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Bivariate Revuz measures and the Feynman-Kac formula

by

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ABSTRACT. – In the first part of the present paper additive and multiplicative functionals of a right Markov process are investigated systematically in the setting of weak duality, by means of bivariate Revuz measures. We first give a representation for such measures. It is proved that additive and multiplicative functionals are uniquely determined by their bivariate Revuz measures and two multiplicative functionals are dual if and only if their bivariate Revuz measures are dual. In the second part we prove that any subprocess of a nearly symmetric Markov process is also nearly symmetric and give a generalized Feynman-Kac formula which describes the relationship between their corresponding Dirichlet forms.

Key words: Markov processes, multiplicative functionals, Revuz measures, Dirichlet forms.

Part I. BIVARIATE REVUZ MEASURES

1. Introduction

In the first section we will set down all definitions and notations enforced throughout this paper. Under the setting of weak duality and some additional conditions, two results of Sharpe (representation of terminal times and

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decomposition of MFs) are given in §2. They are critical throughout this article. Sharpe's canonical measure and Lévy system, which are used to describe discontinuities of the process, are discussed in §3. They are employed to give a representation for bivariate Revuz measures in §4. A generalized Revuz formula is given in §5 and it is especially useful and indispensable in dealing with multiplicative functionals. In §6 we use Kuznetsov measures to prove that a multiplicative functional is uniquely determined by its bivariate Revuz measure and that two multiplicative functionals are dual if and only if their respective bivariate Revuz measures are dual.

In the second part we are going to prove the Feynman-Kac formula for nearly symmetric Markov processes (or non-symmetric Dirichlet forms) and general decreasing multiplicative functionals.

Let

$$X = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, P^x) \quad \text{and} \quad \hat{X} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{\theta}_t, \hat{X}_t, \hat{P}^x),$$

be two right Markov processes on (E, \mathcal{E}) with Borel sub-Markovian semigroups (P_t) and (\hat{P}_t) , respectively, and be in weak duality relative to a fixed σ -finite measure m on E : for all $f, g \in \mathcal{P}\mathcal{E}$,

$$(1.1) \quad (P_t f, g) = (f, \hat{P}_t g), \quad t > 0,$$

where (\cdot, \cdot) denotes the inner product in $L^2(m)$. Walsh [Wa] has shown that under weak duality,

$$(1.2) \quad X_{t-} \text{ exists in } E \text{ for all } t \in]0, \zeta[\text{ a.s. } P^m,$$

and of course the dual assertion holds for \hat{X} . Assuming (1.2) identically on Ω (resp. on $\hat{\Omega}$) relieves us from carrying one more exceptional set everywhere. As is customary, we use the prefix "co" or "co" to describe quantities relative to \hat{X} . Though all definitions and standard results are stated regarding to X , the dual statements also apply to \hat{X} .

Denote by $\text{Exc}^q(X)$ and $\mathcal{S}^q(X)$ (Exc^q or \mathcal{S}^q if no confusion will be caused) the cones of q -excessive measures and functions, respectively.

DEFINITION 1.3. – *A real-valued process $M = (M_t : t \geq 0)$ is called a multiplicative functional (MF) of X if (1) $t \mapsto M_t(\omega)$ is decreasing, right continuous and has values in $[0, 1]$ for each $\omega \in \Omega$; (2) M is adapted, i.e., $M_t \in \mathcal{F}_t$ for any $t \geq 0$; (3) $M_{t+s}(\omega) = M_t(\omega) \cdot M_s(\theta_t \omega)$ for any $s, t \geq 0$ and $\omega \in \Omega$. In addition, an MF M is exact provided for any $t > 0$ and*

every sequence $t_n \downarrow 0$,

$$M_{t-t_n} \circ \theta_{t_n} \rightarrow M_t \text{ a.s. as } n \rightarrow \infty.$$

Let $\text{MF}(X)$ (or MF if no confusion will be caused) be the set of all exact multiplicative functionals of X . For any MF M write $S_M := \inf\{t \geq 0 : M_t = 0\}$, $E_M := \{x \in E : P^x(M_0 = 1) = 1\}$, for the lifetime and the set of permanent points of M , respectively. Let $R := \inf\{t > 0 : X_t \in E_M^c\}$ and $R' := \inf\{t \geq 0 : X_t \in E_M^c\}$. Clearly E_M is also the set of irregular points of S_M , i.e., $E_M = \{x \in E : P^x(S_M > 0) = 1\}$. Let $\text{MF}_+ := \{M \in \text{MF} : S_M > 0 \text{ a.s.}\}$ and $\text{MF}_{++} := \{M \in \text{MF} : M \text{ does not vanish, i.e., } S_M \geq \zeta \text{ a.s.}\}$. If E_M is nearly optional, M is called a right MF . If M is exact, E_M is finely open and consequently M is right.

We define for each $x \in E_M$ a probability measure on Q^x on (Ω, \mathcal{F}^0) by

$$(1.4) \quad Q^x(Z) := P^x \int_{]0, \infty[} Z \circ k_t d(-M_t), \quad Z \in b\mathcal{F}^0$$

where $M_\infty := 0$ and $(k_t)_{t \geq 0}$ are the killing operators on Ω defined by $k_t \omega(s) = \omega(s)$ if $t > s$ and $k_t \omega(s) = \Delta$ if $t \leq s$. If M is right, then (Ω, X_t, Q^x) is also a right Markov process, called the M -subprocess of X and denoted X^M or (X, M) , with state space (E_M, \mathcal{E}_M) . Clearly the semigroup (Q_t) and resolvent (V^q) of X^M are given by

$$(1.5) \quad \begin{aligned} Q_t f(x) &= P^x[f(X_t)M_t]; \\ V^q f(x) &= P^x \int_0^\infty e^{-qt} f(X_t)M_t dt, \end{aligned}$$

and $Q_t(x, \cdot) = V^q(x, \cdot) = 0$ for any $x \notin E_M$. The set of q -excessive functions (resp. excessive measures) of X^M is denoted by $\mathcal{S}^q(M)$ (resp. $\text{Exc}^q(M)$).

An (\mathcal{F}_t) -stopping time $T : \Omega \rightarrow R_+$ is called a terminal time if $T = t + T \circ \theta_t$ identically on $\{t < T\}$. If T is a terminal time, $1_{]0, T[}(t)$ is an MF of X . Write $\text{Exc}^q(T) := \text{Exc}^q(1_{]0, T[})$ and $\mathcal{S}^q(T) := \mathcal{S}^q(1_{]0, T[})$. It is easy to see that S_M is a terminal time if $M \in \text{MF}(X)$.

DEFINITION 1.6. – Let $M \in \text{MF}$. A positive, increasing, right continuous process $A = (A_t : t \geq 0)$ is a raw M -additive functional (of X) provided $A_t < \infty$ for $t < S \wedge \zeta$ and $A_{s+t} = A_t + M_t \cdot A_s \circ \theta_t$ a.s. for each t and s . A raw M -additive functional A is an M -additive functional (of X) if A is adapted.

Let $\text{RAF}(M)$ and $\text{AF}(M)$ denote the sets of raw M -additive functionals and M -additive functionals, respectively. Write RAF for $\text{RAF}(1)$, the set of raw additive functionals, and AF for $\text{AF}(1)$, the set of additive functionals of X . For a terminal time T , $\text{RAF}(T) := \text{RAF}(1_{[0,T]})$ and $\text{AF}(T) := \text{AF}(1_{[0,T]})$.

The energy functional L (of X) is defined on $\text{Exc} \times \mathcal{S}$ by

$$(1.7) \quad L(m, u) = \sup\{\mu(u) : \mu \geq 0 \text{ and } \mu U \leq m\}.$$

Refer to [Ge] for the basic properties of L , among which are $L(\mu U, u) = \mu(u)$ and $L(m, Uf) = m_d(f)$ where m_d is the dissipative part of m .

We now introduce the powerful Kuznetsov measures. Let W be the space of path $w : \mathbb{R} \rightarrow E \cup \{\Delta\}$ that are E -valued and right continuous on an open interval $]\alpha(w), \beta(w)[$ and take the value Δ elsewhere. We denote by $[\Delta]$ the path which constantly equals to Δ . Let $Y = (Y_t : t \in \mathbb{R})$ denote the coordinate process on W , $Y_t(w) = w(t)$. The shifts $\sigma_t : W \rightarrow W$ are defined by $Y_s \circ \sigma_t = Y_{s+t}$. Put $\mathcal{G}_t^0 = \sigma\{Y_s : s \leq t\}$ and $\mathcal{G}^0 = \mathcal{G}_\infty^0$. Then for any $m \in \text{Exc}$, there exists a unique σ -finite measure Q_m on (W, \mathcal{G}^0) not charging $\{[\Delta]\}$ such that if $t_1 < t_2 < \dots < t_n$,

$$(1.8) \quad \begin{aligned} Q_m(\alpha < t_1, Y_{t_1} \in dx_1, \dots, Y_{t_n} \in dx_n, t_n < \beta) \\ = m(dx_1)P_{t_2-t_1}(x_1, dx_2) \cdots P_{t_n-t_{n-1}}(x_{n-1}, dx_n). \end{aligned}$$

The measure Q_m is called the Kuznetsov measure of (X) and m . Clearly and importantly it is translation-invariant; that is, $\sigma_t(Q_m) = Q_m$ for each $t \in \mathbb{R}$.

Notations. – For $M \in \text{MF}$, let $\overline{M}_t := 1 - M_t$. Clearly $\overline{M} \in \text{AF}(M)$. For any process (Z_t) and increasing process (A_t) , we define $(Z_- * A)_t := \int_0^t Z_s - dA_s$ and $(Z * A)_t := \int_0^t Z_s dA_s$. For any $f \in p\mathcal{E}$ and $F \in p\mathcal{E} \times \mathcal{E}$, define $f * A := f(X) * A$, $f_- * A := f(X_-) * A$ and $F * A := F(X_-, X) * A$, where $f(X) := (f(X_t))$, $f(X_-) := (f(X_{-t}))$ and $F(X_-, X) := (F(X_{t-}, X_t))$.

2. Decomposition and weak* duality

We will focus on a special class of multiplicative functionals which have a nice decomposition. From now on we will write *a.s.* for P^m -*a.s.* and drop the m on Revuz measures as they are taken with respect to m if no confusion would be caused.

DEFINITION 2.1. – (i) $M, N \in \text{MF}(X)$ are m -equivalent provided that, for each $t > 0$, $M_t = N_t$ a.s. on $\{\zeta > t\}$. (ii) Let $M \in \text{MF}$ and $A, B \in \text{RAF}(M)$. Then A and B are m -equivalent provided that, for each $t > 0$, $A_t = B_t$ a.s. on $\{\zeta > t\}$.

In view of right continuity of MFs, this is equivalent to the statement that $t \mapsto M_t$ and $t \mapsto N_t$ are identical functions a.s. on $[0, \zeta[$. Hence equality between MFs (resp. M -RAFs) will always be understood to mean m -equivalence.

THEOREM 2.2. – If the following condition holds
 (2.2a) every m -semipolar set is m -polar,
 then any $M \in \text{MF}_+$ has a decomposition

$$(2.3) \quad M_t = \prod_{0 < s \leq t} (1 - \Phi(X_{s-}, X_s)) \exp \left\{ - \int_0^t a(X_s) dA_s \right\} 1_{[0, J_B)}(t),$$

where $\Phi \in \mathcal{E} \times \mathcal{E}$, $0 \leq \Phi < 1$, Φ vanishes on the diagonal D of $E \times E$, $a \in p\mathcal{E}$, A is a continuous additive functional of X , B is a Borel subset of $E \times E$ which is disjoint from D and $S_M = J_B := \inf\{t > 0 : (X_{t-}, X_t) \in B\}$.

The proof mimicks that of [Sh1, Theorem (7.1)]. We will just sketch it here. The Stieltjes logarithm of M

$$(2.4) \quad (\text{slog}M)_t := \int_0^t 1_{\{s < S_M\}} \frac{d(-M_s)}{M_{s-}}$$

is a S_M -additive functional which can be decomposed into the sum of a natural part and a purely discontinuous quasi-left-continuous part. Note that from [GS, (16,21)] two equivalent statements of (2.2a) are

(2.2b) every natural additive functional is a.s. continuous;

(2.2c) if T is a thin natural terminal time, $P^m(T < \zeta) = 0$.

Hence the natural part is continuous and it follows from [GS (16.14)] that $S_M = J_B$ with $B \subset E \times E$ disjoint from D . The purely discontinuous part has common discontinuities as X and must be of the form $\sum_{s \leq t} \Phi(X_{s-}, X_s)$

with $0 \leq \Phi < 1$ since all jumps of $\text{slog}M$ are less than 1.

Weak duality together with (2.2a) is called weak* duality. An example of weak* duality is a pair of nearly symmetric Markov processes (i.e., they satisfy the sector condition) in weak duality. Fitzsimmons [Fi2] proved that in this case every semipolar set is m -polar [Fi2 (4.13)] and m -polar is the same as m -copolar [Fi2 (4.17)]. Hence these two processes are in weak* duality.

Now $M \in \text{MF}_+$ is said to be simple if it has the decomposition (2.3), in which case we write $M \in \text{SMF}_+$. A terminal time $T \in \text{SMF}_+$ means that $T = J_B > 0$ a.s. with $B \in \mathcal{E} \times \mathcal{E}$ which is disjoint from D .

3. Lévy systems and canonical measures

A Lévy system for X is a pair (N, H) , where N is a kernel on (E, \mathcal{E}^u) with $N(x, \{x\}) = 0$ for any $x \in E$ and H is a continuous AF of X having a bounded 1-potential, such that for any $F \in p\mathcal{E}^u \times \mathcal{E}^u$ vanishing on the diagonal and any predictable process Y ,

$$(3.1) \quad P^x \sum_{0 < s \leq t} Y_s F(X_{s-}, X_s) = P^x \int_0^t Y_s N F(X_s) dH_s,$$

where $NF(x) := \int F(x, y)N(x, dy)$. Refer to [Sh, §73] for the existence of Lévy systems.

LEMMA 3.2. – Let $B \in \mathcal{E} \times \mathcal{E}$ and $T := J_B > 0$ a.s. Then $(X_{T-}, X_T) \in B$ a.s. on $\{T < \infty\}$ and

$$(3.3) \quad 1_{\{t < T\}} = \prod_{s \leq t} 1_{B^c}(X_{s-}, X_s)$$

a.s. on Ω .

Proof. – Define the second hit $T^2 := T \circ \theta_T + T$. The statement (3.3) is obviously true on $\{T = \infty\}$. On $\{T < \infty\}$, since $T > 0$ a.s., $T^2 - T = T \circ \theta_T > 0$ a.s. Hence $(X_{T-}, X_T) \in B$ a.s. on $\{T < \infty\}$ by the definition of J_B . Then (3.3) follows directly. ■

THEOREM 3.4. – Let (N, H) be a Lévy system for X and $M \in \text{SMF}_+$ with the decomposition (2.3). Define $\Psi := 1_B + 1_{B^c} \cdot \Phi$ and $N_o := (1 - \Psi(x, y))N(x, dy)$. Then (N_o, H) is a Lévy system of the M -subprocess (X, M) .

Proof. – We have

$$\begin{aligned} Q^x \sum_{0 < s \leq t} Y_s F(X_{s-}, X_s) &= P^x \sum_{0 < s \leq t} Y_s F(X_{s-}, X_s) M_s \\ &= P^x \sum_{0 < s \leq t} Y_s M_{s-} F(X_{s-}, X_s) \frac{M_s}{M_{s-}}. \end{aligned}$$

By (3.2) the discontinuous part M^d of M takes the following form

$$(3.5) \quad M_s^d = \prod_{0 < r \leq s} (1 - \Phi) \cdot 1_{B^c}(X_{r-}, X_r).$$

Hence

$$\frac{M_s}{M_{s-}} = \frac{M_s^d}{M_{s-}^d} = (1 - \Phi) \cdot 1_{B^c}(X_{s-}, X_s) = (1 - \Psi)(X_{s-}, X_s)$$

and then

$$\begin{aligned} Q^x \sum_{0 < s \leq t} Y_s F(X_{s-}, X_s) &= P^x \sum Y_s M_{s-} [F(1 - \Psi)](X_{s-}, X_s) \\ &= P^x \int_0^t Y_s M_{s-} N F(1 - \Psi)(X_s) dH_s \\ &= P^x \int_0^t Y_s M_s N F(1 - \Psi)(X_s) dH_s \\ &= Q^x \int_0^t Y_s N_o F(X_s) dH_s. \end{aligned}$$

By definition, (N_o, H) is a Lévy system of (X, M) . ■

We will next introduce the canonical measure of X relative to m , which was first used in [Sh2]. While the Lévy system describes the discontinuities of the process completely, the canonical measure does this relative to m .

The canonical measure ν is characterized as a σ -finite measure on $E \times E$ which is carried by $E \times E - D$ and such that for any $F \in p\mathcal{E} \times \mathcal{E}$ vanishing on D

$$(3.6) \quad \nu(F) = \lim_{t \rightarrow 0} \frac{1}{t} P^m \sum_{s \leq t} F(X_{s-}, X_s).$$

It follows from (3.1) that

$$(3.7) \quad \nu(dx, dy) = N(x, dy) \rho_H(dx).$$

where (N, H) is a Lévy system of X . Thus (3.4) implies:

COROLLARY 2.8. – *Let $M \in \text{SMF}_+$ and ν^M denote the canonical measure of (X, M) (relative to m). Then $\nu^M = (1 - \Psi)\nu$.*

The transform by a multiplicative functional $M \in \text{MF}$ consists of two steps of killing: a first entrance time $R' = D_{E_M^c}$ and a multiplicative functional in $\text{MF}_+(R')$. The following result gives us a Lévy system of the R' -subprocess of X .

THEOREM 3.9. – *Let (N, H) be a Lévy system of X . Then (N', H) is a Lévy system of the R' -subprocess of X , where $N'(x, dy) = 1_{E_M \times E_M}(x, y) \cdot N(x, dy)$.*

Proof. – Let $(P_r^x : x \in E)$ be the probability distribution of the R' -subprocess. For any $F \in \mathcal{p}\mathcal{E} \times \mathcal{E}$ which is supported on $E_M \times E_M$, a predictable process (Z_t) and $x \in E_M$

$$\begin{aligned}
 & P_r^x \left(\left(\sum_{s \leq t} F(X_{s-}, X_s) Z_s \right) \right) \\
 &= P^x \left(\left(\sum_{s \leq t} F(X_{s-}, X_s) Z_s 1_{]0, R'[}(s) \right) \right) \\
 &= P^x \left(\left(\sum_{s \leq t} F(X_{s-}, X_s) Z_s 1_{]0, R'[}(s) \right) \right) - P^x(F(X_{R'-}, X_{R'}); R' \leq t) \\
 &= P^x \int_0^{t \wedge R'} Z_s N F(X_s) dH_s \\
 &= P^x \int_0^t Z_s 1_{]0, R'[}(s) N F(X_s) dH_s \\
 &= P^x \int_0^t Z_s 1_{]0, R'[}(s) N' F(X_s) dH_s \\
 &= P_r^x \int_0^t Z_s N' F(X_s) dH_s.
 \end{aligned}$$

The third equality holds since $F(X_{R'-}, X_{R'}) = 0$ due to the fine closeness of E_M^c and the fact that $X_{R'} \in E_M^c$. ■

4. A representation of bivariate Revuz measures

Bivariate Revuz measures were also introduced in [Sh1] and will play a critical role in coming sections. We assume that (N, H) is a Lévy system of X and $\nu(dx, dy) = N(x, dy) \rho_H(dx)$ the canonical measure of X relative to m in the sequel.

DEFINITION 4.1. – Let $M \in \text{MF}$ and $A \in \text{RAF}(M)$. The bivariate Revuz measure of A relative to m is defined to be

$$(4.2) \quad \nu_A(F) := \uparrow \lim_{t \downarrow 0} \frac{1}{t} P^m \int_0^t F(X_{s-}, X_s) dA_s, \quad F \in \mathcal{p}\mathcal{E} \times \mathcal{E}.$$

The Revuz measure and left Revuz measure of A relative to m are defined to be

$$(4.3) \quad \begin{aligned} \rho_A(f) &:= \uparrow \lim_{t \downarrow 0} \frac{1}{t} P^m \int_0^t f(X_s) dA_s; \\ \lambda_A(f) &:= \uparrow \lim_{t \downarrow 0} \frac{1}{t} P^m \int_0^t f(X_{s-}) dA_s, \quad f \in p\mathcal{E}. \end{aligned}$$

As is customary, $F(x, y) = 0$ if either $x = \Delta$ or $y = \Delta$. The existence of the limits above can be shown similarly as in [FG2 (2.5)]. Note that we need to assume that $X_{\zeta-}$ exists in (4.3) if A does charge ζ . For any $f, g \in \mathcal{E}$, we write $f(x)g(y)$ as $f \otimes g$. Clearly $\nu_A(1 \otimes \cdot) = \rho_A$, but $\nu_A(\cdot \otimes 1) \neq \lambda_A$ in general. We denote the bivariate Revuz measure, Revuz measure and left Revuz measure of \overline{M} by ν_M, ρ_M and λ_M , respectively. We also define the bivariate potential of $A \in \text{RAF}(M)$ by

$$(4.4) \quad \mathcal{U}_A^q F(x) := P^x \int_0^\infty e^{-qt} F(X_{t-}, X_t) dA_t$$

for any $F \in p\mathcal{E} \times \mathcal{E}$ and $\mathcal{U}_M^q := \mathcal{U}_M^q$ for $M \in \text{MF}$. Denote $U_A^q := \mathcal{U}_A^q(1 \otimes \cdot)$ for $A \in \text{RAF}(M)$, $U_M^q := \mathcal{U}_M^q(1 \otimes \cdot)$, and $P_M^q f(x) := U_M^q f(x)$ for $x \in E_M$; $P_M^q f(x) := f(x)$ for $x \notin E_M$. Clearly $P_M^q = U_M^q$ if $M \in \text{MF}_+$. Recall that $[M]$ denotes the Stieltjes logarithm of $M \in \text{MF}_{++}$ and in this case $[M] \in \text{AF}$. The following lemma is just a collection of facts about bivariate Revuz measures. They can be checked either easily or by mimicking the proofs of the respective results for Revuz measures in [Y11].

LEMMA 4.5. – Let $M \in \text{MF}$ and $A \in \text{RAF}(M)$.

- (4.5a) $\nu_A(F) = \uparrow \lim_q qm \mathcal{U}_A^q F$;
- (4.5b) $\mathcal{U}_A^q F \in \mathcal{S}^q(M)$ and $\nu_A(F) = L^M(m, \mathcal{U}_A F)$ if $m \in \text{Dis}(M)$;
- (4.5c) ν_A does not charge any m -bipolar set, which is a set $B \subset E \times E$ such that either $B \subset E \times B_0$ for some m -polar set B_0 or $B \subset B_1 \times E$ for some m -leftpolar set B_1 ;
- (4.5d) $\lambda_A = \nu_A(\cdot \otimes 1)$ if A does not charge ζ ;
- (4.5e) If A is natural, $\nu_A(F) = \rho_A(F_D) = \lambda_A(F_D)$ for any $F \in p\mathcal{E} \times \mathcal{E}$, where $F_D(x) := F(x, x)$.
- (4.5f) If $M \in \text{MF}_{++}$, then $\nu_M = \nu_{[M]}$.

Now we are going to prove an important representation theorem for bivariate Revuz measures.

THEOREM 4.6. – *Let $M \in \text{SMF}_+$ with the decomposition (2.3). Then the bivariate Revuz measure of M is given by*

$$(4.7) \quad \nu_M(dx, dy) = (1_B + 1_{B^c} \cdot \Phi)(x, y) \cdot \nu(dx, dy) + \delta_{\{y\}}(dx) a(y) \rho_A(dy),$$

where $\delta_{\{y\}}$ is the point mass at y and ν is the canonical measure of X .

Proof. – By (3.2), if $\Psi := 1_B + 1_{B^c} \cdot \Phi$, we have

$$(4.8) \quad M_t = \left[\prod_{s \leq t} (1 - \Psi)(X_{s-}, X_s) \right] \exp \left\{ - \int_0^t a(X_s) dA_s \right\}.$$

Define

$$M_t^d := \prod_{s \leq t} (1 - \Psi)(X_{s-}, X_s);$$

$$M_t^c := \exp \left\{ - \int_0^t a(X_s) dA_s \right\} 1_{[0, S_M](t)}.$$

Clearly $M_t = M_t^d \cdot M_t^c$, $M_{S_M}^d = 0$ and $d(-M_s) = M_s^d d(-M_s^c) + M_{s-}^c d(-M_s^d)$. Then for $F \in \mathcal{pE} \times \mathcal{E}$, since A is continuous,

$$\begin{aligned} \int_0^t F(X_{s-}, X_s) M_s^d d(-M_s^c) &= \int_0^t F(X_{s-}, X_s) M_s^d 1_{\{s < S_M\}} d(-M_s^c) \\ &= \int_0^t F(X_{s-}, X_s) M_s a(X_s) dA_s \\ &= \int_0^t F_D(X_s) M_s a(X_s) dA_s. \end{aligned}$$

Hence by [FG2 (2.22)]

$$(4.9) \quad \lim_{t \rightarrow 0} \frac{1}{t} P^m \int_0^t F(X_{s-}, X_s) M_s^d d(-M_s^c) = \rho_A(F_D \cdot a).$$

On the other hand, since $M_{S_M}^d = 0$ and $\Psi(X_{S_M-}, X_{S_M}) = 1$,

$$\begin{aligned} &\int_0^t F(X_{s-}, X_s) M_{s-}^c d(-M_s^d) \\ &= \sum_{s \leq t} F(X_{s-}, X_s) M_{s-}^c (M_{s-}^d - M_s^d) \\ &= \sum_{s \leq t} F(X_{s-}, X_s) M_{s-}^c M_{s-}^d \cdot \Psi(X_{s-}, X_s). \end{aligned}$$

Then we use the Lévy system formula and [FG2 (2.22)]

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{1}{t} P^m \int_0^t F(X_{s-}, X_s) M_{s-}^c d(-M_s^d) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} P^m \int_0^t M_s N(F\Psi)(X_s) dH_s \\ &= \int \rho_H(dx) N(F\Psi)(x) \\ &= \Psi \cdot \nu(F). \end{aligned}$$

Hence we have

$$(4.10) \quad \nu_M(F) = \Psi \cdot \nu(F) + a \cdot \rho_A(F_D).$$

That proves (4.7) ■

The following corollary is easy to check.

COROLLARY 4.11. – *Let $M \in \text{SMF}_+$. Then we have (a) $\nu_{S_M} = 1_B \cdot \nu$; (b)*

$$\nu^M + 1_{D^c} \cdot \nu_M = \nu.$$

Recall the definition of Stieltjes logarithm of a multiplicative functional in (2.4). Now we are going to compute its bivariate Revuz measure.

PROPOSITION 4.12. – *Let $M \in \text{SMF}_+$ with the decomposition (2.3). Then the bivariate Revuz measure of its Stieltjes logarithm $\text{slog}M$ is given by*

$$(4.13) \quad \nu_{\text{slog}M}(dx, dy) = 1_{B^c} \Phi(x, y) \cdot \nu(dx, dy) + \delta_{\{y\}}(dx) a(y) \rho_A(dy).$$

Proof. – Since M admits the decomposition (2.3), its Stieltjes logarithm $\text{slog}M$ admits the following decomposition

$$(4.14) \quad (\text{slog}M)_t = \sum_{s \leq t} \Phi(X_{s-}, X_s) 1_{\{s < S_M\}} + \int_0^t 1_{\{s < S_M\}} a(X_s) dA_s.$$

Hence for any $F \in p\mathcal{E} \times \mathcal{E}$, using the Lévy system formula, (3.2) and [FG2 (2.22)] we find

$$\begin{aligned} \nu_{\text{slog}M}(F) &= \uparrow \lim_{t \downarrow 0} \frac{1}{t} P^m \int_0^t F(X_{s-}, X_s) d(\text{slog}M)_t \\ &= \uparrow \lim_{t \downarrow 0} \frac{1}{t} P^m \sum_{s \leq t} F(X_{s-}, X_s) \Phi(X_{s-}, X_s) 1_{\{s < S_M\}} \\ &\quad + \uparrow \lim_{t \downarrow 0} \frac{1}{t} P^m \int_0^t F(X_{s-}, X_s) 1_{\{s < S_M\}} a(X_s) dA_s \\ &= \uparrow \lim_{t \downarrow 0} \frac{1}{t} P^m \sum_{s \leq t} F(X_{s-}, X_s) \Phi(X_{s-}, X_s) \prod_{r \leq s} 1_{B^c}(X_{s-}, X_s) \\ &\quad + \rho_A(a \cdot F_D) \\ &= \uparrow \lim_{t \downarrow 0} \frac{1}{t} P^m \int_0^t N(1_{B^c} \Phi F)(X_s) 1_{\{s < S_M\}} dH_s + a \rho_A(F_D) \\ &= 1_{B^c} \cdot \Phi \cdot \nu(F) + a \cdot \rho_A(F_D). \end{aligned}$$

That completes the Proof. ■

Combining (4.7), (4.11a) and (4.13), we can see that if $M \in \text{SMF}_+$, then

$$(4.15) \quad \nu_M = \nu_{\text{slog}M} + \nu_{S_M}.$$

But (4.15) holds even without the assumption that M has a simple decomposition. Actually we have a more general formula. Recall a useful formula [Yi1 (3.12)]. If $M \in \text{MF}_+$, $m \in \text{Dis}$ and $u \in \mathcal{S}$, then

$$(4.16) \quad L(m, u) = L^M(m, u - P_M u).$$

THEOREM 4.17. – (i) Let $M, N \in \text{MF}$ with $E_N \subset E_M$. Then

$$\nu_{MN}^{m^*} = {}^N \nu_M^{m^*} + \nu_N^{m^*},$$

where ${}^N \nu_M^{m^*}$ is the bivariate Revuz measure of M relative to $m^* = m|_{E_N}$ under the N -subprocess of X . (ii) Let $M \in \text{MF}$. Then

$$\nu_M^{m^*} = \nu_{\text{slog}M}^{m^*} + \nu_{S_M}^{m^*}.$$

Proof. – (i) Clearly $E_{MN} = E_N$. For any $F \in p\mathcal{E} \times \mathcal{E}$,

$$\begin{aligned} \nu_{MN}^{m^*}(F) &= \lim_t \frac{1}{t} P^{m^*} \int_0^t F(X_{s-}, X_s) d(-M_s N_s) \\ &= \lim_t \frac{1}{t} P^{m^*} \int_0^t F(X_{s-}, X_s) [N_s d(-M_s) + M_{s-} d(-N_s)]. \end{aligned}$$

But

$$P^{m^*} \int_0^t F(X_{s-}, X_s) N_s d(-M_s) = P_{(N)}^{m^*} \int_0^t F(X_{s-}, X_s) d(-M_s),$$

where $(P_{(N)}^x)$ denotes the probability measures corresponding to the N -subprocess of X . Thus if we write $(M_- * N)_t := M_- * \bar{N}$, then

$$\nu_{MN}^{m^*} = {}^N \nu_M^{m^*} + \nu_{M_- * N}^{m^*}.$$

We claim that the following identity holds

$$(4.18) \quad \mathcal{U}_N = P_M^{(N)} \mathcal{U}_N + \mathcal{U}_{M_- * N} \text{ on } E_N,$$

where $P_M^{(N)} f(x) := P_{(N)}^x \int_0^\infty f(X_t) d(-M_t) = P^x \int_0^\infty f(X_t) N_t d(-M_t)$. In fact for $F \in p\mathcal{E} \times \mathcal{E}$ and $x \in E_N$ with $\mathcal{U}_N F(x) < \infty$

$$\begin{aligned} P_M^{(N)} \mathcal{U}_F(x) &= P_{(N)}^x \int_0^\infty \mathcal{U}_N F(X_t) d(-M_t) \\ &= P^x \int_0^\infty \left(\int_0^\infty F(X_{s-}, X_s) d(-N_s) \right) \circ \theta_t d(-M_t) \\ &= P^x \int_0^\infty \int_t^\infty F(X_{s-}, X_s) d(-N_s) d(-M_t) \\ &= \mathcal{U}_N F(x) - \mathcal{U}_{M_- * N} F(x). \end{aligned}$$

Since $M \in \text{MF}_=(X, N)$, we can use (4.16) and find

$$\nu_N^{m^*}(F) = L^N(m^*, \mathcal{U}_N F) = L^{MN}(m^*, \mathcal{U}_{M_- * N} F) = \nu_{M_- * N}^{m^*}(F).$$

(ii) Let $N = 1_{[0, S_M[}$. Then $M = MN$, $E_N = E_M$ and by (i) we have $\nu_M^{m^*} = {}^N \nu_M^{m^*} + \nu_{S_M}^{m^*}$. but

$${}^N \nu_M^{m^*}(F) = \lim_t \frac{1}{t} P^{m^*} \int_0^t F(X_{s-}, X_s) 1_{\{s < S_M\}} d(-M_s) = \nu_{M_- * \text{slog} M}^{m^*}.$$

Following (4.18) and using the fact that $\text{slog} M \in \text{AF}(S_M)$, it is easy to check that

$$\mathcal{U}_{\text{slog} M} = P_M^{(S_M)} \mathcal{U}_{\text{slog} M} + \mathcal{U}_{M_- * \text{slog} M} \text{ on } E_M.$$

Then using (4.16) again, we have

$${}^N \nu_M^{m^*} = \nu_{M_- * \text{slog} M}^{m^*} = \nu_{\text{slog} M}^{m^*}.$$

That completes the proof. Our last result of this section describes the relationship between the left Revuz measure λ_M and left marginal measure of ν_M .

PROPOSITION 4.19. – Assume that $X_{\zeta-}$ exists in E a.s. and $M_\zeta = 0$. Let $M \in \text{MF}_+$ and κ be the killing measure of X relative to m (the left Revuz measure of ζ)

$$\kappa(f) := \uparrow \lim_{t \rightarrow 0} \frac{1}{t} P^m(f(X_{\zeta-}); \zeta \leq t).$$

Then $\lambda_M = \nu_M(\cdot \otimes 1) + \kappa$.

Proof. – For any $f \in p\mathcal{E}$ we have

$$\begin{aligned} \lambda_M(f) &= \uparrow \lim_{t \rightarrow 0} \int_0^t f(X_{s-}) d(-M_s) \\ &= \nu_M(f \otimes 1) + \uparrow \lim_{t \rightarrow 0} \frac{1}{t} P^m(f(X_{\zeta-})M_{\zeta-}; \zeta \leq t). \end{aligned}$$

The second term is the left Revuz measure of $(\int_0^t M_{s-} d1_{[0,\zeta[}(s)) \in \text{AF}(M)$, which is equal to κ by an argument similar to that used to prove (4.17). ■

5. A generalized Revuz formula

The Revuz formula was first given in [Re1] under strong duality and later proved by in [GS] and [GG] under weak duality. We quote the following form of Revuz formula from [GS (9.9)] and [GG (A8)]. Let $A \in \text{RAF}(X)$ which may charge the life time ζ . Assume that A has a σ -finite left Revuz measure, which is defined as

$$\lambda_A^m(f) := \lim_{t \rightarrow 0} \frac{1}{t} P^m \int_0^t f(X_{s-}) dA_s.$$

(5.1) Hypothesis. $X_{\zeta-}$ exists in E a.s. on $\{\Delta A_\zeta > 0\}$.

Under (5.1) it holds that $\lambda_A^m \hat{U}^q(dx) = (P^x \int_0^\infty e^{-qt} dA_t) m(dx)$. For any $f, g \in p\mathcal{E}$, using (5.1) we have

$$(g, U_A^q f) = \lambda_{f * A} \hat{U}^q f = \nu_A(\hat{U}^q g \otimes f),$$

where $(f * A)_t := \int_0^t f(X_s) dA_s$. Note that $f * A$ never charges ζ and thus (5.2) holds without (5.1).

In this section we will deduce a similar formula for an M -additive functional. Let $M \in \text{MF}(X)$ and $A \in \text{RAF}(M)$ having a σ -finite left Revuz measure relative to $1_{E_M} \cdot m \in \text{Exc}(M)$, and \hat{M} the dual multiplicative functional of M ; namely, $\hat{M} \in \text{MF}(\hat{X})$ such that (X, M) and (\hat{X}, \hat{M}) are in weak duality with respect to m . Write $m^* := m|_{E_M}$ and let (Q_t) and (V^q) (resp., (\hat{Q}_t) and (\hat{V}^q)) be the transition semigroup and resolvent of (X, M) (resp., (\hat{X}, \hat{M})). Here A may charge ζ .

Now we will give the some preliminary lemmas and the generalized Revuz formula. The proofs we present here are analogous to those in [GS].

LEMMA 5.3. – *Let $\phi \in \mathcal{R}^+$ and let η be σ -finite measure on E . Then for each $t > 0$,*

$$(5.4) \quad \int_0^\infty \phi(s) ds E^{\eta Q_s}(f(X_0)A_t) = E^\eta \int_0^\infty dA_r \int_0^\infty f(X_s)\phi(s)1_{[r-t, r)}(s) ds.$$

Proof. – The direct computation gives

$$\begin{aligned} & \int_0^\infty \phi(s) ds E^{\eta Q_s}(f(X_0)A_t) \\ &= \int \phi(s) ds \int_E \int_E \eta(dx) Q_s(x, dy) E^y(f(X_0)A_t) \\ &= \int \phi(s) ds \int \eta(dx) E^x((E^{X_s} f(X_0)A_t)M_s) \\ &= \int \phi(s) ds \int \eta(dx) E^x[(f(X_0)A_t) \circ \theta_s M_s] \\ &= \int \phi(s) ds \int \eta(dx) E^x f(X_s)(A_{t+s} - A_s) \\ &= E^\eta \int_0^\infty dA_r \int_0^\infty f(X_s)\phi(s)1_{[r-t, r)}(s) ds. \end{aligned}$$

That completes the proof. ■

THEOREM 5.5. – *Suppose that P^{m^*} a.s. $t \mapsto f(X_{t-})$ is left continuous on $]0, \infty[$. Then under (5.1)*

$$(5.6) \quad \lim_{t \downarrow 0} \frac{1}{t} P^{m^*} \int_0^t f(X_{s-}) dA_s = \lim_{t \downarrow 0} \frac{1}{t} P^{m^*} (f(X_0)A_t).$$

By the lemma above, the proof of this theorem is exactly the same as the proof of [GS (8.7)]. We won't repeat it here.

THEOREM 5.7. – *Under (5.1), the following Revuz formula holds,*

$$\lambda_A^{m^*} \hat{V}^q(dx) = \left(E^x \int_0^\infty e^{-qt} dA_t \right) m^*(dx).$$

Proof. – Since $\lambda_A^{m^*}$ is σ -finite, it suffices to prove (5.8) when $\lambda_A^{m^*}(E) < \infty$. First notice that nothing will be changed if we replace m^* in the rightside of the above formula by m . Let $f \in \hat{S}^q(M)$ and be bounded. Then $t \mapsto f(X_{t-})$ is left continuous on $]0, \infty[$ a.s. P^{m^*} . Let $C_t(x) = E^x(A_t)$. Thus $C_{t+s} = C_t + Q_t C_s$ and

$$\begin{aligned} \frac{1}{s}(f, C_{t+s} - C_s)_{m^*} &= \frac{1}{s}(f, Q_t C_s)_{m^*} \\ &= \frac{1}{s}(\hat{Q}_t f, C_s)_{m^*} = \frac{1}{s} P^{m^*}(\hat{Q}_t f(X_0) A_s) \end{aligned}$$

By the previous theorem and the fact $e^{-qt} \hat{Q}_t f \in \hat{S}^q(M)$, we have

$$\lim_{s \rightarrow 0} \frac{1}{s}(f, C_{t+s} - C_s) = \lim_{s \rightarrow 0} \frac{1}{s} P^{m^*} \int_0^s \hat{Q}_t f(X_{r-}) dA_r = \lambda_A^{m^*}(\hat{Q}_t f).$$

Therefore $\lambda_A^{m^*}(\hat{Q}_t f)$ is the right derivative of (f, C_t) and

$$(f, C_{t+s} - C_t) = (\hat{Q}_t f, C_s) \leq N \cdot (1, C_s) \leq N \cdot s \lambda_A^{m^*}(E) < \infty,$$

where $N := \|f\|_\infty$. Thus $t \mapsto (f, C_t)$ is absolutely continuous and it follows from $C_0 = 0$ that

$$(5.8) \quad (f, C_t) = \int_0^t \lambda_A^{m^*}(\hat{Q}_s f) ds.$$

For any bounded continuous $h \geq 0$ which supported by E_M , $qV^q h \rightarrow h$ boundedly. Hence (5.8) holds for such h . ■

Finally we will present two useful consequences. Let $L(\hat{u}, v) := L(\hat{u}m, v)$ for $\hat{u} \in \hat{S}$ and $v \in S$. Since L is just the Dirichlet form of X in some sense, we thereby obtain some idea on transformations of Dirichlet forms.

COROLLARY 5.9. – (1) *The following generalized Revuz formula holds without (5.1)*

$$(5.10) \quad (g, U_A^q f) = \nu_A^{m^*}(\hat{V}^q g \otimes f) \quad f, g \in p\mathcal{E}.$$

(2) *If $M \in \text{MF}$ and $S_M > 0$ a.s., then*

$$(5.11) \quad L^M(\hat{u}, v) = L(\hat{u}, v) + \nu_M^m(\hat{u} \otimes v) \quad \text{for } \hat{u} \in \hat{\mathcal{S}}, v \in \mathcal{S}.$$

Proof. – Part (1) is immediate from (5.7) as we indicated in (5.2). (2) Since $\hat{u}m \in \text{Exc}(X)$ for $\hat{u} \in \hat{\mathcal{S}}$, using [Yi (2.5)] and (5.5), we have

$$\begin{aligned} L^M(\hat{u}, v) &= L(\hat{u}, v) + \rho_M^{\hat{u}m}(v) \\ &= L(\hat{u}, v) + \lim_{t \rightarrow 0} \frac{1}{t} P^m \left(\hat{u}(X_0) \int_0^t v(X_s) d(-M_s) \right) \\ &= L(\hat{u}, v) + \lim_{t \rightarrow 0} \frac{1}{t} P^m \int_0^t \hat{u}(X_{s-}) v(X_s) d(-M_s) \\ &= L(\hat{u}, v) + \nu_M^m(\hat{u} \otimes v). \end{aligned}$$

The reason that we can apply (5.6) here is that $(\int_0^t v(X_s) d(-M_s))_{t \geq 0}$ does not charge ζ . ■

6. Uniqueness and dual multiplicative functionals

The following lemma is an m -a.e. version of [BG (IV.2.12)].

LEMMA 6.1. – *Let $A^1, A^2 \in \text{AF}$. Suppose that, for some fixed $q > 0$, $U_{A^1}^q 1 < \infty$ a.e. m , and that for any bound and continuous $f \in p\mathcal{E}$, $U_{A^1}^q f = U_{A^2}^q f$ a.e. m . Then A^1 and A^2 are m -equivalent.*

Proof. – It is clear that a.e. m above can be replaced by q.e., namely, except a set of m -capacity zero. It follows from the proof of [BG, IV(2.12)] that $U_{A^1}^p f = U_{A^2}^p f$ a.e. m for $p \geq q$. Since $q \rightarrow U_{A^i}^q f(x)$ is decreasing and continuous for any $x \in E$, there exists $N \in \mathcal{E}$ with $m(N) = 0$ such that $U_{A^i}^p 1(x) < \infty$ and $U_{A^1}^p f(x) = U_{A^2}^p f(x)$ hold for any $p \geq q$ and $x \notin N$. Thus it follows that A^1 and A^2 are m -equivalent. ■

First we have the following uniqueness result.

PROPOSITION 6.2. – *Suppose that $A^1, A^2 \in \text{AF}$ which are σ -integrable. A^1 and A^2 are m -equivalent if and only if their bivariate Revuz measures*

are the same

$$\nu_{A^1} = \nu_{A^2}.$$

Proof. – The **only if** part is trivial. Now assume that $\nu_{A^1} = \nu_{A^2}$. Since they are σ -finite, there exist a sequence of Borel functions $b_n \in \mathcal{E} \times \mathcal{E}$ such that

- 1) $0 < b_n < 1$ for any n ;
- 2) $b_n \uparrow 1$;
- 3) Both $b_n \cdot \nu_{A^1}$ and $b_n \cdot \nu_{A^2}$ are finite.

Let

$$A_t^{1,n} = \int_0^t b_n(X_{s-}, X_s) dA_s^1$$

$$A_t^{2,n} = \int_0^t b_n(X_{s-}, X_s) dA_s^2.$$

Thus

$$\nu_{A^{1,n}} = b_n \cdot \nu_{A^1} = b_n \cdot \nu_{A^2} = \nu_{A^{2,n}}.$$

From (6.2), it follows that $U_{A^{1,n}}^q f = U_{A^{2,n}}^q f$ a.e. m for any $f \in p\mathcal{E}$. On the other hand, by the Revuz formula,

$$(1, U_{A^{1,n}}^1 1) = \nu_{A^{1,n}}(\hat{U}^1 1 \otimes 1) \leq \nu_{A^{1,n}}(1) < \infty.$$

Thus $U_{A^{1,n}}^1 1 < \infty$ a.e. m . Therefore $A^{1,n}$ and $A^{2,n}$ are m -equivalent and by passing to limit, A^1 and A^2 are m -equivalent. ■

Let $M, N \in \text{MF}$ and S_M, S_N the lifetimes of M, N , respectively. If M and N are m -equivalent, then P^m a.s. $S_M = S_N$, $E_M \triangle E_N$ is m -polar and $\nu_M^{m^*} = \nu_N^{m^*}$ where $m^* := m|_{E_M} = m|_{E_N}$. The following result is a partial answer to its inverse problem.

THEOREM 6.3. – *Assume that $M, N \in \text{SMF}_+$. Then M and N are m -equivalent if and only if $\nu_M = \nu_N$.*

Proof. – It suffices to check the sufficiency. Clearly M and N have the decompositions

$$M_t = \left[\prod_{s \leq t} (1 - \Phi)(X_{s-}, X_s) \right] \exp \left\{ - \int_0^t a(X_s) dA_s \right\};$$

$$N_t = \left[\prod_{s \leq t} (1 - \Psi)(X_{s-}, X_s) \right] \exp \left\{ - \int_0^t b(X_s) dB_s \right\},$$

where

- 1) $S_M = J_K > 0, S_N = J_L > 0$, and both K and L are disjoint from D ;
- 2) $\Phi, \Psi \in p\mathcal{E} \times E, \Phi < 1$ and $\Psi < 1$ everywhere;
- 3) $a, b \in p\mathcal{E}$ and $A, B \in AF$ being continuous.

Then we know that

$$\nu_M|_{D^c} = (1_{K^c}\Phi + 1_K) \cdot \nu, \quad \nu_N|_{D^c} = (1_{L^c}\Psi + 1_L) \cdot \nu.$$

Hence $\nu_M = \nu_N$ implies that

$$(1_{K^c}\Phi + 1_K) \cdot \nu = (1_{L^c}\Psi + 1_L) \cdot \nu.$$

Thus we have

$$1_{K^c}(1 - \Phi) \cdot \nu = 1_{L^c}(1 - \Psi) \cdot \nu.$$

Since $1 - \Phi > 0$ and $1 - \Psi > 0$ everywhere, $\nu(K^c \Delta L^c) = 0$, or equivalently $\nu(K \Delta L) = 0$.

Choose a sequence of Borel subsets (E_n) of $E \times E$ satisfying:

- 1) $E_n \uparrow E \times E - D$;
- 2) $\nu(E_n) < \infty$ for any n ;
- 3) $E_n \subset \{(x, y) : d(x, y) > \frac{1}{n}\}$, where d is a metric on E compatible with the given topology.

Existence of such sequence $\{E_n\}$ is easy to check. Let

$$S_{M,n} := J_{K \cap E_n} \quad \text{and} \quad S_{N,n} := J_{L \cap E_n}.$$

Clearly $\nu\{(K \cap E_n) \Delta (L \cap E_n)\} = 0$. Define

$$K_t^n := \sum_{s \leq t} 1_{K \cap E_n}(X_{s-}, X_s) \quad (= \sum_k 1_{\{S_{M,n}^k \leq t\}})$$

and

$$L_t^n := \sum_{s \leq t} 1_{L \cap E_n}(X_{s-}, X_s) \quad (= \sum_k 1_{\{S_{N,n}^k \leq t\}}),$$

where $S_{M,n}^k$ and $S_{N,n}^k$ are the k^{th} -iterates of $S_{M,n}$ and $S_{N,n}$.

Then $K^n, L^n \in AF$ and

$$\nu_{K^n} = 1_{K \cap E_n} \cdot \nu = 1_{L \cap E_n} \cdot \nu = \nu_{L^n}.$$

From the Lemma (6.2), we know that for any $f, g \in p\mathcal{E}$,

$$(f, U_{K^n}^1 g) = \nu_{K^n}(\hat{U}^1 f \otimes g) = \nu_{L^n}(\hat{U}^1 f \otimes g) = (f, U_{L^n}^1 g),$$

i.e., $U_{K^n}^1 g = U_{L^n}^1 g$ a.e. m .

On the other hand, by the Revuz formula,

$$(1, U_{K^n}^1 1) = \nu_{K^n}(\hat{U}^1 1 \otimes 1) \leq \nu_{K^n}(1) = \nu(K \cap E_n) < \infty.$$

Hence $U_{K^n}^1 1 < \infty$ a.e. m . Therefore K^n and L^n are m -equivalent. But

$$S_{M,n} = \inf\{t > 0 : K_t^n > 0\} \quad \text{and} \quad S_{N,n} = \inf\{t > 0 : L_t^n > 0\}.$$

It follows that P^m a.s. $S_{M,n} = S_{N,n}$ and thus P^m a.s. $S_M = S_N$.

Now we have $\text{slog}M, \text{slog}N \in \text{AF}(S)$, where $S := S_M = S_N$. But by (4.15)

$$\nu_{\text{slog}M} = \nu_{\text{slog}N}.$$

It follows from (6.2) that $\text{slog}M$ and $\text{slog}N$ are m -equivalent. Hence M and N are m -equivalent. ■

Before our next result, we would like to recall some background. Two multiplicative functionals $M \in \text{MF}(X)$ and $\hat{M} \in \text{MF}(\hat{X})$ are said to be dual if their corresponding subprocesses are dual with respect to m . In [Sh1], Sharpe proved that if X and \hat{X} are in strong duality relative to m and $M \in \text{MF}(X)$ and $\hat{N} \in \text{MF}(\hat{X})$, then M and \hat{N} are dual if and only if, 1) the corresponding exact regularizations of S_M and $\hat{S}_{\hat{N}}$ are dual terminal times; 2) the bivariate Revuz measures of the Stieltjes logarithms of M and \hat{N} are dual; that is

$$\nu_{\text{slog}M}(dx, dy) = \hat{\nu}_{\text{slog}\hat{N}}(dy, dx).$$

Here we are going to prove some similar results in weak duality by means of Kuznetsov measures.

Let Q be the Kuznetsov measure of X and m . Ω is identified as a subspace of W , i.e.,

$$(6.4) \quad \Omega = \{w \in W : \alpha(w) = 0, Y_{\alpha^+}(w) \text{ exists in } E\}.$$

Shift operators $(\sigma_t)_{t \in \mathbb{R}}$ and truncated shift operators $(\theta_t)_{t \in \mathbb{R}}$ are defined as

$$\sigma_t w(s) := w(t + s) \quad \text{for any } t, s \in \mathbb{R},$$

$$\theta_t w(s) := w(t + s) \text{ if } s > 0, \quad \theta_t w(s) := \Delta \text{ if } s \leq 0.$$

Clearly $\theta_t|_{\Omega}$ is the shift operator of X and $\theta(\{\alpha < t\}) \subset \Omega$. We also define a reversal operator $\lambda : W \rightarrow W$ as

$$\lambda w(s) := w((-s)-)$$

for any $s \in R$ (write $\hat{w} := \lambda w$ and $\hat{Y}_t(w) = Y_t(\hat{w})$) and the backward shift operator $\hat{\sigma}_t$ on W naturally as

$$\hat{\sigma}_t \hat{w}(s) := \hat{w}(t + s).$$

We have

$$\begin{aligned} \hat{\sigma}_t \circ \lambda w(s) &= \hat{\sigma}_t \hat{w}(s) = \hat{w}(t + s) \\ &= w((-t - s)-) = \sigma_{-t} w((-s)-) \\ &= \lambda \circ \sigma_{-t} w(s). \end{aligned}$$

Hence $\hat{\sigma}_t \circ \lambda = \lambda \circ \sigma_{-t}$. We also have $\hat{\alpha} := \alpha \circ \lambda = -\beta$ and $\hat{\beta} := \beta \circ \lambda = -\alpha$. Similarly, $\hat{\Omega}$ is identified as

$$\hat{\Omega} := \{\hat{w} \in W : \alpha(\hat{w}) = 0, Y_{\alpha+}(\hat{w}) \text{ exists in } E\}$$

or equivalently,

$$w \in \Omega \iff \lambda w \in \hat{\Omega}.$$

Let \hat{Q} be the Kuznetsov measure of \hat{X} and m on W , i.e., for $t_1 < t_2 < \dots < t_n$,

$$\begin{aligned} \hat{Q}(\alpha < t_1, Y_{t_1} \in dx_1, \dots, Y_{t_n} \in dx_n, t_n < \beta) \\ = m(dx_1) \hat{P}_{t_2-t_1}(x_1, dx_2) \cdots \hat{P}_{t_n-t_{n-1}}(x_{n-1}, dx_n). \end{aligned}$$

It is known that \hat{Q} is the reversal of Q , i.e.,

$$(6.5) \quad \hat{Q} = \lambda(Q) = Q \circ \lambda^{-1}.$$

In fact for any $t_1 < t_2 < \dots < t_n$,

$$\begin{aligned} \lambda(Q)(\alpha < t_1, Y_{t_1} \in dx_1, \dots, Y_{t_n} \in dx_n, t_n < \beta) \\ = Q(\alpha < -t_n, Y_{-t_n} \in dx_n, \dots, Y_{-t_1} \in dx_1, -t_1 < \beta) \\ = m(dx_n) P_{t_n-t_{n-1}}(x_n, dx_{n-1}) \cdots P_{t_2-t_1}(x_2, dx_1) \\ = \hat{P}_{t_n-t_{n-1}}(x_{n-1}, dx_n) \cdots \hat{P}_{t_2-t_1}(x_1, dx_2) m(dx_1) \\ = \hat{Q}(\alpha < t_1, Y_{t_1} \in dx_1, \dots, Y_{t_n} \in dx_n, t_n < \beta). \end{aligned}$$

Let $A \in \text{RAF}(X)$ which induces a HRM, denoted by A^* , of Y by the formula

$$(6.6) \quad A^*(w, B) := \uparrow \lim_{t \downarrow \alpha(w)} A(\theta_t w, B - t).$$

Precisely, A^* is carried by $] \alpha, \beta]$ and

$$A^*(ds) \circ \sigma_t = A^*(ds + t).$$

Now define an RM on W

$$(6.7) \quad \hat{A}^*(\hat{w},]s, t[) := A^*(w,] - t, -s[).$$

LEMMA 6.8. – \hat{A}^* is a backward HRM of Y in the sense

$$\hat{A}^*(\hat{\sigma}_t(\hat{w}), ds) = \hat{A}^*(\hat{w}, ds + t).$$

Proof. – Check directly by definition,

$$\begin{aligned} \hat{A}^*(\hat{\sigma}_t(\hat{w}),]r, s[) &= \hat{A}^*(\hat{\sigma}_t \circ \lambda(w),]r, s[) \\ &= \hat{A}^*(\lambda \circ \sigma_{-t}(w),]r, s[) \\ &= A^*(\sigma_{-t}(w),] - s, -r[) \\ &= A^*(w,] - s - t, -r - t[) \\ &= \hat{A}^*(\hat{w},]s, r[+t). \end{aligned}$$

That completes the proof. ■

DEFINITION 6.9. – 1) Let $A \in \text{RAF}(X)$ and $\hat{A} \in \text{RAF}(\hat{X})$. They are said to be dual if \hat{A} is m -equivalent to $\hat{A}^*|_{\hat{\Omega}}$ where \hat{A}^* is defined in (6.7). 2) Two measures μ_1 and μ_2 on $E \times E$ are said to be dual if $\mu_1(dx, dy) = \mu_2(dy, dx)$.

Remark. – The construction of the dual multiplicative functionals is similar. For example, a terminal time T of X induces a stationary terminal time T^* of Y by

$$T^*(w) := \downarrow \lim_{t \downarrow \alpha(w)} (T(\theta_t w) + t).$$

Define the dual of T^* by

$$\hat{T}^*(\hat{w}) := -T^*(w) \quad \text{and set } \hat{T} := \hat{T}^*|_{\hat{\Omega}}.$$

Then \hat{T} is the dual terminal time of T (See [Mi2]). Clearly the birth time α and the death time β are dual.

THEOREM 6.10. – 1) Let $A \in \text{AF}(X)$ and $\hat{A} \in \text{AF}(\hat{X})$ which are σ -integrable. Then A and \hat{A} are dual if and only if ν_A and $\hat{\nu}_{\hat{A}}$ are dual. 2) Furthermore if A and \hat{A} are natural, then they are dual if and only if $\rho_A = \hat{\rho}_{\hat{A}}$.

Proof. – 1) First assume A and \hat{A} are dual. We can identify \hat{A} here with the dual constructed above. Now by [Ge (8.21)], for any $F \in p\mathcal{E} \times \mathcal{E}$,

$$\begin{aligned} \hat{\nu}_{\hat{A}}(F) &= \hat{Q} \int \phi(t)F(Y_{t-}, Y_t)\hat{A}^*(dt) \\ &= Q \int \phi(t)F(Y_{t-}(\hat{w}), Y_t(\hat{w}))\hat{A}^*(\hat{w}, dt) \\ &= Q \int \phi(t)F(Y_{-t}(w), Y_{(-t)-}(w))A^*(w, -dt) \\ &= Q \int \phi(-t)F(Y_t, Y_{t-})A^*(dt) = \nu_A(\hat{F}) \end{aligned}$$

where $\phi \in p\mathcal{R}$ and $\int \phi(t)dt = 1$ and $\hat{F}(x, y) := F(y, x)$.

Conversely, assume that ν_A and $\hat{\nu}_{\hat{A}}$ are dual. Let $\hat{A}' := \hat{A}^*|_{\hat{\Omega}}$. Then we have

$$\hat{\nu}_{\hat{A}'}(dx, dy) = \nu_A(dy, dx) = \hat{\nu}_{\hat{A}}(dx, dy).$$

From (6.2) it follows that \hat{A} and \hat{A}' are m -equivalent. Therefore A and \hat{A} are dual. 2) follows from 1) directly, due to the fact that bivariate Revuz measures of natural additive functionals are carried by the diagonal D . ■

THEOREM 6.11. – *Let $M \in MF_{++}(X)$ and $\hat{M} \in MF_{++}(\hat{X})$. Then the following statements are equivalent.*

- (1) M and \hat{M} are dual;
- (2) $[M]$ and $[\hat{M}]$ are dual;
- (3) ν_M and $\hat{\nu}_{\hat{M}}$ are dual.

Proof. – Since $\nu_M = \nu_{[M]}$ and $\hat{\nu}_{\hat{M}} = \hat{\nu}_{[\hat{M}]}$, the equivalence of (ii) and (iii) follows from (6.10). Now we are going to prove the equivalence of (i) and (iii). First we assume that M and \hat{M} are dual. Using the Revuz formulas (5.2) and (5.10), we find for any bounded $f, g \in p\mathcal{E}$,

$$\begin{aligned} \nu_M(\hat{U}^1 g \otimes V^1 f) &= \nu_{[M]}(\hat{U}^1 g \otimes V^1 f) = (g, U_{[M]}^1 V^1 f) \\ &= (\hat{U}_{[M]}^1 \hat{V}^1 g, f) = (f, \hat{U}_{\hat{M}}^1 \hat{U}^1 g) \\ &= \hat{\nu}_{\hat{M}}(V^1 f \otimes \hat{U}^1 g). \end{aligned}$$

Let $\mu_1(dx) := \nu_M(dx \otimes V^1 f)$ and $\mu_2(dx) := \hat{\nu}_{\hat{M}}(V^1 f \otimes dx)$ for any fixed $f \in bp\mathcal{E}$. Then $\mu_1 \hat{U}^1 = \mu_2 \hat{U}^1$. Pick $h \in \mathcal{E}$ strictly positive such that $m(h) < \infty$. Then $\mu_1 \hat{U}^1(h) = \mu_2 \hat{U}^1(h) = \nu_M(\hat{U}^1 h \otimes V^1 f) = (h, U_{[M]}^1 V^1 f) \leq (h, U^1 f) < \infty$; that is, $\mu_1 \hat{U}^1$ and $\mu_2 \hat{U}^1$ are σ -finite. This implies that $\mu_1 = \mu_2$ by [Ge (2.12)] or $\nu_M(dx \otimes V^1 f) = \hat{\nu}_{\hat{M}}(V^1 f \otimes dx)$.

On the other hand, let $\nu_1(dy) := \nu_M(g \otimes dy)$ and $\nu_2(dy) := \hat{\nu}_{\hat{M}}(dy \otimes g)$ for any fixed $g \in bp\mathcal{E}$. Clearly $\nu_1 V^1 = \nu_2 V^1$ and, using the Revuz formula (6.10) again, $\nu_1 V^1(h) = \nu_2 V^1(h) = \hat{\nu}_{\hat{M}}(V^1 h \otimes g) = (\hat{U}_{\hat{M}}^1 g, h) < \infty$; that is, $\nu_1 V^1$ and $\nu_2 V^1$ are σ -finite. Hence $\nu_1 = \nu_2$; i.e., ν_M and $\hat{\nu}_{\hat{M}}$ are dual.

Conversely, if ν_M and $\hat{\nu}_{\hat{M}}$ are dual, let $N \in MF(X)$ denote the dual of \hat{M} . Then we have

$$\nu_M(dx, dy) = \hat{\nu}_{\hat{N}}(dy, dx) = \nu_N(dx, dy).$$

Hence $\nu_M = \nu_N$. By (6.2) M and N are m -equivalent. Therefore M and \hat{M} are dual. ■

PROPOSITION 6.12. – 1) Let $S \in SMF_+(X)$ and $\hat{S} \in SMF_+(\hat{X})$ be terminal times. Then S and \hat{S} are dual if and only if ν_S and $\hat{\nu}_{\hat{S}}$ are dual. 2) The canonical measures ν and $\hat{\nu}$ of X and \hat{X} are dual.

Proof. – 1) First let S and \hat{S} are dual. There exist Borel subsets $K, L \subset E \times E - D$ such that

$$P^m \text{ a.s. } S = J_K \text{ and } \hat{P}^m \text{ a.s. } \hat{S} = \hat{J}_L.$$

Let

$$G_n := \{(x, y) \in E \times E : d(x, y) > \frac{1}{n}\}.$$

Since $G_n = \hat{G}_n$, J_{G_n} and \hat{J}_{G_n} are dual [Ge3]. Set

$$S_n := J_{K \cap G_n} = S \wedge J_{G_n}$$

and

$$\hat{S}_n := \hat{J}_{L \cap \hat{G}_n} = \hat{S} \wedge \hat{J}_{G_n}.$$

Obviously, S_n and \hat{S}_n are also dual terminal times. Let

$$A_t^n := \sum_{s \leq t} 1_{K \cap G_n}(X_{s-}, X_s) = \sum_k 1_{\{S_n^k \leq t\}}$$

$$\hat{A}_t^n := \sum_{s \leq t} 1_{L \cap G_n}(\hat{X}_{s-}, \hat{X}_s) = \sum_k 1_{\{\hat{S}_n^k \leq t\}},$$

where S_n^k, \hat{S}_n^k are the corresponding k^{th} -iterate. Clearly A^n and \hat{A}^n are dual. By (6.10)

$$\begin{aligned} \nu_S|_{G_n}(dx, dy) &= \nu_{S_n}(dx, dy) \\ &= \nu_{A^n}(dx, dy) = \hat{\nu}_{\hat{A}^n}(dy, dx) \\ &= \hat{\nu}_{\hat{S}_n}(dy, dx) = \hat{\nu}_{\hat{S}}|_{G_n}(dy, dx). \end{aligned}$$

Since n is arbitrary and ν_S and $\hat{\nu}_{\hat{S}}$ do not charge D , it follows that ν_S and $\hat{\nu}_{\hat{S}}$ are dual.

Conversely if ν_S and $\hat{\nu}_{\hat{S}}$ are dual, let $\hat{K} := \{(x, y) : (y, x) \in K\}$ and $\hat{S}' := \hat{J}_{\hat{K}}$. Then S and \hat{S}' are dual and we have

$$\hat{\nu}_{\hat{S}}(dx, dy) = \nu_S(dy, dx) = \hat{\nu}_{\hat{S}'}(dx, dy).$$

Hence $\hat{\nu}_{\hat{S}} = \hat{\nu}_{\hat{S}'}$. By (6.3) $\hat{S} = \hat{S}'$ and S and \hat{S} are dual.

2) Using the notation above, since J_{G_n} and \hat{J}_{G_n} are dual, their bivariate Revuz measures are dual; that is, $1_{G_n} \cdot \nu$ and $1_{G_n} \cdot \hat{\nu}$ are dual. Thus ν and $\hat{\nu}$ are dual. ■

COROLLARY 6.13. – *Let $M \in \text{SMF}_+(X)$ and $\hat{M} \in \text{SMF}_+(\hat{X})$. Then M and \hat{M} are dual if and only if ν_M and $\hat{\nu}_{\hat{M}}$ are dual.*

Proof. – Since duality of M and \hat{M} implies duality of S_M and $\hat{S}_{\hat{M}}$ (see [De]), the necessity is a direct consequence of (6.11) and (6.12). Assume now that ν_M and $\hat{\nu}_{\hat{M}}$ are dual; that is,

$$(1_B + 1_{B^c} \cdot \Phi) \cdot \nu(dx, dy) = (1_{\hat{B}} + 1_{\hat{B}^c} \cdot \hat{\Phi}) \cdot \hat{\nu}(dy, dx),$$

where B and Φ (resp. \hat{B} and $\hat{\Phi}$) are the parameters associated with the decomposition of M (resp. \hat{M}) as in (2.3). Since ν and $\hat{\nu}$ are dual, we find that ν_{S_M} and $\hat{\nu}_{\hat{S}_{\hat{M}}}$ are dual and that $\nu_{\text{slog}M}$ and $\hat{\nu}_{\text{slog}\hat{M}}$ are dual. Thus M and \hat{M} are dual by (6.11) and (6.12). ■

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Part II. THE FEYNMAN-KAC FORMULA

1. Introduction

Though the work on symmetric Markov processes has been completed in [Yi2], the similar result for non-symmetric processes is not just trivial generalization because the method employed in [Fi] or [Yi2] can not be used in non-symmetric case. Given a nearly symmetric process, *i.e.*, one

that satisfies the ‘sector condition’ of Silverstein [Si1, Si2], which would guarantee a nice Dirichlet form associated, and any of its multiplicative functionals, there are two questions to be raised: (1) whether or not is the subprocess still nearly symmetric? (2) if it is, how to describe the Dirichlet form of the subprocess? Our main result in this part is to prove that the subprocess is also nearly symmetric and a similar Feynman-Kac formula holds.

All notations and terminologies without description are inherited from Part I, which we won’t restate here. Only the notion on Dirichlet forms will be given.

Since X is a right process, (P_t) is a strongly continuous semigroup of contractions on $L^2(m)$. Let A denote the strong $L^2(m)$ -infinitesimal generator of (P_t) , with domain $D(A)$. Then $D(A)$ is dense in $L^2(m)$, A is closed, $-A$ is positive: $(f, -Af) \geq 0$ for all $f \in D(A)$. The sector condition is this

$$(1.1) \quad (f, -Ag) \leq K \cdot (f, -Af)^{\frac{1}{2}}(g, -Ag)^{\frac{1}{2}}, \quad f, g \in D(A)$$

for some constant $K > 0$. Of course (1.1) is always valid in the symmetric case. The important consequences of (1.1) are that every semipolar set is m -polar and X and \hat{X} are not only Borel right processes but also m -special standard ones. Thus we can make free use of the results in Part I.

After collecting in §2 some definitions and standard facts concerning Dirichlet forms and multiplicative functionals, which can be found in [MR] and [Fi], we are going to formulate our main result in §3 where the involved MF is assumed to be non-vanishing and in §4 for general MF’s.

2. Dirichlet forms

Let (a, \mathcal{D}_a) be a bilinear form on $L^2(m)$. Define for $u, v \in \mathcal{D}_a$,

$$(2.1) \quad \begin{aligned} \bar{a}(u, v) &:= \frac{1}{2}[a(u, v) + a(v, u)]; \\ a_p(u, v) &:= a(u, v) + p(u, v); \\ [a_p](u) &:= a_p(u, u)^{\frac{1}{2}}. \end{aligned}$$

DEFINITION 2.2. – *A pair (a, \mathcal{D}_a) is called a coercive closed form on $L^2(m)$ if \mathcal{D}_a is a dense linear subspace of $L^2(m)$ and $a : \mathcal{D}_a \times \mathcal{D}_a \rightarrow \mathbb{R}$ is a bilinear form such that the following two conditions hold.*

(i) *Its symmetric part (\bar{a}, \mathcal{D}_a) is a symmetric closed form on $L^2(m)$; that is, $\bar{a} : \mathcal{D}_a \times \mathcal{D}_a \rightarrow \mathbb{R}$ is a positive definite bilinear form and \mathcal{D}_a is complete with respect to the norm $[a_1]$.*

(ii) (a, \mathcal{D}_a) satisfies the sector condition

$$(2.3) \quad |a(u, v)| \leq K \cdot a(u, u)^{\frac{1}{2}} \cdot a(v, v)^{\frac{1}{2}}$$

for a constant $K > 0$.

The sector condition (2.3) is equivalent to

$$(2.3i) \quad |\operatorname{Im} a(u, \bar{u})| \leq K \cdot \operatorname{Re} a(u, \bar{u})$$

for $u \in \mathcal{D}_a + i\mathcal{D}_a$, where \bar{u} is the conjugate of u .

PROPOSITION 2.4. – Let (a, \mathcal{D}_a) be a coercive closed form on $L^2(m)$. Then there exist unique strongly continuous contraction resolvents (G_p) , (\hat{G}_p) on $L^2(m)$ such that $G_p(L^2(m))$, $\hat{G}_p(L^2(m)) \subset \mathcal{D}_a$ and $a_p(u, G_p f) = (u, f) = a_p(\hat{G}_p f, u)$ for all $f \in L^2(m)$, $u \in \mathcal{D}_a$ and $p > 0$.

In this case (G_p) and (\hat{G}_p) are called the resolvents associated with (a, \mathcal{D}_a) .

DEFINITION 2.5. – A coercive closed form (a, \mathcal{D}_a) on $L^2(m)$ is called a Dirichlet form if for all $u \in \mathcal{D}_a$, $u^+ \wedge 1 \in \mathcal{D}_a$ and

$$a(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0, \quad a(u - u^+ \wedge 1, u + u^+ \wedge 1) \geq 0.$$

PROPOSITION 2.6. – Suppose (a, \mathcal{D}_a) is a coercive closed form on $L^2(m)$ with corresponding resolvents (G_p) and (\hat{G}_p) . Then the following two conditions are equivalent.

- (i) (a, \mathcal{D}_a) is a Dirichlet form on $L^2(m)$.
- (ii) (G_p) and (\hat{G}_p) are sub-Markovian.

Now beginning with the process X , we define bilinear forms a_p , $p \geq 0$, by

$$(2.7) \quad a_p(f, g) = (f, -Ag) + p(f, g), \quad f, g \in D(A).$$

We write a for a_0 and $[a_p](f) = a_p(f, f)^{\frac{1}{2}}$ for $f \in D(A)$. Clearly $[a_p]$ is a norm on $D(A)$ for $p > 0$. Let \mathcal{D}_a denote the completion of $D(A)$ relative to $[a_1]$. Owing to (1.1), a extends uniquely to a bilinear form (also denoted a) on \mathcal{D}_a . The pair (a, \mathcal{D}_a) is the Dirichlet form associated with X and

$$(2.8) \quad \mathcal{D}_a = \{u \in L^2(m) : \sup_q q(u, u - qU^{q+1}u) < \infty\}.$$

Let J be the canonical measure of X relative to m . According to [MR], the Dirichlet form a , which is associated with a pair of m -special standard processes, is quasi-regular. By the ‘transfer method’ developed

in [MR (VI)], the Fukushima's decomposition holds. Therefore we have the following decomposition for a on the diagonal

$$(2.9) \quad a(u, u) = a^c(u, u) + \frac{1}{2} \int [u(x) - u(y)]^2 J(dx, dy)$$

for $u \in \mathcal{D}_a$, where a^c is the continuous part of X which includes diffusion and killing.

PROPOSITION 2.10. – *Let $\{f_n\}$ be a sequence in \mathcal{D}_a such that $\sup_n [a_1](f_n) < \infty$ and $f_n \rightarrow f$ a.e. Then $f \in \mathcal{D}_a$ and $f_n \rightarrow f$ weakly in \mathcal{D}_a , i.e., $a_1(f_n, g) \rightarrow a_1(f, g)$ for each $g \in \mathcal{D}_a$.*

3. Non-vanishing case

Unless otherwise stated we assume in this section that $M = (M_t)$ is a multiplicative functional of X which never vanishes before ζ . There exists its dual multiplicative functional \hat{M} , non-vanishing, such that (X, M) and (\hat{X}, \hat{M}) are in weak duality relative to m . Denote their semigroups and resolvents by (Q_t) , (V^q) and (\hat{Q}_t) , (\hat{V}^q) , respectively. The Stieltjes logarithm of M is defined by

$$(3.1) \quad [M]_t := \int_0^t \frac{d(-M_s)}{M_{s-}}.$$

Then $[M]$ is an additive functional of X and we have the following identities

$$(3.2) \quad U^q = V^q + U_{[M]}^q V^q; \quad U_{[M]}^p = U_{[M]}^{q+p} + qU^{q+p}U_{[M]}^p.$$

Since $(1 - M_t) \in \text{AF}(M)$, the bivariate Revuz measure of $(1 - M_t)$ relative to m makes sense and is denoted by ν , which is also called the bivariate Revuz measure of M . Let J be the canonical measure of X relative to m . We know from (I.4.6) that $\nu \leq J$ off the diagonal of $E \times E$. Denote by ρ and λ the marginal measures of ν ; that is, for any $f \in p\mathcal{E}$

$$\rho(f) = \nu(1 \otimes f), \quad \lambda(f) = \nu(f \otimes 1).$$

Finally we list some facts, which may be either easy to see or found in Part I, for handy reference.

(3.3) $\nu = \nu_{[M]}$, where $\nu_{[M]}$ is the bivariate Revuz measure of $[M]$.

(3.4) The Revuz formula $(f, U_{[M]}^q g) = \nu_{[M]}(\hat{U}^q f \otimes g)$ and generalized Revuz formula $(f, U_M^q g) = \nu(\hat{V}^q f \otimes g)$ for $f, g \in p\mathcal{E}$.

(3.5) By Hölder's inequality, we have $|\nu(f \otimes g)| \leq \lambda(f^2)^{\frac{1}{2}} \cdot \rho(g^2)^{\frac{1}{2}}$.

(3.6) The dual statements of all above are true and furthermore ν and $\hat{\nu}$ are dual; that is, $\nu(dx, dy) = \hat{\nu}(dy, dx)$.

Let $(B, D(B))$ be the strong $L^2(m)$ -infinitesimal generator of the semigroup (Q_t) of the M -subprocess. It is known that $D(B) = V^1(L^2(m)) = V^p(L^2(m))$ for $p > 0$.

LEMMA 3.7. – (a) $D(B) \subset \mathcal{D}_a \cap L^2(\rho) \cap L^2(\lambda)$. (b) The following formula holds

$$(3.8) \quad (u, g) = a_p(u, V^p g) + \nu(u \otimes V^p g)$$

for any $u \in \mathcal{D}_a$, $g \in L^2(m)$, $u, g \geq 0$ and $p > 0$.

Proof. – Let $L^2_+(m) := \{f \in L^2(m) : f \geq 0\}$. First of all we will show $V^1(L^2_+(m)) \subset \mathcal{D}_a$. In fact for $u = V^1 f$ with $f \in L^2_+(m)$, using (3.2) we have

$$\begin{aligned} q(u, u - qU^{q+1}u) &= q(u, U^1 f - qU^{q+1}U^1 f - U^1_{[M]}u + qU^{q+1}U^1_{[M]}u) \\ &= q(u, U^{q+1} f - U^{q+1}_{[M]}u) \\ &\leq (u, qU^{q+1} f). \end{aligned}$$

Since $(u, qU^{q+1} f) \rightarrow (u, f)$ as $q \rightarrow \infty$, $\sup_q q(u, u - qU^{q+1}u) < \infty$ and then $u \in \mathcal{D}_a$ by (2.8).

Now for any u, g and p as in (b), we have by (3.2)

$$(u, g) = a_p(u, U^p g)a_p(u, V^p g) + a_p(u, U^p_{[M]} V^p g).$$

Then using (3.4)

$$\begin{aligned} a_p(u, U^p_{[M]} V^p g) &= \lim_q q(u, U^p_{[M]} V^p g - qU^{q+p}U^p_{[M]} V^p g) \\ &= \lim_q (u, qU^{q+p}_{[M]} V^p g) \\ &= \lim_q \nu_{[M]}(q\hat{U}^{q+p}u \otimes V^p g). \end{aligned}$$

Since $u \in \mathcal{D}_a$, $q\hat{U}^{q+p}u \rightarrow u$ q.e. as $q \rightarrow \infty$ and by (3.3) and the generalized Revuz formula (I.5.10)

$$\nu_{[M]}(1 \otimes V^p g) = \nu(1 \otimes V^p g) = \hat{\nu}(V^p g \otimes 1) = (g, \hat{U}^q_M 1) \leq m(g).$$

Thus if, in addition, u is bounded and $g \in L^1(m)$, then $\{q\hat{U}^{q+p}u\}$ is uniformly bounded and $\nu(dx \otimes V^p g)$ is a finite measure, and hence (3.8)

holds by the dominated convergence theorem. Now for $g \in L^1 \cap L^2(m)$ and any $u \in \mathcal{D}_a$, we write $u_n = u \wedge n$ and then $u_n \uparrow u$ and in \mathcal{D}_a . Since u_n is bounded, $(u_n, g) = a_p(u_n, V^p g) + \nu(u_n \otimes V^p g)$. Passing to limit, by the monotone convergence theorem, (3.8) holds in this case. Finally for any $g \in L^2(m)$, we can pick h strictly positive on E and $m(h) < \infty$. Let $g_n = g \wedge (nh)$. Then $g_n \in L^1 \cap L^2(m)$, $g_n \uparrow g$ pointwisely and $(u, g_n) = a_p(u, V^p g_n) + \nu(u \otimes V^p g_n)$. Then $V^p g_n \uparrow V^p g$ by the monotone convergence theorem. Also we have

$$\begin{aligned} a_p(V^p g_n, V^p g_n) &= \lim_q q(V^p g_n, V^p g_n - qU^{q+p}V^p g_n) \\ &\leq \lim_q (V^p g_n, qU^{q+p}g_n) \\ &= (V^p g_n, g_n) \\ &\leq (V^p g, g). \end{aligned}$$

Thus by (2.10), $V^p g_n \rightharpoonup V^p g$ weakly in \mathcal{D}_a and (3.8) holds for positive $u \in \mathcal{D}_a$, $g \in L^2(m)$ and $p > 0$ by another use of the monotone convergence theorem. Now for $f \in L^2_+(m)$, $u = V^p f \in \mathcal{D}_a$ and by (3.8) and (2.9)

$$\begin{aligned} (V^p f, f) &= a_p(u, u) + \nu(u \otimes u) \\ &= a_p^c(u, u) + \frac{1}{2} \int [(u(x) - u(y))]^2 (J - \nu)(dx, dy) \\ &\quad + \frac{1}{2} (\rho(u^2) + \lambda(u^2)). \end{aligned}$$

Thus $\rho(u^2) + \lambda(u^2) < \infty$, i.e., $u \in L^2(\rho) \cap L^2(\lambda)$ and

$$D(B) = V^1(L^2_+(m)) - V^1(L^2_+(m)) \subset \mathcal{D}_a \cap L^2(\rho) \cap L^2(\lambda).$$

That completes the proof. ■

We now define

$$(3.9) \quad \begin{aligned} \mathcal{D}_b &= \mathcal{D}_a \cap L^2(\rho) \cap L^2(\lambda); \\ b(u, v) &= a(u, v) + \nu(u \otimes v), \quad u, v \in \mathcal{D}_b. \end{aligned}$$

By (3.5), (b, \mathcal{D}_b) is a well-defined bilinear form on $L^2(m)$.

THEOREM 3.10. – *The bilinear form (b, \mathcal{D}_b) is a Dirichlet form on $L^2(m)$ associated with the M -subprocess of X . In other words the M -subprocess of a nearly symmetric Markov process with the Dirichlet form (a, \mathcal{D}_a) is also nearly symmetric and its associated Dirichlet form (b, \mathcal{D}_b) is given in (3.9).*

Proof. – First we will show that (b, \mathcal{D}_b) is a coercive closed form on $L^2(m)$. Clearly \mathcal{D}_b is dense in $L^2(m)$ since $D(B) \subset \mathcal{D}_b$ as in (3.7). The decomposition of b on diagonal is

$$\begin{aligned}
 (3.11) \quad b(u, u) &= a(u, u) + \nu(u \otimes u) \\
 &= a^c(u, u) + \frac{1}{2} \int [u(x) - u(y)]^2 J'(dx, dy) \\
 &\quad + \frac{1}{2}(\rho(u^2) + \lambda(u^2)),
 \end{aligned}$$

where $J' = J - \nu|_{E \times E-d}$, the canonical measure of the M -subprocess. Hence $b(u, u) \geq 0$. Now we need to check that \mathcal{D}_b is $[b_1]$ -complete. Let $\{u_n\}$ be any $[b_1]$ -Cauchy sequence in \mathcal{D}_b , i.e., $[b_1](u_n - u_m) \rightarrow 0$. Then by (3.11) $\{u_n\}$ is a $L^2(\rho)$ and $L^2(\lambda)$ -Cauchy sequence. Thus by (3.6), $\nu((u_n - u_m) \otimes (u_n - u_m)) \rightarrow 0$ and $[a_1](u_n - u_m) \rightarrow 0$; that is, $\{u_n\}$ is also a $[a_1]$ -Cauchy sequence in \mathcal{D}_a . There exists $u \in \mathcal{D}_a$ such that $u_n \rightarrow u$ strongly in \mathcal{D}_a . Then $u_n \rightarrow u$ q.e. (at least for a subsequence), and consequently $u_n \rightarrow u$ a.e. ρ and λ . Hence u coincides with the L^2 -limit of $\{u_n\}$ in $L^2(\rho)$ and $L^2(\lambda)$. Therefore $u \in \mathcal{D}_a \cap L^2(\rho) \cap L^2(\lambda) = \mathcal{D}_b$ and $[b_1](u_n - u) \rightarrow 0$.

Next we will prove that (b, \mathcal{D}_b) satisfies the sector condition (2.3i). Let $u = f + ig$ with $f, g \in \mathcal{D}_b$. We need to show that

$$(3.12) \quad |\operatorname{Im} b(u, \bar{u})| \leq K \cdot \operatorname{Re} b(u, \bar{u})$$

for a constant $K > 0$. In fact $b(u, \bar{u}) = a(u, \bar{u}) + \nu(u \otimes \bar{u})$ and

$$\begin{aligned}
 a(u, \bar{u}) &= a(f, f) + a(g, g) + i \operatorname{Im} a(u, \bar{u}) \\
 &= \operatorname{Re} a^c(u, \bar{u}) + i \operatorname{Im} a(u, \bar{u}) \\
 &\quad + \frac{1}{2} \int ([f(x) - f(y)]^2 + [g(x) - g(y)]^2) J(dx, dy) \\
 &= \operatorname{Re} a^c(u, \bar{u}) + i \operatorname{Im} a(u, \bar{u}) \\
 &\quad + \int \left(\frac{1}{2} (|u(x)|^2 + |u(y)|^2) - f(x)f(y) - g(x)g(y) \right) J(dx, dy); \\
 \nu(u \otimes \bar{u}) &= \nu(f \otimes f) + \nu(g \otimes g) + i\nu(g \otimes f - f \otimes g) \\
 &= - \int \left(\frac{1}{2} (|u(x)|^2 + |u(y)|^2) - f(x)f(y) - g(x)g(y) \right) \nu(dx, dy) \\
 &\quad + \frac{1}{2}(\rho(|u|^2) + \lambda(|u|^2)) + i\nu(g \otimes f - f \otimes g).
 \end{aligned}$$

Define

$$\begin{aligned} d &:= \frac{1}{2}(\rho(|u|^2) + \lambda(|u|^2)) < \infty; \\ \nu_o &:= \frac{1}{2}(|u|^2 \otimes 1 + 1 \otimes |u|^2) \cdot \nu; \\ J_o &:= \frac{1}{2}(|u|^2 \otimes 1 + 1 \otimes |u|^2) \cdot J; \\ \text{SIN}[u](x, y) &:= \frac{2(g(x)f(y) - f(x)g(y))}{|u(x)|^2 + |u(y)|^2}; \\ \text{COS}[u](x, y) &:= \frac{2(f(x)f(y) + g(x)g(y))}{|u(x)|^2 + |u(y)|^2}. \end{aligned}$$

Then we have

$$\begin{aligned} a(u, \bar{u}) &= \text{Re } a^c(u, \bar{u}) + J_o(1 - \text{COS}[u]) + i \text{Im } a(u, \bar{u}); \\ b(u, \bar{u}) &= \text{Re } a(u, \bar{u}) + i \text{Im } a(u, \bar{u}) + \text{Re } \nu(u \otimes \bar{u}) \\ (3.13) \quad &+ i \text{Im } \nu(u \otimes \bar{u}) \\ &= \text{Re } a^c(u, \bar{u}) + J_o(1 - \text{COS}[u]) - \nu_o(1 - \text{COS}[u]) + d \\ &\quad i \text{Im } a(u, \bar{u}) + i \nu_o(\text{SIN}[u]), \end{aligned}$$

and clearly $|\text{COS}[u]| \leq 1$, $|\text{SIN}[u]| \leq 1$ and $\nu_o(1) = d$. Now let

$$\begin{aligned} (3.14) \quad G(u) &:= a(u, \bar{u}) - \nu_o(1 - \text{COS}[u]) + \frac{d}{2}; \\ H(u) &:= i \nu_o(\text{SIN}[u]) + \frac{d}{2}. \end{aligned}$$

Then $b(u, \bar{u}) = G(u) + H(u)$ and

$$\begin{aligned} \text{Im } G(u) &= \text{Im } a(u, \bar{u}); \\ \text{Re } G(u) &= \text{Re } a(u, \bar{u}) - \nu_o(1 - \text{COS}[u]) + \frac{d}{2} \\ &= \text{Re } a^c(u, \bar{u}) + (J_o - \nu_o)(1 - \text{COS}[u]) + \frac{d}{2}. \end{aligned}$$

It is easy to see that

$$|\text{Im } H(u)| \leq \nu(|\text{SIN}[u]|) \leq d = 2 \text{Re } H(u).$$

Thus we need only to check that $|\text{Im } G(u)| \leq K \cdot \text{Re } G(u)$ for a constant K . Since a satisfies the sector condition, it suffices to show that $\text{Re } a(u, \bar{u}) \leq K' \cdot \text{Re } G(u)$ for a constant K' .

Denote

$$(3.15) \quad \mathcal{U} = \{u \in \mathcal{D}_b + i\mathcal{D}_b : J_o(1 - \text{COS}[u]) \geq \nu_o(1 - \text{COS}[u]) + 2d\}.$$

Since $0 \leq \nu_o(1 - \text{COS}[u]) \leq 2d$, $\nu_o(1 - \text{COS}[u]) \leq \frac{1}{2}J_o(1 - \text{COS}[u])$ for $u \in \mathcal{U}$ and then

$$(3.16) \quad 2(J_o - \nu_o)(1 - \text{COS}[u]) \geq J_o(1 - \text{COS}[u]).$$

Hence for $u \in \mathcal{U}$, we have

$$\begin{aligned} \text{Re } a(u, \bar{u}) &\leq \text{Re } a^c(u, \bar{u}) + J_o(1 - \text{COS}[u]) \\ &\leq \text{Re } a^c(u, \bar{u}) + 2(J_o - \nu_o)(1 - \text{COS}[u]) \\ &\leq 2 \text{Re } G(u). \end{aligned}$$

On the other hand, if $u \notin \mathcal{U}$,

$$\begin{aligned} \text{Re } a(u, \bar{u}) &\leq \text{Re } a^c(u, \bar{u}) + \nu_o(1 - \text{COS}[u]) + 2d \\ &\leq \text{Re } a^c(u, \bar{u}) + 4d \\ &\leq 8 \left(\text{Re } a^c(u, \bar{u}) + \frac{d}{2} \right) \\ &\leq 8 \text{Re } G(u). \end{aligned}$$

Therefore for any $u \in \mathcal{D}_b + i\mathcal{D}_b$, $\text{Re } a(u, \bar{u}) \leq 8 \text{Re } G(u)$; that is, (b, \mathcal{D}_b) satisfies the sector condition and it is a coercive closed form on $L^2(m)$. By (3.7), it is clear that

$$(3.17) \quad b_p(u, V^p f) = (u, f)$$

for all $u \in \mathcal{D}_b$, $f \in L^2(m)$ and $p > 0$. The dual assertion can be shown similarly. Hence by the uniqueness in (2.4), (V^q) and (\hat{V}^q) are the resolvents associated with (b, \mathcal{D}_b) . Since (V^p) and (\hat{V}^p) are sub-Markovian, it follows from (2.6) that (b, \mathcal{D}_b) is a Dirichlet form on $L^2(m)$, which is associated with the M -subprocess of X . ■

4. General case

In this section we are going to generalize Theorem (3.10) to the most general multiplicative functionals.

THEOREM 4.1. – *Let M be an exact multiplicative functional of X and $m^* := 1_{E_M} \cdot m$. Then the subprocess (X, M) is a nearly m^* -symmetric*

Markov process on $(E_M, \mathcal{E} \cap E_M)$ and its associated Dirichlet form (\mathcal{D}_b, b) is given by

$$(4.2) \quad \begin{aligned} \mathcal{D}_b &= (\mathcal{D}_a)_{E_M} \cap L^2(\rho_M + \lambda_M); \\ b(u, v) &= a(u, v) + \nu_M(u \otimes v), \quad u, v \in \mathcal{D}_b, \end{aligned}$$

where

$$(\mathcal{D}_a)_{E_M} := \{u \in \mathcal{D}_a : u = 0 \text{ q.e. on } E_M^c\}$$

the restricted Dirichlet form of \mathcal{D}_a on E_M .

Proof. – The transformation by M can be completed in three steps: killing X by a hitting time $T_{E_M^c}$, denote the resulting one by X' , killing X' by a positive terminal time S_M of X' , denote the resulting one by X'' , killing X'' by a non-vanishing MF M of X'' . Because of the works on non-vanishing MF's in (3.10) and on restricted Dirichlet forms in [MR], and a connection formula (I.4.16), it suffices to deal with the case of positive terminal times, i.e., $M = 1_{[0, T]}$, where T is a terminal time and $T > 0$ a.s. Then there exists $B \subset E \times E - D$ such that P^m a.s.

$$(4.3) \quad T = J_B := \inf\{t > 0 : (X_{t-}, X_t) \in B\}.$$

We will finishing the proof by two steps. First assume that $B \subset \{(x, y) : d(x, y) > c\}$ for some $c > 0$ where d is a metric on E compatible with the original topology on E . Define $T^{(1)} := T$, $T^{(n+1)} := T \circ \theta_{T^{(n)}} + T^{(n)}$. Since X is right continuous,

$$(4.4) \quad \sum_{s \leq t} 1_B(X_{s-}, X_s) = \sum_n 1_{\{T^{(n)} \leq t\}} < \infty$$

for any $t < \zeta$. Hence for any $0 \leq \delta < 1$, $M_t^\delta := \prod_{s \leq t} [1 - \delta 1_B(X_{s-}, X_s)]$ is a non-vanishing multiplicative functional of X . Let $\delta_n := \frac{n}{n+1}$ for $n \geq 0$, $M_t^n := M_t^{\delta_n}$ and (V_n^q) be the resolvent of the subprocess (X, M^n) . Clearly $V_n^q \downarrow V^q f$ pointwisely for $f \geq 0$ and $\nu_{M^n} = \delta_n \nu_M \uparrow \nu_M$. Define

$$(4.5) \quad \begin{aligned} \mathcal{D}_b &:= \mathcal{D}_a \cap L^2(\rho_M + \lambda_M); \\ b(u, v) &:= a(u, v) + \nu_M(u \otimes v), \quad u, v \in \mathcal{D}_b. \end{aligned}$$

We need only to show that (\mathcal{D}_b, b) is the Dirichlet form and associated with (X, T) . Following the proof of (3.10), it suffices to check the following claims.

(4.6i) $V^1 f \in \mathcal{D}_a$ for any $f \in pL^2(m)$;

(4.6ii) $b_1(u, V^1 f) = (u, f)$ for any $u \in p\mathcal{D}_a, f \in pL^2(m)$.

Since M^n never vanishes, we know that $V_n^1 f \in \mathcal{D}_a$ and

$$(4.7) \quad (u, f) = a_1(u, V_n^1 f) + \nu_{M^n}(u \otimes V_n^1 f).$$

Since $a_1(V_n^1 f, V_n^1 f) \leq (V_n^1 f, f) \leq (U^1 f, f) < \infty$ and $V_n^1 f \downarrow V^1 f$ pointwisely, $V^1 f \in \mathcal{D}_a$ and $V_n^1 f \rightarrow V^1 f$ weakly in (\mathcal{D}_a, a_1) . We also have

$$\nu_M(u \otimes V_1^1 f) = 2\nu_{M^1}(u \otimes V_1^1 f) < \infty.$$

Hence take n to infinity in (4.7) and the dominated convergence theorem gives

$$\begin{aligned} (u, f) &= a_1(u, V^1 f) + \lim_n \nu_M(u \otimes V_n^1 f) \\ &= a_1(u, V^1 f) + \nu_M(u \otimes V^1 f) \\ &= b_1(u, V^1 f). \end{aligned}$$

That completes the proof of the first part. Next we need to check (4.6) for general $B \subset E \times E - D$. Set

$$B_n := B \cap \left\{ (x, y) : d(x, y) > \frac{1}{n} \right\} \quad \text{and} \quad T_n := J_{B_n}.$$

Since $T > 0$ a.s., $\{T_n\}$ well converges decreasingly to T in the sense that for any $\omega \in \Omega$ there exists $N = N(\omega)$ such that $T_n(\omega) = T(\omega)$ for all $n > N$. In fact by (I.3.2) $(X_{T-}, X_T) \in B$. Thus for such an ω , there exists N such that $(X_{T-}, X_T) \in B_N$ and this N guarantees $T_n(\omega) = T(\omega)$ for $n > N$.

Denote by (V_n^q) the resolvent of (X, T_n) . Then $V_n^q f(x) = P^x \int_{]0, T_n[} e^{-qt} f(X_t) dt$ decreases to $P^x \int_{]0, T[} e^{-qt} f(X_t) dt = V^q f(x)$. We can surely apply the result above to T_n , and have

$$(u, f) = a_1(u, V_n^1 f) + \nu_{T_n}(u \otimes V_n^1 f).$$

The similar reasoning gives $V^1 f \in \mathcal{D}_a$ and $V_n^1 f \rightarrow V^1 f$ in (\mathcal{D}_a, a_1) . On the other hand let $\hat{B} = \{(x, y) : (y, x) \in B\}$, $\hat{T} = \hat{J}_{\hat{B}} := \inf\{t > 0 : (\hat{X}_{t-}, \hat{X}_t) \in \hat{B}\}$ and $\hat{T}_n := \hat{J}_{\hat{B}_n}$. Then \hat{T} (resp. \hat{T}_n) is dual to T (resp. T_n), $\hat{\nu}_{\hat{T}}$ (resp. $\hat{\nu}_{\hat{T}_n}$) is dual to and \hat{T}_n well converges to \hat{T} . Using the dual form of generalized Revuz formula (I.5.10) and (3.6) we have

$$\nu_{T_n}(u \otimes V_n^1 f) = \hat{\nu}_{\hat{T}_n}(V_n^1 f \otimes u) = (f, \hat{P}_{\hat{T}_n}^1 u).$$

Now $\hat{P}_{\hat{T}_n}^1 u(x) = \hat{P}^x e^{-\hat{T}_n} u(\hat{X}_{\hat{T}_n})$. By the well convergence of $\{T_n\}$,

$$\hat{P}_{\hat{T}_n}^1 u(x) \longrightarrow \hat{P}^x e^{-\hat{T}} u(\hat{X}_{\hat{T}}) = \hat{P}_{\hat{T}}^1 u(x),$$

i.e., $\hat{P}_{\hat{T}_n}^1 u \longrightarrow \hat{P}_{\hat{T}}^1 u$ pointwisely. Hence for $f \in pL^1(m)$ and bounded u ,

$$(f, \hat{P}_{\hat{T}_n}^1 u) \longrightarrow (f, \hat{P}_{\hat{T}}^1 u).$$

Using the standard techniques, we have this convergence for $u \in p\mathcal{D}_a$ and $f \in pL^2(m)$. Then applying the dual form of (I.5.10) again

$$\begin{aligned} (u, f) &= a_1(u, V^1 f) + (f, \hat{P}_{\hat{T}}^1 u) \\ &= a_1(u, V^1 f) + \hat{\nu}_{\hat{T}}(V^1 f \otimes u) \\ &= a_1(u, V^1 f) + \nu_T(u \otimes V^1 f) \\ &= b_1(u, V^1 f). \end{aligned}$$

That completes the proof. ■

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