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Random walk in a strongly inhomogeneous environment and invasion percolation

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Dedicated to the Memory of Claude Kipnis

ABSTRACT. – Motivated by d-dimensional diffusion in a gradient drift field with small diffusion constant ϵ, we consider an inhomogeneous, but reversible, continuous time nearest neighbor random walk X_t^ϵ on Z^d, or on some other locally finite graph. Let G_n^ϵ be the random subgraph whose edges are the first n distinct edges traversed by X_t^ϵ. We prove that if the strongly inhomogeneous ($\epsilon \to 0$) limit respects some ordering $\mathcal{O}$ of all edges, then $(G_0^\epsilon, G_1^\epsilon, G_2^\epsilon, \ldots)$ converges to invasion percolation for that $\mathcal{O}$.

Key words: Random walk, invasion percolation.

RÉSUMÉ. – Motivés par des diffusions d-dimensionnelles en un champ de dérive gradient avec petit coefficient de diffusion ϵ, nous considérons une marche aléatoire, inhomogène mais réversible, à temps continu et à plus proche voisin, X_t^ϵ sur Z^d ou sur un graphe localement fini. Soit G_n^ϵ le sous-graphe dont les arêtes sont les premières n arêtes distinctes traversées

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par X_t^ϵ. Nous démontrons que si la limite fortement inhomogène, lorsque $\epsilon \to 0$, respecte un ordre $\mathcal{O}$ des arêtes, alors $(G_0^\epsilon, G_1^\epsilon, G_2^\epsilon, \ldots)$ converge vers la percolation par invasion pour cet ordre $\mathcal{O}$.

1. INTRODUCTION

In this section of the paper, we give some historical background to and heuristic motivation for considering the type of strongly inhomogeneous random walk which is our main subject. The motivations discussed here are closely related to the two papers [1], [2] on which one of us (C.M.N.) collaborated with Claude Kipnis; another set of motivations follows from work on the dynamics of disordered systems [3]-[5] which employed concepts introduced in the context of "broken ergodicity" [6], [7], [3]. We defer a discussion of this second group of topics to a later paper, and focus here on the first. We note that a third source of motivations is recent work on metastability in Ising models [8], [9], which led Olivieri and Scoppola to an analysis of strongly inhomogeneous random walks [10] which overlaps the analysis presented here.

Our last encounters with Claude were as workshop participants at Les Houches in the summer of 1992; our first encounters were as visitors to the Courant Institute in the fall of 1982. We now briefly discuss some of the work with Claude from that early period that relates to the present paper. Motivated by the phenomenon, observed in fossil records, known as "punctuated equilibria" (*see* Ref. 2 and its references), we studied [1] the $\epsilon \to 0$ nature of the transition between local minima, y and z, of the function $W(y)$ on R^d, for the stochastic process

$$dY_t = \epsilon^{1/2}\, dB_t - \frac{1}{2}\nabla W(Y_t)\, dt, \tag{1.1}$$

where B_t is the standard d-dimensional Wiener process. In Ref. 1, attention was restricted to $d = 1$ and to adjacent local minima, but there seems to be a common qualitative picture [1], [11]-[16], valid in any d, for "neighboring" pairs y, z of local minima, which we now describe.

As $\epsilon \to 0$, the distribution of waiting times $T_{y,z}^\epsilon$ for a transition to (the neighborhood of) z, starting from (that of) y (and conditional on no transitions occurring meanwhile to any other local minimum z') becomes exponential with mean (to leading exponential order)

$$E(T^{\epsilon}_{y,z}) \sim \exp\left[\frac{1}{\epsilon}\left\{W\Big(s(y,z)\Big) - W(y)\right\}\right]. \tag{1.2}$$

Here $s(y,z) = s(z,y)$ is a saddle point of W, as described below. Prior to the transition, the process spends most of the time near y and the time for the transition itself is negligible on the exponential scale given by Eq. (1.2). In simple cases, the actual transition path, in the limit $\epsilon \to 0$, follows the antideterministic equation $dY_t/dt = \frac{1}{2}\nabla W(Y_t)$, during the uphill climb from y to $s(y,z)$, after which it follows the deterministic equation $dY_t/dt = -\frac{1}{2}\nabla W(Y_t)$ during the downhill segment from $s(y,z)$ to z. In the general case, there will be a sequence of saddle points $s_1, \ldots, s_l$ with uphill antideterministic paths from y to s_1 and from s_{j-1} to s_j for $j \leq$ some k followed by downhill deterministic paths from s_{j-1} to s_j for $j > k$ and from s_l to z. We say that two local minima y and z are *neighbors* if there exists such a sequence of saddle points and paths connecting them. $s(y,z)$ is then defined to be the saddle point which minimizes $\max_j W(s_j)$, over all such saddle point sequences $s_1, \ldots, s_l$. In generic situations, one can restrict attention entirely to saddle points s where the Hessian matrix of W has one strictly negative eigenvalue and $d-1$ strictly positive ones.

We conclude from the above discussion that if Y_t is observed infrequently (*i.e.* only $O(1)$ times between successive transitions, as in Ref. 1), it should have essentially the same $\epsilon \to 0$ behavior as some continuous time random walk (*i.e.* Markov jump process) X^{ϵ}_t with state space

$$\mathcal{V} = \{x : x \text{ is a local minimum for } W\} \tag{1.3}$$

and transitions only between neighboring states. The transition rate from x to y should have the same exponential order as $\left[E(T^{\epsilon}_{xy})\right]^{-1}$:

$$r_{xy}(\epsilon) \sim \exp\left[-(W_{xy} - W_x)/\epsilon\right], \tag{1.4}$$

where $W_x = W(x)$ and $W_{xy} = W_{yx} = W\big(s(x,y)\big)$. Furthermore, since Y^{ϵ}_t is reversible with respect to the density $\exp[-W(y)/\epsilon]$, we will choose rates which are reversible with respect to some density $(\pi_x(\epsilon) : x \in \mathcal{V})$ with similar exponential order:

$$\pi_x(\epsilon) r_{xy}(\epsilon) = \pi_y(\epsilon) r_{xy}(\epsilon), \quad \text{for } x, y \in \mathcal{V} \tag{1.5}$$

$$\pi_x(\epsilon) \sim \exp[-W_x/\epsilon]. \tag{1.6}$$

In the context of evolutionary biology modelling (as in Ref. 2), $W(y)$ describes the "adaptive landscape" which is often thought of as a generic (smooth, for our purposes) function with many local minima, perhaps itself generated by some separate random mechanism. We will suppose that $W(\cdot)$ has the following generic properties: The set of critical points (where $\nabla W = 0$) is countable, the Hessian matrix is nonsingular at every critical point and the W values at critical points are all distinct. The set $\mathcal{V}$ of local minima is nonempty. Each x in $\mathcal{V}$ has only finitely many neighboring y in $\mathcal{V}$, or equivalently, the graph $\mathcal{G}$ with vertex set $\mathcal{V}$ and edges between neighboring pairs is locally finite. $\mathcal{G}$ is a connected graph.

Finally we make an assumption which unfortunately is not generic, but which will simplify our analysis, namely that the saddle points $s(x, y)$ for distinct neighboring pairs x,y of local minima are distinct — and thus there are distinct $W_{xy} = W\big(s(x, y)\big)$ values for the edges of $\mathcal{G}$. The W_{xy}'s define an ordering $\mathcal{O}$ on the edges of $\mathcal{G}$ in which $\{x, y\} < \{x', y'\}$ (in $\mathcal{O}$) if $W_{xy} < W_{x'y'}$. This ordering will play a major role in our analysis.

Remark. – There is a way of avoiding this nongeneric assumption, which we will not pursue here and which creates a number of complications of its own. Namely, one could replace $\mathcal{V}$ by the vertex set consisting of all local minima together with saddle points where the Hessian matrix has a single negative eigenvalue. One would then define neighbors as those vertices connected by a single deterministic (or antideterministic) segment. The rates for an uphill transition from x to y would be of exponential order $\exp\big(-(W_y - W_x)/\epsilon\big)$ and for a downhill segment would be of exponential order one.

If one takes the original diffusion process (1.1) and lets $\epsilon \to 0$ with no scaling of time, one simply gets gradient descent (*i.e.* $dY_t/dt = -\frac{1}{2}\nabla W(Y_t)$) to some local minimum x. If one scales time by $E\big(T_{xy}(\epsilon)\big)$ one can obtain (as in Ref. 1) a jump process with the two states x and y. Our purpose in this paper is to study the limit $\epsilon \to 0$ so as to extract (at least some) global information about all the transitions undergone over all time. Since the various transitions occur on exponentially different timescales, this cannot be done by observing the process directly, for any scaling of time. Our strategy instead will be to observe *the order in which transitions are made for the first time*. This should have a definite limit as $\epsilon \to 0$ which we believe is identical to the corresponding limit for our discretized approximation X_t^ϵ. In the next section we study that limit for X_t^ϵ and explain its dependence on the ordering $\mathcal{O}$; our main result is a theorem stating that the limit is exactly invasion percolation (with respect to $\mathcal{O}$) on the graph $\mathcal{G}$. The definition of invasion percolation is given at the beginning of the next section followed by a statement of our theorem; the proof is given in Section 3.

2. INHOMOGENEOUS RANDOM WALK AS INVASION PERCOLATION

Throughout this section $\mathcal{V}$ will be a nonempty countable set of vertices and $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ will be a graph with vertex set $\mathcal{V}$ and some edge set $\mathcal{E}$. We assume $\mathcal{G}$ is connected and also locally finite, *i.e.*, is such that only finitely many edges touch any vertex x. $\mathcal{O}$ will be some ordering of the edge set $\mathcal{E}$.

In the standard version of invasion percolation [17]-[20], $\mathcal{V} = Z^d$, the edge set $\mathcal{E}$ is $\mathcal{B}^d$ (all nearest neighbor edges of Z^d), and $\mathcal{O}$ is the (a.s. defined) random ordering on $\mathcal{B}^d$ determined by i.i.d. continuous random variables $\{\tilde{W}_e : e \in \mathcal{B}^d\}$ via $e < e'$ (in $\mathcal{O}$) if $\tilde{W}_e < \tilde{W}_{e'}$. Of course the same random ordering would result a.s. from any exchangeable $\tilde{W}_e$'s which are a.s. distinct. We remark that no role will be played in this section by the specific structure of $\mathcal{G}$ or $\mathcal{O}$, including whether or not $\mathcal{O}$ (or $\mathcal{G}$) is random. Thus we will treat $\mathcal{O}$ as a deterministic ordering, and invasion percolation will be the following deterministic growth model.

For a fixed initial vertex $x_0 \in \mathcal{V}$, invasion percolation starting from x_0 is the sequence $G = (G_n = (\mathcal{V}_n, \mathcal{E}_n) : n = 0, 1, 2, \ldots)$ of finite subgraphs of $\mathcal{G}$ determined inductively as follows:

(i) The initial vertex set $\mathcal{V}_0 = \{x_0\}$ and the initial edge set $\mathcal{E}_0$ is empty.

(ii) $\mathcal{E}_{n+1} = \mathcal{E}_n \cup \{e_{n+1}\}$ where e_{n+1} is the edge of lowest $\mathcal{O}$-ordering in $\partial\mathcal{E}_n \equiv \{e \in \mathcal{E} \backslash \mathcal{E}_n : \ e \text{ touches some } y \in \mathcal{V}_n\}$.

(iii) $\mathcal{V}_{n+1} = \{x \in \mathcal{V} : \ x \text{ touches some } e \in \mathcal{E}_{n+1}\}$.

Note that each G_n is connected, that G_n is increasing in n and that the cardinality of $\mathcal{E}_n$ is necessarily n, but the cardinality of $\mathcal{V}_n$ will generally be less than n (unless G_n is a tree) since some e_m (with $m \leq n$) may be the edge between two vertices already in $\mathcal{V}_{m-1}$. If $\mathcal{V}$ is finite, then G_n is defined only for n up to the (finite) cardinality of $\mathcal{E}$. If $\mathcal{V}$ is infinite then G is an infinite sequence, and one may define $G_\infty = (\mathcal{V}_\infty, \mathcal{E}_\infty)$ as the increasing limit of G_n. In general neither $\mathcal{V}_\infty$ nor $\mathcal{E}_\infty$ will be all of $\mathcal{V}$ or $\mathcal{E}$. It is easy to show, for example, in the standard version of invasion percolation that $\mathcal{V}_\infty$ has zero density as a subset of Z^d a.s. if and only if there is no percolation for standard Bernoulli bond percolation on Z^d at its critical point — it is currently an open problem to prove the latter for general d. (For some connections between the nature of $\mathcal{V}_\infty$ and spin glasses, see Ref. 21.)

Our main result is a theorem, presented at the end of this section (with the proof given in Section 3) which states that, under mild conditions, invasion percolation arises as the strongly inhomogeneous ($\epsilon \to 0$) limit of stochastic growth models $G^\epsilon = (G_n^\epsilon : n = 0, 1, 2, \ldots)$ determined by random walks X_t^ϵ on $\mathcal{G}$. For each $\epsilon > 0$, X_t^ϵ will be a continuous time random walk on $\mathcal{G}$, starting at x_0, with a strictly positive rate $r_{xy}(\epsilon)$ for the transition from x to y for each x,$y \in \mathcal{V}$ with $\{x, y\} \in \mathcal{E}$. For concreteness, we take X_t^ϵ to be right-continuous in t. Further hypotheses (in particular, reversibility) will be stated in the theorem below.

To define $G_n^\epsilon = (\mathcal{V}_n^\epsilon, \mathcal{E}_n^\epsilon)$ we first define $\mathcal{V}^\epsilon(s)$ to be the set of all vertices in $\mathcal{V}$, visited by X_t^ϵ for $t \in [0, s]$ and $\mathcal{E}^\epsilon(s)$ to be the set of all edges in $\mathcal{E}$ traversed by X_t^ϵ (in either direction) for $t \in [0, s]$. Let T_n^ϵ be the random time at which the cardinality of $\mathcal{E}^\epsilon(s)$ first reaches n. Then $\mathcal{V}_n^\epsilon = \mathcal{V}^\epsilon(T_n^\epsilon)$ and $\mathcal{E}_n^\epsilon = \mathcal{E}^\epsilon(T_n^\epsilon)$. In other words the sequence G^ϵ describes both the vertices visited and edges traversed by X_t^ϵ in the order they were *first* visited/traversed. The *direction* of first traversal of an edge $\{x, y\}$ is also contained within G^ϵ except in the case where x and y had *both* been visited prior to the traversal. For a brief discussion about the times at which first traversals occur, see the remark at the end of Section 3.

THEOREM. – *Assume that for each $\epsilon > 0$, there exist $\pi_x(\epsilon) > 0$ for $x \in \mathcal{V}$ such that for each $\{x, y\} \in \mathcal{E}$*

$$\pi_x(\epsilon) r_{xy}(\epsilon) = \pi_y(\epsilon) r_{yx}(\epsilon) \left(\equiv R_{\{x,y\}}(\epsilon)\right). \tag{2.1}$$

Assume also that for each distinct e and e′ in $\mathcal{E}$,

$$\lim_{\epsilon \to 0} \frac{R_e(\epsilon)}{R_{e'}(\epsilon)} = \begin{cases} +\infty\ , & \text{if } e < e' \text{ (in } \mathcal{O}), \\ 0\ , & \text{if } e > e' \text{ (in } \mathcal{O}). \end{cases} \tag{2.2}$$

Then the random sequence G^ϵ converges in distribution as $\epsilon \to 0$ to the deterministic invasion percolation sequence G (with the same initial vertex x_0 and the same ordering on the edges).

3. PROOF

G^ϵ converges to G if and only if $(G_0^\epsilon, \ldots, G_n^\epsilon)$ converges to $(G_0, \ldots, G_n)$ for every n. Since the vertices and edges of G_m^ϵ (and G_m) for $m \leq n$ are connected to the starting point x_0 by paths in G containing at most n edges and since $\mathcal{G}$ is locally finite, we may, for each n, replace $\mathcal{G}$ by a finite subgraph of itself (depending on n). It follows that, without loss of generality, we may assume that $\mathcal{G}$ is a *finite* graph.

Related to the continuous time process X_t^ϵ in the standard way is the discrete time Markov chain $\tilde{X}^\epsilon = (\tilde{X}_k^\epsilon : k = 0, 1, \ldots)$ which records the succession of vertices visited by X_t^ϵ (including repeat visits). Clearly G^ϵ depends *only* on $\tilde{X}^\epsilon$ and not on the times spent at successive vertices, which is the extra information contained in X_t^ϵ. Let us denote by $\mathcal{N}_x$ the set of neighbors (in $\mathcal{G}$) of the vertex x. Then the transition matrix for $\tilde{X}^\epsilon$ is

$$P(\tilde{X}_{k+1}^\epsilon = y | \tilde{X}_k^\epsilon = x) = \begin{cases} r_{xy}(\epsilon) / \sum_{z \in \mathcal{N}_x} r_{xz}(\epsilon), & \text{if } y \in \mathcal{N}_x \\ 0, & \text{if } y \notin \mathcal{N}_x \end{cases}. \tag{3.1}$$

Note that the right hand side of Eq. (3.1) is unchanged if the rate r_{xy} is replaced by $\pi_x r_{xy}$ for every x and y (with $\{x, y\} \in \mathcal{E}$). Thus, without loss of generality, we may assume that $\pi_x \equiv 1$ and that the *rates are symmetric*: $r_{xy} = R_{\{x,y\}} = r_{yx}$.

We next state two lemmas, one about random walks and one about invasion percolation. Then we complete the proof of the theorem, using the lemmas, and afterward present the proofs of the lemmas. We end the section with a remark about the time scales for new transitions to occur.

LEMMA 1. – *Let $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ be a finite connected graph with M vertices and let X_t' be a continuous time random walk on $\mathcal{G}'$ with strictly positive symmetric rates, $r_{xy} = r_{yx} = R_{\{x,y\}} > 0$ for each $\{x, y\} \in \mathcal{E}'$. There exist constants C_j $(j = 1, 2, 3, 4)$ in $(0, \infty)$ depending only on $\mathcal{G}'$ (but not on the rates R_e), such that for every $x, y \in \mathcal{V}'$, $e \in \mathcal{E}'$, and $t \geq 0$,*

$$\left| P(X_t' = y | X_0' = x) - \frac{1}{M} \right| \leq C_1 e^{-\rho t / C_2}, \tag{3.2}$$

and

$$P(T_e' > t | X_0' = x) \leq C_3 e^{-\rho t / C_4}, \tag{3.3}$$

where T_e' is the time X_t' first traverses the edge e, and

$$\rho = \min_{e \in \mathcal{E}} R_e. \tag{3.4}$$

LEMMA 2. – *Let $G = \big(G_n = (\mathcal{V}_n, \mathcal{E}_n) : n = 0, 1, 2, \ldots\big)$ be invasion percolation on $G = (\mathcal{V}, \mathcal{E})$ starting from some $x_0 \in \mathcal{V}$, with respect to some ordering $\mathcal{O}$ on $\mathcal{E}$, and let $e_1, e_2, \ldots$ be the sequence of invaded edges. If $e_m > e_{m+1}$ (in $\mathcal{O}$), then $e_m = \{x, y\}$ and $e_{m+1} = \{y, z\}$ for some $x \in \mathcal{V}_{m-1}$, $y \notin \mathcal{V}_{m-1}$, and $z \notin \mathcal{V}_m$; furthermore, z is the vertex in $\mathcal{N}_y$*

which minimizes the $\mathcal{O}$-order of $\{y, z\}$. If $e_m < e_{m+1}$ (in $\mathcal{O}$), then in the subgraph $(\mathcal{V}_{m+1}, \tilde{\mathcal{E}}_{m+1})$ of G_{m+1}, where

$$\tilde{\mathcal{E}}_{m+1} = \{e \in \mathcal{E}_{m+1} : \ e \leq e_{m+1} \ (in\ \mathcal{O})\}, \tag{3.5}$$

e_m and e_{m+1} belong to the same connected component.

Continuing with the proof of the theorem, we denote by $e_1^\epsilon, e_2^\epsilon, \dots$ the sequence of random edges giving the edge set of G_n^ϵ, $\mathcal{E}_n^\epsilon = \{e_1^\epsilon, \dots, e_n^\epsilon\}$. We denote by T_e^ϵ the time when X_t^ϵ first traverses edge e. Our object is to show that for each $n = 1, 2, \dots$

$$P(e_1^\epsilon = e_1, \dots, e_n^\epsilon = e_n) \to 1 \text{ as } \epsilon \to 0. \tag{3.6}$$

We proceed by induction on n.

Let A_n^ϵ denote the event $\{e_1^\epsilon = e_1, \dots, e_n^\epsilon = e_n\}$. For $n = 1$, this is simply the event that the first transition from x_0 is made to the z in $\mathcal{N}_{x_0}$ (call it $\overline{z}$) with minimum $\mathcal{O}$-order of $\{x_0, z\}$. Recalling Eq. (3.1) and Eq. (2.2), we see that

$$P(A_1^\epsilon) = \frac{R_{\{x_0, \overline{z}\}}(\epsilon)}{\sum_{z \in \mathcal{N}_{x_0}} R_{\{x_0, z\}}(\epsilon)} \to 1 \text{ as } \epsilon \to 0. \tag{3.7}$$

Now suppose $P(A_m^\epsilon) \to 1$. We must show that $P(A_{m+1}^\epsilon | A_m^\epsilon) \to 1$. If $e_m > e_{m+1}$ (in $\mathcal{O}$), then by Lemma 2, we have (conditional on A_m^ϵ) that for $t = T_{e_m}$, $X_t^\epsilon = y \notin \mathcal{V}_{m-1}$ and also (again conditional on A_m^ϵ) that A_{m+1}^ϵ contains the event that the next transition (after $T_{e_m}^\epsilon$) is from y to the z in $\mathcal{N}_y$ (call it $\overline{z}$) such that $\{y, z\}$ has minimum $\mathcal{O}$-order. In this case, $P(A_{m+1}^\epsilon | A_m^\epsilon)$ exceeds the expression in (3.7), but with x_0 replaced by y, and so tends to 1 as $\epsilon \to 0$.

Now consider the case where $e_m < e_{m+1}$ (in $\mathcal{O}$). In this case, X_t^ϵ for $t = T_{e_m}$ may not be deterministic (even when conditioned on A_m^ϵ), but it can only be either y' or y'', where $\{y', y''\} = e_m$. Let $\tilde{G}_{m+1}$ denote the connected component of e_m (which also contains e_{m+1}) in the graph $(\mathcal{V}_{m+1}, \tilde{\mathcal{E}}_{m+1})$ of Lemma 2. Let $\partial\tilde{G}_{m+1}$ denote the set of all *edges* in $\mathcal{G}$ which do not belong to $\tilde{G}_{m+1}$, but which touch $\tilde{G}_{m+1}$. Note that every edge in $\partial\tilde{G}_{m+1}$ is of higher $\mathcal{O}$-order than every edge in $\tilde{G}_{m+1}$.

Conditional on A_m^ϵ, A_{m+1}^ϵ contains the event that (after time T_{e_m}) e_{m+1} is traversed before any edge in $\partial\tilde{G}_{m+1}$ is traversed. Let us assign independent Poisson alarm clocks (with rates $R_e(\epsilon)$) to the edges of $\mathcal{G}$ to determine the (possible) transitions of X_t^ϵ. Let B^ϵ denote the event that none of the clocks assigned to the edges in $\partial\tilde{G}_{m+1}$ rings (and hence no edge in $\partial\tilde{G}_{m+1}$ is traversed) during the time interval $[T_{e_m}, T_{e_m} + t(\epsilon)]$, where $t(\epsilon)$ will be chosen later. Then

$$P(B^\epsilon) = \prod_{e \in \partial\tilde{G}_{m+1}} \exp\left(-R_e(\epsilon)t(\epsilon)\right). \tag{3.8}$$

Furthermore, conditional on B^ϵ and $X_t^\epsilon = y$ (a vertex in $\partial\tilde{G}_{m+1}$) for $t = T_{e_m}$, the process X_t^ϵ for $t \in [T_{e_m}, T_{e_m} + t(\epsilon)]$ is identical to one in which $\mathcal{G}$ is replaced by $\tilde{G}_{m+1}$ (with the same rates). Let $\tilde{X}_t^\epsilon$ denote this replacement process and $\tilde{T}_e^\epsilon$ (for e an edge of $\tilde{G}_{m+1}$) denote the time $\tilde{X}_t^\epsilon$ first traverses e. Then, we have (for any choice of $t(\epsilon)$)

$$\begin{aligned} P(A_{m+1}^\epsilon | A_m^\epsilon) &\geq P(B^\epsilon)P(A_{m+1}^\epsilon | A_m^\epsilon \cap B^\epsilon) \\ &\geq P(B^\epsilon) \max_{y \in \{y', y''\}} P(\tilde{T}_{e_{m+1}} \leq t(\epsilon) | \tilde{X}_0^\epsilon = y). \end{aligned} \tag{3.9}$$

It remains to choose $t(\epsilon)$ so that the RHS of (3.9) tends to 1. To do so, we first apply (3.3) of Lemma 1 with $\mathcal{G}' = \tilde{G}_{m+1}$, $X_t' = \tilde{X}_t^\epsilon$, and $T_e' = \tilde{T}_{e_{m+1}}$ so that (3.9) combined with (3.8) yields

$$P(A_{m+1}^\epsilon | A_m^\epsilon) \geq (1 - C_3 e^{-\rho(\epsilon)t(\epsilon)/C_4}) \prod_{e \in \partial\tilde{G}_{m+1}} e^{-R_e(\epsilon)t(\epsilon)}, \tag{3.10}$$

where $\rho(\epsilon)$ is the minimum $R_e(\epsilon)$ for edges in $\tilde{G}_{m+1}$. From (2.2) and the fact that all edges in $\partial\tilde{G}_{m+1}$ exceed in $\mathcal{O}$-order all edges in $\tilde{G}_{m+1}$, we see that $t(\epsilon)$ can be chosen so that

$$R_e(\epsilon)t(\epsilon) \to \begin{cases} +\infty, & \text{if } e \in \tilde{G}_{m+1}, \\ 0, & \text{if } e \in \partial\tilde{G}_{m+1}, \end{cases} \tag{3.11}$$

which forces the RHS of (3.10) to tend to 1, as desired.

Proof of Lemma 1. – For small $\delta > 0$, define the $M \times M$ transition matrix

$$P^{(\delta)}(x,y) = \begin{cases} \delta R_{\{x,y\}}, & \text{if } \{x,y\} \in \mathcal{E}', \\ 1 - \sum_{z \in \mathcal{N}_x} \delta R_{\{x,z\}}, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases} \tag{3.12}$$

Let $\beta_1(\delta)$ denote the second largest eigenvalue of $P^{(\delta)}$ (the largest equals one) and let $X_t^{(\delta)}$ denote the continuous time Markov process which makes transitions according to $P^{(\delta)}$ at times given by a Poisson process of rate

$1/\delta$. Since $P^{(\delta)}$ is symmetric, its invariant distribution is uniform. It then follows from Eq. (1.10) of Ref. 22 that

$$\left|P\big(X_t^{(\delta)} = y | X_0^{(\delta)} = x\big) - \frac{1}{M}\right| \leq (M-1)^{\frac{1}{2}} \exp\big(-t[1-\beta_1(\delta)]/\delta\big). \tag{3.13}$$

On the other hand, Proposition 1 of Ref. 22 gives the bound

$$[1-\beta_1(\delta)]/\delta \geq \left[\frac{1}{M} \max_e \sum_{\gamma_{xy} \ni e} \sum_{e' \in \gamma_{xy}} \frac{1}{R_{e'}}\right]^{-1} \tag{3.14}$$

where the max is over all e in $\mathcal{E}'$ and γ_{xy}, for each ordered pair x, y of vertices, is some path in $\mathcal{G}'$ connecting x and y. The RHS of (3.14) is easily bounded by $\rho/C_2(M)$. Inserting that into (3.13) and letting $\delta \to 0$ yields (3.2).

It remains to obtain (3.3) from (3.2). Let $\hat{X}_t$ denote the process which is the same as X_t' except that the single rate R_e is set to zero. Let y be an endpoint of e which is connected to x in the graph $\hat{G}$ (which is $\mathcal{G}'$ with the single edge e removed). Let U_t denote the amount of time in $[0,t]$ spent by $\hat{X}_t$ at y. Then $\{T_e' > t\}$ is contained in the event that the Poisson clock at e (see the discussion preceding (3.8) above) does not ring during the U_t seconds when $\hat{X}_t$ is at y and thus, for any $s \leq t$,

$$\begin{aligned} P(T_e' > t | X_0' = x) &\leq 1 - P(U_t \geq s)\big(1 - \exp(-R_e s)\big) \\ &\leq 1 - P(U_t \geq s)(1 - e^{-\rho s}). \end{aligned} \tag{3.15}$$

Next we note that by applying (3.2) to $\hat{X}_t$ (restricted to the connected component of x in $\hat{\mathcal{G}}$, which has $\hat{M}$ vertices), we have

$$\begin{aligned} \frac{1}{t} E(U_t | \hat{X}_0 = x) &= \frac{1}{t} \int_0^t P(\hat{X}_s = y | \hat{X}_0 = x)\, ds \\ &\geq \frac{1}{\hat{M}} - \frac{1}{t} \int_0^t \hat{C}_1 e^{-\hat{\rho} s/\hat{C}_2} ds. \end{aligned} \tag{3.16}$$

Noting that $\hat{\rho} \geq \rho$, that $\hat{M} \leq M$ and then considering worst cases of $\hat{C}_1$ and $\hat{C}_2$, we see that there exists a C_5 depending only on $\mathcal{G}'$ so that $E(U_t) \geq \frac{t}{2M}$ for $t \geq C_5/\rho$. Since for $\alpha \in (0,1)$,

$$E(U_t) \leq \alpha t + P(U_t \geq \alpha t) t, \tag{3.17}$$

we may take $\alpha = \frac{1}{4M}$ and conclude that for $T \geq C_5/\rho$,

$$P(U_t \geq t/4M) \geq \frac{1}{4M} \text{ for } T \geq C_5/\rho. \tag{3.18}$$

Thus from (3.15) with $t = C_5/\rho$ and $s = t/4M$,

$$P\Big(T'_e > C_4/\rho \Big| X'_0 = x\Big) \leq 1 - \frac{1}{4M}\bigg(1 - \exp\bigg[-\frac{C_5}{4M}\bigg]\bigg). \tag{3.19}$$

Denote the RHS of (3.19) by $\exp(-C_6)$. Then by a standard Markov process argument,

$$P(T'_e > t|X'_0 = x) \leq \exp\bigg(-C_6\Big[\frac{t}{C_5/\rho} - 1\Big]\bigg), \tag{3.20}$$

which can be rewritten as (3.3).

Proof of Lemma 2. – We first consider the case $e_m > e_{m+1}$ (in $\mathcal{O}$). If both endpoints of e_m were in $\mathcal{V}_{m-1}$, then the edges of $\partial\mathcal{E}_m$ would already all be in $\partial\mathcal{E}_{m-1}$ and so e_{m+1} would be above e_m (in $\mathcal{O}$) since it was not chosen as the m^{th} edge in the invasion; this contradiction shows that $e_m = \{x, y\}$ with $x \in \mathcal{V}_{m-1}$ and $y \notin \mathcal{V}_{m-1}$. Similarly, e_{m+1} cannot be any of the edges already in $\partial\mathcal{E}_{m-1}$ and so it must be an edge in $\partial\mathcal{E}_m \backslash \partial\mathcal{E}_{m-1}$, which forces it to be of the form $\{y, z\}$ with $z \notin \mathcal{V}_m$. e_{m+1} must be the edge touching y, with minimum $\mathcal{O}$-order, or else it wouldn't be invaded immediately after e_m.

Now consider the case $e_m < e_{m+1}$ (in $\mathcal{O}$). We must show that there is a path γ of edges in $\mathcal{E}_m$, all of which are below e_{m+1} in $\mathcal{O}$-order, which connects some vertex of e_m to some vertex of e_{m+1}. Since G_m is connected, and touches both e_m and e_{m+1}, there must be some site-self-avoiding path γ of edges in $\mathcal{E}_m$ connecting a single vertex of e_m to a single vertex of e_{m+1}. We will prove by contradiction that every edge in γ is below e_{m+1} in $\mathcal{O}$-order.

Let e^* be the edge of maximum $\mathcal{O}$-order in γ and suppose that e^* is above both e_m and e_{m+1} in $\mathcal{O}$-order. Let l be the first n ($< m$) such that G_n touches γ. The touched vertex is either between e_m and e^* in γ or else between e_{m+1} and e^* in γ. In the former case, e_m and all edges in γ between e_m and e^* would be invaded prior to e^* being invaded; this would contradict the fact that e^* is invaded before e_m (since e^* is in G_m and $e^* \neq e_m$). Similarly, in the latter case e_{m+1} would be invaded before e^* which is again a contradiction.

Remark. – Although we will not pursue the issue at length in this paper, we make some heuristic comments about the timescales on which first traversals of edges by X_t^ϵ occur as $\epsilon \to 0$ in the context of the reversible rates of (2.1) – (2.2). In the situation of Lemma 1 with symmetric rates, the time T'_{e^*} for first traversal of e^*, the edge with minimum rate among the edges of $\mathcal{G}'$ (assume distinct rates) is of the order of $1/R_{e^*} = 1/\rho$. In a process on $\mathcal{G}'$ with rates which are not symmetric, but rather reversible with respect to $\{\pi_y\}$, let us denote by $\mathcal{G}^*$ the connected component of the initial vertex x in the graph obtained from $\mathcal{G}$ by deleting e^*, and denote by x^* the vertex in $\mathcal{G}^*$ which minimizes π_y (assume for the time being distinct π_y's). The circle of methods used in the proof of Lemma 1 also shows that prior to T'_{e^*}, the process on $\mathcal{G}^*$ reaches equilibrium and thus spends most of its time at vertex x^*. If one replaces the nonsymmetric process on $\mathcal{G}^*$ by the symmetric one with rates $\pi_x r_{xy}/\pi_{x^*} = R_{\{x,y\}}/\pi_{x^*}$, then the rates for transitions from x^* are unchanged, as is the order of the time spent at x^*. We conclude that in a reversible process on $\mathcal{G}'$, starting from x, the time T'_{e^*} is of the order π_{x^*}/R_{e^*}. Combining this with Lemma 2 and the rest of this section, we reach the following conclusion about the times $T^\epsilon_{e_m}$ for first traversals by the process X_t^ϵ (starting at x_0) of the edges $e_1, e_2, \ldots$ (given by invasion percolation for the ordering $\mathcal{O}$). Define $x_1^* = x_0$,

$$\pi_m^* = \begin{cases} \pi_{x_0}, & \text{for } m = 1, \\ \pi_y, & \text{if } e_m \in \{y, z\} < e_{m-1} = \{x, y\}, \\ \min\{\pi_x : x \in G_m^*\}, & \text{otherwise,} \end{cases} \tag{3.21}$$

where G_m^* is the connected component containing e_m of the graph obtained from G_m by deleting all edges e' with $e' \geq e_m$ (in $\mathcal{O}$). Then as $\epsilon \to 0$ (and under the hypotheses (2.1) – (2.2))

$$T^\epsilon_{e_m} - T^\epsilon_{e_{m-1}} \sim \pi_m^*/R_{e_m}, \tag{3.22}$$

in an appropriate distributional sense.

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