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Jumping Markov Processes

by

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ABSTRACT. – This paper is devoted to a systematic study of the basic properties of the so-called *Jumping Markov Processes* (JMP in short). By this we mean a Markov process $X = (X_t)_{t \geq 0}$ taking values in an arbitrary measurable space $(E, \mathcal{E})$, and which is piecewise-deterministic in the sense that it follows a "deterministic" path $X_t = f(t, X_0)$ up to some random time τ_1, at which time it "jumps" to some random value X_{τ_1}, then it follows the path $f(t - \tau_1, X_{\tau_1})$ up to another random time $\tau_2 > \tau_1$, and so on... Such processes had already been studied by M. H. A. Davis [3] in a particular case, but here the emphasis is on the characterization of JMPs, in particular in terms of the structure of the martingales, and on the properties of the basic objects (additive functionals, semimartingales, semimartingale functions) usually associated with Markov processes.

We also introduce a class of Markov processes which we call "purely discontinuous" and appear as suitable limits of JMP's.

RÉSUMÉ. – Cet article est consacré à l'étude systématique des propriétés d'une classe de processus de Markov, que nous appelons JMP (pour "Jumping Markov Processes") et qu'on peut décrire ainsi : un processus de Markov $X = (X_t)_{t \geq 0}$ à valeurs dans un espace mesurable quelconque

$(E, \mathcal{E})$ est un JMP s'il suit une trajectoire « déterministe » $X_t = f(t, X_0)$ jusqu'à un temps d'arrêt τ_1; puis il « saute » à l'instant τ_1 en un point aléatoire X_{τ_1}; puis il suit la trajectoire $f(t - \tau_1, X_{\tau_1})$ jusqu'à un autre temps d'arrêt $\tau_2 > \tau_1$, etc. De tels processus ont déjà été étudiés par M. H. A. Davis [3] dans un cas particulier, mais ici nous mettons l'accent sur la structure de la filtration et d'une série d'objets habituellement associés aux processus de Markov: martingales, semimartingales, fonctionnelles additives, « fonctions semimartingales », etc.

Nous introduisons pour finir une classe plus vaste de processus de Markov, obtenus comme limites convenables des JMP.

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1. INTRODUCTION

1) We consider here the class of continuous-time E-valued homogeneous Markov processes X which are piecewise-deterministic in the following sense: there is a strictly increasing sequence (τ_n) of stopping times such that on each interval (τ_n, τ_{n+1}) the process follows a deterministic curve in the state space E. In order words we have a family of curves $\mathbb{R}_+ \ni t \to f(x, t)$ in E such that

$$X_t = f(X_0, t) \qquad \text{for all} \quad t < \tau_1. \tag{1}$$

Then at the (random) time τ_1 it "jumps" to some (random) value X_{τ_1}; then it moves according to $X_{\tau_1+t} = f(X_{\tau_1}, t)$ for $t < \tau_2 - \tau_1$ (with the same function f), and so on.

The simplest and main example of such processes consists in *step Markov processes*, where X is piecewise-constant (that is $f(x, t) = x$). The structure of these processes is well known: the sequence (X_{τ_n}) constitutes a Markov chain in E, and conditionally on this sequence the variables $\tau_n - \tau_{n-1}$ (with $\tau_0 = 0$) are independent, exponentially distributed with a parameter depending only on $X_{\tau_{n-1}}$.

"Age" processes also fall within this scope, and have been considered a long time ago. For instance if $(Y_n)_{n\geq 0}$ is an increasing random walk (*i.e.* the variables $Y_n - Y_{n-1}$ are i.i.d. and $(0, \infty)$-valued) starting at $Y_0 = 0$, the $\mathbb{R}_+$-valued processes X and X' defined by $X_t = Y_n - t$ and $X'_t = t - Y_{n-1}$ if $Y_{n-1} \leq t < Y_n$ satisfy the property above, with respectively $f(x, t) = (x - t)^+$ and $f'(x, t) = x + t$, and $\tau_n = Y_n$.

Many other processes of this type have been described in the literature, often in connection with renewal theory, Markov renewal theory, queueing theory, models for dams and storage, etc. A large class of such processes have been studied systematically by M. H. A. Davis [3] under the name of *piecewise-deterministic processes*: namely those for which $E = \mathbb{R}^d$ and f is the flow associated with a differential equation $\frac{d\zeta}{dt}(t) = g(\zeta(t))$ and the first jump time τ_1 has a distribution absolutely continuous w.r.t. Lebesgue measure. In this paper we wish to achieve the greatest possible generality (for instance the state space E has no topological structure): so in order to distinguish them form the case studied by Davis we will call them Jumping Markov Processes (JMP in short).

2) JMP's have of course interest of their own, even in the "general" case when no topological structure on E is used. But our main motivation lies in the class of Markov processes that are "limits" in some sense of JMP's: we are thinking about measure-valued branching processes (*see* e.g. [7]) with infinite mass, obtained as limits of usual branching processes where the particles follow deterministic curves between branching times, and also about infinite-dimensional interaction processes.

However, the aim of this paper is somewhat more modest: after defining the class of JMP we study some "basic" properties of these processes, and introduce a first notion of "limits" of JMP's in what we call "purely discontinuous" Markov processes. Because of the applications to branching processes it is necessary to consider the non-homogeneous case. However for the convenience of the reader, we have presented first all results in the homogeneous case, and the non-homogeneous case is quickly considered in Section 6.

3) Let us be more precise. The state space being possibly a non-topological space, the "jump times" τ_n are not really times of jump for the process X, but rather for the *filtration* $(\mathcal{F}_t)$ generated by the process: that is $\mathcal{F}_t \cap \{\tau_n \leq t < \tau_{n+1}\} = \mathcal{F}_{\tau_n} \cap \{\tau_n \leq t < \tau_{n+1}\}$, so the filtration is "constant" on the intervals $[\tau_n, \tau_{n+1})$, and "jumps" at times τ_n, so to speak. Such filtrations have been called *jumping filtrations* and studied in [10], and Section 2 is devoted to recalling and extending some of their main properties.

Our definition of a JMP, at least in the quasi-left continuous case, becomes the following: a strong Markov process having a quasi-left continuous filtration is a JMP iff the filtration is jumping. In the non-quasi-left continuous case the definition is a bit more complicated, due to the intrinsic non-uniqueness of the stopping time τ_1 in (1). In Section 3 we draw consequences of this definition, then construct JMP's starting from the law $G_X(dy, dt)$ of (X_{τ_1}, τ_1) when $X_0 = x$, and finally give some simple criteria for a JMP to be *regular*, that is the explosion time $\tau_\infty = \lim_n \tau_n$ is a.s. infinite.

In Section 4 we exhibit the special form taken in the JMP case by many usual objects in the theory of Markov processes: of particular interest are additive functionals, martingales, semimartingales, and semimartingale functions. A short subsection is devoted to infinitesimal generators.

In Section 5, after some preliminaries about various transformations of Markov processes, we introduce one of the fundamental notions of this paper, namely the so-called *purely discontinuous Markov processes*. This is only a tentative approach of the question, and undoubtedly much more can be said (and perhaps the definition given here is not "optimal" yet, in the sense that it makes the lifetime play too big a rôle).

Finally, as said before, we restate the main results, mostly without proof, in the non-homogeneous setting in Section 6.

2. JUMPING FILTRATIONS

We start with a probability space $(\Omega, \mathcal{F}, P)$ endowed with a right-continuous filtration $(\mathcal{F}_t)$. We use the standard notation of the theory of processes (*see* e.g. Dellacherie-Meyer [4]). In particular, the "stochastic interval" $[\![S, T]\!]$ is the set of all (ω, t) such that $S(\omega) \leq t \leq T(\omega)$ and $t < \infty$ and $[\![S]\!] = [\![S, S]\!]$.

We say that $(\mathcal{F}_t)$ is an *a.s. jumping filtration on* $\mathbb{R}_+$ if there is an increasing sequence (τ_n) of stopping times with $\lim_n \tau_n = \infty$ a.s. and $\tau_0 = 0$, and if for all $t \geq 0$, $n \in \mathbb{N}$ the σ-fields $\mathcal{F}_t$ and $\mathcal{F}_{\tau_n}$ coincide on $\{\tau_n \leq t < \tau_{n+1}\}$ up to P-null sets. Then (τ_n) is called a *jumping sequence*. In [10], these filtrations were simply called "jumping filtrations", and the following was proved:

THEOREM 1. – *a)* $(\mathcal{F}_t)$ *is an a.s. jumping filtration on* $\mathbb{R}_+$ *iff all local martingales are a.s. of locally finite variation.*

b) If $(\mathcal{F}_t)$ *is a quasi-left continuous a.s. jumping filtration, there is a jumping sequence* (τ_n) *such that*

(i) τ_n *is totally inacessible for* $n \geq 1$, *and* $\tau_n < \tau_{n+1}$ *if* $\tau_n < \infty$;

(ii) *every totally inacessible time* T *has* $[\![T]\!] \subseteq \bigcup_{n\geq 1} [\![\tau_n]\!]$ *a.s.;*

(iii) *any other jumping sequence* (τ'_n) *has* $\cup[\![\tau'_n]\!] \supseteq \cup[\![\tau_n]\!]$ *a.s.;*

(iv) *all local martingales are a.s. continuous outside* $\cup[\![\tau_n]\!]$.

c) If $(\mathcal{F}_t)$ *is an a.s. jumping filtration with a jumping sequence satisfying* (i) *above, it is quasi-left continuous.*

In this paper we need to consider "exploding" jumping filtrations, for which the jumping sequence increases to a stopping time which may be finite.

Let us begin with some terminology. A *predictable interval starting at* 0 is a random set of the form $I = \cup [\![0, \tau_n]\!]$, for some increasing sequence (τ_n) of stopping times. If $\delta = \sup(t : t \in I))$, we write $\overset{\circ}{I} =]\!]0, \delta[\![$. A *local martingale on* I is a process M such that each stopped process $M_t^n := M_{t \wedge \tau_n}$ is a local martingale; this does not depend on the particular representation of I as $I = \cup [\![0, \tau_n]\!]$. The process M is *a.s. locally of finite variation on* I if each M^n above has (a.s.) finite variation on compact intervals, or equivalently if for almost all ω, $t \to M_t(\omega)$ has finite variation on any compact subset of $I(\omega)$. The filtration $(\mathcal{F}_t)$ is *quasi-left continuous on* $\overset{\circ}{I}$ if for every predictable time T we have $\mathcal{F}_T \cap \{T \in \overset{\circ}{I}\} = \mathcal{F}_{T-} \cap \{T \in \overset{\circ}{I}\}$ up to null sets, or equivalently if all martingales (or all local martingales on I) have totally inaccessible jumps only on the set $\overset{\circ}{I}$. Finally a *totally inacessible time on* $\overset{\circ}{I}$ is a stopping time T such that $S = T_{\{T \in \overset{\circ}{I}\}}$ (*i.e.* $S = T$ if $T \in \overset{\circ}{I}$ and $S = \infty$ otherwise) is totally inacessible.

DEFINITION 1. – *a*) We say that $(\mathcal{F}_t)$ is a *jumping filtration on* I if there is an increasing sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times with $\tau_0 = 0$, $I = \bigcup_{n \in \mathbb{N}} [\![0, \tau_n]\!]$, and

$$\left.\begin{array}{l} \mathcal{F}_t \cap \{\tau_n \leq t < \tau_{n+1}\} = \mathcal{F}_{\tau_n} \cap \{\tau_n \leq t < \tau_{n+1}\} \quad \text{for all } t \geq 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad n \in \mathbb{N}. \end{array}\right\} \quad (2)$$

b) $(\mathcal{F}_t)$ is an *a.s. jumping filtration on* I if (2) holds up to P-null sets. ■

The sequence (τ_n) above is again called a *jumping sequence*. The usual P-completion (*i.e.* adding to each $\mathcal{F}_t$ all the null sets of the P-completion of $\mathcal{F}_\infty$) of an a.s. jumping filtration on I is a jumping filtration on I. If $I = \mathbb{R}_+$ we recover the notion of an a.s. jumping filtration as it appears in Theorem 1, and this theorem takes the following form in the general case:

THEOREM 2. – *Let I be a predictable interval starting at* 0.

a) $(\mathcal{F}_t)$ *is an a.s. jumping filtration on I iff all local martingales on I are a.s. of locally finite variation on I.*

b) If $(\mathcal{F}_t)$ *is an a.s. jumping filtration on I and is quasi-left continuous on $\overset{\circ}{I}$, there is a jumping sequence* (τ_n) *such that with* $\delta = \lim_n \uparrow \tau_n$*:*

(i) τ_n *is totally inaccessible on* $\overset{\circ}{I}$ *and* $\tau_n < \tau_{n+1}$ *if* $\tau_n < \delta$*;*

(ii) *every totally inaccessible time* T *on* $\overset{\circ}{I}$ *has* $\overset{\circ}{I} \cap [\![T]\!] \subseteq \cup [\![\tau_n]\!]$ *a.s.;*

(iii) *any other jumping sequence* (τ'_n) *has* $\cup [\![\tau'_n]\!] \supseteq \cup [\![\tau_n]\!]$ *a.s.;*

(iv) *all local martingales on* $\overset{\circ}{I}$ *are a.s. continuous outside* $\cup [\![\tau_n]\!]$.

c) If $(\mathcal{F}_t)$ *is an a.s. jumping filtration on* I *with a jumping sequence satisfying* (i) *above, it is quasi-left continuous on* $\overset{\circ}{I}$.

(iii) means that (τ_n) is the (unique) "minimal" jumping sequence for the filtration $(\mathcal{F}_t)$ on I, while (ii) means that it is the "maximal" sequence of totally inaccessible times on $\overset{\circ}{I}$. When the filtration is jumping but not quasi-left continuous on $\overset{\circ}{I}$, there is still a sequence (τ_n) satisfying (i) and (ii), but this is not a jumping sequence for $(\mathcal{F}_t)$.

Proof. – We have $I = \cup [\![0, T_m]\!]$ for some sequence of stopping times T_m. For each n let $\mathcal{F}^m_t = \mathcal{F}_{t \wedge T_m}$, and also $\delta = \lim_n T_n$.

If $(\mathcal{F}_t)$ is an a.s. jumping filtration on I with jumping sequence (τ_n), then each $(\mathcal{F}^m_t)$ is an a.s. jumping filtration on $\mathbb{R}_+$ with jumping sequence $(\tau^m_n)_{n \in \mathbb{N}}$ given by $\tau^m_n = (\tau_n)_{\{\tau_n \leq T_m\}}$. Conversely if each $(\mathcal{F}^m_t)$ is an a.s. jumping filtration on $\mathbb{R}_+$ with jumping sequence $(\tau^m_n)_{n \in \mathbb{N}}$, then $(\mathcal{F}_t)$ is an a.s. jumping filtration on I with jumping sequence (τ_n), where $\tau_0 = 0$, $\tau_{n+1} = \delta \wedge \inf (t > \tau_n \,:\, t \in A)$, and $A = \bigcup_{m \geq 1} \{(\bigcup_{n \geq 1} [\![\tau^m_n]\!]) \cap]\!] T_{m-1}, T_m]\!]\}$.

With these facts, it is now trivial to deduce (*a*) from Theorem 1. For (*b*) and (*c*), it is enough to reproduce the proof of Theorem 2 of [10], upon substituting $\mathcal{I}$ with the class of all totally inaccessible times on $\overset{\circ}{I}$, and q in (3) of [10] with $q \wedge T_q$. ■

Suppose that $(\mathcal{F}_t)$ is a jumping filtration on I, with jumping sequence (τ_n). (2) obviously implies

$$\mathcal{F}_{\tau_{n+1}-} = \mathcal{F}_{\tau_n} \vee \sigma (\tau_{n+1}) \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

A trivial adaptation of Lemma (3.2) and Proposition (3.3) of [8] shows that if (τ_n) is a jumping sequence for the jumping filtration $(\mathcal{F}_t)$ on I, then we have the following:

$$\begin{aligned}&\text{If T is a stopping time, for each } n \in \mathbb{N} \text{ there is a nonnegative}\\ &\mathcal{F}_{\tau_n}\text{-measurable random variable } R_n \text{ with}\\ &T \wedge \tau_{n+1} = (\tau_n + R_n) \wedge \tau_{n+1}.\end{aligned} \tag{4}$$

$$\left.\begin{array}{l}\text{A process } H \text{ is optional on } I \text{ iff for each } n \in \mathbb{N} \text{ there is an} \\ \mathcal{F}_{\tau_n}\text{-measurable process } H^n \text{ with } H_t = H_t^n \text{ on } \{\tau_n \le t < \tau_{n+1}\} \\ \text{and if } H_{\tau_n} \text{ is } \mathcal{F}_{\tau_n}\text{- measurable on } \{\tau_n < \infty\}.\end{array}\right. \tag{5}$$

$$\left.\begin{array}{l}\text{A process } H \text{ is predictable on } I \text{ iff for each } n \in \mathbb{N} \text{ there is an} \\ \mathcal{F}_{\tau_n}\text{-measurable process } H^n \text{ with } H_t = H_t^n \text{ on } \{\tau_n \le t \le \tau_{n+1}\}.\end{array}\right. \tag{6}$$

In the rest of the paper we will apply these results in the particular case where the filtration $(\mathcal{F}_t)$ is the (right-continuous) filtration *generated by a process* X taking its values in a Blackwell measurable space $(E, \mathcal{E})$, under the following two assumptions:

$$X \text{ is } (\mathcal{F}_t)\text{-optional} \tag{7}$$

$$\text{For any finite stopping time } T, \text{ we have } \mathcal{F}_T = \mathcal{F}_{T-} \vee \sigma(X_T). \tag{8}$$

Assume moreover that $(\mathcal{F}_t)$ is a jumping filtration on I, with jumping sequence $(\tau_n)_{n\in\mathbb{N}}$. Introduce the integer-valued random measure μ on $\mathbb{R}_+ \times E$:

$$\mu(\omega,\, dt \times dx) = \sum_{n\ge 1,\, \tau_n(\omega) < \tau_{n+1}(\omega)} \varepsilon_{(\tau_n(\omega), X_{\tau_n}(\omega))}(dt \times dx) \tag{9}$$

(this is a "marked point process"), and the smallest complete filtration $(\mathcal{G}_t)$ for which μ is optional and such that $\mathcal{G}_0 = \mathcal{F}_0$.

LEMMA 1. – *We have $\mathcal{G}_t = \mathcal{F}_t$ in restriction to the set $\{t \in I\}$.*

Proof. – Recall first that τ_n is a $(\mathcal{G}_t)$-stopping time, and that $\mathcal{G}_{\tau_n} = \mathcal{G}_0 \vee \sigma((\tau_p, X_{\tau_p} 1_{\{\tau_p < \delta\}}) : p \le n)$, and also that $(\mathcal{G}_t)$ satisfies (2). Now (3) and (8) yield $\mathcal{F}_{\tau_n} = \mathcal{F}_{\tau_{n-1}} \vee \sigma(\tau_n, X_{\tau_n})$, hence the property that $\mathcal{G}_t = \mathcal{F}_t$ on $\cup_n \{\tau_n \le t < \tau_{n+1}\}$ readily follows from (2) applied to $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ and we have the result. ■

Then we can apply the results of [9] for the structure of martingales. In order to simplify the statements, we set $X_\infty = x_0$ where x_0 is some arbitrary point in E. If $G_n(\omega;\, dx,\, dt)$ denotes the distribution of (X_{τ_n}, τ_n), conditional on $\mathcal{F}_{\tau_{n-1}}$, the compensator of μ is the measure

$$\nu(dt \times dx) = \sum_{n\in\mathbb{N}^*} \frac{G_n(dx \times dt)}{G_n(E \times [t,\, \infty])} 1_{(\tau_{n-1},\, \tau_n]}(t). \tag{10}$$

Further, any local martingale M on I is of the following form for some predictable function U on $\Omega \times \mathbb{R}_+ \times E$:

$$\left.\begin{array}{c} M_t = M_0 + \displaystyle\int_{[0,\,t]\times E} U(s,\, x)\,(\mu(ds \times dx) - \nu(ds \times dx)), \\ t \in I, \end{array}\right\} \tag{11}$$

where outside a null set $|U(s, x)| 1_{[0, t] \times E}(s, x)$ is integrable w.r.t. $|\mu - \nu|$ for all $t \in I$. Finally, if M above is locally square-integrable, its predictable bracket is, with the notation $\hat{U}_t = \int U(t, x)\, \nu(\{t\}, dx)$ and $a_t = \nu(\{t\} \times E)$:

$$\left.\begin{aligned} \langle M, M \rangle_t = \int_{[0, t] \times E} & (U(s, x) - \hat{U}_s)^2\, \nu(ds, dx) \\ & - \sum_{s \leq t} (1 - a_s)\, \hat{U}_s^2, \\ & t \in I. \end{aligned}\right\} \tag{12}$$

3. JUMPING MARKOV PROCESSES: DEFINITION AND CHARACTERIZATION

As said in the introduction, our definition of a JMP (Jumping Markov Process) is not a "constructive" one (starting with (1)), but is in terms of the filtration of the process.

So, unless otherwise stated, we start with a *normal strong Markov process* $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \vartheta_t, P_x)$ (in the setting of Blumenthal-Getoor [1]) with values in a Blackwell space $(E, \mathcal{E})$. Possibly there is a finite lifetime ζ and a cimetary point Δ, and in all cases $X_\infty = \Delta$ by convention. $(\mathcal{F}_t^0)_{t \geq 0}$ is the filtration generated by X, and $(\mathcal{F}_t)$ is the usual Markov completion of $(\mathcal{F}_{t+}^0)$. The transition semi-group of X is $(P_t)_{t \geq 0}$. We also *assume that* (7) *holds* (this is not automatically satisfied, because E is not a topological space and so there is no regularity of the paths $t \to X_t$; note also that the σ-fields $\mathcal{F}_t^0$ are not separable here, although $\mathcal{E}$ is separable).

Now, loosely speaking, the process X is a JMP if $(\mathcal{F}_t)$ is a jumping filtration on $I = \cap [\![0, \tau_n]\!]$ for a jumping sequence (τ_n). The difficulties in formulating a proper definition for a JMP come from the following facts:

1) We do not want to impose $\zeta = \infty$, because we want to accomodate "minimal" JMP's which are killed at the explosion time $\tau_\infty = \lim_n \uparrow \tau_n$.

2) As seen in Section 2, the jumping sequence is not unique, and even the set I on which $(\mathcal{F}_t)$ is a jumping filtration is not unique, while we obviously wish to have I as large as possible.

3) We are mainly interested in the case where $(\mathcal{F}_t)$ is quasi-left continuous on $\overset{\circ}{I}$, a property that is difficult to state when the set I is not known beforehand! (we could assume that it is quasi-left continuous on the whole $\mathbb{R}_+$, but this is a rather serious restriction, since in general the explosion time τ_∞ will be predictable with $\mathcal{F}_{\tau_\infty} \neq \mathcal{F}_{\tau_\infty^-}$ if $\tau_\infty < \infty$).

Before proceeding, we recall some usual terminology for Markov processes: a stopping time is w.r.t. $(\mathcal{F}_t)$, and it is totally inaccessible if it is P_x-totally inaccessible for all $x \in E$; a process is a martingale (semimartingale, etc.) if it is a martingale (semimartingale, etc.) relatively to each measure P_x; a set A is null if $P_x(A) = 0$ for all x;

3.1. Quasi-Hunt jumping Markov processes

Among all equivalent formulations we pick the most intuitive on for a definition:

DEFINITION 2. – The process X is called a *quasi-Hunt jumping Markov process* if $(\mathcal{F}_t)$ is a jumping filtration on a set $I = \cup[\![0, \tau_n]\!]$, with the jumping sequence (τ_n) having the following properties:

(i) τ_n is totally inaccessible (hence $\tau_n > 0$ a.s.) if $n \geq 1$;

(ii) $\tau_n \leq \zeta$ and $\tau_n < \tau_{n+1}$ a.s. on the set $\{\tau_n < \zeta\}$.

Further, X is called *regular* if $\tau_\infty := \lim_n \uparrow \tau_n = \infty$ P_x-a.s. for all $x \in E$. ■

In virtue of Theorem 2 (ii, iii) the above sequence is a.s. unique, and it is called the *canonical jumping sequence* of X. The terminology "quasi-Hunt" refers to the usual definition of a Hunt process, as it is apparent in the following theorem:

THEOREM 3. – *Assume that the lifetime ζ has $P_x(\zeta = \infty) = 1$ for all $x \in E$. The following three properties are equivalent:*

(i) *For each $x \in E$, the P_x-martingales are quasi-left continuous and with locally finite variation.*

(ii) *The filtration $(\mathcal{F}_t)$ is a jumping filtration on $\mathbb{R}_+$, and for each $x \in E$ the P_x-martingales are quasi-left continuous.*

(iii) *X is a regular quasi-Hunt JMP.*

The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) readily follow from Theorem 1, and the implication (i) $\Rightarrow$ (iii) will be proved after the next two results. That (i) $\Rightarrow$ (ii) does *not* immediately follows from Theorem 1, because in (ii) we ask the filtration $(\mathcal{F}_t)$ to be jumping and not only a.s. jumping. The first theorem below gives a set of conditions apparently much weaker than, but in fact equivalent to, those of Definition 2. The second one shows that the τ_n's above have very special properties connected with the fact that X is Markov.

THEOREM 4. – *The process X is a quasi-Hunt JMP iff for every $x \in E$ there is a stopping time S_x such that:*

(i) $S_x > 0$ *P_x-a.s., and the time $\zeta_{\{\zeta < S_x\}}$ is P_x-totally inacessible.*

(ii) *Any P_x-martingale which is constant after S_x is quasi-left continous and with P_x-a.s. locally finite variation;*

(iii) *there is (at least) a P_x-martingale M which has P_x-a.s. at least one jump on the interval $(0, S_x]$, on the set $\{S_x < \infty\}$.*

The necessary condition above is trivial: take $S_x = \tau_1$ ((iii) is satisfied with $M = Y - \tilde{Y}$, where $Y_t = 1_{\{t \geq \tau_1\}}$ and $\tilde{Y}$ is the P_x-compensator of Y). The meanings of (i) and (ii) are clear, and (iii) insures that the σ-field $\mathcal{F}_{S_x}$ in restriction to $\{S_x < \infty\}$ is "sufficiently big".

THEOREM 5. – *Suppose that X is a quasi-Hunt JMP, with the canonical jumping sequence (τ_n).*

a) There exists a totally inaccessible terminal time $\tau \leq \zeta$ such that the sequence (τ_n) is given by $\tau_0 = 0$, $\tau_{n+1} = \tau_n + \tau \circ \vartheta_{\tau_n}$ a.s. Call H_x and G_x the laws of τ and (X_τ, τ) under P_x, and

$$\eta(x) = \inf(t : H_x([t, \infty]) = 0). \tag{13}$$

b) If $x \in E$, H_x has no atom except possibly $\{+\infty\}$.

c) There is a measurable function $f : E \times \mathbb{R}_+ \to E$ such that

$$\left.\begin{array}{ll} f(x, 0) = x, & \\ f(x, t+s) = f(f(x, t), s) & \text{if} \quad t+s < \eta(x), \end{array}\right\} \tag{14}$$

$$\tau = \inf(t : X_t \neq f(x, t)) \quad \text{and} \quad X_\tau \neq f(x, \tau), \quad P_x\text{-a.s.}, \tag{15}$$

$$\left.\begin{array}{ll} H_x((t, t+s]) = H_x((t, \infty])\, H_{f(x,t)}((0, s]) & \text{if} \quad t < \eta(x), \\ s \in [0, \infty]. & \end{array}\right\} \tag{16}$$

d) There exists a probability kernel Γ from E into E_Δ such that for all $x \in E$ we have $\Gamma(x, \{x\}) = 0$ and

$$G_x(dy, dt) = H_x(dt)\,[1_{\{t<\infty\}}\,\Gamma(f(x, t), dy) + 1_{\{t=\infty\}}\,\varepsilon_\Delta(dy)]. \tag{17}$$

Before proving these theorems we state a lemma, which is almost trivial when the σ-fields $\mathcal{F}_t^0$ are separable (because in this case there is a countable family of martingales which "generates" for each x all P_x-martingales), but unfortunately not so in general.

LEMMA 2. – *There is a sequence (M^n) of bounded martingales with the following property: if $x \in E$ and if S_x is a stopping time such* *that all P_x-martingales are quasi-left continuous on $[\![0,\ S_x]\!]$, then for each P_x-martingale M we have*

$$\{t : 0 < t \leq S_x \text{ and } \Delta M_t \neq 0\} \subseteq \bigcup_{n\geq 1}\{t > 0 : \Delta M_t^n \neq 0\} \quad P_x\text{-}a.s.$$

Proof. – For every P_x-martingale M, set $D(M) = \{t > 0 : \Delta M_t \neq 0\}$.

1) First we show that, due to (7) and to the strong Markov property, if h is bounded measurable on E, then $t \to P_{s-t}\, h(X_t)$ is a.s. càdlàg on $[0,\ s]$ (it is known that conversely this implies the strong Markov property). Set

$$Y_t = P_{s-t}\, h(X_t)\, 1_{\{t<s\}} + h(X_s)\, 1_{\{t\geq s\}}.$$

Let M^y be a càdlàg version of the P-martingale $M_t^y = E_y(h(X_s)|\mathcal{F}_t)$. By the strong Markov property $Y_T = M_T^y\, P_y$-a.s. for every finite stopping time T. Since both M^y and Y are $(\mathcal{F}_t)$-optional (the later by (7)), it follows from [4] that they are P_y-indistinguishable. Thus, setting $Y(h,\ s)_t = Y_t$ on the set where Y is càdlàg and $Y(h,\ s)_t = 0$ elsewhere, we obtain an $(\mathcal{F}_t)$-adapted càdlàg process $Y(h,\ s)$ with $Y(h,\ s)_t = E_y(h(X_s)|\mathcal{F}_t)$ for all $y \in E$. The main point here is that $Y(h,\ s)$ *does not depend on* y.

2) More generally, denote by $\mathcal{Z}$ the family of all variables of the form $Z = \prod_{0\leq i\leq n} h_i(X_{t_i})$, where $t_0 = 0 < \cdots < t_n < \infty$ and h_i are bounded measurable on E. If we set $g_j(y) = E_y[\prod_{j+1\leq k\leq n} h_k(X_{t_k - t_{j+1}})]$ and $Z_j = \prod_{0\leq i\leq j} h_i(X_{t_i})$, then

$$Y(Z)_t = \begin{cases} Z_j\, Y(g_j,\ t_{j+1})_t & \text{if } t_j \leq t < t_{j+1} \\ Z & \text{if } t \geq t_n \end{cases} \tag{18}$$

is a càdlàg version of the martingale $E_x(Z|\mathcal{F}_t)$, for all $x \in E$.

3) Next we construct the sequence (M^n) as follows. Let $\mathcal{E}_0$ be a countable algebra generating the σ-field $\mathcal{E}$ on E, and $\mathcal{U}$ be the family of all $\mathcal{E}_0$-measurable functions on E taking only finitely many rational values. The set $\mathcal{U} \times \mathbb{Q}_+$ is countable, and we may write it as a sequence $(h_n,\ s_n)_{n\geq 1}$. Then we set $M^n = Y(h_n,\ s_n)$, and $D = \bigcup_{n\geq 1} D(M^n)$.

4) Let $s \in \mathbb{Q}_+$ and h be bounded measurable on E. There is a sequence g_p in $\mathcal{U}$ such that $g_p(X_s) \to h(X_s)$ in $L^2(P_x)$. Then $Y(g_p,\ s)_t \to Y(h,\ s)_t$ in $L^2(P_x)$, uniformly in $t \in \mathbb{R}_+$. Since the jumps of all $Y(g_p,\ s)$ are in D, we have $D(Y(h,\ s)) \subseteq D \quad P_x$-a.s.

Next let $s > 0$ and h be bounded measurable on E, and $M = Y(h, s)$. If $u \in \mathbb{Q}_+ \cap [0, s]$ we have $M_t = P_{u-t} P_{s-u} h(X_t) = Y(P_{s-u} h, u)_t$ a.s. for $t < u$, hence $[0, u) \cap D(M) \subseteq D\, P_x$-a.s. Since $M_t = h(X_s)$ for $t \geq s$ and since u is arbitrarily close to s, we deduce $D(M) \subseteq D \cup \{s\}\, P_x$-a.s. Using (18) it follows that for $Z \in \mathcal{Z}$ as in Step 2) above, $D(Y(Z)) \subseteq D \cup \{t_1, \ldots, t_n\}\, P_x$-a.s. Now apply the property that $Y(Z)$ is P_x-quasi-left continuous on $[\![0, S_x]\!]$ to obtain

$$D(Y(Z)) \cap [\![0, S_x]\!] \subseteq D \quad P_x\text{-a.s.} \quad \text{if } Z \in \mathcal{Z} \tag{19}$$

5) Finally if M is a P_x-martingale and T is a jump (stopping) time of M, there is bounded P_x-martingale M' with $\Delta M'_T \neq 0\, P_x$-a.s. on $\{T < \infty\}$. There exists a uniformly bounded sequence $Z_n \in \mathcal{Z}$ converging to M'_∞ in $L^2(P_x)$. Then $Y(Z_n)_t \to M'_t$ in $L^2(P_x)$, uniformly in $t \in \mathbb{R}_+$. Since each $Y(Z_n)$ satisfies (19), the same is true for M' and thus $T \in D\, P_x$-a.s. on the set $\{T \leq S_x\}$: hence the lemma is proved. ■

Proof of Theorems 4 and 5. – As said before, for Theorem 4 it only remains to prove the sufficient condition. So we suppose that for each $x \in E$ there is a stopping time S_x having (i, ii, iii) of Theorem 4, and we will prove at the same time that X is a quasi-Hunt JMP and that (*a*, *b*, *c*, *d*) of Theorem 5 hold.

Let (M^n) be as in Lemma 2. Set $\sigma_n = \inf(t : \Delta M^n_t \neq 0)$ and $\tau = \zeta \wedge \inf_n \sigma_n$, and use the notation G_x, H_x, $\eta(x)$ associated with τ as in (*a*).

1 (i) and Theorem 2 imply that

$$(\mathcal{F}_t) \text{ is a } P_x\text{-a.s. jumping filtration on } I_x = [\![0, S_x]\!] \tag{20}$$

Let $(\tau^x_n)_{n \in \mathbb{N}}$ be the jumping sequence constructed in Theorem 2 (*b*). Combining (i), (ii) and (iii) gives a P_x-totally inaccessible time T such that P_x-a.s.: $0 < T \leq S_x$ and $T \leq \zeta$ on the set $\{\zeta < S_x\}$. Since $\tau^x_1 \leq S_x$, Theorem 2 (*b*)-(ii) implies $\tau^x_1 \leq T$ P_x-a.s., thus $\tau^x_1 \leq \zeta$ P_x-a.s., and $\tau^x_1 = T$ P_x-a.s. on the set $\{\tau^x_1 = S_x\}$: hence τ^x_1 is P_x-totally inaccessible, and there is a P_x-martingale M having $\Delta M_{\tau^x_1} = 1$ on $\{\tau^x_1 < \infty\}$.

By Theorem 2 (*b*)-(iv) we have P_x-a.s. $\sigma_n \geq \tau^x_1$, hence $\tau \geq \tau^x_1$ (recall $\tau^x_1 \leq \zeta$). Conversely, Lemma 2 applied to the martingale M constructed above yields $\tau \leq \tau^x_1$ P_x-a.s., so finally:

$$\tau^x_1 = \tau \leq S_x \quad P_x\text{-a.s.}, \quad \text{and } \tau \text{ is } P_x\text{-totally inaccessible.} \tag{21}$$

In particular we deduce that $H_x(\{t\}) = 0$ for all $t < \infty$, and (*b*) holds.

2) (20) and (21) yield a function $f : E \times \mathbb{R}_+ \to E$ with $P_x(X_t \neq f(x, t), \tau > t) = 0$ for all t. Then $E_x[h(X_t) 1_{\{t<\tau\}}] =$

$h \circ f(x, t) H_x((t, \infty])$ for any bounded measurable h. Now, $(x, t) \to E_x[h(X_t) 1_{\{t<\tau\}}]$ is measurable, as well as $(x, t) \to H_x((t, \infty])$ and $x \to \eta(x)$: thus $(x, t) \to h \circ f(x, t) 1_{\{t<\eta(x)\}}$ is measurable. Then, up to changing $f(x, t)$ when $t \geq \eta(x)$ (with η defined by (13)), we can and will assume that $(x, t) \to f(x, t)$ is measurable.

Using again (20) and (21), we have by (4):

$$\text{Any stopping time } T \text{ is } P_x\text{-a.s. constant on } \{T < \tau\}. \tag{22}$$

By (22) and the definition of f, the two optional processes $t \to X_t 1_{\{t<\tau\}}$ and $t \to f(x, t) 1_{\{t<\tau\}}$ coincide P_x-a.s. on $\{T < \tau\}$ for each stopping time T, hence by [4]:

$$X_t = f(x, t) = f(X_0, t) \quad \text{for all } t < \tau, \quad P_x\text{-a.s.} \tag{23}$$

3) Set $\tau' = \inf(t : X_t \neq f(X_0, t))$. Obviously $\mathcal{F}_t^0 \cap \{t < \tau'\} = \sigma(X_0) \cap \{t < \tau'\}$ so exactly as for (22) we get that τ is P_x-a.s. constant on $\{\tau < \tau'\}$, which contradicts (*b*) unless $P_x(\tau \geq \tau') = 1$, and in view of (23) we deduce $P_x(\tau = \tau') = 1$, and in particular we have the first part of (15).

Fix $t \geq 0$ and set $g(x, s) = f(x, s) 1_{\{s \leq t\}} + f(f(x, t), s - t) 1_{\{s>t\}}$ and $\tau'' = \inf(s : X_s \neq g(X_0, s))$. The same argument as above shows $P_x(\tau \geq \tau'') = 1$. Markov property at time t and the first part of (15) imply that P_x-a.s., $\tau'' = \tau$ if $\tau < t$ and $\tau'' = t + \tau \circ \vartheta_t$ if $\tau > t$. Since τ is $P_{f(x,t)}$-totally inaccessible and $P_x(\tau = t) = 0$, it follows that τ'' is P_x-totally inaccessible; then Theorem 2 (*b*)-(ii) and (21) yield $P_x(\tau'' \geq \tau) = 1$, so finally $P_x(\tau'' = \tau) = 1$ and

$$\tau = t + \tau \circ \vartheta_t \quad P_x\text{-a.s. on } \{\tau > t\}. \tag{24}$$

Combining this, the first part of (15) and the Markov property at time t on the set $\{\tau > t\}$, we get

$$G_x(A \times (t, t+s]) = H_x((t, \infty]) G_{f(x,t)}(A \times (0, s]) \quad \text{for} \quad t < \eta(x), \tag{25}$$

for all $s \in [0, \infty]$, $A \in \mathcal{E}_\Delta$, and (16) follows by taking $A = E_\Delta$.

Further, $P_x(\tau'' = \tau) = 1$ yields $g(x, t+s) = f(x, t+s)$ P_x-a.s. on $\{\tau > t + s\}$; since f and g are deterministic, we deduce the second part of (14), while the first part is obvious. Further, the strong Markov property at time τ and (23) imply $X_{\tau+u} = f(X_\tau, u)$ P_x-a.s. for all u small enough on $\{\tau < \zeta\}$. Then on the set $A = \{\tau < \zeta, X_\tau = f(x, \tau)\}$ we have $\tau < \eta(x)$ P_x-a.s., hence $X_s = f(x, s)$ P_x-a.s. for all $s \leq \tau + \varepsilon$ by (14), for some $\varepsilon > 0$ (depending on ω): this contradicts the first part of (15), unless $P_x(A) = 0$, hence the second part of (15) and (*c*) is proved.

4) (24) means that τ is a terminal time. By construction $\tau \leq \zeta$, and (*b*) implies $P_x(\tau > 0) = 1$ for all $x \in E$.

Set $\tau_0 = 0$, $\tau_n = \tau_{n-1} + \tau \circ \vartheta_{\tau_{n-1}}$ as in (*a*). By the strong Markov property at τ_n, we have that τ_{n+1} is P_x-totally inaccessible for all $x \in E$ and that $X_t = f(X_{\tau_n}, t - \tau_n)$ for all $t \in [\tau_n, \tau_{n+1})$, P_x-a.s. for all $x \in E$. Since $\mathcal{F}_0$ contains all null sets, we readily deduce that $(\mathcal{F}_t)$ satisfies (2). Therefore X is a quasi-Hunt JMP (*see* Definition 2) with canonical jumping sequence (τ_n), that is the sufficient part of Theorem 4 is proved, as well as (*a*) of Theorem 5.

5) It remains to prove (*d*). If $A \in \mathcal{E}_\Delta$ we define a measurable function $(x, t) \to Z(x, t, A)$ by

$$Z(x, t, A) = \liminf_{s \in \mathbb{Q},\, s \downarrow 0} \frac{G_x(A \times (t, t+s])}{H_x((t, t+s])}$$

(with $0/0 = 0$). Since $t \to H_x((t, \infty])$ is continuous, Lebesgue derivation Theorem implies

$$G_x(A \times B) = \int_B H_x(dt)\, Z(x, t, A) \quad \text{for all } B \in \mathcal{R}_+, \qquad A \in \mathcal{E}_\Delta.$$

Further, (25) yields $Z(x, t, A) = Z(f(x, t), 0, A)$ if $t < \eta(x)$, so

$$G_x(A \times B) = \int_B H_x(dt)\, Z(f(x, t), 0, A) \quad \text{for all } B \in \mathcal{R}_+, \qquad A \in \mathcal{E}_\Delta.$$

Now the σ-additivity of $A \to G(A \times B)$ and the Blackwell property of $(E, \mathcal{E})$ allow to obtain a probability kernel Γ from E into E_Δ such that $Z(f(x, t), 0, A) = \Gamma(f(x, t), A)$ H_x-a.s. in t on $[0, \infty)$ (exactly like for the construction of the Lévy kernel of a Hunt process). Since $X_\infty = \Delta$, we deduce that (17) holds, and that one may choose Γ such that $\Gamma(x, \{x\}) = 0$ follows from the last property in (15). ■

Proof of Theorem 3. – We only have to prove that (i) $\Rightarrow$ (iii). Now, (i) implies the conditions of Theorem 4, with $S_x = \infty$, so X is a quasi-Hunt JMP.

To check the regularity of X, we consider the jumping sequence (τ_n) of Theorem 5. Fix $x \in E$. For each n there is a P_x-martingale M^n having exactly one jump of size 1 at time τ_n on the set $\{\tau_n < \infty\}$ and continuous elsewhere. These martingales are pairwise orthogonal, because $\tau_n < \tau_{n+1}$ if $\tau_n < \infty$. Hence the series $M = \sum \frac{1}{n} M^n$ converges in $L^2(P_x)$. The variation of $t \to M_t$ is infinite on the set $\{\tau_\infty < \infty\}$, so $P_x(\tau_\infty < \infty) = 0$ and we are finished. ■

Remark 1. – For a quasi-Hunt JMP the set $I = \cup [\![0, \tau_n]\!]$ is uniquely determined (up to an evanescent set): this follows from Theorem 5, or from the argument of the previous proof. This is in contrast with what we will see for general JMP's below. ■

3.2. General jumping Markov processes

When we drop the quasi-left continuity of the filtration, we have to be more careful: the proper definition of a general JMP makes use of the specific form of the jumping sequence explicited in Theorem 5, and it goes as follows:

DEFINITION 3. – The process X is called a τ-JMP if τ is a terminal time with $\tau \leq \zeta$ and $P_x(\tau > 0) = 1$ for all $x \in E$ and such that, if $\tau_0 = 0$ and $\tau_{n+1} = \tau_n + \tau \circ \vartheta_{\tau_n}$, $(\mathcal{F}_t)$ is a jumping filtration on $I = \cup_n [\![0, \tau_n]\!]$ with jumping sequence $(\tau_n)_{n \in \mathbb{N}}$.

Further, it is called *regular* if $\tau_\infty := \lim\limits_n \uparrow \tau_n = \infty$ P_x-a.s. for all $x \in E$. ■

Theorems 4 and 5 have the following versior of τ-JMP:

THEOREM 6. – *Let τ be a terminal time with $\tau \leq \zeta$ and $P_x(\tau > 0) = 1$ for all $x \in E$.*

a) If for all t we have $\mathcal{F}_t \cap \{t < \tau\} = \mathcal{F}_0 \cap \{t < \tau\}$ up to null sets, then X is a τ-JMP.

b) If X is a τ-JMP, call H_x and G_x the law of τ and (X_τ, τ) under P_x for $x \in E$ and define $\eta(x)$ by (13). *Then there is a measurable function $f : E \times \mathbb{R}_+ \to E$ satisfying* (14) *and*

$$\left.\begin{aligned} &\tau = \eta(x) \wedge \inf(t : X_t \neq f(x, t)) \\ &\text{and} \\ &X_\tau \neq f(x, \tau) \quad \text{on } \{\tau < \eta(x)\}, \quad P_x\text{-a.s.} \end{aligned}\right\} \tag{26}$$

$$G_x(A \times (t, t+s]) = H_x((t, \infty])\, G_{f(x,t)}(A \times (0, s]) \quad \text{for} \quad t < \eta(x). \tag{27}$$

Furthermore there is a factorization

$$G_x(dy, dt) = H_x(dt)\, G'(x, t; dy) \tag{28}$$

where G' is a probability kernel from $E \times [0, \infty]$ into E_Δ such that

$$G'(x, \infty; \cdot) = \varepsilon_\Delta(\cdot), \qquad G'(x, t; \{f(x, t)\}) = 0 \quad \text{if } t < \eta(x). \tag{29}$$

c) Under the assumptions of (b), *if $H_x(\{t\}) = 0$ for all $t < \infty$, then τ is totally inaccessible and X is a quasi-Hunt JMP.*

Observe that in the quasi-Hunt case, (*b*) above reduces to (*c*)-(*d*) of Theorem 5 (in this case $P_x(\tau < \eta(x)) = 1$ if $\eta(x) < \infty$: so (26) and (15) are the same).

Proof. – *a*) Define by induction $\tau_0 = 0$, $\tau_{n+1} = \tau_n + \tau \circ \vartheta_{\tau_n}$. By hypothesis there is a function $f : E \times \mathbb{R}_+ \to E$ such that $X_t = f(x, t)$ P_x-a.s. on $\{t < \tau\}$. We see as in Step 2) of the proof of Theorem 4 that there is a version of f which is measurable and that (23) holds. The strong Markov property at time τ_n yields $X_t = f(X_{\tau_n}, t - \tau_n)$ a.s. for all $t \in [\tau_n, \tau_{n+1})$. Then the same argument than in Step 4) shows that $(\mathcal{F}_t)$ is a jumping filtration with jumping sequence (τ_n).

b) In view of (5) and (7) there is a measurable function: $f : E \times \mathbb{R}_+ \to E$ such that $X_t = f(X_0, t)$ if $t < \tau$. The first part of (14) is obvious. (27) and the second part of (14) readily follow from the Markov property applied at time t on the set $\{t < \tau\}$ and from the fact that τ is a terminal time.

Set $\tau' = \inf(t : X_t \neq f(X_0, t))$. We have $\tau \leq \tau'$ and $\mathcal{F}_t \cap \{t < \tau'\} = \mathcal{F}_0 \cap \{t < \tau'\}$ for all t. Thus $\tau = \tau' \wedge u_x$ P_x-a.s. by (4) for some constant u_x; from the definition of η we deduce $\eta(x) \leq u_x$ and thus $\tau = \tau' \wedge \eta(x)$ P_x-a.s., which is the first part of (26). On the set $A = \{\tau < \eta(x) \wedge \zeta, X_\tau = f(x, \tau)\}$ we have P_x-a.s. $X_{\tau+u} = f(X_\tau, u) = f(x, \tau + u)$ for all u small enough, by the strong Markov property applied at time τ and (14): this contradicts the first part of (26), unless $P_x(A) = 0$, which yields the second part of (26). Finally, there are always factorizations (28), and we may choose one satisfying (29) by $X_\infty = \Delta$ and (26).

c) For a jumping filtration the first jump time $\tau_1 = \tau$ is P_x-totally inaccessible iff its law is diffuse in restriction to $\mathbb{R}_+$, hence the claim. ■

Remark 2. – We cannot write $G'(x, t; \cdot) = \Gamma(f(x, t), \cdot)$ for some probability kernel Γ on E in general: first because (27) is true only if $t < \eta(x)$ while here we may have $H_x(\{\eta(x)\}) > 0$ and $\eta(x) < \infty$; second, even if $H_x(\{\eta(x)\}) = 0$ when $\eta(x) < \infty$, we cannot use Lebesgue derivation theorem as in Step 6) of the proof of Theorem 5 because $t \to H_x((t, \infty])$ may be discontinuous. ■

Remark 3. – If X is τ-JMP (possibly quasi-Hunt) it is also a σ-JMP if $\sigma = \tau \wedge \eta'(X_0)$ and η' is a function: $E \to [0, \infty]$ having $\eta'(x) = t + \eta'(f(x, t))$ when $\eta'(x) > t$. And of course it is possible that $P_x(\tau \neq \sigma) > 0$.

When X is a quasi-Hunt JMP, there is a "maximal" stopping time τ for which it is a τ-JMP , in the sense that if it is also a σ-JMP then $\sigma \leq \tau$ a.s.

(take the totally inaccessible time τ occuring in Theorem 5). This is not the case for general JMP's, as seen in the following example:

Take $E = (-\infty,\, 0) \cup \{1,\, 2\}$ and consider a càdlàg E-valued process X whose law for each starting point x is:

$$P_x\,(X_t = x,\, \forall\, t \geq 0) = 1 \quad \text{if } x \in \{1,\, 2\},$$

$$P_x\,(X_t = x + t,\ \forall\, t < -x,\ X_t = i,\ \forall\, t \geq -x) = \frac{1}{2} \quad \text{if } x < 0 \ \text{ and } \ i = 1,\, 2.$$

Then X is a τ-JMP with $\tau = \inf\,(t\,:\,X_{t-} = 0)$. With the previous notation, we have $H_x = \varepsilon_\infty$ for $x \in \{1,\, 2\}$ and $G_x(dy,\, dt) = \frac{1}{2}\,\varepsilon_{-x}\,(dt)\,(\varepsilon_1\,(dy) + \varepsilon_2\,(dy))$ if $x < 0$, and for the function f we can take f_α below, with α arbitrary in E:

$$f_\alpha\,(x,\, t) = \begin{cases} x + t & \text{if } t < -x \ \text{ and } \ x < 0 \\ \alpha & \text{if } t \geq -x \ \text{ and } \ x < 0 \\ & \text{if } t \geq 0 \ \text{ and } \ x \in \{1,\, 2\} \end{cases}$$

Now set

$$\tau' = \begin{cases} \infty & \text{if } X_0 = 2 \\ \inf\,(t\,:\,X_t = 2) & \text{if } X_0 \neq 2, \end{cases}$$

$$\tau'' = \begin{cases} \infty & \text{if } X_0 = 1 \\ \inf\,(t\,:\,X_t = 1) & \text{if } X_0 \neq 1. \end{cases}$$

Then X is also a τ'-JMP with the function $f = f_1$ above, and a τ''-JMP with the function $f = f_2$. But $\tau' \vee \tau'' = \infty$ a.s., and X is not a $\tau' \vee \tau'$-JMP. In this example there is no maximal terminal time σ such that X is a σ-JMP. ■

COROLLARY 1. – *Assume that there is a measurable function $f\,:\,E \times \mathbb{R}_+ \to E$ such that $\tau := \inf\,(t\,:\,X_t \neq f\,(X_0,\, t))$ satisfies $P_x\,(\tau > 0) = 1$ for all $x \in E$. We call H_x the law of τ under P_x, and define $\eta\,(x)$ by* (13). *Then under either one of the following conditions:*

a) τ is a terminal time,

b) f satisfies (14),

c) $H_x\,(\{t\}) = 0$ for all $t < \infty$,

the process X is a τ-JMP, and it is quasi-Hunt under (*c*).

Proof. – $(a) \Rightarrow (b)$ by the Markov property at time t on $\{t < \tau\}$. $(b) \Rightarrow (a)$ is obvious. Since clearly $\mathcal{F}_t \cap \{t < \tau\} = \mathcal{F}_0 \cap \{t < \tau\}$, the result under (a) or (b) follows then from Theorem 6(a).

Finally assume (c). Using once more $\mathcal{F}_t \cap \{t < \tau\} = \mathcal{F}_0 \cap \{t < \tau\}$, we see that τ is P_x-totally inaccessible for all $x \in E$. Then one can reproduce Step 3) of the proof of Theorem 4 to the effect that (b) holds. ∎

Example. – Step Markov Processes. According to the classical notion, we say that X is a *step Markov process* if the stopping time $\tau = \inf(t : X_t \neq X_0)$ is a.s. strictly positive and if $X_\tau \neq \Delta$ a.s. on the set $\{\tau < \infty\}$.

Such a process is obviously a τ-JMP (Condition (b) of Corollary 1 is met with $f(x, t) = x$ for all $t \geq 0$). Further, τ has an exponential distribution with some parameter $a(x) \geq 0$ under P_x ($a(x) = 0$ means that $\tau = \infty$ P_x-a.s.). This is easily checked, since (27) yields in this case $H_x((t+s, \infty]) = H_x((t, \infty]) H_x((s, \infty])$ for $t < \eta(x)$: this give $\eta(x) = \infty$ and $H_x((t, \infty]) = e^{-a(x)t}$ for some $a(x) \geq 0$. Then in fact X is a quasi-Hunt JMP.

3.3. The enclosed Markov chain and a construction of JMPs

We start with a definition.

DEFINITION 4. – The *characteristics* of the τ-JMP X are the pair (f, G) of Theorem 6. ∎

With a pair (f, G) we always associate $H_x(dt) = G_x(E_\Delta, dt)$ and $\eta(x)$ defined by (13). We also complement G by setting

$$G_\Delta(\cdot) = \varepsilon_{(\Delta, 0)}(\cdot) \tag{30}$$

(so if (f, G) are the characteristics of the τ-JMP X, G_x is the law of (X_τ, τ) under P_x for all $x \in E_\Delta$, provided we set $\tau = 0$ on the set $\{\zeta = 0\}$).

Suppose that X is a τ-JMP with characteristics (f, G) and with the jumping sequence (τ_n) of Definition 3 (or of Theorem 5 in the quasi-Hunt case). Set $\sigma_n = \tau_n - \tau_{n-1}$. We have $(X_{\tau_{n+1}}, \sigma_{n+1}) = (X_\tau, \tau) \circ \vartheta_{\tau_n}$, hence the strong Markov property of X immediately yields:

THEOREM 7. – *a) The process (X_{τ_n}, σ_n) is a Markov chain with transition $Q'(x, s, \cdot) = G_x(\cdot)$* (and (X_{τ_n}) is also a Markov chain).

b) The process (X_{τ_n}, τ_n) is a Markov chain with transition $Q(x, s; dy, dt) = G_x(dy, s+dt)$ (it is a so-called "Markov random walk").

Now it is natural to look at the "converse" problem of constructing a JMP with given characteristics (especially in view of the intuitive definition of JMPs as given in the introduction). To this effect, exactly as for step Markov processes, we will use Theorem 7.

So we start with a pair (f, G) where f is a measurable function from $E \times \mathbb{R}_+$ into E and $G_x(dy, dt)$ is a probability kernel from E_Δ into $E_\Delta \times (0, \infty]$. This pair is called *admissible* if it satisfies (14), (27), (29) and (30). As seen before the characteristics of a JMP are admissible, and we presently show the converse: any admissible pair (f, G) are the characteristics of some JMP.

The method of construction closely follows the classical method for constructing a step Markov process when one knows the parameters $a(x)$ of the holding times and the transition kernel Γ: essentially we construct a Markov chain (Y_n, τ_n) with transition $Q'(x, s; dy, dt) = G_x(dy, s + dt)$ and set

$$X_t = \begin{cases} f(Y_n, t - \tau_n) & \text{if } \tau_n \leq t < \tau_{n+1} \quad \text{and} \quad Y_0 \neq \Delta, \\ \Delta & \text{if } t \geq \zeta := \lim_n \tau_n \quad \text{or} \quad Y_0 = \Delta. \end{cases} \tag{31}$$

Let us be more precise. Set $\tilde{E} = E_\Delta \times [0, \infty]$. Call Ω the space of all sequences $(Y_n, \sigma_n)_{n \in \mathbb{N}}$ with values in $\tilde{E}$ and such that $Y_{n+1} = \Delta$ if $Y_n = \Delta$, with the canonical filtration $(\mathcal{G}_n)_{n \in \mathbb{N}}$. Denote by P_x the unique probability measure on Ω for which (Y_n, σ_n) is a Markov chain with transition $Q(x, s; dy, dt) = G_x(dy, dt)$ and starting at $(Y_0, \sigma_0) = (x, 0)$. Then set $\tau_0 = 0$, $\tau_n = \sigma_1 + \ldots + \sigma_n$ if $n \geq 1$, and define X by (31).

THEOREM 8. – *If (f, G) is admissible, X is a τ-JMP with $\tau = \tau_1$.*

All conditions for admissibility are necessary in an obvious way, except perhaps the last property of (29): this property insures that τ_1 is indeed a stopping time relatively to the filtration generated by X.

Proof. – Call ∂ the point of Ω for which $Y_n = \Delta$ and $\sigma_n = 0$ for all $n \geq 0$. In view of (14) the family of maps $\vartheta_t : \Omega \to \Omega$ defined below is a semi-group having $X_{t+s} = X_t \circ \vartheta_s$: we set $\vartheta_t(\omega) = \partial$ if $t \geq \zeta(\omega)$ or if $Y_0(w) = \Delta$; if $\omega = (Y_n, \sigma_n)_{n \in \mathbb{N}}$ and $\tau_p \leq t < \tau_{p+1}$ and $Y_0 \neq \Delta$ then $\vartheta_t(\omega) = (Y'_n, \sigma'_n)_{n \in \mathbb{N}}$ with $Y'_n = Y_{p+n}$, $\sigma'_0 = 0$, $\sigma'_1 = \tau'_{p+1} - t$ and $\sigma'_n = \sigma_{p+n}$ for $n \geq 2$. Let $(\mathcal{F}_t^0)$ be the filtration generated by X, and $(\mathcal{F}_t)$ be the completion of $(\mathcal{F}_t^0)$ w.r.t. all P_x.

If $\tau = \tau_1$ then $\tau = t + \tau \circ \vartheta_t$ on $\{t < \tau\}$ and $\tau_{n+1} = \tau_n + \tau \circ \vartheta_{\tau_n}$. Hence τ is a terminal time relatively to the smallest filtration $(\mathcal{H}_t)$ w.r.t. which all τ_n are stopping times and Y_n is $\mathcal{F}_{\tau_n}$-measurable. Clearly $\mathcal{F}_t^0 \subseteq \mathcal{H}_t$. Conversely if $\tau' = \inf(t : X_t \neq f(X_0, t))$ then $\tau' \geq \tau$ by (31), while (29) implies $\tau' \leq \tau$ P_x-a.s.: hence τ is an $(\mathcal{F}_t)$-stopping time, hence all τ_n's as well and $\mathcal{H}_t \subseteq \mathcal{F}_t$. Further, it is obvious that $(\mathcal{H}_t)$, hence $(\mathcal{F}_t)$ as well, are jumping filtrations with jump times τ_n.

It remains to prove the strong Markov property of X, which is obvious for stopping times $T \geq \zeta$. We start by proving

$$E_x [Z \circ \vartheta_t 1_{\{t<\tau\}}] = E_x [E_{f(x,t)} (Z) 1_{\{t<\tau\}}]. \tag{32}$$

It is enough to prove this for $Z = g(X_\tau, \tau) Z' \circ \vartheta_\tau$, because $\mathcal{F} = \vartheta_\tau^{-1}(\mathcal{F}) \vee \sigma(X_\tau, \tau)$. Since τ is terminal, $Z \circ \vartheta_t = Z$ on $\{ t < \tau \}$, and $\vartheta_t = \vartheta$, where ϑ is the shift of the chain (Y_n, σ_n). Furthermore Z' does not depend on σ_0 (recall (31)), so due to the special form of transition Q, the law of Z' conditional on (Y_0, σ_0) does not depend on σ_0. Therefore the Markov property for the chain (Y_n, σ_n) and (3.8) yield:

$$\begin{aligned} E_x [Z \circ \vartheta_t 1_{\{t<\tau\}}] &= E_x [1_{\{t<\tau\}} g(X_\tau, \tau) E_{X_\tau}(Z')] \\ &= P_x(t<\tau) E_{f(x,t)} [g(X_\tau, \tau) E_{X_\tau}(Z')] \\ &= P_x(t<\tau) E_{f(x,t)}(Z) = E_x [E_{f(x,t)}(Z) 1_{\{t<\tau\}}]. \end{aligned}$$

Since $(\mathcal{F}_t)$ is a jumping filtration, for any stopping time T there are $\mathcal{F}_{\tau_n}$-measurable variables ρ_n with $T = \tau_n + \rho_n$ on $A_n = \{ \tau_n \leq T < \tau_{n+1} \}$, and if $B \in \mathcal{F}_T$ there is $B_n \in \mathcal{F}_{\tau_n}$ with $B \cap A_n = B_n \cap \{ \tau_n \leq T \} \cap \{ \rho_n < \tau \circ \vartheta_{\tau_n} \}$. Thus

$$\begin{aligned} &E_x [1_{B\cap A_n} Z \circ \vartheta_T] \\ &\quad = E_x [1_{\{\tau_n \leq T\}\cap B_n} Z \circ \vartheta_{\tau_n+\rho_n} 1_{\{\rho_n < \tau\circ\vartheta_{\tau_n}\}}] \\ &\quad = \int P_x(d\omega) 1_{\{\tau_n\leq T\}\cap B_n}(\omega) E_{X_{\tau_n}(\omega)} [Z \circ \vartheta_{\rho_n(w)} 1_{\{\rho_n(\omega)<\tau\}}] \end{aligned}$$

(using the strong Markov property at τ_n, which follows from $\vartheta_{\tau_n} = \vartheta^n$). By (32), this is equal to

$$\begin{aligned} &= \int P_x(d\omega) 1_{\{\tau_n\leq T\}\cap B_n}(\omega) E_{X_{\tau_n}} [E_{f(X_{\tau_n}, \rho_n)(\omega)}(Z) 1_{\{\rho_n(\omega)<\tau\}}] \\ &= E_x [1_{\{\tau_n\leq T\}\cap B_n} 1_{\{\rho_n<\tau\circ\vartheta_{\tau_n}\}} E_{f(X_{\tau_n}, \rho_n)}(Z)] = E_x [1_{B\cap A_n} E_{X_T}(Z)] \end{aligned}$$

and the strong Markov property holds for T on A_n. ■

3.4. The semi-group equation

If X is a τ-JMP with characteristics (f, G) and jumping sequence (τ_n) and $\tau_\infty = \lim_n \uparrow \tau_n$, we set

$$X_t^* = \begin{cases} X_t & \text{if } t < \tau_\infty \\ \Delta & \text{if } t \geq \tau_\infty. \end{cases} \tag{33}$$

This process X^* is called the *minimal* process associated with X, and it is again a τ-JMP, with the same (f, G) as X (this is obvious). Denoting by (P_t) and (P_t^*) the submarkovian transition semi-groups of X and X^* (on E), we readily obtain from the strong Markov property at time τ on $\{\tau \le t\}$ that both of them satisfy

$$P_t(x, A) = H_x((t, \infty]) 1_A(f(x, t)) + \int_{E\times[0, t]} G_x(dy, ds) P_{t-s}(y, A). \tag{34}$$

THEOREM 9. – *a) Let (f, G) be an admissible pair. Then equation (34) has solutions that are sub-markovian semi-groups on E, and among these there is a minimal solution (P_t^*) (in the sense that $P_t^*(x, A) \le P_t(x, A)$ for all t, A), which is the semi-group of the process constructed in Theorem 8.*

b) If X is a τ-JMP with characteristics (f, G), then (P_t^*) is the semi-group of the minimal process X^* associated with X by (33).

Proof. – *a)* Let X be the process of Theorem 8, with its semi-group (P_t^*). Set $P_t^{(n)}(x, A) = E_x[1_A(X_t) 1_{\{t<\tau_n\}}]$. Then $P_t^{(0)}(x, A) = H_x((t, \infty]) 1_A(f(x, t))$ and, by the Markov property at τ:

$$P_t^{(n+1)}(x, A) = H_x((t, \infty]) 1_A(f(x, t)) + \int_{E\times[0, t]} G_x(dy, ds) P_{t-s}^{(n)}(y, A).$$

By the monotone class theorem $P_t^{(n)}(x, A) \uparrow P_t^*(x, A)$. Let (P_t) be another semi-group solution to (34). We have $P_t \ge P_t^{(0)}$ and by induction on n we see that $P_t \ge P_t^{(n)}$ for all n, hence $P_t \ge P_t^*$ and the claims are proved.

b) Now let X be any τ-JMP with characteristics (f, G), and X^* the associated minimal process. Then $P_t^{(n)}(x, A) = E_x[1_A(X_t^* 1_{\{t<\tau_n\}}]$ by induction on n, hence (P_t^*) is the semi-group of X^*. ■

3.5. Regularity

In this section we study the regularity of the τ-JMP X, in terms of the characteristics (f, G). For any nonnegative function g on $E \times \mathbb{R}_+$, set

$$\xi_t = \xi(g)_t = \sum_{n\in\mathbb{N}} g(X_{\tau_n}, t + \tau_n) 1_{\{\tau_n<\infty\}}.$$

Consider also the Markov renewal equation associated with the transition Q of Theorem 7 and the function g above:

$$u(t, x) = g(t, x) + \int_{E\times[0, \infty)} G_x(dy, ds) u(y, t+s) \quad \text{for} \quad x \in E, \qquad t < \infty \tag{35}$$

(*i.e.* $u = g + \bar{Q}\,u$, where $\bar{Q}$ is the sub-markovian transition on $E \times \mathbb{R}_+$ given by $\bar{Q}\,(x,\ s;\ dy,\ dt) = Q\,(x,\ s;\ dy,\ dt)\, 1_{E \times [0,\,\infty)}\,(y,\ t)$ and with Q like in Theorem 7).

THEOREM 10. – *a) Equation* (37) *admits nonnegative (possibly infinite) solutions, and among all these there is a minimal solution denoted by* $\hat{u}$.

b) The minimal solution satisfies for $x \in E$*:*

$$\hat{u}\,(x,\ t) = E_x\,(\xi_t). \tag{36}$$

$$\hat{u}\,(x,\ t) < \infty \quad \Rightarrow \quad E_x\,[\hat{u}\,(X_{\tau_n},\ t + \tau_n)] \to 0 \quad \text{as } n \to \infty. \tag{37}$$

Proof. – Define $\hat{u}$ by (36), *i.e.* $\hat{u} = \sum_{n \in \mathbb{N}} \bar{Q}^n\, g$. That $\hat{u}$ satisfies (35) is obvious. Let u be any other nonnegative solution. Then $u = g + \bar{Q}\, g + \cdots + \bar{Q}^n\, g + \bar{Q}^{n+1}\, u$ for all n, hence $u \geq \hat{u}$: this proves (*a*) and that $\hat{u}$ is the minimal solution. For simplicity we write $g\,(x,\ \infty) = 0$.

By Theorem 7, $E_x\,[\hat{u}\,(X_{\tau_n},\ t + \tau_n)] = E_x\,(\xi_t^n)$ where $\xi_t^n = \sum_{p=n}^{\infty} g\,(X_{\tau_p},\ t + \tau_p)$. If $\hat{u}\,(x,\ t) < \infty$, then ξ_t is P_x-integrable and $\xi_t^n \to 0$ as $n \to \infty$ and $\xi_t^n \leq \xi_t$. Then Lebesgue Theorem yields (37). ■

Below we denote by $\hat{u}_t$ the minimal solution of (35) for $g\,(x,\ s) = 1_{\{q \leq t\}}$.

THEOREM 11. – *a) If* $\hat{u}_t\,(x,\ 0) < \infty$ *for all* $x \in E$, $t \in \mathbb{R}_+$, *then* X *is regular and* $\hat{u}_t\,(x,\ s) < \infty$ *for all* $s,\ t \geq 0$.

b) The set $S_0 = \{(x,\ s) \in E \times \mathbb{R}_+ \ :\ \hat{u}_t\,(x,\ s) < \infty,\ \forall\, t \in \mathbb{R}_+\}$ *is an invariant set for the submarkovian transition* $\bar{Q}$ (i.e. $\bar{Q}\,(x,\ s;\ S_0) = 1$ for all $(x,\ s) \in S_0$).

Proof. – *a)* Since $\hat{u}_t\,(x,\ 0) = E_x\,(\sum_{n \in \mathbb{N}} 1_{\{\tau_n \leq t\}})$, the first claim is obvious, and the second one follows from $\hat{u}_t\,(x,\ s) = E_x\,(\sum_{n \in \mathbb{N}} 1_{\{\tau_n \leq (t-s) \vee 0\}}) \leq \hat{u}_t\,(x,\ 0)$.

b) We have $\hat{u}_t\,(x,\ s) = 1_{\{s \leq t\}} + \bar{Q}\,\hat{u}_t\,(s,\ x)$. Hence if $(x,\ s) \in S_0$ we have $\bar{Q}\,(x,\ s;\ A_t) = 0$ where $A_t = \{\,(x,\ s)\ :\ \hat{u}_t\,(x,\ s) < \infty\,\}$. Since $t \to \hat{u}_t\,(x, s)$ is increasing we have $S_0 = \bigcap_{n \in \mathbb{N}} A_n$, and the claim follows. ■

Now we give two necessary and sufficient conditions for regularity. The first one is in terms of the above sub-markovian transition $\bar{Q}$ on $E \times \mathbb{R}_+$, the second one in terms of the submarkovian transition G^λ on E defined by

$$G^\lambda\,(x,\ A) = \int_{A \times [0,\,\infty)} G_x(dy,\ dt)\, e^{-\lambda t} = E_x\,[1_A\,(X_\tau)\, e^{-\lambda \tau}].$$

These conditions are extensions of well-known regularity conditions for step Markov processes (with the same proof).

THEOREM 12. – *The τ-jumping Markov process X is regular iff one of the following two conditions holds* (where $\lambda > 0$ is arbitrary):

a) $\bar{Q}$ *has no non-trivial nonnegative bounded harmonic function* $\bar{u}$ *satisfying* $\sup_{x \in E} \bar{u}(x, s) \to 0$ *as* $s \to \infty$.

b) G^λ *has no non-trivial nonnegative bounded harmonic function.*

Proof. – *a*) Assume first X regular, and let $\bar{u}$ be a bounded nonnegative harmonic function for $\bar{Q}$ with $v(s) := \sup_x \bar{u}(x, s) \to 0$ as $s \to \infty$. We naturally set $\bar{u}(x, \infty) = 0$. Then $\bar{u}(X_{\tau_n}, t + \tau_n)$ is a bounded martingale under P_x, converging to a limit χ_t having $E_x(\chi_t) = \bar{u}(x, t)$. Now $\bar{u}(X_{\tau_n}, t + \tau_n) \leq v(t + \tau_n) \to 0$ P_x-a.s. since $\tau_n \to \infty$ P_x-a.s. It follows that $\chi_t = 0$ P_x-a.s. and thus $\bar{u} = 0$.

Conversely if X is not regular, $\bar{u}_n(x, t) = E_x[\exp - (t + \tau_n)]$ decreases to the nonnegative bounded function $\bar{u}(x, t) = E_x(e^{-t-\zeta})$ and there is at least a value $x \in E$ with $\bar{u}(x, t) > 0$ for all $t \geq 0$. Now $\overline{Q}\,\bar{u}_n = \bar{u}_{n+1}$, and $\overline{Q}\,\bar{u} = \bar{u}$ follows.

b) The strong Markov property yields $E_x[f(X_{\tau_n}) \exp - \lambda \tau_n] = (G^\lambda)^n f(x)$. If f is bounded and X is regular, we deduce $(G^\lambda)^n f(x) \to 0$, so if further f is harmonic we have $f(x) = 0$ for all $x \in E$. Conversely if X is not regular, $f_n(x) = (G^\lambda)^n 1(x)$ decreases to the bounded nonnegative function $f(x) = E_x(e^{-\lambda\zeta})$, and there is least a value $x \in E$ with $f(x) > 0$. Now $f_{n+1} = G^\lambda f_n$, hence $f = G^\lambda f$. ■

As a trivial consequence, we obtain that if $\alpha := \inf_{x \in E} H_x([\delta, \infty]) > 0$ for some $\delta > 0$, then X is regular: indeed in this case $G^\lambda 1(x) \leq \alpha\, e^{-\lambda\delta} < 1$, which obviously implies Property (*b*) of the previous theorem.

4. MARTINGALES, SEMIMARTINGALES, INFINITESIMAL GENERATOR

In this section we study the traditional objects of interest in Markov processes, such as martingales, semimartingales, additive functionals, etc. Apart from the first and very simple subsection, we deal only with quasi-Hunt JMPs: the case of general τ-JMPs is much more technical and presumably not very important for applications.

4.1. Martingales of τ-JMP

Here we suppose that X is a τ-JMP with characteristics (f, G) and jumping sequence (τ_n) and $\tau_\infty = \lim_n \uparrow \tau_n$.

Since $(X_{\tau_\infty + t})$ is again a JMP (by the strong Markov property at τ_∞) with an explosion time τ'_∞ and $(X_{\tau_\infty + \tau'_\infty + t})$ is a JMP with explosion time τ''_∞, etc., by transfinite induction and use of Theorem 2 we have

THEOREM 13. – *Any (local) P_x-martingale of a τ-JMP is a compensated sum of jumps (however, these martingales may have infinite variation on finite intervals $[0, t]$, if $t \geq \tau_\infty$.*

Consider now the integer-valued random measure μ given by (9). Recall that (7) holds by hypothesis, and (8) is easily deduced from (26). In view of (10) and of Theorem 7, its compensator under each measure P_x given by

$$\nu(dt, dy) = \sum_{n \in \mathbb{N}} \frac{G_{X_{\tau_n}}(dy, dt - \tau_n)}{H_{X_{\tau_n}}([t - \tau_n, \infty])} 1_{(\tau_n, \tau_{n+1}]}(t).$$

Then (11) and the fact that a semimartingale is a local martingale plus a process of locally finite variation yield:

THEOREM 14. – *a) If M is a P_x-local martingale on $I = \cup [\![0, \tau_n]\!]$ there exists a predictable function U on $\Omega \times \mathbb{R}_+ \times E$ such that*

$$\int_{(0, t] \times E} |U(s, y)| \, |\mu - \nu|(ds, dy) < \infty \quad P_x\text{-a.s.} \quad \text{on } \{t \in I\}, \tag{38}$$

$$M_t = M_0 + \int_{(0, t] \times E} |U(s, y)(\mu - \nu)(ds, dy) \quad \text{on } \{t \in I\}. \tag{39}$$

Conversely, for any predictable function U satisfying (38) *the formula* (39) *defines a P_x-local martingale on I.*

b) A process is a semimartingale on I iff it is a.s. of finite variation on any compact subset of I.

4.2. Local characteristics of a quasi-Hunt JMP

In all the rest of Section 4 we suppose that X is a quasi-Hunt JMP, with characteristics (f, G). We have the canonical jumping sequence (τ_n), with $\tau = \tau_1$. Recall $H_x(dt) = G_x(E_\Delta \times dt)$ and $\eta(x)$ given by (13), and that (14), (15), (16) and (17) hold.

The measure G used above is not a "good" characteristic, in the sense that is not "local". So we introduce below two other characteristics. One governs the "size of jumps" and is the kernel Γ of (17). The other governs the "rate of jumps" and is defined by

$$\ell_x(t) = -\log H_x((t, \infty]) \qquad (\text{with } \log(0) = -\infty). \tag{40}$$

DEFINITION 5. – The *local characteristics* of the quasi-Hunt JMP X are the triple (f, ℓ, Γ) defined above. ■

This triple is *admissible* in the sense that

a) $(x, t) \to \ell_x(t)$ is a measurable map from $E \times \mathbb{R}_+$ into $[0, \infty]$, with $\ell_x(0) = 0$ and $t \to \ell_x(t)$ continuous increasing;

b) f is measurable from $E \times \mathbb{R}_+$ into E and satisfies (14), with $\eta(x) = \inf(t : \ell_x(t) = \infty)$;

c) ℓ satisfies $\ell_x(t+s) = \ell_x(t) + \ell_{f(x,t)}(s)$ if $s, t \in \mathbb{R}_+$;

d) Γ is a probability kernel on E with $\Gamma(x, \{x\}) = 0$.

((*a*) is obvious by (40), which also implies that η as given by (13) is also $\eta(x) = \inf(t : \ell_x(t) = \infty)$; then (*b*) and (*d*) come from Theorem 5, and (*c*) comes from (16) when $t < \eta(x)$ and is obvious for $t \geq \eta(x)$).

Conversely, suppose that (f, ℓ, Γ) is admissible. Define H_x by $H_x((t, \infty]) = e^{-\ell_x(t)}$ and G by (17) for $x \in E$ and (30) if $x = \Delta$. One readily checks that the pair (f, G) is admissible in the sense of § 3.3 and H_x has no atom on $\mathbb{R}_+$: hence there is a quasi-Hunt JMP having (f, ℓ, Γ) for local characteristics (apply Theorem 8).

Let us write some of the previous results with the help of the local characteristics. First, the semi-group equation (34) becomes

$$\begin{aligned} P_t(x, A) &= e^{-\ell_x(t)} 1_A(f(x, t)) \\ &\quad + \int_0^t e^{-\ell_x(s)} d\ell_x(s) \int_E \Gamma(f(x, s), dy) P_{t-s}(y, A). \end{aligned} \tag{41}$$

Second, the predictable measure ν takes the form

$$\nu(dt, dy) = \sum_{n \in \mathbb{N}} d\ell_{X_{\tau_n}}(t - \tau_n) \Gamma(f(X_{\tau_n}, t - \tau_n), dy) 1_{(\tau_n, \tau_{n+1}]}(t). \tag{42}$$

We even have a simpler expression. Introduce a continuous increasing adapted process L as follows:

$$\left. \begin{aligned} &L_0 = 0, \qquad L_t = L_{\tau_n} + \ell_{X_{\tau_n}}(t - \tau_n) \quad \text{for } \tau_n < t \leq \tau_{n+1}, \\ &L_t = \lim_n \uparrow L_{\tau_n} \qquad \text{if} \quad t \geq \tau_\infty. \end{aligned} \right\} \tag{43}$$

Then we have

$$\nu(dt, dy) = dL_t \Gamma(X_t, dy). \tag{44}$$

Example. – *Step Markov processes.* The quasi-Hunt JMP X is a step Markov process iff its local characteristics (f, ℓ, Γ) have $f(x, t) = x$ and $\ell_x(t) = a(x)t$, where $a(x)$ is the parameter of the holding time at point x. Then (41) is the usual forward Kolmogorov equation, and (44) becomes:

$$\nu(dt, dy) = a(X_t)\Gamma(X_t, dy)\, dt\, 1_{\{t<\tau_\infty\}}.$$

4.3. Additive functionals

By *additive functional* we mean a càdlàg adapted process A such that for all $x \in E$ and all stopping times T:

$$A_{T+s} = A_T + A_s \circ \vartheta_T \quad a.s. \text{ on } \{T < \infty\} \tag{45}$$

(the usual terminology is "strong additive functional"). We say that A is an *additive functional on* $I = \bigcup_n [\![0, \tau_n]\!]$ if it is adapted, càdlàg on I, and (45) holds on $\{T + s \in I\}$. We also set

$$X_{\tau_n}^- = f(X_{\tau_{n-1}}, \tau_n - \tau_{n-1}) \quad \text{on } \{\tau_n < \infty\} \tag{46}$$

(observe that $X_{\tau_n}^- \neq X_{\tau_n}$ a.s. on $\{\tau_n < \infty\}$, and that $\Gamma(\bar{X}_{\tau_n}, \cdot)$ is the law of X_{τ_n}, conditional on $\mathcal{F}_{\tau_n^-}$).

THEOREM 15. – *Assume that X is a quasi-Hunt JMP.*

a) With every additive functional A on I are associated a measurable function $a : E \times \mathbb{R}_+ \to \mathbb{R}$ with

$$\left.\begin{aligned} a(x, 0) &= 0, \\ a(x, t+s) &= a(x, t) + a(f(x, t), s) \qquad \text{if} \quad t + s < \eta(x), \end{aligned}\right\} \tag{47}$$

$$t \to a(x, t) \text{ is càdlàg on } [0, \eta(x)), \tag{48}$$

and a measurable function $\bar{a} : E \times E \to \mathbb{R}$, such that outside a null set A is determined on I by induction, starting with $A_0 = 0$, by

$$\left.\begin{aligned} A_t &= A_{\tau_n} + a(X_{\tau_n}, t - \tau_n) && \text{if} \quad \tau_n \le t < \tau_{n+1}, \\ A_{\tau_{n+1}} &= A_{\tau_{n+1}-} + \bar{a}(X_{\tau_{n+1}}^-, X_{\tau_{n+1}}) && \text{if} \quad \tau_{n+1} < \infty. \end{aligned}\right\} \tag{49}$$

b) Conversely if a and $\bar{a}$ *are as above,* (49) *defines an additive functional A on I.*

Proof. – *a*) By additivity, it suffices to prove (49) for $n = 0$. By (5) there is a measurable function $a : E \times \mathbb{R}_+ \to \mathbb{R}$ such that $A_t = a(X_0, t)$ for all $t < \tau$. Since A is càdlàg, (48) holds, while (47) follows from additivity, and we have the first part of (49) for $n = 0$.

Next, $\mathcal{F}_\tau$ equals $\sigma(X_0, X_\tau, \tau)$ up to null sets, hence there is a measurable function $b : E \times E \times \mathbb{R}_+ \to \mathbb{R}$ such that $\Delta A_\tau = b(X_0, X_\tau, \tau)$ a.s. on $\{\tau < \infty\}$. By additivity we get $\Delta A_\tau = \Delta A_\tau \circ \vartheta_t$ a.s. on $\{\tau > t\}$, hence

$$b(x, y, t) = b(f(x, t), y, s - t) \quad G_x\text{-a.s. in } (y, s) \in E \times (t, \infty) \tag{50}$$

We define a transition kernel from E into $E \times E \times \mathbb{R}_+$ by

$$\bar{G}_x(B) = \int G_x(dy, ds)\, 1_B(y, b(x, y, s), s). \tag{51}$$

It factorizes as $\bar{G}_x(dy, dz, dt) = H_x(dt)\, K(x, t; dy, dz)$, and

$$K(x, t; dy, dz) = \Gamma(f(x, t), dy)\, \varepsilon_{b(x, y, t)}(dz), \tag{52}$$

and by (27) and (50) the kernel $\bar{G}_x$ satisfies (27) as well. Then we can reproduce Step 5) of the proof of Theorem 4 with $\bar{G}_x$ instead of G_x, to obtain a probability kernel $\bar{\Gamma}$ from E into $E \times E$ such that a version of K is $K(x, t; dy, dz) = \bar{\Gamma}(f(x, t); dy, dz)$. Comparing to (52), we obtain a factorization $\bar{\Gamma}(u; dy, dz) = \Gamma(u, dy)\, \varepsilon_{\bar{a}(u, y)}(dz)$ for some measurable function $\bar{a} : E \times E \to \mathbb{R}$. Thus $\bar{G}_x(B) = \int G_x(dy, ds)\, 1_B(y, \bar{a}(f(x, t), y), s)$ and comparing to (51) yields $b(y, b(x, y, s), s) = \bar{a}(f(x, s), y)$ G_x-a.s. in (y, s): so $\Delta A_\tau = \bar{a}(X_\tau^-, X_\tau)$ P_x-a.s. on $\{\tau < \infty\}$ for all $x \in E$, and the second half of (49) holds for $n = 0$.

b) This is obvious. ■

For example, the process L of (43) is an additive functional on I, with $a(x, t) = \ell_x(t)$ and $\bar{a} = 0$.

Remark 4. – We can write the second half of (49) differently: there is another measurable function $\bar{a}'$ on $E \times E$ such that

$$A_{\tau_{n+1}} = A_{\tau_n} + a(X_{\tau_n}, \tau_{n+1} - \tau_n) + \bar{a}'(X_{\tau_{n+1}}^-, X_{\tau_{n+1}}) \tag{53}$$

(same proof as above, upon substituting b with a function b' satisfying $A_\tau = b'(X_0, X_\tau, \tau)$). Now, if A is an additive functional on I which is not càdlàg, we still have the same result, provided we drop (48) and the second half of (49) is replaced by (53). ■

Remark 5. – If X is not quasi-Hunt, the same properties hold except that the second half of (49) reads as

$$A_{\tau_{n+1}} = A_{\tau_{n+1}-} + b(X_{\tau_n}, X_{\tau_{n+1}}, \tau_{n+1} - \tau_n)$$

where b is a measurable function: $E \times E \times \mathbb{R}_+ \to \mathbb{R}$ satisfying (50). ■

If A is continuous, then $\bar{a} = 0$ and we replace (48) by the continuity of $a(x, \cdot)$ on $[0, \eta(x))$. If A in non-decreasing (or equivalently nonnegative), then a is non-decreasing (equivalently nonnegative) and $\bar{a}$ is nonnegative. The following shows that one can approximate functions a satisfying (47) and (48) by functions having (47) and further are continuous in time:

THEOREM 16. – *a) If a satisfies* (47) *and* (48) *on* $\mathbb{R}_+$, *the following functions satisfy* (47) *and converge to a as $h \to 0$:*

$$a_h(x, t) = \frac{1}{h} \int_0^t a(f(s, x), h)\, ds.$$

b) If $g : E \to \mathbb{R}$ is measurable and $t \to g(f(x, t))$ is locally bounded, the function $a(x, t) = \int_0^t g \circ f(x, s)\, ds$ satisfies (47) *and is continuous in t.*

c) If the sequence a_n of functions converges to a and satisfies (47), *and* (48) *uniformly in n, then a satisfies* (47) *and* (48).

Proof. – Observing that $a_h(x, t) = \frac{1}{h} \int_0^t [a(x, s+h) - a(x, s)]\, ds$, the first claim of ($a$) is obvious, and the second claim of (a) follows from

$$a_h(x, t) = \frac{1}{h}\left[\int_t^{t+h} a(x, s)\, ds - \int_0^h a(x, s)\, ds\right]$$

and from the right-continuity of $a(x, \cdot)$. (b) and (c) are obvious. ■

4.4. Martingales and semimartingales

A) Martingales have the structure described in § 4.1. Since now ν is given by (42) or (44), we can be more explicit. Let M be a P_x-local martingale on I. Using first (44), we have a predictable function U on $\Omega \times \mathbb{R}_+ \times E$ such that

$$M_t = M_0 + \sum_{n:\tau_n \leq t} U(\tau_n, X_{\tau_n}) - \int_0^t dL_s \int_E \Gamma(X_s, dy)\, U(s, y)$$

P_x-a.s. on $\{t \in I\}$, and both the sum and the integral above are P_x-a.s. absolutely convergent on $\{t \in I\}$.

Secondly, using (42) and (6), there are $\mathcal{F}_{\tau_n} \otimes \mathcal{R}_+ \otimes \mathcal{E}$-measurable function U_n on $\Omega \times \mathbb{R}_+ \times E$ such that M is defined P_x-a.s. (by induction, starting with M_0) on I by

$$\left.\begin{array}{l} M_t = M_{\tau_n} - \displaystyle\int_0^{t-\tau_n} d\ell_{X_{\tau_n}}(s) \int_E \Gamma(f(X_{\tau_n}, s), dy)\, U_n(s, y) \\ \quad \text{if } \tau_n \leq t < \tau_{n+1} \\ M_{\tau_{n+1}} = M_{\tau_{n+1}-} + U_n(\cdot, \tau_{n+1} - \tau_n, X_{\tau_{n+1}}) \\ \quad \text{if } \tau_{n+1} < \infty \end{array}\right\} \quad (54)$$

with an a.s. absolutely convergent integrals in the first formula above.

DEFINITION 6. – The quasi-Hunt JMP X is called *quasi-Ito* if for each $x \in E$ the predictable brackets of the (locally) square-integrable P_x-martingales are absolutely continuous w.r.t. Lebesgue measure on I. ■

THEOREM 17. – *The quasi-Hunt JMP X is quasi-Ito iff there is a measurable nonnegative functions λ on E such that*

$$\ell_x(t) = \int_0^t \lambda(f(x, s))\, ds. \quad (55)$$

Proof. – Under (55), $\nu(dt, E) \ll dt$ by (42), and X is quasi-Ito by (12). Conversely assume that X is quasi-Ito. The function $U = 1$ satisfies (38) and the martingale M defined by (39) has

$$\langle M, M\rangle_t = \nu((0, t] \times E) = \ell_x(t) \quad P_x\text{-a.s.} \quad \text{on } \{t < \tau\}.$$

Then $d\ell_x(t) \ll dt$, and by the same argument than in Step 5) of the proof of Theorem 4 we get (55). ■

B) Now we study martingales and semimartingales which are additive.

THEOREM 18. – *Assume that X is a quasi-Hunt JMP, and let A be an additive functional on I, and a, $\bar{a}$ associated with it as in Theorem* 15.

a) A is a semimartingale on I iff

$$t \to a(x, t) \text{ is càdlàg with locally finite variation on } [0, \eta(x)). \quad (56)$$

b) A is a local martingale on I iff we have

$$\left.\begin{array}{r} \displaystyle\int_0^t d\ell_x(s) \int \Gamma(f(x, s), dy)|\bar{a}(f(x, s), y)| < \infty \\ \text{if} \quad t < \eta(x), \end{array}\right\} \quad (57)$$

$$a(x, t) = -\int_0^t d\ell_x(s) \int_0^t \Gamma(f(x, s), dy)\, \bar{a}(f(x, s), y). \quad (58)$$

In this case, a version of the function U_n *in* (54) *is*

$$U_n(\omega, t, y) = \bar{a}(f(X_{\tau_n}(\omega), t), y), \tag{59}$$

c) A *is a special semimartingale on* I *iff we have* (56) *and* (57). *In this case the canonical decomposition* $A = B + M$ *of* A *does not depend on the measure* P_x, *and* M *is an additive local martingale on* I, *and* B *is the additive functional on* I *defined by* $B_0 = 0$ *and*

$$B_t = B_{\tau_n} + a'(X_{\tau_n}, t - \tau_n) \quad \text{if} \quad \tau_n \leq t \leq \tau_{n+1}, \tag{60}$$

where

$$a'(x, t) = a(x, t) + \int_0^t d\ell_x(s) \int \Gamma(f(x, s), dy)\, \bar{a}(f(x, s), y). \tag{61}$$

There is a perhaps more pleasant way to explain the canonical decomposition in (*c*) above: in fact if A is a special semimartingale on I, it is also (because of (*a*) above) a.s. a càdlàg process with locally integrable variation on I and the term B in (*c*) is the *compensator of* A. We can decompose A into the sum of two other additive functionals on I:

$$A_t^d = \sum_{n:\tau_n \leq t} \Delta A_{\tau_n}, \qquad A^c = A - A^d.$$

The associated pairs are respectively $a^d = 0$, $\bar{a}^d = \bar{a}$ for A^d, $a^c = a$, $\bar{a}^c = 0$ for A^c. We can apply (*c*) above separately to A^c, and A^d, and use (42), to obtain

COROLLARY 2. – *Under the above assumptions,* A^c *is predictable and the compensator of* A *is*

$$B_t = A_t^c + \int_{[0,\, t] \times E} \bar{a}(X_s, y)\, \nu(ds, dy) \quad \text{for } t \in I.$$

Proof of Theorem 18. – *a*) Necessity is obvious, and sufficiency readily follows from (49).

b) Assume first that A is a local martingale on I. Writing (54) for $M = A$ we get that M is continuous on $I \backslash \cup [\![\tau_n]\!]$, and comparing with (58) gives $U_n(\cdot, \tau_{n+1} - \tau_n, X_{\tau_{n+1}}) = \bar{a}(X_{\tau_{n+1}}^-, X_{\tau_{n+1}})$ on $\{\tau_{n+1} < \infty\}$. The function U_n given by (59) satisfies this and is $\mathcal{F}_{\tau_n} \otimes \mathcal{R}_+ \otimes \mathcal{E}$-measurable, hence (since a compensated sum of jumps is characterized by its jumps) it can be used in (54) and the function U in (39) is $U(\cdot, t, y) = \sum_{n \geq 0} U_n(\cdot, t - \tau_n, y)\, 1_{(\tau_n, \tau_{n+1}]}(t)$. Then (57) follows from (38) written for $t < \tau$, while (58) follows from comparing the first halves of (49) and (54).

For the converse, define U_n by (59) and U as above. (38) follows from (57). Further $M = A$ satisfies (54) (or equivalently (49)) by (58), (59) and the additivity of A. Then A is a local martingale on I.

c) Assume first that A is a special semimartingale on I. That its canonical decomposition $A = B + M$ does not depend on P_x and gives a predictable additive functional B on I and a local martingale M on I follows from [2] (recall that here "additive" means "strongly additive"; the fact that we are restricted to the interval I makes no differences with additive functionals on $\mathbb{R}_+$, because τ is a terminal time).

Call $(a', \bar{a}')$ and $(a'', \bar{a}'')$ the terms associated with B and M in Theorem 15. Clearly $a = a' + a''$ and $\bar{a} = \bar{a}' + \bar{a}''$. Since B is predictable we have $\Delta B_{\tau_n} = 0$, hence $\Delta M_{\tau_n} = \Delta A_{\tau_n}$, on $\{\tau_n < \infty\}$. Therefore $\bar{a}'' = \bar{a}$ and $\bar{a}' = 0$. Applying (*b*), we obtain (57) and also that a'' is given by the right-hand side of (58). Therefore $a' = a - a''$ satisfies (61), and (60) follows from (49) written for B, with a' and $\bar{a}' = 0$.

Conversely, assume (56) and (57). Set $\bar{a}'' = \bar{a}$ and define a'' by the right-hand side of (58): the pair $(a'', \bar{a}'')$ is associated with an additive local martingale M and I by (*b*). If a' is given by (61), (60) defines a predictable process B on I. Since the function a' satisfies also (56) and (47), B is additive. Then $A = B + M$ is a special semimartingale. ■

C) *Lévy measure*. – We cannot speak about the proper Lévy measure of the process X here, because this process is not càdlàg. However, in a sense it is natural to define the Lévy measure (or "Levy system") as the compensator ν of the random measure μ of (9), that is the measure given by (47) or (44).

In particular if h is a nonnegative measurable function on $E \times E$ such that the process $A_t = \sum_{n \geq 1} h(X_{\tau_n}^-, X_{\tau_n})$ is locally integrable on I (recall (46) for $X_{\tau_n}^-$), its compensator is the process

$$B_t = \int_0^t dL_s \int_E \Gamma(X_s, dy)\, h(X_s, y)$$

(this follows e.g. from Corollary 2, upon observing that with the notation of this corollary we have $A^c = 0$ and $\bar{a} = h$).

4.5. Semimartingale functions

We say that a measurable function g on E is a *semimartigale* (resp. *special semimartingale*, resp. *martingale*) *function* of X if $g(X)$ is a P_x-semimartingale (resp. special semimartingale, resp. local martingale) on I, for all $x \in E$.

THEOREM 19. – *Assume that* X *is a quasi-Hunt JMP.*

a) g *is a semimartingale function* iff, *with* $\hat{g}(x, t) = g \circ f(x, t)$,

$$\hat{g}(x, \cdot) \text{ is càdlàg with locally finite variation on } [0, \eta(x)). \tag{62}$$

b) g *is a special semimartingale function* iff *we have* (62) *and*

$$\left.\begin{array}{c}\displaystyle\int_0^t d\ell_x(s) \int_E \Gamma(f(x, s), dy)|g(y) - g \circ f(x, s)| < \infty \\ \text{if} \quad t < \eta(x).\end{array}\right\} \tag{63}$$

In this case the predictable process B *in the canonical decomposition* $g(X) = B + M$ *is given by* (60) *(starting with* $B_0 = 0$*), where*

$$\begin{aligned} a'(x, t) &= g \circ f(x, t) - g(x) \\ &\quad + \int_0^t d\ell_x(s) \int_E \Gamma(f(x, s), dy)\,[g(y) - g \circ f(x, s)]. \end{aligned} \tag{64}$$

c) g *is a martingale function iff we have* (62), (63), *and*

$$g \circ f(x, t) = g(x) + \int_0^t d\ell_x(s) \int_E \Gamma(f(x, s), dy)\,[g(y) - g \circ f(x, s)]$$
$$\text{if} \quad t < \eta(x).$$

Proof. – *a*) $A_t = g(X_t) - g(X_0)$ is always additive, so the first part of (49) holds with $a(x, t) = \hat{g}(x, t) - \hat{g}(x, 0)$. Then (*a*) follows from Theorem 18 (*a*).

b) Assume (62). We also have the second half of (49) with some measurable function $\bar{a}$ on $E \times E$. Then $\bar{a}(X_\tau^-, X_\tau) = g(X_\tau) - \hat{g}(X_0, \tau-)$ a.s., and since the law of X_τ conditional on $\mathcal{F}_{\tau-}$, on $\{\tau < \infty\}$, is $\Gamma(X_\tau^-, \cdot)$, by taking the conditional expectation we get

$$\begin{aligned} &\int \Gamma(X_\tau^-, dz)|\bar{a}(X_\tau^-, z)| \\ &\quad = \int \Gamma(X_\tau^-, dz)|g(z) - \hat{g}(x, \tau^-)| \quad P_x\text{-a.s.} \quad \text{on } \{\tau < \infty\} \end{aligned}$$

and the same without absolute values if the above is finite. This can also be written as

$$\int \Gamma(f(x,s),dz)|\bar{a}(f(x,s),z)|$$
$$= \int \Gamma(f(x,s),dz)|g(z)-\hat{g}(x,s-)|d\ell_x\text{-a.s.} \quad \text{in } s \quad \text{on } [0,\eta(x)),$$

and since ℓ_x is continuous we can replace $\hat{g}(x,s-)$ by $\hat{g}(x,s)=g\circ f(x,s)$ above. Therefore (63) and (57) coincide, hence the necessary and sufficient condition. Further, (64) and (61) also coincide, so the last claim follows from Theorem 18 (*c*).

c) g is a martingale function iff it is a special semimartingale function with $B=0$ in the canonical decomposition $g(X)=B+M$: hence (*b*) ⇒ (*c*). ■

4.6. Infinitesimal generator

Recall that (P_t) is the (sub-markovian) transition semi-group of X on E. We denote by $\mathbb{B}_0$ the set of all bounded measurable functions g on E such that $P_t\,g\to g$ pointwise as $t\to 0$, and by $(\mathcal{A},\mathcal{D}_{\mathcal{A}})$ the (weak) infinitesimal generator of (P_t) considered as a semi-group acting on $\mathbb{B}_0$ (*cf.* Dynkin [6]). We begin with a trivial lemma:

LEMMA 3. – $\mathbb{B}_0$ *is the set of all bounded measurable functions* g *on* E *such that* $g\circ f(x,t)\to g(x)$ *as* $t\to 0$ *for all* $x\in E$, *and in this case* $t\to g\circ f(x,t)$ *is right-continuous on* $[0,\eta(x))$.

Proof. – The first claim comes from the formula

$$P_t\,g(x)=g\circ f(x,t)\,H_x((t,\infty])+E_x(g(X_t)\,1_{\{\tau\le t\}}),$$

and the second claim comes from the first one and from (14). ■

THEOREM 20. – *If* $g\in\mathcal{D}_{\mathcal{A}}$ *we have* (with I_d the identity operator):

$$\begin{aligned}\mathcal{A}\,g(x)&=\lim_{t\to 0}\frac{1}{t}\Big[H_x((t,\infty])\,g\circ f(x,t)-g(x)\\&\quad+\int_0^t H_x(ds)\,(\Gamma g)\circ f(x,s)\Big]\\&=\lim_{t\to 0}\frac{1}{t}\Big[g\circ f(x,t)-g(x)\\&\quad+\int_0^t H_x(ds)\,[(\Gamma-I_d)\,g]\circ f(x,s)\Big].\end{aligned}\tag{65}$$

Proof. – Let $g \in \mathcal{D}_{\mathcal{A}}$. The process

$$M_t^g = g(X_t) - g(X_0) - \int_0^t \mathcal{A}\, g(X_s)\, ds \tag{66}$$

is a bounded càd martingale, so $E_x(M^g_{t\wedge\tau}) = 0$. Further

$$E_x[g(X_{t\wedge\tau}) - g(X_0)] = H_x((t, \infty])\, g\circ f(x, t) - g(x) + \int_0^t H_x(ds)\, (\Gamma g)\circ f(x, s),$$

$$\frac{1}{t} E_x\left[\int_0^{t\wedge\tau} \mathcal{A}\, g(X_s)\, ds\right] = \frac{1}{t}\int_0^t ds\, E_X[\mathcal{A}\, g(X_s)\, 1_{\{s<\tau\}}]$$
$$= \frac{1}{t}\int_0^t ds\, (\mathcal{A}\, g)\circ f(x, s)\, H_x((s, \infty]) \to \mathcal{A}\, g(x)$$

because $\mathcal{A}\, g \in \mathbb{B}_0$, hence $(\mathcal{A}\, g)\circ f(x, s)\, H_x((s, \infty]) \to \mathcal{A}\, g(x)$ as $s \to 0$ by Lemma 3. (65) follows by letting $t \to 0$ in $\frac{1}{t} E_x(M^g_{t\wedge\tau}) = 0$. ■

If $g \in \mathcal{D}_{\mathcal{A}}$ then g is a special semimartingale function (recall (66)), so we can also use Theorem 19: in fact, in the canonical decomposition $g(X) = B + M$ we must have $B_t = \int_0^t \mathcal{A}\, g(X_s)\, ds$. Using (60) and (64) for $t < \tau$, we get

$$\int_0^t (\mathcal{A}\, g)\circ f(x, s)\, ds = g\circ f(x, s) - g(x) + \int_0^t d\ell_x(s)\, [(\Gamma - I_d)\, g]\circ f(x, s) \quad \text{if} \quad t < \eta(x) \tag{67}$$

This is a complicated relation, which relates the measures ds, $d\ell_x(s)$, and $d_s[g\circ f(x, s)]$ (recall that g satisfies (62) here). When X is quasi-Ito we have (55), and (67) simplifies somewhat:

THEOREM 21. – *a) If X is quasi-Ito and if $g \in \mathcal{D}_{\mathcal{A}}$, then*

$$g\circ f(x, t) = g(x) + \int_0^t [\mathcal{A}\, g - \lambda(\Gamma - I_d)\, g]\circ f(x, s)\, ds \qquad \text{if} \quad t < \eta(x).$$

b) If X is quasi-Ito and if $\lambda\circ f(x, t) \to \lambda(x)$ as $t \to 0$ *for all $x \in E$, then for any $g \in \mathcal{D}_{\mathcal{A}}$ such that $\Gamma g \in \mathbb{B}_0$ we have*

$$\mathcal{A}\, g(x) = \lambda(x)\, [\Gamma g(x) - g(x)] + \lim_{t\to 0} \frac{1}{t}\, [g\circ f(x, t) - g(x)].$$

(a) is deduced from either (65) or (67). This implies in particular that if $g \in \mathcal{D}_A$, then $t \to g \circ f(x, t)$ is differentiable from the right at $t = 0$.

Exemple. – Step Markov process. – If X is a step Markov process, $f(x, t) = x$ and $\lambda(x) = a(x)$. So $\mathcal{A} g(x) = a(x)[\Gamma g(x) - g(x)]$ if $g \in \mathcal{D}_{\mathcal{A}}$ and $\Gamma g \in \mathbb{B}_0$. ■

We cannot go much further in the examination of the infinitesimal generator (which, for instance, is not determined by the characteristics if the process is not regular). Note that all previous results (Theorems 20 and 21 and (67)) still holds for *extended generator* (again introduced by Dynkin [6]): this is the linear operator $(\mathcal{A}', \mathcal{D}_{\mathcal{A}})$, where $\mathcal{D}_{\mathcal{A}'}$ is the set of bounded measurable functions g on E such that there exists a bounded measurable function $\mathcal{A}' g$ such that (compare to (66)):

$$M_t^{\prime g} = g(X_t) - g(X_0) - \int_0^t \mathcal{A}' g(X_s)\, ds$$

is a martingale on $I = \cup [\![0, \tau_n]\!]$ (the restriction to I is *ad hoc* for JMP).

In fact, considering the additive functional L on I defined by (43), which plays a basic rôle here, we can also consider the *L-extended generator* (introduced by Kunita [11]): this is the linear operator $(\mathcal{A}'', \mathcal{D}_{\mathcal{A}''})$, where $\mathcal{D}_{\mathcal{A}''}$ is the set of bounded measurable functions g on E such that there exists a bounded measurable function $\mathcal{A}'' g$ such that

$$M_t^{\prime\prime g} = g(X_t) - g(X_0) - \int_0^t \mathcal{A}'' g(X_s)\, dL_s$$

is a martingale on I. Here again we have $\mathcal{D}_{\mathcal{A}''} \subseteq \mathbb{B}_0$. But now, without any condition on X (except being a quasi-Hunt JMP), we have the following, which easily follows from Theorem 19:

THEOREM 22. – *Assume that X is a quasi-Hunt JMP. Then a bounded measurable function g on E belongs to $\mathcal{D}_{\mathcal{A}''}$ iff there is a bounded measurable function $K g$ such that*

$$g \circ f(x, t) = g(x) + \int_0^t (K g) \circ f(x, s)\, d\ell_x(s) \qquad \text{if} \quad t < \eta(x),$$

and in this case $\mathcal{A}'' g(x) = K g(x) + \Gamma g(x) - g(x)$.

5. PURELY DISCONTINUOUS MARKOV PROCESSES

In this section we introduce a sort of generalization of JMP's, which we call "purely discontinuous Markov processes". This supposes that our

Markov process has càdlàg paths, and so we assume in this section that E is a Polish space with a metric ρ and the Borel σ-field $\mathcal{E}$. In a sense, those processes are the "natural" limits of JMP's.

Roughly speaking, a purely discontinuous Markov process X is as follows: for $0 < \varepsilon < \delta$ we throw away all jumps of X of size between ε and δ; letting $\varepsilon \to 0$ we should obtain a limit in law (for J1 topology) Y^δ which of course has only jumps of size bigger than δ, and which should be a *JMP*; then letting in turn $\delta \to 0$ we should have weak convergence of Y^δ to the original process X. When for example X is a diffusion with jumps, the fact that Y^δ is a JMP implies that the diffusion term should not actually paper.

Since we want to remain within the Markov setting, throwing away jumps needs some work, which is done in § 5.1. Because of technical questions related to finite lifetime and non-regularity for JMP's, it is unfortunately not true that all JMP's are purely discontinuous, but all "decent" ones are. So what follows is a first attempt to some theory which is not well rounded up yet.

5.1. Transformations of a process

The first two transformations below (continuation and killing) are well known in much greater generality than what we quickly recall here, for the convenience of the reader. We start with a normal strong Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \vartheta_t, X_t, P_x)$ with lifetime ζ and càdlàg paths $t \to X_t$ on $[0, \zeta)$, so (7) is met. It is not a restriction to assume also that there is a unique point ∂ in Ω such that $\zeta(\partial) = 0$, so that $\vartheta_t(\omega) = \partial$ if $t \geq \zeta(\omega)$.

A) *Continuation.* – Consider a transition probability N from $(\Omega, \mathcal{F})$ into $(E_\Delta, \mathcal{E}_\Delta)$ such that

$$\left.\begin{array}{lll} N(\omega, \cdot) = \varepsilon_\Delta(\cdot) & \text{if} & \zeta(\omega) = 0 \quad \text{or} \quad \zeta(\omega) = \infty \\ N(\vartheta_t \omega, \cdot) = N(\omega, \cdot) & \text{if} & t < \zeta(\omega) \end{array}\right\}$$

According to Meyer [13] we can construct a new normal strong Markov process $\tilde{X} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{X}_t, \tilde{\vartheta}_t, \tilde{P}_x)$, called the *minimal continuation of X associated with the kernel N*, as follows: set $\tilde{\Omega} = \Omega^{\mathbb{N}^*}$, $\tilde{\mathcal{F}} = \mathcal{F}^{\otimes \mathbb{N}^*}$. We denote by $\mathcal{A}_n$ the class of cylindrical sets in $\tilde{\mathcal{F}}$ of the form $A = \prod_{i \in \mathbb{N}^*} A_i$ with $A_i \in \mathcal{F}$ and $A_i = \Omega$ for all $i \geq n+1$. For $x \in E_\Delta$ we define $\tilde{P}_x$ on $\bigcup \mathcal{A}_n$ by induction on n, starting with $\tilde{P}_x(A) = P_x(A_1)$ if $A = \prod A_i \in \mathcal{A}_1$

and setting for $A = \prod A_i \in \mathcal{A}_n$ if we know $\tilde{P}_x$ on $\mathcal{A}_{n-1}$:

$$\tilde{P}_x(A) = \int P_x(d\omega)\, 1_{A_1}(\omega) \int N(\omega, dy)\, \tilde{P}_y(A')$$

where $A' = \prod_{i \geq 1} A_{i+1}$. As is well known, this extends uniquely into a probability kernel $\tilde{P}_x$ from $(E_\Delta, \mathcal{E}_\Delta)$ into $(\tilde{\Omega}, \tilde{\mathcal{F}})$. It remains to define the process $\tilde{X}$ and the shifts $\tilde{\vartheta}_t$ by setting for $\tilde{\omega} = (\omega_n)_{n\geq 1}$:

$$\zeta_0(\tilde{\omega}) = 0, \qquad \zeta_n(\tilde{\omega}) = \zeta(\omega_1) + \cdots + \zeta(\omega_n) \qquad \text{if} \quad n \geq 1,$$

$$\tilde{X}_t(\tilde{\omega}) = \begin{cases} X_{t-\zeta_n(\tilde{\omega})}(\omega_{n+1}) & \text{if} \quad \zeta_n(\tilde{\omega}) \leq t < \zeta_{n+1}(\tilde{\omega}), \\ \Delta & \text{if} \quad t \geq \tilde{\zeta}(\tilde{\omega}) := \lim_n \uparrow \zeta_n(\tilde{\omega}), \end{cases}$$

$$\tilde{\vartheta}_t(\tilde{\omega}) = \begin{cases} (\vartheta_{t-\tilde{\zeta}_n(\tilde{\omega})}(\omega_{n+1}), \omega_{n+2}, \ldots) & \text{if} \quad \zeta_n(\tilde{\omega}) \leq t < \zeta_{n+1}(\tilde{\omega}), \\ (\partial, \partial, \ldots) & \text{if} \quad t \geq \tilde{\zeta}(\tilde{\omega}). \end{cases}$$

Finally we denote by $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ the usual Markov completion of the filtration generated by $\tilde{X}$, as $(\mathcal{F}_t)_{t\geq 0}$ is associated with X. An important feature of $\tilde{X}$ is that it is defined on an extension of the original space Ω, and that, with some obvious abuse of notation:

$$\left.\begin{array}{lll} \tilde{X}_t = X_t & \text{if} & t < \zeta, \\ \tilde{\mathbb{P}}_x(A) = P_x(A) & \text{if} & A \in \mathcal{F}_t \cap \{t < \zeta\}. \end{array}\right\} \tag{68}$$

A particular case is when

$$\left.\begin{array}{l} \Omega_\zeta = \{0 < \zeta < \infty,\ X_{\zeta-} \text{ exists}\} \\ N(\omega, dx) = 1_{\Omega_\zeta}(\omega)\, \Phi(X_{\zeta-}(\omega), dx) + 1_{\Omega\setminus\Omega_\zeta}(\omega)\, \varepsilon_\Delta(dx) \end{array}\right\} \tag{69}$$

where Φ is some probability kernel on E. A continuation of X associated with N as above will be simply called *continuation associated with* Φ.

B) *Killing*. – Now we consider a terminal time σ of X, having $\sigma \leq \zeta$ and $P_x(\sigma > 0) = 1$ for all $x \in E$. Set

$$(\vartheta_t^*(\omega), X_t^*(\omega)) = \begin{cases} (\vartheta_t(\omega), X_t(\omega)) & \text{if} \quad t < \sigma(\omega) \\ (\partial, \Delta) & \text{if} \quad t \geq \sigma(\omega). \end{cases}$$

It is well known that the process $X^* = (\Omega, \mathcal{F}, \mathcal{F}_t, \vartheta_t^*, X_t^*, P_x)$ is a normal strong Markov process with lifetime $\zeta^* = \sigma$, called the *process killed at time* σ.

C) *Throwing out jumps.* – Let C be a measurable subset of $E \times E$ such that $\sigma^C = \zeta \wedge \inf(t > 0 : (X_{t-}, X_t) \in C)$ (a terminal time) has $P_x(\sigma^C > 0) = 1$ for all $x \in E$.

DEFINITION 7. – The *process obtained from* X *by throwing out jumps inside* C is the minimal continuation X^C associated with the kernel $\Phi(x, dy) = \varepsilon_x(dy)$ (remember (69)) of the process X^{*^C} which is X killed at σ^C. ■

Now we look at what happens when our càdlàg Markov process is also a τ-JMP. The function $f(x, \cdot)$ of Theorem 6 is obviously càdlàg on $[0, \eta(x))$. In the Hunt case, we have the following:

a) If X is Hunt and is a quasi-Hunt JMP, the function $f(x, \cdot)$ is continuous on $[0, \eta(x))$ (otherwise there would be predictable jumps). If X is Hunt and is a *regular* τ-JMP, it also a quasi-Hunt JMP (we do not know whether this is true in the non-regular case).

b) If X is a *regular* quasi-Hunt JMP with $f(x, \cdot)$ continuous on $[0, \eta(x))$ it is a Hunt process (if it is not regular, it may have a predictable jump or no left-hand limit at the first explosion time).

THEOREM 23. – *Assume that* X *is a* τ*-JMP with characteristics* (f, G), *having the following:*

a) $f(x, \cdot)$ *is continuous on* $[0, \eta(x))$.

b) $P_x(\tau = \eta(x)) = 0$ *if* $\eta(x) < \infty$.

Let C *be a measurable subset of* $E \times E$ *which does not intersect the diagonal. Then* X^C *is a JMP with characteristics* (f, G^C), *where* G^C *satisfies*

$$\begin{aligned} G_x^C(A \times [0, t]) = & \int_{(A \cap E) \times [0, t]} G_x(dy, du)\, 1_{C^c}(f(x, u), y) \\ & + \int_{E_\Delta \times [0, t]} G_x(dy, du)\, [1_C(f(x, u), y) \\ & + 1_{\{\Delta\}}(y)]\, G_{f(x, u)}^C(A \times [0, t - u]) \end{aligned} \tag{70}$$

$$\left.\begin{aligned} & \eta^C(x) := \inf(t : G_x^C(E_\Delta \times [0, t]) = 1) = \eta(x) \\ & \text{and} \\ & G_x^C(E \times \{\eta(x)\}) = 0. \end{aligned}\right\} \tag{71}$$

Further, for each $A \in \mathcal{E}_\Delta$, $(x, t) \to G_x^C(A \times [0, t])$ *is the unique bounded measurable solution of Equation* (70) *on* $\{(x, t) : x \in E, 0 \leq t < \eta(x)\}$.

In particular if X is a quasi-Hunt JMP, (*a*) and (*b*) are satisfied. Further $G_x(E_\Delta, \cdot)$ is diffuse, so $G_x^C(E_\Delta, \cdot)$ also by (70): therefore X^C is again a quasi-Hunt JMP.

Proof. – Since X is continuous on $\overset{\circ}{I} \backslash (\cup [\![\tau_n]\!])$, obviously $X' := X^{*^C}$ is a τ-JMP with the same function f and the same time τ than X.

For the construction of the process $\tilde{X} = X^C$, use the notation of § A. Let τ^C be the first jump time of X^C, and ζ' be the lifetime of X', and assume that we start at x. Three cases are possible:

$\tau < \zeta'$: then $(f(x, \tau), X_\tau) \notin C$, $X_\tau \in E$, $\tau^C = \tau$, $X^C_{\tau^C} = X_\tau$;

$\tau = \zeta' < \infty$: then $(f(x, \tau), X_\tau) \in C$, $\tau^C = \tau + \tau^C \circ \vartheta_\tau$, $X^C_\tau = f(x, \tau)$;

$\tau = \zeta' = \infty$: then $X_\tau = \Delta$, $\tau^C = \infty$.

If G^C_x denotes the law of $(X^C_{\tau^C}, \tau^C)$ (starting at x) then (70) easily follows from the Markov property for X^C at time τ.

Since $\tau^C \geq \tau$ we have $\eta^C(x) \geq \eta(x)$, so (71) holds when $\eta(x) = \infty$. Assume now $\eta(x) < \infty$. Set $\sigma_0 = 0$, $\sigma_n(\tilde{\omega}) = \zeta'(\omega_1) + \cdots + \zeta'(\omega_n)$ (notation of § A). Suppose that

$$\text{if } \tau^C > \sigma_n \text{ then } \sigma_n < \eta(x) \text{ and } X^C_t = f(x, t) \text{ for all } t \leq \sigma_n \tag{72}$$

P^C_n-a.s., for some n. If $\tau^C > \sigma_{n+1}$ we have $\zeta'(\omega_{n+1}) = \tau(\omega_{n+1}) < \infty$ and $X^C_t = f(x, t)$ for $t \subseteq \sigma_{n+1}$ P^C_x-a.s. (use (14)); (*b*) implies $\tau(\omega_{n+1}) < \eta \circ f(x, \sigma_n)$ P^C_x-a.s. and since $\eta(x) = \sigma_n = \eta \circ f(x, \sigma_n)$ if $\eta(x) > \sigma_n$ by (27), it follows that (72) holds for $n + 1$. Since (72) is obvious for $n = 0$, it holds for all n. Thus if $\sigma_n < \tau^C \leq \sigma_{n+1}$ we have $\tau^C = \sigma_n + \tau(\omega_{n+1}) < \eta(x)$. Further $\tau^C \leq \sigma_\infty := \lim_n \sigma_n$ (= the lifetime of X^C). It follows from all this that $\tau^C \leq \eta(x)$ and that $\tau^C = \sigma_\infty$ if $\tau^C = \eta(x)$, P^C_x-a.s.: this gives $\eta^C(x) \leq \eta(x)$, hence $\eta^C(x) = \eta(x)$, and also the last part of (71): hence (71) is always satisfied.

It remains to prove the uniqueness. Fix $A \in \mathcal{E}_\Delta$, $x \in E$ and $T < \eta(x)$. Call $g(s, t)$ the first integral in (70) with $f(x, s)$ instead of x, and

$$K(s, du) = \int_{E_\Delta} G_{f(s, x)}(dy, du)\,[1_C(f(x, s+u), y) + 1_{\{\Delta\}}(y)].$$

Let $h(s, t) = G^C_{f(x, s)}(A \times [0, t])$. If $S_T = \{(s, t) : s \geq 0, t \geq 0, s+t \leq T\}$, in view of (14), (70) with $f(x, s)$ instead of x gives

$$h(s, t) = g(s, t) + \int_0^t K(s, du)\, h(s+u, t-u) \qquad \text{if} \quad (s, t) \in S_T. \tag{73}$$

(27) yields $K(s, [0, T - s]) = \dfrac{K(0, (s, T])}{G_x((s, \infty])} \leq \dfrac{G_x((s, T])}{G_x((s, \infty])} \leq G_x([0, T]) < 1$ by definition of T. Hence if we consider g, h as elements of the set $\mathcal{B}$ of all bounded Borel functions on S_T, endowed with the

uniform norm, we can write (73) as $h = g + K h$ where K is an operator on $\mathcal{B}$ with norm strictly smaller than 1. Then (73) has a unique bounded measurable solution on S_T and since T is arbitrary in $[0, \eta(x))$ this gives the last claim. ■

5.2. Purely discontinuous Markov processes

In all this subsection, $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \vartheta_t, X_t, P_x)$ is a càdlàg normal strong Markov process taking values in the Polish space E, with lifetime ζ.

DEFINITION 8. – X is called *deterministic* if there is a measurable function $f : E \times \mathbb{R}_+ \to E$ such that $P_x(X_t = f(x, t) \forall t < \zeta) = 1$ for all $x \in E$. ■

Note that a deterministic process X in the above sense is not properly speaking "deterministic", since the lifetime ζ is random. It is a ζ-JMP by Corollary 1, and one may choose $f(x, \cdot)$ to be càdlàg. Further X is a Hunt process iff $f(x, \cdot)$ is continuous and ζ is totally inacessible.

Below we shall consider weak convergence for Markov processes, which is defined as follows. Denote by $\mathbb{D}([0, t])$ the space of all càdlàg function from $[0, t]$ into E, endowed with the J1 topology. Let $X^n = (\Omega^n, \mathcal{F}^n, \mathcal{F}^n_t, \vartheta^n_t, X^n_t, P^n_x)$ be a sequence of E-valued normal strong Markov càdlàg processes, with lifetimes ζ^n. We say that X^n *weakly converges to* X, and write $X^n \xrightarrow{w} W$, if for all $t > 0$, $x \in E$, and all bounded continuous function φ on $\mathbb{D}([0, t])$,

$$E^n_x[\varphi(X^n) 1_{\{\zeta^n > t\}}] \to E_x[\varphi(X) 1_{\{\zeta > t\}}] \tag{74}$$

where of course $\varphi(X^n)$ is the function φ evaluated for the restriction of X^n to the interval $[0, t]$.

With the notation of Definition 7, we write $X^{\varepsilon, \delta} = X^C$ when $C = \{(x, y) : \varepsilon < \rho(x, y) \leq \delta\}$, and $X^\varepsilon = X^{\varepsilon, \infty}$.

DEFINITION 9. – *a*) The process X is called *purely discontinuous* if for every $\delta \in (0, \infty)$ there is a quasi-Hunt JMP Y^δ such that

$$X^{\varepsilon, \delta} \xrightarrow{w} Y^\delta \quad \text{as } \varepsilon \to 0; \tag{75}$$

$$Y^\delta \xrightarrow{w} X \quad \text{as } \delta \to 0. \tag{76}$$

b) The process X is called *weakly purely discontinuous* if there exists a *deterministic* process Y such that $X^\varepsilon \xrightarrow{w} Y$ as $\varepsilon \to 0$. ■

We provide first some general results for these processes.

THEOREM 24. – *A purely discontinuous process is weakly purely discontinuous.*

Proof. – By Theorem 5 a quasi-Hunt JMP which is continuous up to its lifetime is deterministic in the sense of Definition 8. Hence it is enough to prove that the process Y^∞ of (75) with $\delta = \infty$ is continuous. Now, the jumps of $X^\varepsilon = X^{\varepsilon,\infty}$ are smaller than ε (*i.e.* $\rho(X^\varepsilon_{t-}, X^\varepsilon_t) \le \varepsilon$), hence by (75) the jumps of Y^∞ are also smaller than ε on $(0, \zeta^\infty)$, where ζ^∞ is the lifetime of Y^∞. Since $\varepsilon > 0$ is arbitrary, Y^∞ is continuous on $[0, \zeta^\infty)$, hence the result. ■

THEOREM 25. – *If X is a τ-JMP such that $\eta(x) = \infty$ and that $f(x, \cdot)$ is continuous for all $x \in E$, then it is weakly purely discontinuous.*

Proof. – By Theorem 23 we know that X^ε is a τ^ε-JMP for some stopping time τ^ε. Denote by H^ε_x the law of τ^ε when $X^\varepsilon_0 = x$, and $h^\varepsilon_x(t) = H^\varepsilon_x([0,t])$, and $\alpha^\varepsilon_x(t) = \int G_x(dy, du)\, 1_{\{u\le t,\, \rho(f(x,u), y)\le\varepsilon\}}$ (notation of Theorem 6 for X), and $\beta_x(t) = P_x(\tau \le t)$. Then (70) yields

$$h^\varepsilon_t(x) \le \alpha^\varepsilon_x(t) + \beta_x(t) \sup_{0\le u\le t} h^\varepsilon_{f(x,u)}(t-u). \tag{77}$$

(27) yields $1-h^\varepsilon_x(t) = [1-h^\varepsilon_x(u)]\,[1-h^\varepsilon_{f(x,u)}(t-u)] \le 1-h^\varepsilon_{f(x,u)}(t-u)$, hence (77) implies $h^\varepsilon_x(t) \le \alpha^\varepsilon_x(t) + \beta_x(t)\, h^\varepsilon_x(t)$, hence $h^\varepsilon_x(t) \le \alpha^\varepsilon_x(t)/[1-\beta_x(t)]$. If we fix $t < \infty$, we have $\beta_x(t) < 1$ because $\eta(x) = \infty$, and $\alpha^\varepsilon_x(t) \to 0$ as $\varepsilon \to 0$ because of (29), thus $h^\varepsilon_x(t) \to 0$ as $\varepsilon \to 0$. This readily implies that $X^\varepsilon \xrightarrow{w} Y$ where Y is the (obviously deterministic) process having $P_x(Y_t = f(x, t)\, \forall t \ge 0) = 1$. ■

THEOREM 26. – *If X is a quasi-Hunt JMP such that X and each $X^{0,\delta}$ are regular, then X is purely discontinuous* (we have already mentionned after Theorem 23 that $X^{0,\delta}$ is a quasi-Hunt JMP).

Proof. – *a*) Fix $\delta \in (0, \infty]$. In a first step, we construct the continuation $X^{\varepsilon,\delta}$ (for $\varepsilon \in [0, \delta)$) of the process X killed at the first time there is a jump of size in $(\varepsilon, \delta]$, based on X itself and not on the killed process. This is a trivial modification of the procedure described in § A, which goes as follows.

We start with the regular quasi-Hunt JMP $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \vartheta_t, P_x)$. Set $\sigma^\varepsilon = \inf(t : \varepsilon < \rho(X_{t-} X_t) \le \delta)$, which is the lifetime of the killed process (since X is regular, $P_x(\zeta = \infty) = 1$ if $x \in E$ and X is càdlàg on $\mathbb{R}_+$). By convention $X_{\infty-} = \Delta$. Let $\tilde\Omega$, $\tilde{\mathcal{F}}$, $\mathcal{A}_n$ be as in § A. If $A = \prod A_i \in \mathcal{A}_n$ we set

$$P^{\varepsilon,\delta}_x(A) = \int P_x(d\omega_1)\, P_{X_{\sigma^\varepsilon-}(\omega_1)}(d\omega_2)\ldots \times P_{X_{\sigma^\varepsilon-}(\omega_{n-1})}(d\omega_n) \prod_{1\le i\le n} 1_{A_i}(\omega_i), \tag{78}$$

which extends uniquely into a probability kernel $P_x^{\varepsilon,\delta}$ from $(E_\Delta, \mathcal{E}_\Delta)$ into $(\tilde\Omega, \tilde{\mathcal{F}})$. The process $X^{\varepsilon,\delta}$ and shifts $\vartheta_t^{\varepsilon,\delta}$ are defined by setting for $\tilde\omega = (\omega_n)_{n\geq 1}$:

$$\zeta_0^\varepsilon(\tilde\omega) = 0, \qquad \zeta_n^\varepsilon(\tilde\omega) = \sigma^\varepsilon(\omega_1) + \cdots + \sigma^\varepsilon(\omega_n) \qquad \text{if} \quad n \geq 1,$$

$$X_t^{\varepsilon,\delta}(\tilde\omega) = \begin{cases} X_{t-\zeta_n^\varepsilon(\tilde\omega)}(\omega_{n+1}) & \text{if } \zeta_n^\varepsilon(\tilde\omega) \leq t < \zeta_{n+1}^\varepsilon(\tilde\omega), \\ \Delta & \text{if } t \geq \zeta^\varepsilon(\tilde\omega) := 1 \lim_n \uparrow \zeta_n^\varepsilon(\tilde\omega), \end{cases}$$

$$\tilde\vartheta_t^{\varepsilon,\delta}(\tilde\omega) = \begin{cases} (\vartheta_{t-\zeta_n^\varepsilon(\tilde\omega)}(\omega_{n+1}), \omega_{n+2}, \ldots) & \text{if } \zeta_n^\varepsilon(\tilde\omega) \leq t < \zeta_{n+1}(\tilde\omega), \\ (\partial, \partial, \ldots) & \text{if } t \geq \zeta^\varepsilon(\tilde\omega). \end{cases}$$

Finally we denote by $(\mathcal{F}_t^{\varepsilon,\delta})_{t\geq 0}$ the usual Markov completion of the filtration generated by $X^{\varepsilon,\delta}$. Then $X^{\varepsilon,\delta} = (\tilde\Omega, \tilde{\mathcal{F}}, \mathcal{F}_t^{\varepsilon,\delta}, X_t^{\varepsilon,\delta}, \vartheta_t^{\varepsilon,\delta}, P_x^{\varepsilon,\delta})$ is clearly a version of the wanted process.

b) Next we compare $X^{\varepsilon,\delta}$ and $X^{0,\delta}$. Set $C(n, \varepsilon) = \{\tilde\omega = (\omega_p)_{p\geq 1} : \sigma^\varepsilon(\omega_i) = \sigma^0(\omega_i) \text{ for } 1 \leq i \leq n\}$, and $D(n, t, \varepsilon) = C(n, \varepsilon) \cap \{\zeta_n^0 > t\}$. From the construction above,

$$\zeta_i^\varepsilon = \zeta_i^0, \qquad \forall\, i \leq n \quad \text{and} \quad X_s^{\varepsilon,\delta} = X_s^{0,\delta} \qquad \forall\, s \leq t,$$
$$\text{on } D(n, t, \varepsilon),$$
$$P_x^{\varepsilon,\delta}(A \cap C(n, \varepsilon)) = P_x^{0,\delta}(A \cap C(n, \varepsilon)) \qquad \text{if} \quad A \in \mathcal{A}_n.$$

Therefore if φ is a bounded function on $\mathbb{D}([0, t])$, we have

$$E_x^{\varepsilon,\delta}[\varphi(X^{\varepsilon,\delta})\, 1_{D(n,t,\varepsilon)}] = E_x^{0,\delta}[\varphi(X^{0,\delta})\, 1_{D(n,t,\varepsilon)}]. \tag{79}$$

Furthermore, for $\vartheta = 0$ and $\vartheta = \varepsilon$, we have $D(n, t, \varepsilon) \subseteq \{\zeta^\vartheta > t\}$, hence

$$\begin{aligned} &|E_x^{\vartheta,\delta}[\varphi(X^{\vartheta,\delta})\, 1_{\{\zeta^\vartheta > t\}}] - E_x^{\vartheta,\delta}[\varphi(X^{\vartheta,\delta})\, 1_{D(n,t,\varepsilon)}]| \\ &\quad \leq \|\varphi\| P_x^{\vartheta,\delta}(D(n, t, \varepsilon)^c) \end{aligned} \tag{80}$$

c) We have $\sigma^0 \leq \sigma^\varepsilon$, and $P_x(\sigma^0 < \sigma^\varepsilon) \to 0$ as $\varepsilon \to 0$. Then (78) yields $P_x^{0,\delta}(C(i, \varepsilon) \backslash C(i+1, \varepsilon)) \to 0$ as $\varepsilon \to 0$ for all i, hence $P_x^{0,\delta}(C(n, \varepsilon)) \to 1$. Further $P_x^{0,\delta}(\zeta^0 = \infty) = 1$ if $x \in E$ since $X^{0,\delta}$ is regular by hypothesis. Hence

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} P_x^{0,\delta}(D(n, t, \varepsilon)) = 1.$$

Finally $P_x^{\varepsilon,\delta}(D(n,t,\varepsilon)) = P_x^{0,\delta}(D(n,t,\varepsilon))$ by (79), so (79) and (80) yield $E_x^{\varepsilon,\delta}[\varphi(X^{e,\delta})\, 1_{\{\zeta^\varepsilon > t\}}] \to E_x^{0,\delta}[\varphi(X^{0,\delta})\, 1_{\{\zeta^0 > t\}}]$ as $\varepsilon \to 0$, for every bounded function φ on $\mathbb{D}([0, t])$: that is, we have (75) with $Y^\delta = X^{0,\delta}$.

d) It remains to prove (76). Set $T_\delta = \inf(t : 0 < \rho(X_{t-}, X_t) < \delta)$. In view of (68) we have $X_t^{0,\delta} = X_t$ for all $t < T_\delta$. Since X is regular, $T_\delta \to \infty$ P_x-a.s. for all $x \in E$, so (76) is obvious. ■

Now we consider various examples of quasi-Hunt JMP which are or are not purely discontinuous. The first one is a step Markov process which is not purely discontinuous (note that by Theorem 25, every step Markov process is weakly purely discontinuous).

Example 1. – We consider the countable state space $E = \{1/i : i \in \mathbb{N}^*\} \cup \{0\}$ endowed with the metric induced by the usual metric on $\mathbb{R}$. Take for X the minimal step Markov process with jump transition matrix $\pi = (\pi_{x,y} : x, y \in E)$ and parameters of holding times $(a_x : x \in E)$ given by $\pi_{1/i, 1/(i+1)} = 1$ and $a_{1/i} = i$ if $i \in \mathbb{N}^*$, and $a_0 = 1$, $\pi_{0,1} = 1$. The lifetime is $\zeta = \lim_{\varepsilon \to 0} \inf(t : 0 < X_t < \varepsilon)$.

Let $0 < \varepsilon < \delta < 1/2$, let n and m be the integers such that $\frac{1}{n} - \frac{1}{n+1} \leq \varepsilon < \frac{1}{n-1} - \frac{1}{n}$ and $\frac{1}{m} - \frac{1}{m+1} \leq \delta < \frac{1}{m-1} - \frac{1}{m}$. Then $X^{\varepsilon,\delta}$ is the conservative minimal step Markov process with jump transition matrix π, and $a_{1/i}^{\varepsilon,\delta} = i$ for $i \leq m-1$ or $i \geq n$, and $a_{1/i}^{\varepsilon,\delta} = 0$ if $m \leq i \leq n-1$ and $a_0^{\varepsilon,\delta} = 1$. Clearly $X^{\varepsilon,\delta} \xrightarrow{w} X^{0,\delta}$ as $\varepsilon \to 0$, where $X^{0,\delta}$ is the conservative minimal step Markov process with the same π and $a_{1/i}^{0,\delta} = i$ for $i \leq m-1$ and $a_{1/i}^{0,\delta} = 0$ if $i \geq m$ and $a_{1/i}^{0,\delta} = 1$. Since each $X^{0,\delta}$ is conservative while X is not, we cannot have $X^{0,\delta} \xrightarrow{w} X$ as $\delta \to 0$ because (74) fails for the function $\varphi = 1$ on $\mathbb{D}([0, t])$. ■

The next example shows that in Theorem 26 the assumption that each $X^{0,\delta}$ is regular is not necessary for the conclusion to hold.

Example 2. – Let X be an $\mathbb{N}$-valued minimal step Markov processes with jump transition matrix π and parameters of holding times $(a_1 : i \in \mathbb{N})$ given by $a_{2i+1} = 0$ and $a_{2i} = 1/i$ for $i \in \mathbb{N}$ (the odd integers are absorbing points), and $\pi_{i,i+1} = \pi_{i,i+2} = 1/2$. Then X is regular, and X_t converges to a limit X_∞ as $t \to \infty$ with

$$P_i(X_\infty = i) = 1 \qquad \text{if} \quad i \text{ is odd;}$$
$$P_i(X_\infty = i + 1 + 2k) = 2^{-k-1} \qquad \text{if} \quad i \text{ is even.}$$

Endow $\mathbb{N}$ with the usual metric. Then $X^{\varepsilon,\delta}$ is a minimal Markov process with the following characteristics (recall $\varepsilon < \delta$ below):

a) If $0 \leq \varepsilon < 1$, $\delta > 2$: each point is absorbing.

b) If $0 \leq \varepsilon < 1 < \delta \leq 2$: each odd integer is absorbing; the holding time in the even state i is $2a_i$, and with probability 1 the process jumps from i to $i+2$ if i is even.

c) If $1 \leq \varepsilon < 2 < \delta$: each odd integer is absorbing; the holding time in the even state i is $2\, a_i$, and with probability 1 the process jumps from i to $i+1$ if i is even.

d) In all other cases, $X^{\varepsilon, \delta} = X$.

Clearly X is purely discontinuous. However the process $X^{\varepsilon, \delta}$ for $0 \leq \varepsilon < 1 < \delta \leq 2$ is not conservative. ■

The next example is a regular quasi-Hunt JMP which is not weakly purely discontinuous.

Example 3. – The state space is $E = (0, \infty) \times [0, \infty)$. We consider the regular quasi-Hunt JMP X with characteristics (f, H, Γ) given by

$$f((x, 0), t) = \left(\frac{x}{1 - tx}, 0\right) \quad \text{for} \quad 0 \leq t < 1/x,$$
$$f((x, y), t) = (xt, yt) \quad \text{if} \quad y > 0,$$

$$H_{(x, 0)}(dt) = x\, 1_{(0, 1/x)}(t)\, dt, \qquad H_{(x, y)}(dt) = \varepsilon_\infty(dt) \qquad \text{if} \quad y > 0,$$

$$\Gamma((x, y), dx', dy') = \varepsilon_{(x, 1/x)}(dx', dy').$$

These characteristics are admissible. We have $\eta(x, 0) = 1/x$ and $\eta(x, y) = \infty$ if $y > 0$. A more concrete description of X is as follows: if $X_0 = (x, y)$ with $y > 0$ then $X_t = (tx, ty)$ for all t. If $X_0 = (x, 0)$, then $X_t = \left(\frac{x}{1 - tx}, 0\right)$ for $t < \tau$ and τ is uniformly distributed on $(0, 1/x)$ and $X_\tau = (Y, Z)$ where $Y = \frac{x}{1 - \tau x}$ and $Z = 1/Y$, and $X_{\tau + t} = (tY, tZ)$ for all $t \geq 0$.

If $\varepsilon > 0$ the process X^ε is again a regular quasi-Hunt JMP, whose characteristics are $(f, H^\varepsilon, \Gamma)$, with

$$H^\varepsilon_{(x, 0)}(dt) = \frac{1}{\varepsilon}\, 1_{(1/x - \varepsilon, 1/x)}(t)\, dt$$

if $x < 1/\varepsilon$ and $H^\varepsilon_{(x, y)} = H_{(x, y)}$ otherwise. Then if we start at $(x, 0)$, the processes X^ε do not converge in law as $\varepsilon \to 0$, because the limit, if it existed, would explose at time $t = 1/x$.

In this example, Theorem 25 fails because $\eta(x, 0) < \infty$, and Theorem 26 fails because the process $X^{0, \delta}$ is non-regular. ■

We end this section by checking which real-valued processes with stationary independent increments (in short PIIS) are purely discontinuous in the sense of Definition 9. Fix a continuous "truncation" function, *i.e.*

a continuous function h on $\mathbb{R}$ with compact support and $h(x) = x$ on a neighbourhood of 0. Recall that the *characteristics* of a PIIS X are a triple (b, c, F) with $b \in \mathbb{R}$, $c \in \mathbb{R}_+$ and F (the Lévy measure) a measure on $\mathbb{R}$ integrating $|x|^2 \wedge 1$ and with $F(\{0\}) = 0$, and they are related to X via $E(e^{iuX_t}) = e^{t\varphi(u)}$ with

$$\varphi(u) = iub - c^2 u^2/2 + \int F(dx)(e^{iux} - 1 - iuh(x)).$$

THEOREM 27. – *Let X be a PIIS with characteristics (b, c, F). There is equivalence between:*

a) X is purely discontinuous.

b) X is weakly purely discontinuous.

c) We have $c = 0$ and

$$\int F(dx)\, h(x)\, 1_{\{|x|>\varepsilon\}} \text{ converges in } \mathbb{R} \text{ to some limit } \beta \text{ as } \varepsilon \to 0. \tag{81}$$

(c) above is fulfilled if X has locally finite variation (*i.e.* $c = 0$ and F integrates $|x| \wedge 1$). It is also fulfilled if $c = 0$ and F is symmetric about 0 (e.g. for symmetric stable processes).

Proof. – For $0 < \varepsilon < \delta \leq \infty$ the process $X^{\varepsilon,\delta}$ is again a PIIS with characteristics $(b^{\varepsilon,\delta}, c^{\varepsilon,\delta}, F^{\varepsilon,\delta})$ given by

$$\left.\begin{aligned} b^{\varepsilon,\delta} &= b - \int F(dx)\, h(x)\, 1_{\{\varepsilon<|x|\leq\delta\}}, \qquad c^{\varepsilon,\delta} = c, \\ F^{\varepsilon,\delta}(dx) &= F(dx)\, 1_{\{\varepsilon<|x|\leq\delta\}^c}. \end{aligned}\right\} \tag{82}$$

Let us introduce the second "modified" characteristics:

$$\tilde{c} = c + F(h^2), \qquad \tilde{c}^{\varepsilon,\delta} = c + F^{\varepsilon,\delta}(h^2).$$

Let us first study (b). As is well known, $X^\varepsilon = X^{\varepsilon,\infty}$ converges weakly to a limit Y (necessarily another PIIS, with characteristics (b', c', F')) iff

$$b^{\varepsilon,\infty} \to b', \qquad \tilde{c}^{\varepsilon,\infty} \to \tilde{c}' := c' + F'(h^2), \qquad F^{\varepsilon,\infty}(g) \to F'(g) \tag{83}$$

for all continuous bounded g which are 0 in a neighbourhood of 0, as $\varepsilon \to 0$. Now, (83) holds iff (81) holds, in which case $b' = b - \beta$, $c' = c$ and $F' = 0$. Further, Y is deterministic in the sense of Definition 8 iff $c' = 0$, $F' = 0$. Therefore (b) $\Leftrightarrow$ (c).

Now, assume (c). If $\delta > 0$ we clearly have

$$b^{\varepsilon,\delta} \to b^{0,\delta} := b - \beta + \int F(dx)\, h(x)\, 1_{\{|x|>\delta\}},$$

$$\tilde{c}^{\varepsilon,\delta} \to \tilde{c}^{0,\delta} := F^{0,\delta}(h^2), \qquad F^{\varepsilon,\delta}(g) \to F^{0,\delta}(g)$$

for g as in (83), as $\varepsilon \to 0$ ($F^{0,\delta}$ is defined by (82)). Therefore $X^{\varepsilon,\delta} \xrightarrow{w} Y^\delta$, where Y^δ is a PIIS with characteristics $(b^{0,\delta}, 0, F^{0,\delta})$. Since $F^{0,\delta}(\mathbb{R}) < \infty$, Y^δ is a compound Poisson process with linear drift, and so is a quasi-Hunt JMP. Finally, as $\delta \to 0$ we have

$$b^{0,\delta} \to b, \qquad \tilde{c}^{0,\delta} \to \tilde{c}, \qquad F^{0,\delta}(g) \to F(g)$$

and thus $Y^\delta \xrightarrow{w} X$. Thus we have (*a*). Finally (*a*) $\Rightarrow$ (*b*) by Theorem 24. ∎

6. NON-HOMOGENEOUS JUMPING MARKOV PROCESSES

In this section we wish to consider the JMP property in the non-homogeneous case. We start with a *non-homogeneous normal strong Markov process* $X = (\Omega, \mathcal{F}, \mathcal{F}_t^s, X_t, P_{s,x})$ taking values in the Blackwell space $(E, \mathcal{E})$: let us briefly recall some essential properties here; *see* Dynkin [5] for a general account on the subject. We have a measurable space $(\Omega, \mathcal{F})$ and a transition probability $P_{s,x}(d\omega)$ from $\mathbb{R}_+ \times E_\Delta$ into $(\Omega, \mathcal{F})$; X is an E_Δ-valued process with lifetime ζ ($X_t = \Delta \Leftrightarrow t \geq \zeta$); we consider the σ-fields $\mathcal{F}_t^{0,s} = \sigma(X_r : s \leq r \leq t)$ for $s \leq t$, and we denote by $(\mathcal{F}_t^s)_{t \geq s}$ the usual completion of the filtration $(\mathcal{F}_{t+}^{0,s})_{t \geq s}$ relative to the family of measures $\int \mu(dx) P_{s,x}$, where μ runs through all finite measures on E. Property (7) is replaced by:

$$\text{For each } s \geq 0, \text{ the process } (X_t)_{t \geq s} \text{ is optional w.r.t. } (\mathcal{F}_t^s)_{t \geq s}. \tag{84}$$

The *normally* of the process is $P_{s,x}(X_s = x) = 1$. The *strong Markov property* is

$$E_{s,x}[g(X_t)|\mathcal{F}_T^s] = \int P(T, X_T, t, dy)\, g(y) \quad \text{on } \{T \leq t\} \tag{85}$$

for any bounded measurable g and any $(\mathcal{F}_t^s)_{t \geq s}$-stopping time $T \geq s$, where the family $(P(s, x, t, dy) : s \leq t)$ is the non-homogeneous transition semi-group of X defined by $P(s, x, t, A) = P_{s,x}(X_t \in A)$. As mentionned in the proof of Lemma 2 for the homogeneous case, if the process X is Markov (*i.e.* it satisfies (85) for T deterministic) and has (84), then it is strong Markov iff the processes $r \to P(r, X_r, t, A)$ are $P_{s,x}$-a.s. càdlàg for $r \in [s, t]$ (for all $s < t$, $A \in \mathcal{E}$): this is *regularity* in Dynkin's sense, not to be confused with the regularity of a JMP as introduced before.

It is well known that the associated *space-time process* is a normal strong *homogeneous* process, and we say that X is a JMP if the associated space-time process is a JMP (*see* eG;G; Mayer [12]). However it is just as simple to rewrite everything in the non-homogeneous case than to translate the results from the space-time process to the original non-homogeneous process X.

6.1. Quasi-Hunt jumping Markov processes

We consider only the quasi-Hunt case. Of course the filtration $(\mathcal{F}_t^s)_{t\geq s}$, which "starts" at time s, is said to be *jumping* if it satisfies the conditions of Definition 1 with $\tau_0 = s$ instead of $\tau_0 = 0$.

DEFINITION 10. – The process X is called a *quasi-Hunt jumping Markov process* if for each $s \geq 0$ the filtration $(\mathcal{F}_t^s)_{t\geq s}$ is a jumping filtration on a set $I^s = \bigcup_n [\![s,\ \tau_n^s]\!]$, with a jumping sequence (τ_n^s) having the following properties:

(i) τ_n^s is $P_{s,x}$-totally inaccessible for all $x \in E$, $n \geq 1$;

(ii) $\tau_n^s \leq \zeta$ and $\tau_n^s < \tau_{n+1}^s$ on the set $\{\tau_n^s < \zeta\}$, $P_{s,x}$-a.s. for all $x \in E$. Further, X is called *regular* if $\tau_\infty^s := \lim_n \uparrow \tau_n^s = \infty$ $P_{s,x}$-a.s. for all $x \in E$, $s \geq 0$. ■

The results of § 3.1 read as follows in the non-homogeneous case:

THEOREM 28. – *Assume that the lifetime ζ has $P_{s,x}(\zeta = \infty) = 1$ for all $s \geq 0$, $x \in E$. The following three properties are equivalent:*

(i) *For all $s \geq 0$, $x \in E$, the $P_{s,x}$-martingales (w.r.t. the filtration $(\mathcal{F}_t^s)_{t\geq s}$) are quasi-left continuous and with locally finite variation.*

(ii) *For each $s \geq 0$ the $(\mathcal{F}_t^s)_{t\geq s}$ filtration is a jumping filtration on $[s, \infty)$, and for each $x \in E$ the $P_{s,x}$-martingales are quasi-left continuous.*

(iii) *X is a regular quasi-Hunt JMP.*

THEOREM 29. – *The process X is a quasi-Hunt JMP iff for all $s \geq 0$, $x \in E$ there is a stopping time S_x^s such that:*

(i) *$P_{s,x}(S_x^s > s) = 1$ and $\zeta_{\{\zeta < S_x^s\}}$ is $P_{s,x}$-totally inacessible;*

(ii) *any $P_{s,x}$-martingale which is constant after S_x^s is quasi-left continuous with $P_{s,x}$-a.s. locally finite variation;*

(iii) *there is (at least) a $P_{s,x}$-martingale M which has $P_{s,x}$-a.s. at least one jump on the interval $(0,\ S_x^s]$, on the set $\{S_x^s < \infty\}$.*

The sequences $(\tau_n^s)_{n\in\mathbb{N}}$ are of course related, and to describe the relationship we introduce an *ad-hoc* terminology. An optional subset D of $\Omega \times \mathbb{R}_+$ is of *type A* if the sections $D_\omega = \{t : (\omega, t) \in D\}$ are closed, at most countable, and $\inf(s \in D_\omega : s > t) > t$ for all $t \in D_\omega$. If one denotes by $\mathcal{O}$ the class of all countable ordinals, a random set D is of type A iff it has the form $D = \bigcup_{\alpha\in\mathcal{O}} [\![\tau_\alpha]\!]$ where the τ_α's are stopping times, with $\tau_{\alpha+1} > \tau_\alpha$ if $\tau_\alpha < \infty$, and $\tau_\alpha = \sup_{\beta<\alpha} \tau_\beta$ when α is a limit ordinal (recall that the section $[\![\tau]\!]_\omega$ is empty when $\tau(\omega) = \infty$).

THEOREM 30. – *Suppose that X is a quasi-Hunt JMP. There is an $(\mathcal{F}_t^0)_{t\geq 0}$-optional set D of type A contained in $[\![0, \zeta]\!]$ such that, if we set $\tau_0^s = s$, $\tau_{n+1}^s = s \vee [\zeta \wedge \inf(t \in D : t > \tau_n^s)]$ for $n \geq 1$, and if $H_{s,x}$ and $G_{s,x}$ are the laws of τ_1^s and $(X_{\tau_1^s}, \tau_1^s)$ and*

$$\eta(s, x) = \inf(t : H_{s,x}([t, \infty]) = 0) \qquad (\text{so } \eta(s, x) > s), \tag{86}$$

we have the following properties:

a) $(\mathcal{F}_t^s)_{t\geq s}$ is a jumping filtration on $I^s = \bigcup [\![s, \tau_n^s]\!]$ with jumping sequence $(\tau_n^s)_{n\in\mathbb{N}}$, and the τ_n^s are $P_{s,x}$-totally inaccessible for all $x \in E$, $n \geq 1$.

b) For all $s \geq 0$, $x \in E$, then $H_{s,x}$ has no atom except perhaps $\{+\infty\}$.

c) There is a measurable function $f : \mathbb{R}_+ \times E \times \mathbb{R}_+ \to E$ such that

$$\left.\begin{aligned} f(s, x, s) &= x, \\ f(s, x, u) &= f(t, f(s, x, t), u) \quad \text{if} \quad s < t < u < \eta(s, x), \end{aligned}\right\} \tag{87}$$

$$\left.\begin{aligned} &\tau_1^s = \inf(t > s : X_t \neq f(s, x, t)) \\ &\text{and} \\ &X_{\tau_1^s} \neq f(s, x, \tau_1^s) \quad \text{on } \{\tau_1^s < \infty\}, \qquad P_{s,x}\text{-a.s.} \\ &\quad H_{s,x}((t, u]) = H_{s,x}((t, \infty])\, H_{t, f(s,x,t)}((t, u]) \\ &\qquad\qquad \text{if } s < t < \eta(s, x). \end{aligned}\right\} \tag{88}$$

d) There exists a probability kernel Γ from $\mathbb{R}_+ \times E$ into E_Δ such that for all $x \in E$ we have $\Gamma(t, x, \{x\}) = 0$ and

$$\begin{aligned} G_{s,x}(dy, dt) = H_{s,x}(dt)\, [1_{\{t<\infty\}}\, \Gamma(t, f(s, x, t), dy) \\ + 1_{\{t=\infty\}}\, \varepsilon_\Delta(dy)]. \end{aligned} \tag{89}$$

An homogeneous quasi-Hunt JMP is a quasi-Hunt JMP in the sense of Definition 10. If we compare Theorems 5 and 30 in the homogeneous case, we have $\tau_n = \tau_n^0$ and $I = I^0$, and the set D of Theorem 30 is $D = \bigcup_{\alpha\in\mathcal{O}} [\![\tau_\alpha]\!]$ where the τ_α's are defined by induction as $\tau_0 = 0$, $\tau_{\alpha+1} = \tau_\alpha + \tau \circ \vartheta_{\tau_\alpha}$ and $\tau_\alpha = \sup_{\beta<\alpha} \tau_\beta$ if β is a limit ordinal.

In the non-homogeneous case, the property for each τ_1^s to be a "terminal time" is expressed by the fact that if $\tau_1^s > t$, then $\tau_1^s = \tau_1^t$.

Proof of Theorems 28, 29, 30. – 1) First we observe that Lemma 2 remains valid, with the following formulation: for each $s \geq 0$ there is a sequence $(M^{n,s})$ of bounded $(\mathcal{F}_t^s)_{t\geq s}$-martingales (*i.e.* martingales for $P_{s,x}$ for all $x \in E$) such that, if $x \in E$ and S_x^s is an $(\mathcal{F}_t^s)_{t\geq s}$-stopping time such that all $P_{s,x}$-martingales are quasi-left continuous on $[\![s, S_x^s]\!]$, then any $P_{s,x}$-martingale M has

$$\{t : s < t \leq S_x^s,\ \Delta M_t \neq 0\} \subseteq \bigcup_n \{t > s : \Delta M_t^{n,s} \neq 0\} \quad P_{s,x}\text{-a.s.}$$

(the proof is the same, using the comments after (85)).

Now, if $M = (M_t)_{t\geq s}$ is a $P_{s,x}$-martingale and $r > s$, then $M = (M_t)_{t\geq r}$ is a $P_{r, X_r(\omega)}$-martingale for $P_{s,x}$-almost all ω with $X_r(\omega) \in E$; therefore we have also $\{t : s < t \leq S_x^s,\ \Delta M_t \neq 0\} \subseteq \bigcup_n \bigcup_{r\in\mathbb{Q},\, r>s} \{t > r : \Delta M_t^{n,r} \neq 0\}$ $P_{s,x}$-a.s.

2) Next we reproduce the proof of Theorems 4 and 5 with the following changes: first, set $\sigma_n^s = \inf(t > s : \Delta M_t^{n,s} \neq 0)$ and $\tau^s = \zeta \wedge \inf_{n\geq 1,\, r\in\mathbb{Q}\cap(s,\infty)} \sigma_n^r$. Then $(\omega, s) \to \tau^s(\omega)$ is measurable, and (20) becomes

$$(\mathcal{F}_t^s)_{s\geq t} \text{ is a } P_{s,x}\text{-a.s. jumping filtration on } I_x^s = [\![0, S_x^s]\!], \tag{20'}$$

with a jumping sequence $(\tau_n^{s,x})_{n\in\mathbb{N}}$, and instead of (21) we get

$$s < \tau_1^{s,x} = \tau^s \leq S_x^s\ P_{s,x}\text{-a.s., and } \tau^s \text{ is } \mathrm{P}_{s,x}\text{-totally inacessible.} \tag{21'}$$

In Step 2), we have a function $f : \mathbb{R}_+ \times E \times \mathbb{R}_+ \to E$ such that for all $s \leq t$, $x \in E$, $P_{s,x}(X_t \neq f(s, x, t),\ s \leq t < \tau^s) = 0$, and since $(\omega, s) \to \tau^s(\omega)$ is measurable we deduce the measurability of f, and thus (23) becomes

$$X_t = f(s, x, t) = f(s, X_s, t) \qquad P_{s,x}\text{-a.s. for all } t \in [s, \tau^s). \tag{23'}$$

In Step 3) we set $\tau'^s = \inf(t > s : X_t \neq f(s, X_s, t))$ and get $P_{s,x}(\tau'^s = \tau^s) = 1$. Next if $t \geq s$ we set $g(x, u) = f(s, x, u)\,1_{\{u>t\}} + f(t, f(s, x, t), u)\,1_{\{u>t\}}$ and $\tau'' = \inf(u \geq s : X_u \neq g(X_s, u))$. Then $P_{s,x}(\tau^s \geq \tau'') = 1$ and $\tau'' = \tau^s$ if $\tau^s < t$, and by the Markov property $\tau'' = \tau^t$ if $\tau^s \geq t$. Then τ'' is $P_{s,x}$-totally inaccessible and finally $P_{s,x}(\tau'' = \tau^s) = 1$. Hence we get instead of (24):

$$\tau^s = \tau^t \quad P_{s,x}\text{-a.s. on } \{\tau^s > t\}. \tag{24'}$$

Then we get, with $G_{s,x}$ being the law of (X_{τ^s}, τ^s) under $P_{s,x}$:

$$G_{s,x}(A \times (t, u]) = H_{s,x}((t, \infty]) \, G_{t, f(s,x,t)}(A \times (t, u]) \quad \text{for } s < t < \eta(s, x), \tag{25'}$$

and the rest of Step 3) to obtain (*c*) is essentially unchanged.

Step 4) is replaced by the following: we set $D' = \bigcup_{s \in \mathbb{Q}_+} [\![\tau^s]\!]$ and D is the closure of D', and we define τ_n^s as in the statement of Theorem 30. Then using the strong Markov property at each τ_n^s we deduce as in Step 4) that (*a*) of Theorem 30 holds, and Step 5) is unchanged to obtain (*d*) of Theorem 30.

3) Finally Theorem 28 is proved exactly as Theorem 3. ■

Corollary 1 becomes (with the same proof):

COROLLARY 3. – *Assume that there is a measurable function* $f : \mathbb{R}_+ \times E \times \mathbb{R}_+ \to E$ *such that* $\tau^s = \inf(t \geq s : X_t \neq f(s, X_s, t))$ *satisfies* $\tau^s > s$ $P_{s,x}$-a.s. *We call* $H_{s,x}$ *the law of* τ^s *under* $P_{s,x}$, *and we define* $\eta(s, x)$ *by* (86). *Then if either* f *satisfies* (87) *or* $H_{s,x}(\{t\}) = 0$ *for all* $t < \infty$, *the process* X *is a quasi-Hunt JMP.*

The first claim of Theorem 7 has no counterpart, but the second claim is:

THEOREM 31. – *If* X *is a quasi-Hunt JMP, for any* $s \geq 0$ *the process* $(X_{\tau_n^s}, \tau_n^s)$ *is an homogeneous Markov chain, with transition probability* $Q(x, t; dy, du) = G_{t,x}(dy, du)$ *if* $x \in E$ *and* $Q(\Delta, t; dy, du) = \varepsilon_{(\Delta, t)}(dy, du)$ *independent from* s. *Further*

$$X_t = f(\tau_n^s, X_{\tau_n^s}, t) \quad P_{s,x}\text{-a.s.} \quad \text{for all } t \text{ with } \tau_n^s \leq t < \tau_{n+1}^s \tag{90}$$

The pair (f, G), or equivalently the triple (f, H, Γ), will be called the *characteristics* of X, and it is *admissible* in the sense that it satisfies (87), (88), (89) with $\Gamma(t, x, \{x\}) = 0$, and $H_{s,x}(\{t\}) = 0$ for all $t < \infty$.

Conversely if we start with an admissible pair (f, G) we can construct a Markov chain (Y_n, τ_n) with transition Q as in the previous theorem, and set

$$X_t = \begin{cases} f(\tau_n, Y_n, t) & \text{if} \quad \tau_n \leq t < \tau_{n+1} \quad \text{and } Y_0 \neq \Delta, \\ \Delta & \text{if} \quad t \geq \tau_\infty := \lim_n \tau_n \quad \text{or } Y_0 = \Delta. \end{cases} \tag{91}$$

Then we have as in Theorem 8 (the technical details being similar):

THEOREM 32. – *Under the above assumptions,* X *is a quasi-Hunt JMP with lifetime* $\zeta = \tau_\infty$ *and characteristics* (f, G).

If X is a quasi-Hunt JMP, its submarkovian transition semi-group $P(s, x, t, dy)$ (on E) satisfies the following equation, similar to (34):

$$P(s, x, t, A) = H_{s,x}((t, \infty])\, 1_A(f(s, x, t)) + \int_0^t H_{s,x}(du) \int_E \Gamma(u, f(s, x, u)\, dy)\, P(u, y, t, A).$$

Conversely, if we start with an admissible pair (f, G), this equation for any given $s \geq 0$ has solutions, and there is a minimal one $P^*(s, x, t, dy)$. Further, $P^*(0, x, t, E) = 1$ for all $x \in E, t \geq 0$ implies $P^*(s, x, t, E) = 1$ for all $s \leq t$, $x \in E$, and this is necessary and sufficient for the corresponding JMP to be regular. Another necessary and sufficient condition for regularity is given by Theorem 12 (a), with the same $\bar{Q}$ (and Q as in Theorem 31).

6.2. Additive functionals, martingales, semimartingales

Now we quickly review the main results of Section 4 for a non-homogeneous quasi-Hunt JMP X, and we use the notation of Theorem 30. We do not provide proofs, which are the same as in the homogeneous case.

Additive functionals. – By *additive functional on* I we mean a family $A = (A_t^s : t \in I^s)$ of processes, such that for every $s \geq 0$ we have:

(i) the process $A_t^s 1_{I^s}(t)$ is adapted to $(\mathcal{F}_t^s)_{t \geq s}$, càdlàg, with $A_s^s = 0$;

(ii) $(s, \omega, t) \to A_t^s(\omega)$ is measurable;

(iii) for every $(\mathcal{F}_t^s)_{t \geq s}$-stopping time $T \geq s$ we have

$$A_t^s = A_T^s + A_t^T \quad P_{s,x}\text{-a.s. for all } x \in E \text{ on the set } \{ s \leq T \leq t \}.$$

For all $s \geq 0$, $n \geq 1$, we set (recall (46)):

$$X_{\tau_n^s}^- = f(\tau_{n-1}^s, X_{\tau_{n-1}^s}, \tau_n^s) \quad \text{on} \quad \{ \tau_{n-1}^s < \tau_n^s < \infty \}$$

(One easily checks that if $\tau_n^s = \tau_m^t$, then $X_{\tau_n^s}^- = X_{\tau_m^t}^-$ on the set $\{ \tau_{n-1}^s < \tau_n^s < \infty \} \cap \{ \tau_{m-1}^t < \tau_m^t < \infty \}$: hence X^- is a "process" on the subset of all isolated points of the set D of Theorem 30). Theorem 15 becomes:

THEOREM 33. – *Assume that X is a quasi-Hunt JMP.*

a) With every additive functional A on I are associated a measurable function $a : \mathbb{R}_+ \times E \times \mathbb{R}_+ \to \mathbb{R}$ with

$$a(s, x, s) = 0, \qquad a(s, x, u) = a(s, x, t) + a(t, f(s, x, t), u)$$

$$\text{if } s \leq t \leq u < \eta(s, x), \qquad t \to a(s, x, t) \text{ is càdlàg on } [s, \eta(s, x)),$$

and a measurable function $\bar{a} : \mathbb{R}_+ \times E \times E_\Delta \to \mathbb{R}$, *such that outside a* $P_{s,x}$*-null set* A^s *is determined on* I^s *by induction, starting with* $A^s_s = 0$, *by*

$$\left.\begin{array}{ll} A^s_t = A^s_{\tau^s_n} + a(\tau^s_n, X_{\tau^s_n}, t) & \text{if } \tau^s_n \le t < \tau^s_{n+1}, \\ A^s_{\tau^s_{n+1}} = A^s_{\tau^s_{n+1}-} + \bar{a}(\tau^s_{n+1}, X^-_{\tau^s_{n+1}}, X_{\tau^s_{n+1}}) & \text{if } \tau^s_n < \tau^s_{n+1} < \infty. \end{array}\right\} \tag{92}$$

b) Conversely if a *and* $\bar{a}$ *are as above,* (92) *defines an additive functional* A *on* I.

The basic example of additive functional is associated with the function:

$$\ell(s, x, t) = -\log H_{s,x}((t, \infty]) \qquad (\text{with } \log(0) = -\infty).$$

It is the increasing continuous additive functional $L = (L^s_t)$ defined by (92) with $a = \ell$ and $\bar{a} = 0$, that is

$$L^s_s = s, \qquad L^s_t = L^s_{\tau^s_n} + \ell(\tau^s_n, X_{\tau^s_n}, t) \quad \text{for } \tau^s_n < t \le \tau^s_{n+1}.$$

Consider the following family of integer-valued random measures on $\mathbb{R}_+ \times E_\Delta$:

$$\mu^s(dt, dx) = \sum_{n \ge 1 : \tau^s_{n-1} < \tau^s_n < \infty} \varepsilon_{(\tau^s_n, X_{\tau^s_n})}(dt, dx).$$

Then μ^s is $(\mathcal{F}^s_t)_{s \ge t}$-optional and its $P_{s,x}$-compensator of each x is

$$\nu^s(dt, dy) = dL^s_t \, \Gamma(t, X_t, dy) \, 1_{I^s}(t).$$

Martingales. – Any $P_{s,x}$-local martingale is a compensated sum of jumps, and has locally finite variation on $I^s = \bigcup_n [\![s, \tau^s_n]\!]$. Any $P_{s,x}$-semimartingale has locally finite variation on I^s. If M is a $P_{s,x}$-local martingale on I^s, there is an $(\mathcal{F}^s_t)_{t \ge s}$-predictable function on $\Omega \times [s, \infty) \times E_\Delta$ such that for $t \in I^s$:

$$M_t = M_s + \sum_{u \in D \cap I^s} U(u, X_u) - \int_s^t dL^s_u \int_E \Gamma(u, X_u, dy) \, U(u, y). \tag{93}$$

It also has a second representation, with $\mathcal{F}^s_{\tau_n} \otimes \mathcal{R}_+ \otimes \mathcal{E}_\Delta$-measurable functions U_n on $\Omega \times \mathbb{R}_+ \times E_\Delta$ (by induction, starting from M_s):

$$\left.\begin{array}{l} M_t = M_{\tau^s_n} - \displaystyle\int_0^t d_u \, \ell(\tau^n_s, X_{\tau^s_n}, u) \\ \qquad \times \displaystyle\int_E \Gamma(u, f(\tau^s_n, X_{\tau^s_n}, u), dy) \, U_n(n, y) \\ \qquad\qquad \text{if } \quad \tau^s_n \le t < \tau^s_{n+1} \\ M_{\tau^s_{n+1}} = M_{\tau^s_{n+1}-} + U_n(\cdot, \tau^s_{n+1}, X_{\tau^s_{n+1}}) \\ \qquad \text{if } \quad \tau^s_n < \tau^s_{n+1} < \infty \end{array}\right\} \tag{94}$$

In both (93) and (94), all integrals are $P_{s,x}$-a.s. absolutely convergent. At this point, it is worthwile to note that (90) yields $(\mathcal{F}^s_{\tau^s_n}) = \sigma(\tau^s_p, X_{\tau^s_p} : p \leq n)$ up to $P_{s,x}$-null sets. Then we can take $U_n(\cdot, u, y) = g_{n+1}(\tau^s_1, X_{\tau^s_1}, \cdots, \tau^s_n, X_{\tau^s_1}, u, y)$ for some measurable function g_{n+1} on $(\mathbb{R}_+ \times E_\Delta)^{n+1}$, so the first half of (94) is also

$$M_t = M_{\tau^s_n} - \int_{(s,t]\times E_\Delta} G_{\tau^s_n, X_{\tau^s_n}}(dy, du)\, h_{n+1}(\tau^s_1, X_{\tau^s_1}, \ldots, \tau^s_n, X_{\tau^s_n}, u, y)$$

for

$$h_{n+1}(t_1, x_1, \ldots, t_{n+1}, x_{n+1}) = g_{n+1}(t_1, x_1, \ldots, t_{n+1}, x_{n+1})/H_{t_n, x_n}((t_{n+1}, \infty]).$$

Semimartingales. – An additive functional $A = (A^s_t)$ on I is called a *semimartingale* (*special semimartingale, local martingale*) if for each $s \geq 0$ the process $(A^s_t)_{t\in I^s}$ is a $P_{s,x}$-semimartingale (special semimartingale, local martingale) on I^s for all $x \in E$.

THEOREM 34. – *Assume that* X *is a quasi-Hunt JMP, and let* $A = (A^s_t)$ *be an additive functional on* I, *and* a, $\bar{a}$ *associated with it as in Theorem* 33.

a) A *is a semimartingale on* I *iff*

$$t \to a(s, x, t) \text{ is càdlàg with locally finite variation on } [s, \eta(s, x)). \tag{95}$$

b) A *is a local martingale on* I *iff we have the next two properties:*

$$\left.\begin{aligned} &\int_0^t d_u \ell(s, x, u) \int \Gamma(u, f(s, x, u), dy) \\ &\quad \times |\bar{a}(u, f(s, x, u), y)| < \infty \qquad \text{if} \quad t < \eta(s, x), \end{aligned}\right\} \tag{96}$$

$$a(s,x,t) = -\int_0^t d_u\ell(s,x,u) \int \Gamma(u, f(s,x,u), dy)\, \bar{a}(u, f(s,x,u), y).$$

c) A *is a special semimartingale on* I *iff we have* (95) *and* (96). *In this case the canonical decomposition* $A^s = B^s + M^s$ *of* A^s *is the same for all measures* $P_{s,x}$ $(x \in E)$ *and* $M = (M^s_t)$ *is an additive local martingale on* I, *and* $B = (B^s_t)$ *is the additive functional on* I *defined by* $B^s_s = 0$ *and*

$$B^s_t = B^s_{\tau^s_n} + a'(\tau^s_n, X_{\tau^s_n}, t) \qquad \text{if} \quad \tau^s_n \leq t \leq \tau^s_{n+1}$$

where

$$a'(s, x, t) = a(s, x, t) + \int_0^t d_u \ell(s, x, u) \times \int \Gamma(s, f(s, x, u), dy)\, \bar{a}(u, f(s, x, u), y). \tag{97}$$

In view of the definition of the τ_n^s's in Theorem 30, the following is obvious:

COROLLARY 4. – *If X is a regular quasi-Hunt JMP, an additive functional $A = (A_t^s)$ on I is a semimartingale (resp. special semimartingale, local martingale) iff the process $(A_t^0)_{t \geq 0}$ is a $P_{0,x}$-semimartingale (resp. special semimartingale, local martingale) for all $x \in E$.*

Semimartingale functions. – We say that a measurable function $g : \mathbb{R}_+ \times E_\Delta \to \mathbb{R}$ is a *semimartingale (special semimartingale, martingale) function* of X if the family $A_t^s = g(t, X_t) - g(s, X_s)$ is a semimartingale (special semimartingale, local martingale) additive functional on I, in the sense given above. Note that (A_t^s) has always the properties of an additive functional on I, except perhaps that it is not càdlàg. Observe also that if X is regular, then it amounts to saying that $t \to g(t, X_t)$ is a $P_{0,x}$-semimartingale (special semimartingale, local martingale) for all $x \in E$.

THEOREM 35. – *Assume that X is a quasi-Hunt JMP.*

a) g is a semimartingale function iff, with $\hat{g}(s, x, t) = g(t, f(s, x, t))$,

$$\hat{g}(s, x, \cdot) \text{ is càdlàg with locally finite variation on } [s, \eta(s, x)). \tag{98}$$

b) g is a special semimartingale function iff we have (98) *and*

$$\int_0^t d_u \ell(s, x, u) \int_{E_\Delta} \Gamma(u, f(s, x, u), dy) \times |g(u, y) - g(u, f(s, x, u))| < \infty \quad \text{if} \quad t < \eta(s, x). \tag{99}$$

In this case the predictable compensator B^s of $(g(t, X_t) - g(s, X_s))_{t \geq s}$ under each $P_{s,x}$ is given by (97)*, with $B_s^s = 0$ and*

$$a'(s, x, t) = g(t, f(s, x, t)) - g(s, x) + \int_0^t d_u \ell(s, x, u) \int_{E_\Delta} \Gamma(u, f(s, x, u), dy) \times [g(u, y) - g(u, f(s, x, u))].$$

c) g is a martingale function iff we have (97), (98) *and $a' = 0$ in* (99).

6.3. Transformation of the phase space

In this section we consider again a non-homogeneous normal strong Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t^s, X_t, P_{s,x})$, with values in $(E, \mathcal{E})$.

We consider another Blackwell space $(E', \mathcal{E}')$ and a measurable map $\mathcal{U} : \mathbb{R}_+ \times E_\Delta \to E'_\Delta$ with $\mathcal{U}(t, \Delta) = \Delta$, and the process $X'_t = \mathcal{U}(t, X_t)$. The process X' is of course not Markov in general, but when it is and when X is a JMP, then obviously X' will be a JMP as well (because it generates filtrations $(\mathcal{F}'^s_t)$ which are smaller than $(\mathcal{F}^s_t)$).

To begin with, it is clear that the process X' cannot take at time s an initial value x' which does not belong to the image of E by $\mathcal{U}(s, \cdot)$. So in order to be able to start with any initial value, we assume that $\mathcal{U}(s, \cdot)$ is surjective for each $s \geq 0$.

In fact there is a rather simple set of conditions on the characteristics of X when it its a JMP, which insures that X' is also a JMP.

THEOREM 36. – *Assume that X is a regular quasi-Hunt JMP with characteristics (f, H, Γ). Let the following conditions be fulfilled:*

a) There is a probability kernel $H'_{s,x}$, from $\mathbb{R}_+ \times E'$ into $[0, \infty]$ with $H'_{s, \mathcal{U}(s,x)} = H_{s,x}$ for all $s \geq 0$, $x \in E$.

b) There is a measurable function $f' : \mathbb{R}_+ \times E' \times \mathbb{R}_+ \to E'$ with $f'(s, \mathcal{U}(s, x), t) = \mathcal{U}(t; f(s, x, t))$ for all $x \in E$, $0 \leq s \leq t < \eta(s, x)$.

c) There is a probability kernel Γ' from $\mathbb{R}_+ \times E'$ into E'_Δ such that $\Gamma'(t, \mathcal{U}(t, x), \cdot)$ is the image of $\Gamma(t, x, \cdot)$ by the map $\mathcal{U}(t, \cdot)$ for all $x \in E$, $t \geq 0$, and that $\Gamma'(t, x', \{x'\}) = 0$ for all $t \geq 0$, $x' \in E'$.

Then X' is a regular quasi-Hunt JMP with characteristics (f', H', Γ').

Proof. – The first thing to do is to show that the triple (f', H', Γ') is admissible (*see* after Theorem 31), and this comes from elementary computations (note that $\eta'(s, \mathcal{U}(s, x)) = \eta(s, x)$). Associate G' with (f', H', Γ') by (89), and Q' with G' as in Theorem 31. If g' is a bounded measurable function on $E' \times [0, \infty]$, we set $g(x, s) = g'(\mathcal{U}(s, x), s)$ (with $\mathcal{U}(\infty, x) = \mathcal{U}(s, \Delta) = \Delta$) and another elementary computation shows that $(Q' g')(\mathcal{U}(s, x), s) = (Q g)(x, s)$. Then if $\tau_n = \tau_n^0$ and $\mathcal{F}_t = \mathcal{F}_t^0$, we have by Theorem 31:

$$\begin{aligned} E_{s,x}[g'(X'_{\tau_{n+1}}, \tau_{n+1})|\mathcal{F}_{\tau_n}] &= E_{s,x}[g(X_{\tau_{n+1}}, \tau_{n+1})|\mathcal{F}_{\tau_n}] \\ &= (Q g)(X_{\tau_n}, \tau_n) = (Q' g')(X'_{\tau_n}, \tau_n). \end{aligned}$$

Hence $(X'_{\tau_n}, \tau_n)_{n \geq 0}$ is an homogeneous Markov chain under each $P_{s,x}$, with transition probability Q'. Further, (90) implies

$$X'_t = \mathcal{U}(t, f(\tau_n, X_{\tau_n}, t)) = f'(\tau_n, X'_{\tau_n}, t) \qquad \text{if} \quad \tau_n \leq t < \tau_{n+1}.$$

That is, X' satisfies the first half of (91) with $Y_n = X'_{\tau_n}$, and the second half is empty here since X is regular. Then Theorem 32 gives the result. ■

Remark 6. – This result is not true when X is not regular. Indeed in this case the conditions of Theorem 36 tell nothing about what happens at the explosion time τ_∞, and it may happen that X' is not a Markov process. ■

Remark 7. – Conversely it is easy to see that if X is regular, and if X' is a Markov process, then (a) and (b) above are met and (c) is "almost" true, in the sense that $\Gamma'(t, \mathcal{U}(t, x), \cdot)$ is the image of $\Gamma(t, x), \cdot)$ by the map $\mathcal{U}(t, \cdot)$ for $H_{s,x}(dt)$-almost t, for all $x \in E$, $s \geq 0$ (we could indeed replace (c) in Theorem 36 by this slightly weaker condition). ■

By analogy with the homogeneous case, we call *quasi-Hunt non-homogeneous step Markov process* a quasi-Hunt JMP whose first characteristic is $f(s, x, t) = x$, or in other word a non-homogeneous strong Markov process which is right-continuous for the discrete topology of E and whose jump times are totally inacessible.

COROLLARY 5. – *Assume that X is a regular quasi-Hunt JMP with characteristics (f, H, Γ), such that for all $0 \leq s < t$ the function $x \to f(s, x, t)$ admits an inverse $f^{-1}(s, x, t)$ that $f(s, x, u) = f(t, f(s, x, t), u)$ for all $0 \leq s < t < u < \infty$* (*i.e.* the second part of (87) is true with $\eta(s, x) = \infty$) *and that $\mathcal{U}(t, x) = f^{-1}(0, x, t)$ is measurable on $\mathbb{R}_+ \times E$.*

Then $X'_t = \mathcal{U}(t, X_t)$ is a quasi-Hunt non-homogeneous step Markov process with characteristics (f', H', Γ') given by $f'(s, x, t) = x$ and

$$H'_{s,x} = H_{s, f(0, x, s)}, \qquad \Gamma'(t, x, A) = \Gamma(t, f(0, x, t), \{y : U(t, y) \in A\}).$$

Proof. – One readily checks that the conditions (a) and (c) of Theorem 31 are met. for (b), we first note that $f(0, x, t) = f(s, f(0, x, s), t)$ (true for all $0 \leq s \leq t < \infty$) implies that $f(s, y, t) = f(0, \mathcal{U}(s, y), t)$, hence $y \to f(s, y, t)$ is invertible with $f^{-1}(s, y, t) = f(0, \mathcal{U}(t, y), s)$. Then $\mathcal{U}(t, y) = \mathcal{U}(s, f^{-1}(s, y, t))$, and thus $\mathcal{U}(t, f(s, x, t)) = \mathcal{U}(s, x)$: hence ($b$) is satisfied as well. ■

Remark 8. – If (87) is true only for $u < \eta(s, x)$, there is still a version of this corollary, involving a more complicated definition for the transformation $\mathcal{U}$. ■

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