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Antisymmetric functionals of reversible Markov processes

by

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This paper is dedicated to the memory of Claude Kipnis.

ABSTRACT. – We prove the central limit theorem and invariance principle for antisymmetric functionals of ergodic reversible Markov processes under extremely mild conditions. Moreover, the proof is based upon an elementary, direct decomposition of the functional, into the sum of a martingale term and an asymptotically negligible term, by an analysis relying almost solely on symmetry considerations.

Key words: Central limit theorem, reversible Markov processes.

RÉSUMÉ. – Nous prouvons, sous des conditions extrêmement faibles, un théorème central limite et un principe d'invariance pour des processus de Markov réversibles et ergodiques. De plus, la preuve est basée sur une décomposition directe et élémentaire de la fonctionnelle en une somme d'une martingale et d'un terme asymptotiquement négligeable, grâce à une analyse utilisant presque uniquement des considérations de symétrie.

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1. INTRODUCTION

In this paper we derive the central limit theorem for symmetric random motions in a random environment by an argument based almost solely on symmetry considerations.

Consider a stochastic process $X(t) \in \mathbb{R}$ [or $\mathbb{R}^d$], where $t \in \mathbb{Z}$ (discrete-time case) or $t \in \mathbb{R}$ (continuous-time case). We shall say that this process obeys the central limit theorem (CLT) if there exists a $D \geq 0$ such that as $t \to \infty$ the rescaled process converges to a normal random variable with mean 0 and variance D,

$$\frac{X(t)}{\sqrt{t}} \to \mathcal{N}(0, D) \tag{1.1}$$

in the sense of convergence in distribution. In almost all examples which naturally arise, it follows immediately from the argument for the CLT that as $\varepsilon \to 0$

$$X^{(\varepsilon)}(t) \equiv \varepsilon X([\varepsilon^{-2} t]) \to W_D(t), \tag{1.2}$$

where W_D is the Wiener process with variance Dt and the convergence is in the sense of finite dimensional distributions. $X^{(\varepsilon)}$ is the process viewed on macroscopic length and time scales. (For the case of discrete time the "[ ]" designates the greatest integer function.)

By CLT I will mean, in fact, both of the above. Slightly stronger than the CLT but usually satisfied together with it is the invariance principle (IP):

$$X^{(\varepsilon)} \to W_D \tag{1.3}$$

in the sense of weak convergence of path-space measures for processes with paths in $D[0, \infty)$. The IP amounts to the CLT combined with "tightness." (Whenever we are concerned with the IP, it should be understood without further ado that, whenever appropriate, modifications which are right continuous with left limits have been taken.)

A frequent starting point for the asymptotic analysis of a process $X(t)$ is the observation that an ergodic square-integrable martingale $M(t)$ with stationary increments satisfies the IP [1]. Thus one obtains the CLT or the IP for a process $X(t)$ by suitably approximating it by a martingale.

A typical example [2] of the sort of process we wish to consider is provided by a random walker in the random bond model. A particle starts, at $t = 0$, at the origin of the lattice $\mathbb{Z}^d$ and randomly jumps to nearest neighbor sites; to each nearest neighbor bond b is assigned a jump rate $a(b) > 0$, the rate at which the particle jumps across bond b. The jump rates are random variables defining a translation invariant random field which is assumed to be ergodic under translations. $X(t)$ is the position of the particle at time t. (For other examples, see [4].)

The following two sections are intended as background and motivation for the analysis and results of Sections 4 (discrete time) and 5-7 (continuous time).

2. THE STANDARD DECOMPOSITION

For the random bond model it is not difficult to see that $X(t)$ satisfies the *standard decomposition*

$$X(t) = \int_0^t \phi(\xi_s)\, ds + \mathcal{M}(t), \tag{2.1}$$

where the first term on the right is the integrated drift (formally $\phi(\xi_0) = (E(dX \mid \xi_0)/dt)_{t=0} = \nabla a$), $\mathcal{M}(t)$ is a square-integrable martingale with stationary increments, and ξ_t is the environment process, the environment seen by the random walker, *i.e.*, translated so that the random walker is at the origin. ξ_t is a reversible Markov process, ergodic and reversible with respect to the probability measure μ defining the random field of bonds, *i.e.*, the translation invariant probability measure on environments.

More generally, it typically happens that the displacement $X(t)$ for a symmetric random motion in a random environment can be written in the form (2.1) for some function ϕ of the state ξ of a reversible Markov process ξ_t. (The reversibility of ξ_t is a direct reflection of the symmetry of the random motion together with the translation invariance of the random environment.) We shall thus consider a general Markov process ξ_t, ergodic and reversible with respect to some probability measure μ. We shall denote the path space measure for the process starting with distribution μ by P_μ and the corresponding expectation by E_μ. For the case of continuous time, we denote the negative generator of this process by L, a positive self-adjoint operator on $L^2(\mu)$. We shall denote the inner product on $L^2(\mu)$ by $(\ ,\ )$.

It is clear that the approximation of a process $X(t)$ of the form (2.1) by a martingale can be reduced to the approximation of the first term $S(t)$

$$S(t) = \int_0^t \phi(\xi_s)\, ds \tag{2.2}$$

by a martingale. It can be shown [4] that for $\phi \in L^1(\mu)$

$$\limsup_{t\to\infty} \frac{E_\mu(S(t)^2)}{t} < \infty \quad \text{if and only if } \phi \in H_{-1}, \tag{2.3}$$

where

$$H_{-1} \equiv H_{-1}(L) = \overline{\{\phi \in L^2(\mu) \mid \|\phi\|_{-1}^2 \equiv (\phi, L^{-1}\phi) < \infty\}} \tag{2.4}$$

with the overbar denoting the completion in $\|\ \ \|_{-1}$.

Kipnis and Varadhan [2] (*see* also [3]) have shown that a symmetric functional $S(t)$ of the form (2.2) satisfies the CLT, and, in fact, even the IP, provided $\phi \in L^2(\mu) \cap H_{-1}$. They do this by establishing the decomposition

$$S(t) = R(t) + N(t) \tag{2.5}$$

where $N(t)$ is a square integrable martingale with stationary increments and $R(t)$ is asymptotically small in the sense that

$$\frac{R(t)}{\sqrt{t}} \to 0, \quad \text{as } t \to \infty, \tag{2.6}$$

in $L^2(P_\mu)$, which yields the CLT for $S(t)$, and, indeed, that

$$\sup_{0 \le s \le t} \frac{|R(s)|}{\sqrt{t}} \to 0, \tag{2.7}$$

in P_μ-probability, which yields the IP. The same conclusions clearly hold as well for $X(t)$ satisfying (2.1), since by (2.1) and (2.5)

$$X(t) = R(t) + M(t) \tag{2.8}$$

where $M(t) = N(t) + \mathcal{M}(t)$ is also a square-integrable martingale with stationary increments.

Kipnis and Varadhan thus establish the IP for a large class of symmetric random motions in random environments by showing that the drift ϕ appearing in the standard decomposition (2.1) satisfies

$$\phi \in L^2(\mu) \cap H_{-1}. \tag{2.9}$$

3. ANTISYMMETRIC FUNCTIONALS

It is shown in [4] that it is, in fact, not necessary to check (2.9) for the drift arising from a symmetric random motion in a random environment – it turns out that this condition is automatically satisfied by such a ϕ. This is because of a fundamental symmetry obeyed by such processes, namely, that they are antisymmetric under time-reversal. More precisely, the displacement $X(t) = X_{[0,t]}$ where X_{I} is an *antisymmetric functional* of the ergodic reversible Markov process ξ_t: For each interval

$\mathbf{I} \subset \mathbb{R}$, $X_{\mathbf{I}} \in \mathcal{F}_{\mathbf{I}} = \sigma\{\xi_t, t \in I\}$ and the family $\{X_{\mathbf{I}}\}$ is *covariant, additive, and antisymmetric,*

$$X_{\theta \mathbf{I}}(\xi) = X_{\mathbf{I}}(\theta \xi) \tag{3.1}$$

for any time-translation θ,

$$X_{[a,\,c]} = X_{[a,\,b]} + X_{[b,\,c]} \tag{3.2}$$

and

$$X_{\mathbf{I}}(R_{\mathbf{I}}\,\xi) = -X_{\mathbf{I}}(\xi) \tag{3.3}$$

where $R_{\mathbf{I}}$ denotes times-reversal about the midpoint of **I**. (See [4] for details.)

(Observe that the displacement in the random bond model is of this form, with $X_{\mathbf{I}}$ the displacement over the time interval **I**. This is determined by the "jumps" in the environment during this interval, and hence by ξ_t, $t \in \mathbf{I}$. It is antisymmetric since the time-reversal of the motion leads to a reversal in sign of the displacement.)

Consider a process $X(t)$ arising from an antisymmetric functional of a reversible Markov process, as described above, and suppose that $X(t)$ satisfies the standard decomposition

$$X(t) = S(t) + \mathcal{M}(t). \tag{3.4}$$

Since $S(t)$ is of the form (2.2) it is clearly symmetric under time-reversal, *i.e.*, under $R_{t/2} \equiv R_{[0,\,t]}$. Moreover, since ξ_t is reversible with respect to μ, P_μ is invariant under time-reversal, *i.e.*, under R_t for any t. Squaring $\mathcal{M}(t) = X(t) - S(t)$ and noting that the cross term $-2\,X(t)\,S(t)$ is antisymmetric and hence has vanishing expectation with respect to P_μ, we find that

$$E_\mu(\mathcal{M}(t)^2) = E_\mu(X(t)^2) + E_\mu(S(t)^2) \tag{3.5}$$

so that

$$\frac{E_\mu(S(t)^2)}{t} \leq \frac{E_\mu(\mathcal{M}(t)^2)}{t} = K < \infty \tag{3.6}$$

where K does not depend upon t. Thus, by (2.3), $\phi \in H_{-1}$ and the IP follows provided $\phi \in L^2(\mu)$. Moreover, it is shown in [4] that the condition $\phi \in L^1(\mu) \cap H_{-1}$ is sufficient for the CLT for $X(t)$ and that under an additional mild technical condition one also obtains the IP.

4. THE DIRECT DECOMPOSITION

The preceding argument has always left me a little unhappy. What seems to me vaguely unsettling about it is the conjunction of the following facts: (1) It employs the standard decomposition to transform the analysis of $X(t)$ to that of $S(t)$. (2) In general, the analysis of a symmetric functional $S(t)$ of the form (2.2) requires the condition that $\phi \in H_{-1}$. (3) There is, however, no need to demand any corresponding condition on $X(t)$ – since the $S(t)$'s which arise, as described, from antisymmetric $X(t)$'s automatically satisfy the relevant condition. It would seem, therefore, that there should somehow exist an analysis of $X(t)$ leading directly to the decomposition (2.8) into the sum of an asymptotically negligible term and a martingale, which entirely avoids the standard decomposition and $S(t)$. (Note also in this regard that, formally, one can at least imagine the situation in which the standard decomposition fails in the sense that $\mathcal{M}(t)$ is not square-integrable, $\phi \notin H_{-1}$, and (2.5) is satisfied but with the martingale $N(t)$ also failing to be square-integrable, in such a manner that the "divergences" in these two martingales cancel so that the sum $M(t)$, in eq. (2.8), is square-integrable and the CLT follows.)

We show here that such a direct analysis does in fact exist. The remainder of this section will be devoted to the presentation of this analysis for the discrete-time case, for which the CLT for antisymmetric functionals involves a cleaner statement, with fewer conditions – essentially none beyond the symmetry conditions – than for continuous time. We begin with a statement of this CLT, in fact IP, first proven in [4].

THEOREM 4.1. – *Let ξ_n, $n \in \mathbb{Z}$, be a Markov process with state space Γ, ergodic and reversible with respect to the probability measure μ on Γ. Let P_μ be the measure on path space describing the process starting from μ. Suppose X is an antisymmetric function on $\Gamma \times \Gamma$,*

$$X(\xi, \eta) = -X(\eta, \xi) \tag{4.1}$$

and that

$$X_i \equiv X(\xi_{i-1}, \xi_i) \in L^2(P_\mu). \tag{4.2}$$

Then

$$X(t) = \sum_{i=1}^{t} X_i \tag{4.3}$$

obeys the IP.

Note that X_i is an odd function of the i-th jump $\xi_{i-1} \to \xi_i$. Just as for continuous time, many discrete-time models can naturally be cast into a form to which the above theorem may be applied.

The key idea for obtaining the CLT part of this theorem by a direct analysis is to observe that

$$X_1 \in L^2 (P_\mu \restriction \mathcal{F}_{[0,1]})_{\text{antisymmetric}} \equiv \mathcal{H} \tag{4.4}$$

and thus to focus on the antisymmetric subspace $\mathcal{H}$ of $L^2 (P_\mu \restriction \mathcal{F}_{[0,1]})$, *i.e.*, the closed subspace of functions $f(\xi_0, \xi_1)$ which reverse sign under the interchange $(\xi_0, \xi_1) \to (\xi_1, \xi_0)$.

The simplest elements of $\mathcal{H}$ are of the form

$$\Delta\psi \equiv \psi(\xi_1) - \psi(\xi_0) \tag{4.5}$$

for $\psi \in L^2(\mu)$. We therefore form the L^2-closure $\mathcal{H}_0$ of the set of elements of this form,

$$\mathcal{H}_0 \equiv \overline{\{\Delta\psi \mid \psi \in L^2(\mu)\}}, \tag{4.6}$$

the "null" subspace of $\mathcal{H}$, and form the corresponding orthogonal decomposition of $\mathcal{H}$

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_m \tag{4.7}$$

into the null subspace and its orthogonal complement $\mathcal{H}_m \equiv \mathcal{H}_0^\perp$ in $\mathcal{H}$.

The critical observation is now that $\mathcal{H}_m$ is precisely the set of antisymmetric square-integrable martingale differences, *i.e.*, the subset of elements $M \in \mathcal{H}$ satisfying

$$E_\mu(M \mid \mathcal{F}_0) = 0. \tag{4.8}$$

In fact, for $M \in \mathcal{H}_m$ we have that for all $\psi \in L^2(\mu)$

$$\begin{aligned} 0 &= E_\mu(M(\psi(\xi_1) - \psi(\xi_0))) \\ &= E_\mu(R(M)\,R(\psi(\xi_1))) - E_\mu(M\,\psi(\xi_0)) = -2\,E_\mu(M\,\psi(\xi_0)) \end{aligned} \tag{4.9}$$

where R implements the interchange $(\xi_0, \xi_1) \to (\xi_1, \xi_0)$ and where the symmetry of P_μ has been used. (4.8) follows immediately from (4.9) (and conversely).

We thus decompose X_1 according to the decomposition (4.7)

$$X_1 = \psi(\xi_1) - \psi(\xi_0) + M_1 \tag{4.10}$$

into a null term $\Delta\psi$ and a martingale difference $M_1 \equiv M_X(\xi_0, \xi_1)$. Replacing (ξ_0, ξ_1) by (ξ_{i-1}, ξ_i) we similarly decompose

$$X_i = \psi(\xi_i) - \psi(\xi_{i-1}) + M_i \tag{4.11}$$

so that upon adding we obtain that

$$X(t) = \psi(\xi_t) - \psi(\xi_0) + M(t) \tag{4.12}$$

where $M(t)$ is a square-integrable martingale with stationary increments and $\psi(\xi_t) - \psi(\xi_0)$ is evidently of order unity (as $t \to \infty$). Thus by invoking the decomposition (4.7) we seem to have obtained the CLT with apparently no work at all.

We should point out immediately that the argument just presented is a swindle. The trouble is that the projection of X_1 onto $\mathcal{H}_0$ need not be in $\Delta L^2(\mu)$; it need only be in the completion, and we cannot be sure that the sums of the time-translates of such elements are also of order unity – in fact, they of course are not! We must thus be more careful and focus on the structure of $\mathcal{H}_0$.

The crucial observation at this point is that for any $\psi \in L^2(\mu)$

$$||\Delta\psi||^2 \equiv E_\mu((\psi(\xi_1) - \psi(\xi_0))^2) = 2(\psi, (I-P)\psi) \equiv 2||\psi||_1^2, \tag{4.13}$$

where P is the transition probability for the Markov process ξ_n and is self-adjoint by reversibility, I is the identity, and the norm on the right is the discrete-time $H_1 \equiv H_1(I-P)$-norm. Therefore Δ, after dividing by $\sqrt{2}$, extends to a unitary from H_1, the completion of $L^2(\mu)$ in the norm $||\ \ ||_1$, onto $\mathcal{H}_0$. In particular, we find that

$$\mathcal{H}_0 = \Delta H_1. \tag{4.14}$$

It follows that there exists a $\psi \in H_1$ such that in place of (4.10) we have that

$$X_1 = \Delta\psi + M_1 \tag{4.15}$$

and in place of (4.11) we have that

$$X_i = \Delta_i\psi + M_i \tag{4.16}$$

where $\Delta_i \equiv \Delta\psi(\xi_{i-1}, \xi_i)$ and $M_i \equiv M_X(\xi_{i-1}, \xi_i)$.

For $\psi \in L^2(\mu)$ define

$$\Delta_{[0,t]}\psi \equiv \psi(\xi_t) - \psi(\xi_0). \tag{4.17}$$

This satisfies

$$\begin{aligned} \| \Delta_{[0,t]} \psi \|^2 &= 2(\psi, (I - P^t)\psi) \\ &= 2t(\psi, [(I - P^t)/t]\psi) \le 2t \|\psi\|_1^2 \end{aligned} \tag{4.18}$$

since

$$0 \le (I - P^t)/t \le I - P. \tag{4.19}$$

Thus $\Delta_{[0,t]}$ extends by continuity to all of H_1, and the full (4.18) remains valid for this extension (since the middle term of (4.18) is continuous as a function of $\psi \in H_1$).

Moreover, the relation

$$\sum_{i=1}^{t} \Delta_i \psi = \Delta_{[0,t]} \psi, \tag{4.20}$$

obviously true for $\psi \in L^2(\mu)$, extends by continuity to all $\psi \in H_1$. Thus, summing (4.16), we have in place of (4.12) that

$$X(t) = \Delta_{[0,t]} \psi + M(t). \tag{4.21}$$

Since for $|x| \le 1$, $\lim_{t \to \infty} (1 - x^t)/t = 0$, it follows from (4.18) and (4.19), using the spectral representation [4] for P on H_1, that by the dominated convergence theorem $R(t) \equiv \Delta_{[0,t]} \psi$ satisfies (2.6), completing the proof of the CLT part of Theorem 4.1.

5. CONTINUOUS-TIME DIRECT DECOMPOSITION

The extension to continuous time of the direct decomposition of Section 4 is reasonably straightforward. Let $X_{\mathbf{I}}$ be a square-integrable antisymmetric functional of the ergodic and reversible (continuous-time) Markov process ξ_t, as described at the beginning of Section 3. We note that for each interval $\mathbf{I} = [a, b]$,

$$X_{\mathbf{I}} \in L^2(P_\mu \restriction \mathcal{F}_{\mathbf{I}})_{\text{antisymmetric}} \equiv \mathcal{H}^{\mathbf{I}}, \tag{5.1}$$

the kernel of $R_{\mathbf{I}} + I$ acting on $L^2(P_\mu \restriction \mathcal{F}_{\mathbf{I}})$.

For $\psi \in L^2(\mu)$ define

$$\Delta_{\mathbf{I}} \psi \equiv \psi(\xi_b) - \psi(\xi_a) \tag{5.2}$$

and let

$$\mathcal{H}_0^{\mathbf{I}} \equiv \overline{\{\Delta_{\mathbf{I}} \psi \mid \psi \in L^2(\mu)\}}. \tag{5.3}$$

Since

$$\| \Delta_{\mathbf{I}} \psi \|^2 = 2(\psi, (I - e^{-Lt})\psi) \equiv 2 \| \psi \|_{(t)}^2 \tag{5.4}$$

where $t = b - a$, and since the norms $\| \quad \|_{(t)}$ are equivalent for all $t > 0$, it follows that $\Delta_{\mathbf{I}}$ extends to a continuous bijection $H_{(1)} \to \mathcal{H}_0^{\mathbf{I}}$ satisfying (5.4), where $H_{(1)} \equiv H_1(I - e^{-L})$, the completion of $L^2(\mu)$ in $\| \quad \|_{(1)}$. In particular $\mathcal{H}_0^{\mathbf{I}} = \Delta_{\mathbf{I}} H_{(1)}$.

Let $\mathcal{H}_m^{\mathbf{I}} = (\mathcal{H}_0^{\mathbf{I}})^{\perp}$, the orthogonal complement of $\mathcal{H}_0^{\mathbf{I}}$ in $\mathcal{H}^{\mathbf{I}}$. For $M \in \mathcal{H}^{\mathbf{I}}$, we have that $M \in \mathcal{H}_m^{\mathbf{I}}$ if and only if $E_\mu(M \mid \mathcal{F}_a) = 0$. Corresponding to the decomposition

$$\mathcal{H}^{\mathbf{I}} = \mathcal{H}_0^{\mathbf{I}} \oplus \mathcal{H}_m^{\mathbf{I}} \tag{5.5}$$

we decompose

$$X_{\mathbf{I}} = \Delta_{\mathbf{I}} \psi + M_{\mathbf{I}}. \tag{5.6}$$

Using the covariance, eq. (3.1), and additivity, eq. (3.2), of $X_{\mathbf{I}}$ and the similar additivity of $\Delta_{\mathbf{I}}$ (which follows by continuity from the obvious additivity on $\psi \in L^2(\mu)$), we see that ψ does not depend upon the choice of the interval $\mathbf{I}$ for $t = b - a$ rational. Moreover, if X_I is cadlag in the sense that $X_{[0,t]}$ has a modification which is right continuous with left limits, or, more generally, if $X_{\mathbf{I}}$ is continuous in the sense that $X_{[0,t]} \to 0$ in P_μ-probability as $t \to 0$, then this independence holds for all t.

We then find that there exists a $\psi \in H_{(1)}$ such that for all $t \geq 0$

$$X(t) = \Delta_{[0,t]} \psi + M(t) \tag{5.7}$$

where $M(t) = M_{[0,t]}$ is a square-integrable martingale. Moreover, it follows from eq. (5.4) and the fact that for $x \geq 0$ and $t \geq 1$, we have $0 \leq \dfrac{1 - e^{-xt}}{t} \leq 1 - e^{-x}$, that $R(t) \equiv \Delta_{[0,t]} \psi$ satisfies (2.6). We have thus established Theorem 5.1 below, which improves the CLT part of Theorem 2.2 of reference [4] by completely eliminating the need for hypotheses concerning the existence or properties of the drift $\phi(\xi_t)$.

THEOREM 5.1. – *Let $X_{\mathbf{I}}$ be an antisymmetric functional of an ergodic reversible Markov process as specified at the beginning of Section 3, and suppose that $X_{\mathbf{I}}$ is square-integrable and that $\mathbf{I} \to X_{\mathbf{I}}$ is continuous in probability. Then $X(t) \equiv X_{[0,t]}$ obeys the CLT.*

We remark that the decomposition (2.8) of a process $X(t)$ into any process $R(t)$ satisfying (2.6) and a square-integrable martingale with stationary increments is unique: Forming the difference of two such decompositions of $X(t)$ and using the fact that the variance of a square-integrable martingale with stationary increments is linear in t, we find upon taking $t \to \infty$ that the difference of the corresponding martingales must vanish. In particular, for $X(t)$ as described in Theorem 5.1 we have found the form of this unique $R(t)$, namely, $R(t) = \Delta_{[0,t]}\,\psi$ for some $\psi \in H_{(1)}$.

6. SYMMETRIC FUNCTIONALS, ANTISYMMETRIC FUNCTIONALS, AND THE ASSOCIATED MARTINGALE

We would like to make contact with the drift ϕ, *i.e.*, with the standard decomposition (3.4), in as much generality as possible. We begin by remarking that, just as with the decomposition (2.8), the decomposition (3.4) of a process $X(t)$ into any process $S(t)$ satisfying

$$\frac{S(t)}{\sqrt{t}} \to 0, \quad \text{as } t \to 0, \tag{6.1}$$

in $L^2(P_\mu)$ and a square-integrable martingale with stationary increments is unique. This unique $S(t)$, if it exists, may be regarded as – and will be called – the generalized integrated drift. We will use the decomposition (5.7) to establish the existence of $S(t)$ with great generality, and to arrive at its form.

We begin by noting [4] that for any $\phi \in L^2(\mu) \cap H_{-1}$, with $H_{-1} = H_{-1}(L)$,

$$\begin{aligned} E_\mu\left[\left(\int_0^t \phi(\xi_s)\,ds\right)^2\right] &= 2\int_0^t \int_0^s (\phi, e^{-L\tau}\phi)\,d\tau\,ds \\ &= 2\int_0^t \left(\phi, \frac{I - e^{-Ls}}{L}\phi\right) ds \le 2t\,\|\phi\|_{-1}^2. \end{aligned} \tag{6.2}$$

Thus the integral from 0 to t extends by continuity to an operator $\int_0^t : H_{-1} \to L^2(P_\mu)$ satisfying

$$\left\| \int_0^t \phi \right\|^2 = 2\int_0^t (\phi, I - e^{-Ls}\phi)_{-1}\,ds \le 2t\,\|\phi\|_{-1}^2. \tag{6.3}$$

Observe that since, for $x > 0$, $1 \ge 1 - e^{-xs} \searrow 0$ as $s \to 0$, we have for all $\phi \in H_{-1}$ that as $t \to 0$

$$\frac{\int_0^t \phi}{\sqrt{t}} \to 0 \tag{6.4}$$

in $L^2(P_\mu)$.

Let

$$H_1 \equiv H_1(L) = \overline{\{\psi \in L^2(\mu) \mid \|\psi\|_1^2 \equiv (\psi, L\psi) < \infty\}} \tag{6.5}$$

where the overbar denotes the completion in $\|\ \ \|_1$. Since $0 \leq \frac{I - e^{-Lt}}{t} \leq L$ for $t > 0$, we have that $\|\ \ \|_{(t)} \leq t\|\ \ \|_1$ and that $H_1 \subset H_{(1)}$. In particular, we have from eq. (5.4) that for $\psi \in H_1$

$$\|\Delta_{[0,t]}\psi\|^2 \leq 2t\|\psi\|_1^2. \tag{6.6}$$

Moreover, since $0 \leq \frac{1 - e^{-xt}}{t} \nearrow x$ as $t \searrow 0$ for $x > 0$, it follows from (5.4) using the monotone convergence theorem that for $\psi \in H_{(1)}$ we have that $\frac{\|\Delta_{[0,t]}\psi\|^2}{t}$ is increasing as $t \searrow 0$ and that the limit is $< \infty$ if and only if $\psi \in H_1$ (in which case this limit is $2\|\psi\|_1^2$).

Note also that $L : H_1 \to H_{-1}$ is unitary. Thus, from eq. (6.3),

$$\left\| \int_0^t L\psi \right\|^2 \leq 2t\|\psi\|_1^2. \tag{6.7}$$

For $\psi \in H_1$ define

$$M^\psi(t) = \Delta_{[0,t]}\psi + \int_0^t L\psi. \tag{6.8}$$

When ψ belongs to the $L^2(\mu)$-domain of L, which is dense in H_1, it is well known, and one may easily check, that $M^\psi(t)$ is a square-integrable martingale. From eqs. (6.6) and (6.7) it follows by continuity that this is in fact true for all $\psi \in H_1$. We note in passing that for any $\phi \in H_{-1}$, the process $\int_0^t \phi = \Delta_{[0,t]}\psi - M^\psi(t)$ with $\psi = -L^{-1}\phi$, and hence obeys the CLT.

Combining the preceding observations, and using eqs. (5.7) and (6.8) and the fact that the variance of a square-integrable martingale with stationary increments is linear in t, we arrive at the following result:

THEOREM 6.1. – *Let $X_{\mathbf{I}}$ be an antisymmetric functional of an ergodic reversible Markov process as specified at the beginning of Section 3, and suppose that $\|X_{\mathbf{I}}\|^2 = O(|\mathbf{I}|)$ as $|\mathbf{I}| \to 0$. Then the generalized integrated drift $S(t)$ exists for the process $X(t) \equiv X_{[0,t]}$ and is given by $S(t) = \int_0^t \phi$ where $\phi = -L^{-1}\psi$ with ψ given by* (5.7).

7. THE INVARIANCE PRINCIPLE

We mentioned in Section 2 that Kipnis and Varadhan [2] have shown that if $\phi \in L^2(\mu) \cap H_{-1}$, then $S(t) = \int_0^t \phi$ satisfies (2.5) with $R(t)$ obeying (2.7), a result which is extended in reference [4] to all $\phi \in L^1(\mu) \cap H_{-1}$ also satisfying

$$\sup_{0 \le t \le 1} \left| \int_0^t \phi \right| \in L^2(P_\mu). \tag{7.1}$$

Now the argument in [4] in fact shows, if we use the fact $M^\psi(t)$ is a square-integrable martingale with stationary increments for all $\psi \in H_1$, that the condition $\phi \in L^1(\mu) \cap H_{-1}$ can be relaxed to the condition that $\phi \in H_{-1}$ and is such that $\int_0^t \phi$ has a modification which is right-continuous with left limits.

If $X_{\mathbf{I}}$ and ϕ are as in Theorem 6.1, it follows from the the Doob-Kolmogorov inequality [5] that (7.1) is equivalent to

$$\sup_{0 \le t \le 1} |X_{[0,t]}| \in L^2(P_\mu). \tag{7.2}$$

Moreover, the condition that $\int_0^t \phi$ has a modification which is right-continuous with left limits is equivalent to the analogous condition on $X_{[0,t]}$ – since this condition is always satisfied by the martingale $\mathcal{M}(t)$ [6]. We have thus arrived at the following result:

THEOREM 7.1. – *Let $X_{\mathbf{I}}$ be an antisymmetric functional of an ergodic reversible Markov process as specified at the beginning of Section 3, and suppose that $\| X_{\mathbf{I}} \|^2 = O(|\mathbf{I}|)$ as $|\mathbf{I}| \to 0$, that $\sup_{0 \le t \le 1} |X_{[0,t]}| \in L^2(P_\mu)$, and that $X_{[0,t]}$ has a modification which is right-continuous with left limits. Then $X(t) \equiv X_{[0,t]}$ obeys the IP.*

In view of the sentiments which motivated this paper, expressed at the beginning of Section 4, we should not be happy with the proof of this theorem, and, indeed, we are not! However, we have not been able to avoid the indirect analysis exploiting the beautiful estimate of Kipnis and Varadhan [2]: For all $\psi \in L^2(\mu) \cap H_1$

$$P_\mu\Big(\sup_{0 \le s \text{ rational } \le t} |\psi(\xi_s)| > \eta \Big) \le 3\eta^{-1} \sqrt{\|\psi\|^2 + t\|\psi\|_1^2}. \tag{7.3}$$

Notice that for a general $\psi \in H_1$, for which $\psi(\xi_s)$ need have no meaning, there can be no such estimate. However, what we would like to have, in view of (5.7), is a simple, direct estimate on the sup of $R(t) = \Delta_{[0,t]}\psi$, and not, of course, an estimate involving $\psi(\xi_t)$ itself. If ψ is given by (5.7) for X_{I} satisfying the hypotheses of Theorem 7.1, then $R(t) = \Delta_{[0,t]}\psi$ does, in fact, obey eq. (2.7) – by (the proof of) Theorem 7.1 and the uniqueness described at the end of Section 5. But we don't know how to see this directly. Nor do we know the precise condition on a $\psi \in H_1$ under which $R(t) = \Delta_{[0,t]}\psi$ satisfies (2.7) (at least with the sup restricted to rational s).

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