v).$$

We will say that $\Lambda_{k-i}[\phi; z_{i+1}, \ldots, z_k; u]$ is obtained from ϕ by adjunction of $z_{i+1}, \ldots, z_k; u$. We note that when $\phi \equiv 1$ we have for all $i < k - 1$

$$(6.12)\quad \Lambda_{k-i}[1; z_{i+1}, \ldots, z_k; u](v) = g(z_{i+1} + v - u) \prod_{j=i+2}^{k} g(z_j)\mathcal{D}_{z_j} 1 = 0$$

whereas

$$(6.13)\quad \Lambda_1[1; z_k; u](v) = g(z_k + v - u).$$

Therefore we have

$$(6.14)\quad \begin{aligned} &E\{\gamma_{k,\varepsilon}(x_2, x_3, \ldots, x_k) \mid \mathcal{F}_t\} \\ &= \gamma_{k,\varepsilon}(x_2, x_3, \ldots, x_k; t) \\ &\quad + \gamma_{k-1,\varepsilon}(x_2, x_3, \ldots, x_{k-1}; \Lambda_{1,\varepsilon}[1; x_k; Y_t]; t). \end{aligned}$$

We will say that a function $\phi(y_1, \ldots, y_n; z)$ is an admissable function of z with auxilliary parameters $y_1, \ldots, y_n$ if it can be written in the form

$$(6.15)\quad \varphi(y_1, \ldots, y_n; z) = \prod_{j \in B_0} g(y_j)\mathcal{D}_{y_j} \prod_{i=1}^{p} g(z + \sum_{v \in B_i} \pm y_v)$$

where $B_i \subseteq \{1, \ldots, n\}$, $\forall i = 0, 1, \ldots, p$, and $\{1, \ldots, n\} = \bigcup_{i=1}^{p} B_i$. Here p is an arbitrary positive integer. If $\varphi(y_1, \ldots, y_n; z)$ is of the above form we will say that $\varphi(y_1, \ldots, y_n; z)$ is of weight $|B_0| + p$. Note that the weight of $\varphi(y_1, \ldots, y_n; z)$ is the number of g factors in (6.15). We will also consider the function $\varphi(z) \equiv 1$ to be an admissable function of z (of weight 0 and with no auxilliary parameters).

If $\varphi(y_1, \ldots, y_n; z)$ is an admissable function of z with auxilliary parameters $y_1, \ldots, y_n$ we will use the notation $\varphi_\varepsilon(y_1, \ldots, y_n; z)$ to denote the function in which some of the auxilliary variables have been smoothed.

More precisely, we will say that $\varphi_\varepsilon(y_1,\ldots,y_n;z)$ is a totally ε-smoothed version of $\varphi(y_1,\ldots,y_n;z)$ if

$$\text{(6.16)}\quad \varphi_\varepsilon(y_1,\ldots,y_n;z) = \int \varphi(y_1+\varepsilon z_1,\ldots,y_n+\varepsilon z_n;z)\prod_{i\in A} f(z_i)\,dz_i \prod_{i\in A^c} d\mu_0(z_i)$$

for some subset $A\subseteq\{1,\ldots,n\}$ such that (with the notation of (6.15)) $B_0\subseteq A$ and $B_i\bigcap A\neq\emptyset$ for all $i=1,\ldots,p$, *i.e.* we require that each g factor in (6.15) contain at least one element of the set y_j, $j\in A$. Here μ_0 is the Dirac measure which puts unit mass at the origin. It would be more precise to refer to the function defined in (6.16) as $\varphi_{\varepsilon,A}(y_1,\ldots,y_n;z)$, but in order to avoid further cluttering the notation, and because the actual nature of the set A will be irrelevant for us, we shall simply drop it from the notation.

It is easy to see that $\varphi_\varepsilon(y_1,\ldots,y_n;z)$ is continuous in $\varepsilon,y_1,\ldots,y_n,z$ for $\varepsilon>0$, and therefore $\gamma_{i,\varepsilon}(x;\varphi_\varepsilon(y;\cdot);t)$ is continuous in ε,x,y for $\varepsilon>0$. Thus, as with $\gamma_{\varepsilon,k}(x;t)$, saying that $\gamma_{i,\varepsilon}(x;\varphi_\varepsilon(y;\cdot);t)$ converges locally uniformly or converges rationally locally uniformly on $(\mathbf{R}^m)^{n+i-1}\times\mathbf{R}_+$ as $\varepsilon\to0$ are equivalent.

The next lemma assembles some facts about adjunction which follow easily from the definitions.

LEMMA 7. – *Let $\varphi(y_1,\ldots,y_n;z)$ be an admissable function of z of weight q and auxilliary parameters $y_1,\ldots,y_n$, and let $\Lambda_{k-i}[\varphi(y;\cdot);x_{i+1},\ldots,x_k;u]$ denote the function in* (6.11) *obtained from $\varphi(y_1,\ldots,y_n;z)$ by adjunction of $x_{i+1},\ldots,x_k;u$. Then:*

1. *$\Lambda_{k-i}[\varphi(y;\cdot);x_{i+1},\ldots,x_k;u](z)$ is an admissable function of z of weight $q+k-i$ and auxilliary parameters $y_1,\ldots,y_n,x_{i+1},\ldots,x_k,u$.*

2. *If $\varphi_\varepsilon(y_1,\ldots,y_n;z)$ is a totally ε-smoothed version of $\varphi(y_1,\ldots,y_n;z)$, then $\Lambda_{k-i,\varepsilon}[\varphi_\varepsilon(y;\cdot);x_{i+1},\ldots,x_k;u]$ defined in* (6.10) *is a totally ε-smoothed version of $\Lambda_{k-i}[\varphi(y;\cdot);x_{i+1},\ldots,x_k;u]$.*

The next lemma generalizes Theorems 4 and 5.

LEMMA 8. – *Let $\varphi(y;z)$ be an admissable function of z of weight $k-i$ and auxilliary parameters $y=(y_1,\ldots,y_j)$ and let $\varphi_\varepsilon(y;z)$ be a totally ε-smoothed version of $\varphi(y;z)$. Then under the assumptions of theorem 4, there exists $\delta>0$ such that for each n and $M<\infty$ we can find $c_{n,M}<\infty$ such that*

$$\text{(6.17)}\qquad E\left(\left\{\sup_{F_M}\frac{|\gamma_{i,\varepsilon}(x;\varphi_\varepsilon(y;\cdot))-\gamma_{i,\varepsilon'}(x';\varphi_{\varepsilon'}(y';\cdot))|}{|(\varepsilon,x,y)-(\varepsilon',x',y')|^\delta}\right\}^n\right)\le c_{n,M}$$

where $\sup_{F_M}$ *is taken over all dyadic rational pairs* $(\varepsilon, x, y) \neq (\varepsilon', x', y')$ *such that* $0 < \varepsilon, \varepsilon' \leq 1$ *and* $|x|, |x'|, |y|, |y'| \leq M$.

Proof of Lemma 8. – According to Theorem 2.1, chapter 1 of [12], it suffices to show that there exists $\delta > 0$ such that for each n and $M < \infty$ we can find $c_{n,M} < \infty$ such that

$$E(|\gamma_{i,\varepsilon}(x; \varphi_\varepsilon(y; \cdot)) - \gamma_{i,\varepsilon'}(x'; \varphi_{\varepsilon'}(y'; \cdot))|^n) \leq c_{n,M}|(\varepsilon, x, y) - (\varepsilon', x', y')|^{\delta n} \tag{6.18}$$

for all $(\varepsilon, x, y), (\varepsilon', x', y')$ such that $0 < \varepsilon, \varepsilon' \leq 1$ and $|x|, |x'|, |y|, |y'| \leq M$. This follows as in the proof of Theorems 4 and 5 once we realize that the condition that $\varphi(y; z)$ be an admissable function of z of weight $k - i$ is precisely what is needed to guarantee (5.15). This completes the proof of Lemma 8. □

Proof of Theorem 6 *(continued).* – We will show by induction on $i = 0, 1, \ldots, k$ that $\gamma_{i,\varepsilon}(x; \varphi_\varepsilon(y; \cdot); t)$ converges locally uniformly in $(x, y, t) \in (\mathbf{R}^m)^{j+i-1} \times [0, \zeta)$ as $\varepsilon \to 0$ for all admissable functions $\varphi(y; z)$ of z of weight $k - i$ and auxilliary parameters $y = (y_1, \ldots, y_j)$. The case $i = k$ and $\varphi(y; z) \equiv 1$ will prove our theorem.

Consider first the case of $i = 0$. We have to show that if $\varphi(y; z)$ is an admissable function of z of weight k and auxilliary parameters $y = (y_1, \ldots, y_j)$ and $\varphi_\varepsilon(y; z)$ is a totally ε-smoothed version of $\varphi(y; z)$, then

$$\gamma_{0,\varepsilon}(x; \varphi_\varepsilon(y; \cdot); t) \equiv \int \varphi_\varepsilon(y; z)\, dz$$

converges locally uniformly in $y \in (\mathbf{R}^m)^j$ as $\varepsilon \to 0$. But it follows from (5.15) as in the proof of Theorem 4 that

$$\int \varphi(y; z)\, dz$$

is continuous in $y \in (\mathbf{R}^m)^j$, hence recalling the definition (6.16) of $\varphi_\varepsilon(y; z)$ we see that $\int \varphi_\varepsilon(y; z)\, dz$ converges to $\int \varphi(y; z)\, dz$ locally uniformly in $y \in (\mathbf{R}^m)^j$ as $\varepsilon \to 0$.

Assume now that for all $p < i$, and for all admissable functions $\Phi(y; z)$ of z of weight $k - p$ and auxilliary parameters $y = (y_1, \ldots, y_{j'})$ we have that $\gamma_{p,\varepsilon}(x; \Phi_\varepsilon(y; \cdot); t)$ converges locally uniformly in $(x, y, t) \in (\mathbf{R}^m)^{j'+p-1} \times [0, \zeta)$ as $\varepsilon \to 0$ for any totally ε-smoothed version $\Phi_\varepsilon(y; z)$ of $\Phi(y; z)$. Let us show that if $\varphi(y; z)$ is an admissable functions of z of weight $k-i$ and auxilliary parameters $y=(y_1,\ldots,y_j)$, and $\varphi_\varepsilon(y;z)$ is a totally ε-smoothed version of $\varphi(y;z)$, then $\gamma_{i,\varepsilon}(x;\varphi_\varepsilon(y;\cdot);t)$ converges locally uniformly in $(x,y,t)\in(\mathbf{R}^m)^{j+i-1}\times[0,\zeta)$ as $\varepsilon\to 0$.

With F_M as in Lemma 8, let F_M^m; $m=1,2,\ldots$ be an exhaustion of F_M by a sequence of finite symmetric subsets. (A set F of pairs (a,b) is symmetric if $(a,b)\in F\Rightarrow(b,a)\in F$). Let us define the right continuous martingale

$$\Gamma_{i,\varepsilon}(x;\varphi_\varepsilon(y;\cdot);t)=E\{\gamma_{i,\varepsilon}(x;\varphi_\varepsilon(y;\cdot))\,|\,\mathcal{F}_t\}. \tag{6.19}$$

By Lemma 8 applied to the right continuous submartingale

$$A_t^m=\sup_{F_M^m}\frac{\Gamma_{i,\varepsilon}(x;\varphi_\varepsilon(y;\cdot);t)-\Gamma_{i,\varepsilon'}(x';\varphi_{\varepsilon'}(y';\cdot);t)}{|(\varepsilon,x,y)-(\varepsilon',x',y')|^\delta}. \tag{6.20}$$

we have that

$$\begin{aligned}E\left(\left\{\sup_t\sup_{F_M^m}\frac{|\Gamma_{i,\varepsilon}(x;\varphi_\varepsilon(y;\cdot);t)-\Gamma_{i,\varepsilon'}(x';\varphi_{\varepsilon'}(y';\cdot);t)|}{|(\varepsilon,x,y)-(\varepsilon',x',y')|^\delta}\right\}^n\right)\\ \le E\left(\left\{\sup_{F_M^m}\frac{|\gamma_{i,\varepsilon}(x;\varphi_\varepsilon(y;\cdot))-\gamma_{i,\varepsilon'}(x';\varphi_{\varepsilon'}(y';\cdot))|}{|(\varepsilon,x,y)-(\varepsilon',x',y')|^\delta}\right\}^n\right)\\ \le E\left(\left\{\sup_{F_M}\frac{|\gamma_{i,\varepsilon}(x;\varphi_\varepsilon(y;\cdot))-\gamma_{i,\varepsilon'}(x';\varphi_{\varepsilon'}(y';\cdot))|}{|(\varepsilon,x,y)-(\varepsilon',x',y')|^\delta}\right\}^n\right)\\ \le c_{n,M}.\end{aligned} \tag{6.21}$$

Hence

$$\begin{aligned}E\left(\left\{\sup_t\sup_{F_M}\frac{|\Gamma_{i,\varepsilon}(x;\varphi_\varepsilon(y;\cdot);t)-\Gamma_{i,\varepsilon'}(x';\varphi_{\varepsilon'}(y';\cdot);t)|}{|(\varepsilon,x,y)-(\varepsilon',x',y')|^\delta}\right\}^n\right)\\ \le c_{n,M}.\end{aligned} \tag{6.22}$$

In particular this shows that

$$\sup_t\sup_{F_{1,M}}|\Gamma_{i,\varepsilon}(x;\varphi_\varepsilon(y;\cdot);t)-\Gamma_{i,\varepsilon'}(x;\varphi_{\varepsilon'}(y;\cdot);t)|\le C(\omega)|\varepsilon-\varepsilon'|^\delta \tag{6.23}$$

where $F_{1,M}$ denotes the set of dyadic rational $(x,y)\in(\mathbf{R}^m)^{j+i-1}$ with $|x|,|y|\le M$. Thus, $\Gamma_{i,\varepsilon}(x;\varphi_\varepsilon(y;\cdot);t)$ converges rationally locally uniformly on $(\mathbf{R}^m)^{j+i-1}\times\mathbf{R}_+$ as $\varepsilon\to 0$ with probability 1.

By (6.9)

$$\begin{aligned}(6.24)\quad \Gamma_{i,\varepsilon}(x;\varphi_\varepsilon(y;\cdot);t) &= \gamma_{i,\varepsilon}(x_2,x_3,\dots,x_i;\varphi_\varepsilon(y;\cdot);t)\\ &\quad+\sum_{p=0}^{i-1}\gamma_{p,\varepsilon}(x_2,x_3,\dots,x_p;\Lambda_{i-p,\varepsilon}[\varphi_\varepsilon(y;\cdot);x_{p+1},\dots,x_i;Y_t];t)\end{aligned}$$

Hence to show that $\gamma_{i,e}(x;\varphi_\varepsilon(y;\cdot);t)$ converges locally uniformly on $(\mathbf{R}^m)^{j+i-1}\times[0,\zeta)$ as $\varepsilon\to 0$ with probability 1 it suffices to show that for each $p<i$

$$\gamma_{p,\varepsilon}(x_2,x_3,\dots,x_p;\Lambda_{i-p,\varepsilon}[\varphi_\varepsilon(y;\cdot);x_{p+1},\dots,x_i;Y_t];t)$$

converges locally uniformly on $(\mathbf{R}^m)^{j+i-1}\times[0,\zeta)$ as $\varepsilon\to 0$ with probability 1. However, by Lemma 7 $\Lambda_{i-p,\varepsilon}[\varphi_\varepsilon(y;\cdot);x_{p+1},\dots,x_i;u]$ is a totally ε-smoothed version of $\Lambda_{i-p}[\varphi(y;\cdot);x_{p+1},\dots,x_i;u]$, and the latter is an admissable function of weight $k-p$ with auxilliary variables $y,x_{p+1},\dots,x_i,u$. Therefore, by our induction assumption,

$$\gamma_{p,\varepsilon}(x_2,x_3,\dots,x_p;\Lambda_{i-p,\varepsilon}[\varphi_\varepsilon(y;\cdot);x_{p+1},\dots,x_i;u];t)$$

converges locally uniformly in $(x,y,u,t)\in(\mathbf{R}^m)^{j+i}\times[0,\zeta)$ as $\varepsilon\to 0$ with probability 1. Since Y_t is locally bounded on $[0,\zeta)$, this completes proof of Theorem 6. □

THEOREM 7. – *Under the assumptions of theorem 4, $\alpha_{\varepsilon,k}(x;t)$ converges locally uniformly on $(\mathbf{R}^m-\{0\})^{k-1}\times\mathbf{R}_+$ as $\varepsilon\to 0$, with probability 1. Hence*

$$(6.25)\qquad \alpha_k(x;t)\stackrel{\text{def}}{=}\lim_{\varepsilon\to 0}\alpha_{\varepsilon,k}(x;t)$$

is almost surely continuous in $(x,t)\in(\mathbf{R}^m-\{0\})^{k-1}\times\mathbf{R}_+$.

Furthermore,

$$(6.26)\qquad \gamma_k(x;t)=\sum_{A\subseteq\{2,\dots,k\}}(-1)^{|A|}\Big(\prod_{j\in A}g(x_j)\Big)\alpha_{k-|A|}(x_{A^c};t)$$

for all $(x,t)\in(\mathbf{R}^m-\{0\})^{k-1}\times\mathbf{R}_+$.

In addition, the occupation density formula holds a.s.:

$$\int \Phi(x_2, \dots, x_k)\alpha_k(x_2, \dots, x_k; t)\, dx_2 \dots dx_k = \int \cdots \int_{\{0 \le t_1 \le \cdots \le t_k \le t\}} \times \Phi(X_{t_2} - X_{t_1}, \dots, X_{t_k} - X_{t_{k-1}})\, dt_1 \dots dt_k \tag{6.27}$$

for all bounded Borel measurable functions Φ *on* $(\mathbf{R}^m)^{k-1}$.

Remark 3. – The occupation density formula (6.27) shows that $\alpha_k(x; t)$ is independent of the particular f used to define $\alpha_{k,\varepsilon}(x; t)$.

Proof. – This follows easily by induction from Theorem 6 using (6.1). (6.27) follows by the methods used to prove (4.3) of Theorem 3. □

7. THE SYMMETRIC STABLE PROCESSES

In this section we prove Theorems 1 and 2 of the Introduction by verifying that the conditions of our Theorems 3-7 are satisfied by the symmetric stable processes in $\mathbf{R}^2$.

Proof of Theorems 1 *and* 2. – It is well known that $g(x) \sim \frac{c}{|x|^{2-\beta}}$ for $|x| \sim 0$. By (2.10), and (7.15) of Rosen [13] we have for $|x| \neq 0$

$$g(x) \le r_{2-\beta,3}(x) \tag{7.1}$$

$$\left| \prod_{i=1}^{m} \Delta_{a_i} g(x) \right| \le c \frac{\prod_{i=1}^{m} |a_i|}{|x|^m} r_{2-\beta,3}(x) \tag{7.2}$$

for $|a_i| \le |x|/4m$ where $r_{a,b}(x)$ denotes a function which is $\sim \frac{c}{|x|^a}$ for $|x|$ small, and $O(\frac{c}{|x|^b})$ for $|x|$ large.

Take h to be a symmetric monotone decreasing $r_{2-\beta+2\delta,3}$ where $\delta > 0$ is chosen sufficiently small so that $(2k-1)(2-\beta+2\delta) < 2$, so that (5.8) will be satisified. We first prove (5.9) . If $|a_i| \le |x|/4m$ for all $1 \le i \le u+d$, this follows easily from (7.2). In general, let $A = \{i;\ |a_i| \le |x|/4m\}$, and expand $(\prod_{i=1}^{u+d} \Delta_{a_i})g(x)$ as a sum of terms involving $(\prod_{i \in A} \Delta_{a_i})\, g$ evaluated at points $x + b$ where b involves the $a_i \in A^c$. For $i \in A^c$'s we use the fact that $g(a_i) \le cg(x) \le c|x|^\delta h(x) \le c|x|^\delta(h(x) \vee 1) \le$

$c|a_i|^\delta(h(x) \vee 1)$, and similarly $1 \le c|a_i|^\delta/|x|^\delta$. For each b, we now divide the indices in A into two groups, setting $A_b = \{i \in A;\ |a_i| \le |x+b|/4m\}$, expand $(\prod_{i\in A} \Delta_{a_i})\, g(x+b)$ as a sum of terms involving $(\prod_{i\in A_b} \Delta_{a_i})\, g$ evaluated at points $x+b+d$ where d involves the $a_i \in A_b^c$. For $i \in A_b^c$ we use the fact that $g(a_i) \le c|a_i|^\delta(h(x+b) \vee 1)$ and $1 \le c|a_i|^\delta/|x+b|^\delta$ as above. This proceedure, iterated a finite number of times, will complete the proof of (5.9).

To prove (5.6) it suffices to note that for the symmetric stable processes $g(x)$ is monotone decreasing in $|x|$. This is obvious for Brownian motion, and for the general case we use the formula

$$g(x) = \int_0^\infty \bar{g}(s) p_s(x)\, ds$$

which comes from subordination, where $p_s(x)$ is the transition density for Brownian motion in the plane (obviously monotone in $|x|$) and $\bar{g}(s)$ is the 1-potential density for the stable subordinator of index $\beta/2$. *See* (2.30) of [2]. (5.7) then follows easily from (7.2) and the fact, already mentioned above, that $g(x) \sim \frac{c}{|x|^{2-\beta}}$ for $|x| \sim 0$. □

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