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by

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ABSTRACT. – We give optimal estimates of the L^∞-norm of the heat diffusion kernel on a unimodular Lie group.

RÉSUMÉ. – On donne des estimations pour la norme L^∞ du noyau de la chaleur sur un groupe de Lie unimodulaire.

0. INTRODUCTION

Let G be a locally compact group and let $\mu \in \mathbb{P}(G)$, then $\| \mu \|_{2\to 2} = e^{-\lambda}$ where $\lambda \geq 0$, here we denote by $\| \mu \|_{p\to q}$ the $L^p (G; d^r g) \to L^q (G; d^r g)$ norm of the operator $f \mapsto f * \mu$ where $d^r g$ the right invariant measure on G. The number $\lambda = \lambda (\mu)$ will be called the spectral gap of μ. (We shall use that terminology even for measures that are not symmetric and do not satisfy $\mu (g) = \mu (g^{-1})$). It is well known that when G is connected and when $d\mu (g) = f (g)\, dg$ is given by a continuous density f, then the number $\lambda (\mu)$ is either zero for all such measures, and we then say that G is amenable, or λ is always non zero. It is important to recall that a connected Lie group G is amenable if and only if its quotient by the radical Q (*cf.* [7]) $G/Q = \Sigma$ is compact. Let us finally recall the definition of the second moment of μ:

$$E (\mu) = \int_G | g |^2 \, d\mu (g)$$

where $|g| = d(g, e)$ is the "distance" in G from g to the neutral element e (*cf.* [3], for precise definitions).

Let now Σ be a real connected non compact semisimple Lie group and let $\Sigma = KAN$ be the Iwasawa decomposition of Σ where K contains the center Z, and is such that K/Z is compact , $A \cong \mathbb{R}^d$ $(d = 1, 2, \ldots)$ and N is nilpotent. Let us also denote by p the number of indivisible positive roots of the action of A on N (*i.e.* $1/2$ of any of these roots is not a root). Let finally $r = 0, 1, \ldots$ be the rank of the center $Z \cong \mathbb{Z}^r \times F$ where F is finite abelian group. I shall, in what follows, denote by

$$q = q(\Sigma) = d + 2p + r. \tag{0.1}$$

The significance of the integer q lies in the following well known theorem of Ph. Bougerol (*cf.* [2]).

THEOREM (Ph. Bougerol). – *Let Σ be a real semisimple non compact Lie group as above and let us assume that the center of Σ is finite. Let $d\mu(g) = f(g)\,dg$ be a probability measure with finite second moment and with an L^1 density and let us denote by $d\mu^n(g) = f_n(g)\,dg$ the n^{th} convolution power of μ. Let us further assume that $\bigcup_{n\geq 1} \operatorname{supp} \mu^n = G$. For every compact subset $C \subset\subset G$ we then have:*

$$\int_C f_n(g)\,dg = 0\left(e^{-\lambda n}\, n^{-q/2}\right)$$

where λ is the spectral gap of μ.

Observe that $\sup_{k_1, k_2 \in K} |k_1 g k_2| \leq |g| + C$, $g \in \Sigma = KAN$. This implies that the above μ has "un moment d'ordre deux" in the sense of [2]. (Observe also that the left distance that we use on $\Sigma = NAK$ (*cf.* [3]) can be assumed K-biinvariant. That distance induces therefore on the subgroup AN a new left distance that is equivalent to the intrinsic left group distance of AN).

Let now G be an arbitrary real connected Lie group, let $Q \subset G$ be its radical (*cf.* [7]) which is a closed connected subgroup. We shall assume throughout that $G/Q = \Sigma$ is non compact in other words we shall assume that G is non amenable. let also $\gamma(n) =$ Haar measure in Q of Ω^n where $\Omega = \Omega^{-1} \subset Q$ is a compact Nhd of e in Q. $\gamma(n)$ is the growth function of Q and we always have either $C^{-1}\, n^D \leq \gamma(n) \leq C\, n^D$ $(n \geq 1)$ for some $C > 0$ and $D = 0, 1, 2, \ldots$, if Q is of polynomial growth, or we have $\gamma(n) \geq C\, e^{Cn}$ $(n \geq 1)$ for some $(C > 0)$, if Q is of exponential growth. The number $D = D(G)$ only depends on G and is independent of the particular choice of Ω (*cf.* [17]).

For a Lie group as above we can consider a left invariant subelliptic Laplacien $\Delta = -\Sigma X_j^2$ and the corresponding Heat diffusion semigroup $e^{-t\Delta}$. The corresponding convolution kernel ϕ_t can then be defined by (*cf.* [3], [10])

$$e^{-t\Delta} f(x) = \int_G \phi_t (y^{-1} x) f(y) \, dy; \qquad f \in C_0^\infty (G).$$

To avoid unecessary complications let us assume from here onwards that G is unimodular and let us define $d\mu(g) = \phi_1(g)\, dg$. The above Theorem applies to such a measure (*cf.* [3]) and it is interesting to observe that in that case the spectral gap of μ has the following geometric interpretation.

$$\lambda = \inf_{0 \neq f \in C_0^\infty} \frac{\| \nabla f \|_2^2}{\| f \|_2^2} \tag{0.2}$$

where

$$\| \nabla f \|_2^2 = (\Delta f, f) = \sum_f \int_G | X_j f |^2 \, dg.$$

In this paper I shall prove the following theorem that improves previous results of [1], [2].

THEOREM 1. – *Let G be a connected unimodular, non amenable, real Lie group and let Δ and $\phi_t(g)$ be as above let λ be the spectral gap of Δ as defined in* (0.2). *Let finally Q denote the radical of G.*

If Q is of polynomial growth and if $D = D(Q)$ is as above we have

$$\| \phi_t \|_\infty = 0 \, (e^{-\lambda t} \, t^{-q/2 - D/2})$$

where $q = q(G/Q)$ is defined as in (0.1).

If Q is of exponential growth there exists $c > 0$ such that

$$\| \phi_t \|_\infty = 0 \, (e^{-\lambda t - ct^{1/3}}).$$

To clarify the above theorem the following remarks are in order:

(i) For every open subset $\Omega \subset G$, by the local Harnack principle *cf.* [3], there exists $C > 0$ such that

$$\| \phi_t \|_\infty = \phi_t (e) \leq C \int_\Omega \phi_{t+1} (x) \, dx, \qquad t > 1$$

(ii) The estimates given by the theorem are unimprovable in the sense that they are sharp when G is soluble or when G is semisimple without center *cf.* [3], [2]. We shall come back to this question at the end of this paper.

(iii) Let G be a locally compact group and let $H \subset G$ be a closed normal subgroup that is amenable. Let further $\mu \in \mathbb{P}(G)$ and $\overset{\circ}{\mu} = \check{\pi}(\mu)$ be the image of μ by the canonical projection. Then λ the spectral gap of μ in G satisfies $\lambda = \overset{\circ}{\lambda}$ where $\overset{\circ}{\lambda}$ is the spectral gap of $\overset{\circ}{\mu}$ in G/H (*cf.* [4] and (5-1) bellow). This remark applies in particular to $Q \subset G$ where Q is as in our theorem and to $Z \subset \Sigma$ where Z is the center of the semisimple group Σ.

Let us now go back to a canonical Laplacian Δ on a real connected Lie group and observe (*cf.* [3]) that there exists $\delta = 1, 2, \ldots$ (if $G \neq \{e\}$) and $C > 0$ such that

$$C^{-1}\, t^{-\delta/2} \le \phi_t(e) \le C\, t^{-\delta/2}; \qquad 0 < t < 1. \tag{0.3}$$

Let us also recall that we can define for every $\alpha \in \mathbb{C}$, $Re\,\alpha > 0$ and λ as in (0.2)

$$(\Delta - \lambda)^{-\alpha/2} = C_\alpha \int_0^\infty t^{\alpha/2-1}\, e^{-t\Delta}\, e^{\lambda t}\, dt. \tag{0.4}$$

More general functions of $\Delta - \lambda$ can also be defined by:

$$m(\Delta - \lambda) = \int_0^\infty m(x)\, dE_x \tag{0.5}$$

where $\Delta - \lambda = \int_0^\infty x\, dE_x$ is the spectral decomposition of $\Delta - \lambda$. From Theorem 1 we obtain easily the following natural generalization of classical results of Hardy and Littlewood (*cf.* [6]).

COROLLARY. – *Let G as in the theorem and let us assume that its radical Q is of exponential growth. Let further Δ, λ and δ be as* (0.2) (0.3) *and $L < p \le 2 \le q < +\infty$, $\alpha \in \mathbb{C}$, $a = Re\,\alpha > 0$ then the operator:*

$$(\Delta - \lambda)^{-\alpha/2} : L^p(G) \to L^q(G)$$

is bounded if and only if $1/p - 1/q \le a/\delta$.

The corollary extends to the more general operators (0.5) provided that $|m(x)| = 0(x^{-a/2})$. The analogue of our corollary when $Q = \{e\}$ has been proved in [1].

The proof of our theorem will be given in a slightly more general context. For G as in the theorem we shall consider $d\mu(g) = f(g)\,dg$ where $0 < f \in C(G)$, $E(\mu) < +\infty$, and the corresponding convolution powers $d\mu^n(g) = f_n(g)dg$. For every $C \subset\subset G$ we shall then show that $\int_C f_n(x)dx$ is either $0(e^{-\lambda(\mu)\,n}\, n^{-q/2-D/2})$ or $0(e^{-\lambda(\mu)\,n - cn^{1/3}})$ $(c > 0)$ as the case may be. This clearly contains our theorem since $0 < \phi_1(\bullet) \in C(G)$, and $E(\phi_1) = E(\mu) < +\infty$.

Let us now recall that $\phi_t(e) = \|T_t\|_{1\to\infty}$, where we denote throughout by $\|R\|_{p\to q}$ the operator norm $R : L^p(G) \to L^q(G)$. The Theorem 1 admits the following generalisation.

THEOREM 2. – *Let G, Q, λ, p, d, r, Δ and $T_t = e^{-t\Delta}$ be as in* (0.1) *and as in Theorem* 1. *Let us further assume that Q is of polynomial growth and that $D = D(Q)$ is as in Theorem* 1, *then for all* $2 < q \leq +\infty$ *we have:*

$$\|T_t\|_{2\to q} \leq C\, e^{-\lambda t}\, t^{-1/2\,(p+d/2)}\, t^{-(D+r)/2\,(1/2-1/q)}; \qquad t \geq 1 \tag{0.6}$$

where C is independent of t.

To understand the way the exponent of t is made up in (0.6), one should consider the case where $D = r = 0$. In that case the estimate (0.6) is a consequence of the Kunze-Stein phenomenon (*cf.* [13]). This basic observation comes from [1]. The interest of the estimate (0.6) lies in the fact that it is optimal. This is easely seen on product groups (as in § 6) for Laplacians that "split".

From the estimate (0.6) one easily deduces that if $\alpha \in \mathbb{C}$; $Re\,\alpha \geq 0$ and if $2 < q < +\infty$ are such that

$$2\,Re\,\alpha < p + d/2 + (D+r)\,(1/2 - 1/q),\ 1/2 - 1/q \leq \frac{Re\,\alpha}{\delta};$$

[with δ as in (0.3)] then the mapping:

$$(\Delta - \lambda)^{-\alpha/2} : L^2(G) \to L^q(G) \tag{0.7}$$

with $\lambda > 0$ as in (0.2) is bounded. If we dualise (0.7) and combine this with (0.7) we obtain a corresponding range of α's for which $(\Delta - \lambda)^{-\alpha/2}$ is $L^p \to L^q$ bounded with $p \neq q$, $1 < p \leq 2 \leq q < +\infty$. The estimate (0.7) generalizes results of [1].

The question naturally arises as to what happens when G is not unimodular. The right invariant Haar measure is given then by $d^r g = m(g)\, d^l g$ where $d^l g = dg$ is the left invariant Haar measure and $m(g)$ is the modular function normalised by $m(e) = 1$. For such a group we can still consider $\Delta = \Sigma X_j^2$ where X_j are left invariant fields that generate the Lie algebra of G. What is however natural here is to consider instead $\tilde{\Delta} = m^{1/2}\,\Delta m^{-1/2}$ which is now a left invariant operator on G that is in addition self-adjoint with respect to the left measure (*cf.* [5]).

The spectral gap of $\tilde{\Delta}$ is then given by:

$$\lambda = \inf_{f \in C_0^\infty} \left\{ \int_G \sum_j |X_j f|^2\, d^r g;\ \int_G |f|^2\, d^r g = 1 \right\} \tag{0.8}$$

(*cf.* [5]). In [5] I have considered the semigroup $\tilde{T}_t = e^{-t\tilde{\Delta}}$ and the powers $(\tilde{\Delta} - \lambda)^{\alpha}$, $\alpha \in \mathbb{C}$ and in [10] I have stated without proof the fact that:

$$\| \tilde{T}_t \|_{1\to\infty} = 0\,(e^{-\lambda t - ct^{1/3}}); \qquad t \geq 1 \tag{0.9}$$

for some $c > 0$ provided that G is a (C) group (I shall refer the reader to [10], [11], [5] for the definition of the (C) condition). A proof of (0.9) was given in [10] only in the case when $\lambda = 0$. The proofs of [10] can however be adapted to give a proof of the estimate (0.9) in the general case (*i.e.* $\lambda \geq 0$). The details of the proof will appear in a forthcoming paper. The following Hardy-Littlewood type of theorem follows at once from (0.9).

THEOREM 3. – *Let G be a Lie group that satisfies the (C) condition but is not necessarily assumed to be unimodular. Let Δ and $\tilde{\Delta}$ be as above and let $\lambda \geq 0$ and $\delta = 1, 2, \ldots$ be as in* (0.8) *and* (0.3) *respectively. Let finally $1 < p \leq 2 \leq q < +\infty$, $\alpha \in \mathbb{C}$, $Re\,\alpha \geq 0$. Then the mapping:*

$$(\tilde{\Delta} - \lambda)^{-\alpha/2} : L^p\,(G;\,dg) \to L^q\,(G;\,dg)$$

is bounded if and only if:

$$1/p - 1/q \leq \frac{Re\,\alpha}{\delta}.$$

1. THE ACTION OF G/H ON H

Throughout this section we shall denote by G some locally compact group and by $H \subset G$ a closed normal subgroup. We shall say that G/H acts on H if there exists $\alpha : G/H \to \mathrm{Aut}\,(H)$ an algebraic homomorphism (α is not necessarily assumed to be continuous) and $S \subset G$ a locally bounded Borel section of the canonical projection $\pi : G \to G/H$ (*i.e.* $\pi^{-1}\,(C) \cap S$ is relatively compact for all compact subsets $C \subset\subset G/H$ and $\pi\,|_S$ is $(1-1)$ and onto $S \to G/H$) such that

$$\sigma^{-1}\,h\,\sigma = \alpha\,(\pi\,(\sigma))\,(h); \quad \sigma \in S, \quad h \in H.$$

When $H \subset G$ is a central subgroup, then G/H acts on H trivially (simply set α = Identity automorphism of H).

When G is a semidirect product of H with another closed subgroup K (*i.e.* when there exists $K \subset G$ a closed subgroup such that $H \cap K = \{e\}$ and $HK = G$) then again G/H acts on H for it suffices to set $S = K$. This is in particular the case when G is a simply connected real Lie group and $H = Q$ is its radical. Indeed in this case $G = Q \lambda M$ where M is some Levi subgroup $M \cong G/Q$. M is then simply connected and semisimple (*cf.* [7]).

The situation is more complicated when G is connected but not necessarily simply connected real Lie group. The analytic subgroup that corresponds to the radical of the Lie algebra is again a closed connected subgroup $Q \subset G$ and we can find $M_1 \subset G$ a (not necessarily closed) Levi subgroup which can be identified to a semisimple Lie group (*cf.* [7]). Let us denote by M the universal covering group of M_1. The inner automorphisms in G induce then an action of M on G and on Q and the semidirect product $\tilde{G} = Q \lambda M$ can be defined, $\tilde{G}$ is then a covering group of G. We shall denote by $\theta : \tilde{G} \to G$ the corresponding covering map. Ker θ lies then in the center of $\tilde{G}$.

Let us denote by $Z(M)$ the center of M (which is a discrete closed subgroup), then there exists $Z \subset Z(M)$ a subgroup that is of finite index *i.e.* $[Z(M) : Z] < +\infty$ and such that the action of Z on Q is trivial. Here is a proof of this fact. First of all we have $Z(M) = Z(M_1) \times \ldots Z(M_k)$ where $M_1 \times \ldots \times M_k = M$ is the decomposition of M into simple factors. Observe next that it is enough to consider the linear action Ad of $Z(M)$ on q the Lie algebra of Q. By Schur's Lemma for each $z \in Z(M_i)$ $(i = 1, \ldots, k)$ Ad (z) gives rise to a scalar matrix on each component of the representation Ad on $GL(q)$. Since, by semisimplicity, the determinant of this matrix is one, it follows that this scalar matrix can be identified with a root of unity. Our assertion follows.

The situation we have is now this:

$$\begin{tikzcd}[column sep=small, row sep=small] & \tilde{H} = \pi^{-1}(Z) \arrow[dr, hook] & \\ \operatorname{Ker}\theta \cap \tilde{H} \arrow[ur, hook] \arrow[dr, hook] & & \tilde{G} \\ & \operatorname{Ker}\theta \arrow[ur, hook] & \end{tikzcd}$$

where $\pi : Q \lambda M \to M$ denotes the canonical projection.

If we quotient by Ker $\theta \cap \tilde{H}$ we obtain

$$H_1 = \tilde{H}/\operatorname{Ker}\theta \cap \tilde{H} \subseteq G_1 = \tilde{G}/\operatorname{Ker}\theta \cap \tilde{H} \to \tilde{G}/\operatorname{Ker}\theta = G$$

H_1 is clearly a closed normal subgroup of G_1. Since on the other hand Ker θ is central in $\tilde{G}$ we have Ker $\theta \subset \pi^{-1}(Z(M))$ and since $\tilde{H} = \pi^{-1}(Z) \subset \pi^{-1}(Z(M))$ is of finite index it follows that Ker $\theta \cap \tilde{H}$ is of finite index in Ker θ. This means that G_1 is a finite cover of G (in the esoteric terminology of the subject one says that G_1 is *isogenic* with G). We also have:

$$G_1/H_1 \cong \tilde{G}/\tilde{H} \cong M/Z$$

and therefore G_1/H_1 is a semisimple group with finite center. M now acts canonically by inner automorphism on $\tilde{G}$ this action stabilises every element of $\operatorname{Ker}\theta$ furthermore the action of every element of Z on $\tilde{G}$ is trivial. We obtain thus canonical mappings: $M \to \text{Inner Aut}(\tilde{G})$; $M \to \text{Inner Aut}(G_1)$; $M/Z \to \text{Inner Aut}(\tilde{G})$; $M/Z \to \text{Inner Aut}(G_1)$ and it is very easy to verify that the induced action:

$$\alpha : G_1/H_1 \cong M/Z \to \text{Aut}(H_1)$$

satisfies the conditions given at the beginning of this section.

We shall collect all the information obtained in this section in the following

PROPOSITION. – *Let G be a real connected Lie group. There exists then G_1 a Lie group that is isogenic to G and $H_1 \subset G_1$ a closed normal subgroup such that G_1/H_1 is semisimple with finite center and such that G_1/H_1 acts on H_1 in the sense defined at the beginning of this section. Furthermore H_1 is isomorphic with $Q_1 \times D$ where Q_1 is a finite extension of Q and $D \cong \mathbb{Z}^r$ with $r =$* rank *of the center of G/Q.*

To see the last point observe, that by our construction, there exists D_1 a subgroup of finite index of $Z(M)$ such that $\tilde{H} = Q \times D_1 \subset Q \lambda M = \tilde{G}$. It is furthermore clear that $\operatorname{Ker}\theta \cap Q = \{e\}$ and therefore $H_1 = Q\,\theta(D_1)$. Here $\theta(D_1)$ is a finitely generated, but not necessarily closed, subgroup that is central in G_1.

It follows that $H_1/Q \cong \mathbb{Z}^r \times F$ where F a finite abelian. Let us denote by $Q_1 = \kappa^{-1}(F)$ where $\kappa : H_1 \to H_1/Q \cong \mathbb{Z}^r \times F$. Let us further choose $\zeta_1, \ldots, \zeta_r \in \theta(D_1)$ such that $\kappa(\zeta_j)$ $j = 1, \ldots, r$ are free generators of $\mathbb{Z}^r$ and set $D = Gp(\zeta_1, \ldots, \zeta_r)$. It is then clear that $H_1 = Q_1 \times D$. Finally $r =$ rank of center of $G_1/Q_1 =$ rank of center of G_1/Q because the center of G_1/H_1 is finite. But then, by the isogeny between G and G_1, r is also the rank of the center of G/Q.

2. EQUIVALENT MEASURES AND THE NASH INEQUALITIES

Let G be a unimodular compactly generated locally compact group. We shall define first an equivalence relation on the set of probability measures $\mathbb{P}(G)$ of G. We shall write $\mu \sim \nu \in \mathbb{P}(G)$ if there exists $g \in G$ or $\alpha \in \text{Aut}(G)$ such that one (or several) of the following relations hold

$$\mu = \delta_g * \nu; \qquad \mu = \nu * \delta_g; \qquad \mu = \check{\alpha}(\nu) \qquad \text{et} \qquad \check{\alpha}(h) = h.$$

$\check{\alpha}$ is the mapping induced by α on measures, h is the Haar measure on G, and δ_g is the point mass at g. We shall say that two measures

$\mu_1, \mu_2 \in \mathbb{P}(G)$ are equivalent and write $\mu_1 \approx \mu_2$ if there exists $p \geq 1$ and $\nu_1, \nu_2, \ldots, \nu_p \in \mathbb{P}(G)$ such that $\mu_1 \sim \nu_1 \sim \ldots \nu_p \sim \mu_2$. It is of course clear that for two equivalent measures $\mu \approx \nu$ we have $\| \mu \|_{p \to q} = \| \nu \|_{p \to q}$.

Let $\mu \in \mathbb{P}(G)$ and $n > 0$ we shall then define $N_n(\mu) = \inf C$ (*i.e.* the optimal C) among the numbers $C > 0$ that satisfy

$$\| \mu * f \|_2^{2(1+2/n)} \leq C [\| f \|_2^2 - \| \mu * f \|_2^2] \| f \|_1^{4/n}; \qquad f \in C_0(G).$$

We shall of course set $N_n(\mu) = +\infty$ if no such $C > 0$ exist. Let us assume that G is such that $\gamma(n) \geq C n^D$ (*cf.* § 0 for the definition of γ) for some $D > 2$ and let $\Omega = \Omega^{-1} \subset G$ be some open symmetric generating (*i.e.* $\bigcup_n \Omega^n = G$) neighbourhood of e in G and $C, \varepsilon_0 > 0$. Let further $d\mu = f\, dg$ be a probability measure and let us assume that the density f satisfies

$$\text{(2.1)} \qquad \| f \|_\infty \leq C; \qquad f(g) \geq \varepsilon_0 \quad \forall\, g \in g_0\, \Omega \quad \text{for } some \ g_0 \in G.$$

Then $N_D(\mu) \leq C_1 = C_1(C, \varepsilon_0, \Omega)$. It is important to observe that $C_1(C, \varepsilon_0, \Omega)$ is independent of g_0 (*cf.* [3], [8]). Up to the above "$\approx$" equivalence relation between measures, we can replace $g_0\, \Omega$ by $\Omega\, g_0$ in (2.1). This fact and the freedom of choice for g_0 is basic for the understanding of (3.2) further down, and for the proofs of our theorems. If we relax the condition $D > 2$ then the same fact holds for measures that satisfy (2.1) with arbitrary $D > 0$ provided that we assume in addition that the second moment of the measure $E = \int_G |g|^2\, d\mu(g) < +\infty$ is finite. The constant C_1 depends then also on E. We shall not need this refinement in this paper and shall therefore not give the proof.

When $\gamma(n) \geq C_0\, e^{c_0 n}$ for some $C_0, c_0 > 0$, then the measures $\mu \in \mathbb{P}(G)$ for which (2.1) holds satisfy the stronger inequality (*cf.* [3]):

$$\| f \|_2^2 \leq C \lambda^2 [\| f \|_2^2 - \| \mu * f \|_2^2] + C_2\, e^{-c_2 \lambda} \| f \|_1^2; \qquad (N_\infty) \qquad f \in C_0(G) \qquad \lambda > 1$$

where C_2, c_2 only depend on G (in fact only on C_0, c_0). Inequalities of the form (N_∞) where considered for the first time in [9]. [To extract (N_∞) from [3] and (2.1) we may assume that $g_0 = e$. But then with the notations of [3], VII.5, the Dirichlet form $\| f \|_2^2 - \| \mu * f \|^2 = ((\delta - \check{\mu} * \mu) f, f)$ clearly dominates $\| (I - T_0)^{1/2} f \|_2^2$]. For any $\mu \in \mathbb{P}(G)$ we shall define $N_\infty(\mu) = \inf C$ where the inf is taken among the numbers $C > 0$ for which (N_∞) holds with the convention that $N_\infty(\mu) = +\infty$ if not such a number exists.

It is evident from the definition that $N_n(\mu) = N_n(\delta_g * \mu)$ for all $n \in]0, +\infty]$, $\mu \in \mathbb{P}(G)$, $g \in G$. It can also be easily verified that for any unimodular automorphism $\alpha \in \operatorname{Aut}(G)$ (*i.e.* $\check{\alpha}(h) = h$) we have $N_n(\mu) = N_n(\check{\alpha}(\mu))$. Combining these two remarks, and the fact that $x \mapsto gxg^{-1}$ is a unimodular automorphism of G, we deduce that

$$\mu,\ \nu \in \mathbb{P}(G) \quad \mu \approx \nu \Rightarrow N_n(\mu) = N_n(\nu) \quad \forall\, n \in]0, +\infty].$$

Let us now consider n measures $\mu_1, \ldots, \mu_n \in \mathbb{P}(G)$ and a subsequence $1 \leq n_1 < \ldots < n_k \leq n$ that satisfies

$$\| \mu_{n_j} \|_{1\to 2} \leq C; \qquad N_\nu(\mu_{n_j}) \leq C \qquad j = 1, \ldots, k \tag{2.2}$$

for some fixed $\nu \in]0, +\infty]$. If $0 < \nu < +\infty$ we shall conclude from (2.2) that:

$$\| \mu_1 * \ldots * \mu_n \|_{1\to 2} \leq C_3\, k^{-\nu/4} \tag{2.3}$$

if $\nu = +\infty$ we shall conclude that:

$$\| \mu_1 * \ldots * \mu_n \|_{1\to 2} \leq C_3\, e^{-c_3\, k^{1/3}} \tag{2.4}$$

where C_3, $c_3 > 0$ only depend on ν and C in (2.2).

Let $t_j = \| \mu_j * \mu_{j-1} * \ldots * \mu_1 * f \|_2^2$ for some fixed $f \in C_0(G)$ with $\| f \|_1 = 1$ and let $\tau_j = t_{n_j}$ $(1 \leq j \leq k)$. Then clearly the sequence $t_1 \geq t_2 \geq \ldots$ is non increasing and therefore when $\nu = +\infty$ we have

$$\tau_1 \leq C; \qquad \tau_p \leq C\,\lambda^2\,(\tau_p - \tau_{p+1}) + C_2\, e^{-c_2\,\lambda}; \qquad \lambda > 1. \tag{2.5}$$

The inequalities (2.5) can easily be "integrated" (*cf.* [3]) and (2.4) follows.

When $\nu < +\infty$ the proof is a trifle more subtle. Let $t(x)$ be a continuous function for $x \in [n_1, n]$ that satisfies $t(j) = t_j$ $j = n_1, n_1 + 1, \ldots n$ and is piecewise linear for x between two integers. It is clear that we have:

$$\frac{d}{dx} t(x) \leq 0; \qquad t(x) \leq C,\, x \in [n_1, n]; \qquad \frac{d}{dx} t(x) \leq -C\,(t(x))^{1+2/\nu},$$
$$x \in [n_p, n_p + 1] \qquad p = 1, \ldots, k.$$

Substituting $\xi(x) = 1/t(x)$ we obtain

$$\frac{d}{dx}(\xi(x))^{2/\nu} \geq C(x); \qquad x \in [n_1, n] \tag{2.6}$$

where $C(x) \geq 0$ and $C(x) \geq C > 0$ if $x \in [n_p, n_{p+1}]$ $p = 1, \ldots, k$. If we integrate the differential inequality (2.6) we obtain (2.3).

Let now G be a unimodular compactly generated locally compact group. Let us further consider $\mu_1, \ldots, \mu_n \in \mathbb{P}(G)$ n measures on G and a subsequence $1 \leq n_1 < \ldots < n_k \leq n$ such each measure $d\mu_{n_j}(g) = f_j(g)\,dg$ is given by a density that satisfies (2.1) for some fixed C, ε_0 and $\Omega \subset G$. Let further $\nu_1, \ldots, \nu_n \in \mathbb{P}(G)$ be another sequence of measures such that $\nu_j \approx \mu_j$ (equivalence in the sense given at the beginning of this section). Let us also assume that the growth function of G satisfies $\gamma(n) \geq C\,n^D$ for some $C > 0$, $D > 2$. We can conclude then from the above conditions that

$$\|\nu_1 * \ldots * \nu_n\|_{1\to\infty} \leq C\,k^{-D/2}. \tag{2.7}$$

Indeed for any $1 \leq s \leq n$ we have

$$\|\nu_1 * \ldots * \nu_n\|_{1\to\infty} \leq \|\nu_s * \ldots * \nu_n\|_{1\to 2}\,\|\check{\nu}_{s-1} * \ldots * \check{\nu}_1\|_{1\to 2}$$

where the $\check{}$ is defined by $\check{\nu}(x) = \nu(x^{-1})$ (the $\check{}$ induces the adjoint convolution operator). If we choose $1 \leq s \leq n$ so that it "cuts" the subsequence $n_1 \ldots < n_k$ "about half way" and if we bare in mind the fact that the conditions (2.1) are, up to equivalence, stable by the involution $x \to x^{-1}$ we see that (2.7) is an immediate consequence of (2.3). Similarly we see from (2.4) that if $\gamma(n) \geq C\,e^{cn}$ for some C, $c > 0$, then

$$\|\nu_1 * \ldots * \nu_n\|_{1\to\infty} \leq C\,e^{-ck^{1/3}}. \tag{2.8}$$

3. THE DISINTEGRATION OF A MEASURE

In this section we shall place ourselves in the context of a locally compact unimodular group G with a closed normal subgroup $H \subset G$ such that there exists (as in paragraph 1) $\alpha : G/H \to \mathrm{Aut}(H)$ an algebraic homomorphism and $S \subset G$ a locally bounded Borel section of $\pi : G \to G/H$ for which $\alpha(\pi(s))\,h = s^{-1}\,hs$ ($\forall\, s \in S$, $h \in H$). We shall also assume that G/H is unimodular.

Let $\mu \in \mathbb{P}(G)$ we can then disintegrate that measure along the cosets of H.

$$\mu = \int_{G/H} \mu_x\,d\,\mathring{\mu}\,(x)$$

where $\mathring{\mu} = \check{\pi}(\mu) \in \mathbb{P}(G/H)$ is the image of μ induced by the mapping π and $\mu_x \in \mathbb{P}(\pi^{-1}(x))$ ($x \in G/H$). The measure μ_x is defined only for

$\overset{\circ}{\mu}$-almost all $x \in G/H$. The Borel section S can then be used to identify $\pi^{-1}(x)$ with H $(: \pi^{-1}(x) = H\, s \leftrightarrow H$ for $s \in S \cap \pi^{-1}(x))$. We can therefore identify μ_x with a measure on H.

We shall now make the additional hypothesis that the measure $d\mu(g) = f(g)\, dg$ is given by a continuous positive density $f(g) > 0$ $(g \in G)$. It follows then that $d\overset{\circ}{\mu} = \overset{\circ}{f}\, d\overset{\circ}{g} \in L^1(G/H)$ and for *every* $x \in G/H$ the measure $d\mu_x(h) = f_x(h)\, dh$ is given by a continuous density on H (we use the above identification to set $\mu_x \in \mathbb{P}(H)$). Given any $\varepsilon > 0$ it is easy to see that we can find $A \subset\subset G/H$ a compact subset such that $\overset{\circ}{\mu}(G/H \backslash A) \leq \varepsilon$ and we can also find, C, $\varepsilon_0 > 0$ and $\Omega \subset H$ as in (2.1) such that for each $x \in A$ the measure μ_x satisfie (2.1) (the g_0 of (2.1) depends on x, and we chose A such that $\overset{\circ}{f} \leq C$ on A).

We can now apply (3.1) to the convolution power μ^n of μ and obtain

$$\mu^n = \int_{G/H} \mu_x^{(n)}\, d\overset{\circ}{\mu}{}^n(x) \tag{3.1$'$}$$

where $\overset{\circ}{\mu}{}^n$ is the convolution power of $\overset{\circ}{\mu}$ on G/H and where $\mu_x^{(n)}$ can be identified to a measure in $\mathbb{P}(H)$.

We shall now consider Ω the path space $(\omega = (S_1, S_2, \ldots) \in \Omega)$ of the left invariant random walk on G/H $S_n = X_1 X_2 \ldots X_n$ with independent increment X_j given by $\mathbb{P}(X_j \in dx) = d\overset{\circ}{\mu}(x)$. Using probabilisitc notations we can then write $\mu_x^{(n)}$ as a conditional expectation:

$$\mu_x^{(n)} = \mathbb{E}(\tilde{\mu}_{X_1} * \tilde{\mu}_{X_2} * \ldots * \tilde{\mu}_{X_n} // S_n = x) \tag{3.2}$$

where $\mu_{X_j} \approx \tilde{\mu}_{X_j}$ $(\mu_{X_j} = \mu_x$ as in (3.1) with $x = X_j)$ and where $\approx$, the equivalence relation, is random (*i.e.* the chain $\mu_{X_j} \sim \nu_1 \sim \ldots \sim \tilde{\mu}_{X_j}$ depends on the path ω). The formula (3.2) is basic for us and it has already been used crucially in [8], [10], [11]. The main observation that is used for the proof of (3.2) is the fact that all the inner automorphisms $h \to g^{-1} hg$ $(g \in G,\ h \in H)$ are unimodular on H (this is a consequence of the unimodularity of G). Let us now fix $\varphi \in C_0(G)$ and let us use (3.2). We see that for any decomposition $\Omega = \Omega_1 \cup \Omega_2$ we have

$$\begin{aligned} |\langle \mu^n, \varphi \rangle| \leq{} & C\, \mathbb{E}[\| \tilde{\mu}_{X_1} * \ldots * \tilde{\mu}_{X_n} \|_{1 \to \infty} I(\Omega_1)\, I(S_n \in K)] \\ & + C\, \mathbb{E}[I(\Omega_2)\, I(S_n \in K)] \end{aligned} \tag{3.3}$$

where $K \subset\subset G/H$ and $C > 0$ depend on φ and $I(\bullet)$ denotes the characteristic function of a set. It is this inequality, that for a proper choice of the decomposition $\Omega_1 \cup \Omega_2$, will give the appropriate estimate of $|\langle \mu^n, \varphi \rangle|$ and the proof of our theorem.

4. PROOF OF THEOREM 1

We shall place ourselves once more in the context of a locally compact unimodular group G with a closed compactly generated subgroup H that satisfies the conditions of § 3. All the notations introduced up to now, and especially in § 3, will be preserved.

We shall fix $d\mu = f\, dg$ a probability measure given by a continuous density. Let $0 < \eta < 1$ be some small number to be chosen latter, let $1 \leq n$ be an integer and let the $A \subset\subset G/H$, that was considered in § 3, be chosen such that the following subset of the path space:

$$\Omega_1 = \left\{ [\text{number of } j\text{'s},\ 1 \leq j \leq n,\ \text{ for which } X_j \in A] \geq \frac{1}{2}\, n \right\}$$

satisfies

$$\mathbf{P}(\Omega_1) \geq 1 - \eta^n.$$

This is clearly possible if $\mathbf{P}(G/H \backslash A) \leq \varepsilon$ small enough (the proof of this fact is an elementary calculation involving Bernouille coefficients $\binom{n}{j} \varepsilon^j (1-\varepsilon)^{n-j}$ and will be left as an exercice for the reader). The partition of the space $\Omega = \Omega_1 \cup \Omega_2$ will be then defined by setting $\Omega_2 = \Omega \backslash \Omega_1$.

For fixed $\varphi \in C_0(G)$ as in (3.3) the second member of the right hand side of (3.3) can be estimated by $C\,\eta^n$. The first member of the right hand side of (3.3) can be estimated by

$$C \sup_{\omega \in \Omega_1} \| \tilde{\mu}_{X_1} * \ldots * \tilde{\mu}_{X_n} \|_{1 \to \infty}\, \mathbf{P}(S_n \in K). \tag{4.1}$$

Now by our definition of Ω_1 and (2.8) we see that if H is of exponential volume growth then

$$\Lambda_n = \sup_{\omega \in \Omega_1} \| \tilde{\mu}_{X_1} * \ldots * \tilde{\mu}_{X_n} \|_{1 \to \infty} \leq C\, e^{-cn^{1/3}}, \qquad n \geq 1 \tag{4.2}$$

for some C, $c > 0$. Similarly by (2.7), if we assume that $\gamma_H(n) \geq cn^D$, $c > 0$, $D > 2$, we can assert that there exists $C > 0$ such that

$$\Lambda_n \leq C\, n^{-D/2}, \qquad n \geq 1. \tag{4.3}$$

We are now in a position to complete the proof of Theorem 1. The first step is to use the consideration of § 1 and replace, if necessary, G by an isogenic group G_1 (this clearly does not affect the conclusion of Theorem 1) such that G_1 and $G_1 \supset H_1$ satisfy the conditions of the proposition of § 1. The next reduction is to be able to assume, in the case when Q is of polynomial growth, that $D = D(Q) > 2$. This is done by the standard trick (*cf.* [12]) of replacing, if necessary, G by $G \times \mathbb{R}^3$ and Δ by $\Delta+$ standard Laplacian on $\mathbb{R}^3$.

Having done these reductions we consider $d\mu_1 = f_1\, dg$ with $0 < f_1 \in C(G_1)$ and $E(\mu_1) < +\infty$ and apply the (4.1) and (4.2) or (4.3) to the subgroup H_1. The estimate (3.3) together Bougerol's theorem for the semisimple group G_1/H_1 that we apply to $d\mu = f\, dg \in L^1(G_1/H_1)$ where $f = \overset{\circ}{f}_1 > 0$ and $E(\mu) < +\infty$ completes the proof of Theorem 1.

Let us finally give the proof of the corollary. Towards that we shall decompose the integral in (0.4) as follows:

$$I = \int_0^\infty t^{\alpha/2-1}\, e^{-t\Delta}\, e^{\lambda t}\, dt = \int_0^1 + \int_1^\infty = I_1 + I_2.$$

It is clear that $\| I_1 \|_{p\to q} < +\infty$ for $1/p - 1/q \leq \dfrac{Re\, \alpha}{\delta}$ (*cf.* [3]).

When Q is of exponential growth, by our theorem, we have (*cf.* [5]) $\| e^{-t\Delta}\, e^{\lambda t} \|_{2\to q} = 0\,(t^{-A})$ for $t > 1$ and any $A > 0$, $q > 2$. Therefore $\| I_2 \|_{2\to q} < +\infty$ for any $q > 2$. The conclusion is that $\| I \|_{2\to q} < +\infty$ as long as $q > 2$ and $1/2 - 1/q \leq \dfrac{Re\, \alpha}{\delta}$. The corollary then follows by duality.

The boundedness of the more general operators (0.5) follows by factorising these operators in the obvious way $L^p \to L^2 \to L^2 \to L^q$.

5. PROOF OF THEOREM 2

The basis of the proof of the estimate (0.6) is once more the disintegration formula (3.1). Let G, H, $\mu \in \mathbb{P}(G)$ be as in § 3 and let us disintegrate μ^n as in (3.1)$'$, then for any $1 \leq p,\ q \leq +\infty$ we have

$$\| \mu \|_{p\to q} \leq \| \overset{\circ}{\mu} \|_{p\to q} \sup_x \| \mu_x \|_{p\to q} \tag{5.1}$$

where here $\| \mu \|_{p\to q}$ is the $L^p \to L^q$ convolution norm on G, $\| \overset{\circ}{\mu} \|_{p\to q}$ is the convolution norm of $\overset{\circ}{\mu}$ on G/H and $\| \mu_x \|_{p\to q}$ is the convolution norm on H. To prove (5.1) we use very heavily the unimodularity of the groups. The details will be left as an exercise for the reader.

Let now Ω have the same meaning as in § 3 and let us assume that $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$ be a covering of Ω by (not necessarily disjoint) subsets. With the same notations as in § 3 and 4, we shall then define for each fixed $n \geq 1$

$$\mu_i = \mathbb{E}\,(\tilde{\mu}_{X_1} * \ldots * \tilde{\mu}_{X_n}\, I\,(\Omega_i)); \qquad i = 1,\, 2,\, \ldots$$

It is clear of course that:

$$\| \mu^n \|_{p\to q} \leq \sum_{i=1}^{\infty} \| \mu_i \|_{p\to q}$$

and from (5.1) it follows that

$$\| \mu_i \|_{p\to q} \leq \sup_{\omega\in\Omega_i} \| \tilde{\mu}_{X_1} * \ldots * \tilde{\mu}_{X_n} \|_{p\to q} \| \overset{\circ}{\mu}_i \|_{p\to q} \tag{5.2}$$

where $\overset{\circ}{\mu}_i$ is the projection of μ_i on G/H and is given by:

$$d\,\overset{\circ}{\mu}_i\,(\overset{\circ}{g}) = \mathbb{P}_e\,[S_n \in d\,\overset{\circ}{g};\ \omega \in \Omega_i];\ \overset{\circ}{g}\in G/H \quad i = 1,\, 2,\, \ldots$$

In the above construction the set Ω_1 will be defined exactly as in § 4 for some $A \subset G/H$ large enough. Let us also assume for the moment that the volume growth of H satisfies $\gamma_H\,(t) \geq c\,t^L$, $(t > 1)$ for some $L > 2$. For $i = 1$ the first factor on the right hand side of (5.2) can then be estimated by

$$C\,n^{-\frac{L}{2}\,(1/p-1/q)}. \tag{5.3}$$

This is a consequence of (2.7) and interpolation. To estimate the second factor of the right hand side of (5.2) for any $i = 1,\, 2,\, \ldots$ we shall make the additional assumption that G/H is semisimple with finite center and that $d\,\overset{\circ}{\mu}_i\,(g) = f_n^{(i)}\,(g)\,dg$. The Kunze-Stein phenomenon (*cf.* [13]) gives then the estimate (for $p = 2$, $q > 2$)

$$\| \overset{\circ}{\mu}_i \|_{2\to q} \leq C\, \| f_n^{(i)} \|_{L^2\,(G/H)}.$$

This will be used for $i = 1$ and

$$d\mu = \phi_1\,dg \tag{5.4}$$

where ϕ_t is a Heat diffusion kernel as in § 0. We obtain

$$\| \overset{\circ}{\mu}_1 \|_{2\to q} \leq C \| \overset{\circ}{\phi}_n \|_{L^2(G/H)} = C \overset{\circ}{\phi}_{2n}(e)^{1/2} \leq C e^{-\lambda n} n^{-\frac{1}{2}(p+d/2)}$$

which together with (5.3) gives

$$\| \mu_1 \|_{2\to q} \leq C e^{-\lambda n} n^{-\frac{1}{2}(p+d/2)} n^{-\frac{L}{2}(1/2-1/q)}. \tag{5.5}$$

It remains to control the contributions of the terms μ_i, $i = 2, \ldots$ To be able to do this we have to choose $\Omega_2, \ldots, \Omega_n, \ldots$ appropriately. For $i = 2, \ldots$ we choose

$$\omega \in \Omega_i \quad \Leftrightarrow \quad \omega \notin \Omega_{i-1}; \qquad \inf_{1\leq k\leq n} |X_k(\omega)| \leq i.$$

We use here the notation $| \overset{\circ}{g} | = d(e, \overset{\circ}{g})$, ($\overset{\circ}{g} \in G/H$) for the canonical distance on G/H (*cf.* [3]). It is clear that

$$\mathbf{P}(\inf_{1\leq k\leq n} |X_k| \geq p) \leq \{\overset{\circ}{\mu}(| \overset{\circ}{g} | \geq p)\}^n.$$

From this and the standard Gaussian estimate on $\phi_1(g)$ (*cf.* [3]), it follows that with μ as in (5.4) we have

$$\| \overset{\circ}{\mu}_{p+1} \| \leq C e^{-cnp^2}.$$

Since in our case we have $d\overset{\circ}{\mu}^n(g) = \overset{\circ}{\phi}_n(g)\, dg$ we can conclude also that $d\overset{\circ}{\mu}_i = \overset{\circ}{\psi}_i\, d\overset{\circ}{g}$ and

$$\| \overset{\circ}{\mu}_i \|_{2\to q} \leq C \| \overset{\circ}{\psi}_i \|_{L^2(G/H)} \leq C \| \overset{\circ}{\phi}_n \|_\infty^{1/2} e^{-cn(i-1)^2} \tag{5.6}$$
$$\leq C e^{-cn(i-1)^2}.$$

Now for $i = 2, \ldots$ the first factor of the right hand side of (5.2) can be estimated by

$$\sup_{\omega\in\Omega_i} (\inf_{1\leq k\leq n} \| \tilde{\mu}_{X_k} \|_{p\to q}) \leq \sup_{\omega\in\Omega_i} \inf_{1\leq k\leq n} \| f_{X_k} \|_\infty^C \tag{5.7}$$

where as in § 3 we denote by $d\mu_x(h) = f_x(h)\, dh$ ($h \in H$, $x \in G/H$). To see this we simply use the log-convexity of the $\| \ \|_p$ and the fact that each $\tilde{\mu}_{X_k}$ is a probability measure. To estimate (5.7) we use the following two inequalities

$$\| \phi_1 \|_\infty \leq C; \qquad \overset{\circ}{\phi}_1(\overset{\circ}{g}) \geq C \exp(-c | \overset{\circ}{g} |^2), \qquad \overset{\circ}{g} \in G/H$$

(*cf.* [3]). It follows that the right hand side of (5.7) can be estimated by

$$C \exp(C i^2). \tag{5.8}$$

We can now complete the proof of our theorem. We start by choosing A very large so that $\mathbb{P}(\Omega_1) \geq 1 - \eta^n$ with an η very small to be chosen later. This means that $\| \overset{\circ}{\mu}_i \| \leq \eta^n$ $(i = 2, \ldots)$ and therefore just as in (5.6) $\| \overset{\circ}{\mu}_i \|_{2\to q} \leq C\, \eta^n$. We then fix some $k = 2, \ldots$ and estimate $\| \mu_i \|_{2\to q}$ using (5.2) (5.6) and (5.8) for $i = k + 1, \ldots$ For k large enough we obtain thus the estimate:

$$\sum_{i\geq 2} \| \mu_i \|_{2\to q} \leq C\, e^{-C k^2 n} + C\, \eta^n.$$

For an appropriate choice of η and k the above estimate together with (5.5) completes the proof of (0.6).

In the above proof we have of course used the special structure of the group G which had to satisfy the conditions of § 3 and be such that G/H is semisimple with finite center. In addition the volume growth of H was assumed to satisfy $\gamma_H(t) \geq ct^L$, $(t > 1)$ with $L = D + r > 2$. Here D and r are as in Theorem 2. At this stage we shall invoque the proposition of § 1 which shows that up to isogeny we can assume that we are in the above situation (possibly with $D + r = 0, 1, 2$). The next observation is that the conclusion of Theorem 2 is stable by isogeny. In fact the conclusion of this theorem is even stable by taking the quotient by a compact subgroup (*i.e.* passing from G to G/K with K compact. To see this we use the Harnack principle and average over K). To deal with the exceptional cases $D + r = 0, 1, 2$, we use once more the usual trick of jacking up the dimension by replacing the group G by $G \times \mathbb{R}^3$ as we did in § 4. This completes the proof of Theorem 2. The proof of (0.7) follows then immediately by the same argument as at the end of § 4.

6. THE LOWER ESTIMATES

The results in this final section are not sharp, we shall therefore be brief. What we shall show is that if $G = Q \times \Sigma$ is a direct product of its radical Q, which will be assumed to have polynomial volume growth, and of some semisimple group Σ, and if Δ, λ, q, r, D, ϕ_t are as in § 0, then there exists $C > 0$ such that

$$\phi_t(e) \geq C\, e^{-\lambda t}\, t^{-\frac{q+D}{2}} (\log t)^{-\frac{D+r}{2}}, \qquad t > 1.$$

This shows that the estimates obtained in Theorem 1 are essentially unimprovable.

The proof of this estimate is an easy consequence of the following general principle: let G be a general Lie group and let $H \subset G$ a closed normal subgroup that is of polynomial growth (this implies that H is amenable but H will not be assumed to be connected). We shall further assume that $d_H(x, y)$ $(x, y \in H)$, the intrinsic distance in H is equivalent (for large distances *cf.* [3]) to the induced distance by the embeding $H \subset G$. To be more explicit if we denote by $d_G(x, y)$, $(x, y \in G)$ the canonical left invariant distance on G (*cf.* [3]) then there exists $C > 0$ such that

$$d_G(x, y) \geq C^{-1} d_H(x, y); \qquad x, y \in H \qquad d_H(x, y) \geq C.$$

This phenomenon is rather rare (*cf.* [11]), but (6.1) does hold in the following two cases:

Case A: $G \cong H \times G/H$ *i.e.* H is a direct factor. The verification is trivial.

Case B: G is semisimple and $H = Z$ is the discrete center of G. (6.1) is then not trivial and the verification relies on the fact that if $G = NAK$ then $Z \subset K$ and K/Z is compact. To prove (6.1) we first project $G \to G/AN \cong K$ and obtain a Z invariant (but *not* K invariant) distance on K. Then we use the compactness of K/Z. The details will be left to the reader (*cf.* [14], [15]) where the above result is proved when the above distance is Riemannian).

For a group and a subgroup $G \supset H$ as above and Δ some subelliptic sublaplacian I shall denote by $\phi_t(g) = \phi_t^G(g)$ and $\overset{\circ}{\phi}_t(\overset{\circ}{g}) = \overset{\circ}{\phi}{}_t^{G/H}(\overset{\circ}{g})$, $\overset{\circ}{g} \in G/H$ the corresponding (canonically induced) heat diffusion convolution kernels ($\overset{\circ}{\phi}$ is induced by the projected laplacian $\overset{\circ}{\Delta} = d\pi(\Delta)$ on G/H where $\pi : G \to G/H$ is the canonical projection).

It is clear that

$$\int_G \phi_t(h)\, dh = \overset{\circ}{\phi}_t(e).$$

It is also known that

$$\phi_t(g) \leq C e^{-\lambda t} \exp\left(-\frac{d_G^2(e, g)}{C t}\right); \qquad t \geq 1$$

for some $C > 0$ (*cf.* [3]). From this it follows that there exists $C_0 > 0$ such that

$$\int_{|h| \geq C_0 \sqrt{t \log t}} \phi_t(h)\, dh \leq 1/10\ \overset{\circ}{\phi}_t(e)$$

provided that $\overset{\circ}{\phi}_t(e)$ verifies a lower estimate of the form:

$$\overset{\circ}{\phi}_t(e) \geq C^{-1}\, t^{-C}\, e^{-\lambda t}; \qquad t \geq 1 \tag{6.2}$$

for some $C > 0$. Assuming that this happens, then we immediately deduce that

$$\phi_t(e) = \sup_{h \in H} \phi_t(h) \geq C\ \overset{\circ}{\phi}_t(e)\, [\mathrm{Vol}_H\,(\text{Ball of radius } c_0\, \sqrt{t \log t})]^{-1}$$

If we apply the above procedure first in case B and then in case A, we obtain the required lower estimate. The estimate (6.2) (when $G/H = \Sigma/Z$ is a semisimple group with finite center), that is needed to complete the proof, has been proved in [2].

In fact for the above group $G = Q \times \Sigma$ one can improve the above lower estimate to the sharp result $\phi_t(e) \geq C\, e^{-\lambda t}\, t^{-\frac{q+D}{2}}$ $(t > 1)$. The proof is however considerably more difficult. It will be given elsewhere.

Observe finally that for more general groups (e.g. semidirect products $Q \lambda S$) the situation is very different and the upper estimate of our Theorem 1 can, in some cases, be improved dramatically. We shall publish a complete solution of the problem in a forthcoming paper (*cf.* [16]).

As a final remark I would like to observe that much of what has been proved in this paper automatically extend to more general sublaplacians of the form $\Delta = \Sigma X_j^2 + X_0$ and to measure that are not symmetric. One then of course has to define $e^{-\lambda} = \lim \| \mu^n \|_{2\to 2}^{1/n}$. This question however will be taken up again in a future paper.

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