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Large deviation probabilities for some rescaled superprocesses

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ABSTRACT. – We consider a class of rescaled superprocesses and derive a full large deviation principle with a "good" convex rate functional defined on the measure state space. The rate functional is identified as the Legendre transform of a log-Laplace functional. The latter is described by solutions of an explosive reaction-diffusion equation (cumulant equation) which is discussed in some detail. In the special case that the motion component in the model is suppressed, the variational problem is explicitly solved showing in particular that as a rule the rate functional is not strongly convex and not continuous.

Keywords: Large deviation, superprocess, cumulant equation, rate functional.

RÉSUMÉ. – Nous considérons une classe de superprocessus en changement d'échelle et nous obtenons un principe de grands écarts par rapport à une « bonne » fonctionnelle convexe définie sur une espace d'états de mesures. La fonctionnelle d'intensité est la transformée de Legendre d'une

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fonctionnelle de type Log-Laplace. Cette dernière est caractérisée à l'aide des solutions d'une équation de réaction-diffusion explosive (l'équation cumulante) qui est étudiée en détail. Dans le cas particulier où la composante de dérive du modèle est supprimée, le problème variationnel est résolu explicitement, illustrant par le fait même que la fonctionnelle d'intensité n'est ni fortement convexe ni continue.

1. INTRODUCTION

1.1. Motivation

Since the pioneering paper of Liemant (1969), much has been done in the field of *spatially distributed branching models of infinite populations*: equilibrium theory, convergence theorems, scaling properties, hydrodynamics, sample path properties, random media effects – to mention only some main topics. However, to our knowledge there are only a few papers dealing with *large deviation aspects.* (Generally speaking, large deviation probabilities are of particular interest in statistical physics, in models in random media, and in other respect; the relatively simple branching models may serve as a certain test case only.)

Cox and Griffeath (1985) considered the critical binary branching Brownian motion starting with a homogeneous Poisson particle system of density one and studied in dimensions $d \geqq 3$ the asymptotics of the (logarithmic) large deviation probabilities

$$\log \mathcal{P}r\left(t^{-1}\int_0^t ds\, N_s(B) > (1+\varepsilon)\, l(B)\right)$$

as $t \to \infty$ where $N_s(B)$ counts the number of particles at time s in the bounded Borel set $B \subset \mathbb{R}^d$ of volume $l(B)$, and $\varepsilon > 0$ has to be *sufficiently small.* This last condition has its origin in the method they use based on cumulants: It guarantees the convergence of some power series expansions. Also, in recent manuscripts of Lee (1993) and Iscoe and Lee (1993) similar restrictions enter into some large deviation probabilities for closely related occupation time processes; the only exception is a dimension $d = 3$ result, where a steepness argument could be used.

To remove such "disturbing" conditions was our primary motivation to look for large deviation properties in infinite branching models. From a technical point of view, one has to take into account that in such branching models exponential moments are *infinite* as a rule.

In the present note we are concerned with large deviation probabilities $\log \mathcal{P}r\,(X^K(t) \in A)$ as $K \to \infty$, where X^K refers to a branching process appropriately *scaled* in time, space and mass, t is a fixed macroscopic time point and A is any open or closed set in the state space of the scaled processes. Here we restrict our main attention to *supercritical* dimensions d, *i.e.* to those dimensions where the unscaled process has steady states. Under a critical rescaling we prove a *full* large deviation result.

From the variety of possible choices we decided to work with a measure-valued branching model (*Dawson-Watanabe process, superprocess),* which reduces the number of relevant approximations forced by the scaling and which simplifies the use of some analytical tools.

We feel it is reasonable to assume that the reader is somewhat familiar with the concept of a superprocess, or is willing to consult for example the recent survey of Dawson (1993) which contains a full account. Regarding the technical framework for the class of superprocesses we have in mind, the paper is intended to be self-contained. Since we allow a fairly general motion component in the model, it seems natural to use some functional analytic approach.

In the remainder of this introduction we will describe the model, formulate the main result and provide some heuristic background leading to a dimensionally independent reformulation of the problem.

1.2. Preliminaries

Fix a dimension $d \geqq 1$, a "motion index" $\alpha \in (0, 2]$, constants a_1, a_2 satisfying $d < a_1 \leqq d + \alpha$, $a_2 > 0$, write $a := [a_1, a_2]$, and introduce the *reference function*

$$\varphi_a(y) := 1/(1 + a_2|y|^2)^{a_1/2}, \qquad y \in \mathbb{R}^d.$$

Let Φ denote the linear space of all real-valued *continuous* functions φ defined on $\mathbb{R}^d$ with the property that the ratio $\varphi(y)/\varphi_a(y)$ *converges* to a finite limit as $|y| \to \infty$. In Φ we introduce the norm

$$\|\varphi\| := \sup_{y \in \mathbb{R}} |\varphi(y)/\varphi_a(y)|, \qquad \varphi \in \Phi.$$

Then Φ is a separable Banach space. Note that $\mathcal{C}^{\text{comp}} \subset \Phi \subset \mathcal{C}_0$ where $\mathcal{C}^{\text{comp}}$ and $\mathcal{C}_0 = \mathcal{C}_0[\mathbb{R}^d]$ are the spaces of all continuous functions with compact support or vanishing at infinity, respectively, both equipped with the supremum norm $\|.\|_\infty$ of uniform convergence. Moreover, the embedding of Φ into $\mathcal{C}_0$ is continuous, since $\|\varphi\|_\infty \leqq \|\varphi\|$, $\varphi \in \Phi$.

To Φ we introduce the "dual" set $\mathcal{M}_a$ of all (locally finite non-negative) measures μ defined on $\mathbb{R}^d$ such that $(\mu, \varphi_a) < +\infty$, or equivalently, $(\mu, \varphi) < +\infty$ for all $\varphi \in \Phi_+$. We endow this set $\mathcal{M}_a$ of *tempered measures* with the *a-vague topology.* By definition, this is the coarsest topology such that all real functions $\mu \mapsto (\mu, \varphi)$, $\varphi \in \mathcal{C}_+^{\text{comp}} \cup \{\varphi_a\}$, are continuous. Hence all the mappings $\mu \mapsto (\mu, \varphi)$, $\varphi \in \Phi$, are continuous. Note that the Lebesgue measure l belongs to this set $\mathcal{M}_a$. Next we include also $a_2 = 0$ in which case $\varphi_a = 1$, $\|.\| = \|.\|_\infty$ and where $\mathcal{M}_a$ degenerates to the set of all *finite* measures endowed with the *weak* topology. (This is actually the reason why the constant a_2 was introduced.)

In the following the symbols A_+ and A_- refer to the sets of all non-negative respectively non-positive members of a set A. Integrals $\int m(dx) f(x)$ are written as (m, f).

1.3. Model

Let $X = [X, \mathbb{P}_{s,\mu}^{\kappa,\rho}; s \in \mathbb{R}_+, \mu \in \mathcal{M}_a]$ denote the (critical continuous) *superstable motion* on $\mathbb{R}^d$ with *motion index* $\alpha \in (0, 2]$, "*diffusion*" constant $\kappa \geqq 0$, and constant *branching rate* $\rho \geqq 0$, related via its Laplace transition functionals

$$\left.\begin{array}{l} \mathbb{E}_{s,\mu}^{\kappa,\rho} \exp(\mathrm{X}(t), \varphi) = \exp(\mu, u_\varphi(t-s)), \\ 0 \leqq s \leqq t, \qquad \mu \in \mathcal{M}_a, \qquad \varphi \in \Phi_- \end{array}\right\} \tag{1.3.1}$$

to the solutions $u = u_\varphi$ of the non-linear differential equation

$$\left.\begin{array}{l} \dfrac{\partial}{\partial t} u(t, y) = \kappa \Delta_\alpha u(t, y) + \rho u^2(t, y), \qquad t > 0, \qquad y \in \mathbb{R}^d, \\ u(0+, .) = \varphi \in \Phi_-. \end{array}\right\} \tag{1.3.2}$$

Here $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian acting on the space coordinate y.

In other words, this *time-homogeneous Markov process* X lives in $\mathcal{M}_a$. Given the state $X(s) = \mu$ at an initial time s, the Laplace functional of the random measure $X(t)$, $t > s$, is described by means of the solutions

$u = u_\varphi$ of the non-linear equation (1.3.2). Here the test functions φ of the Laplace functional enter as Cauchy initial conditions.

For readers not familiar with superprocesses we recommend considering both components in the model separately. If $\rho = 0$, the population mass is only smeared out by the stable flow with generator $\kappa\Delta_\alpha$ (the heat flow in the case $\alpha = 2$). On the other hand if $\kappa = 0$ all "differentially small" portions of mass $X(s, dy)$ fluctuate independently (at different space points y) in time according to the stochastic equation

$$d\zeta_r = \sqrt{2\,\rho\zeta_t}\, d\mathcal{W}_t, \qquad t \geqq s, \tag{1.3.3}$$

(starting in $\zeta_s = X(s, dy)$ for y "fixed" and with $\mathcal{W}$ a standard Wiener process in $\mathbb{R}$). This equation describes the simplest critical continuous state Galton-Watson process (Lamperti process) with "branching rate" ρ. (We restrict to the simplest continuous state branching component since later we use the finiteness of exponential moments "around the origin".) Superimposing both components leads heuristically to the superprocess X, and in this way one gets a rough idea of how X behaves.

We also mention that such superprocesses serve as *diffusion approximation* for high density branching particle models, where the particles have a small mass, move independently according to symmetric α-stable motions and split critically with finite variance but with a large rate.

For constants $\gamma > 0$ and $K > 0$ we define the *scaled processes* X^K:

$$\left.\begin{aligned} (X^K(t), \varphi) &:= (X(K^\gamma t), \varphi^K), \qquad t \geqq 0,\\ \varphi^K &:= K^{-d}\varphi(./K), \qquad \varphi \in \Phi; \end{aligned}\right\} \tag{1.3.4}$$

that is we speed up the time by a factor K^γ (with γ specified later), contract the space and rescale the mass, both by the factor K^{-d}. So X^K describes the mass on a large space-time scale. There will be an interplay between the parameter γ, the scaling properties of the spatial α-stable motions and the dimension d of space. This results in a variety of different behaviors of the scaled processes X^K as $K \to \infty$ as we will now review.

1.4. Basic Ergodic Theory

We distinguish between several parameter constellations. First consider the situation of a *critical scaling* by which we mean that $\gamma = \alpha \wedge d$ holds:

($\star$) In the case of a *subcritical dimension, i. e.* if $d < \alpha$, or more explicitly, $\gamma = d = 1 < \alpha$, the scaled processes X^K converge in distribution to X but the latter defined with diffusion constant $\kappa = 0$ (*i. e.* the motion component disappears), provided that the initial measures $X^K(0)$ converge in law to some $X(0)$. If $\rho > 0$, this means, that the scaling will catch clumps, which in the limit are located in Poissonian points; the sizes of the clumps are independent, fluctuate according to (1.3.3), and for a fixed macroscopic time point t, are exponentially distributed. In other words, this limit can be viewed as a collection of independent copies of processes fluctuating according to the stochastic equation (1.3.3), with initial states ζ_0 according to the limiting initial measure $X(0, dy)$. For details concerning this *time-space-mass scaling limit theorem* we refer to Dawson and Fleischmann (1988).

($\star\star$) In the situation of a *critical dimension, i. e.* if $d = \alpha$, or more explicitly, if $\gamma = d = \alpha = 1$ or 2, the superprocess is *self-similar, i. e.* that X^K coincides in distribution with X, provided that the initial states $X^K(0)$ and $X(0)$ coincides in law (e.g., if $X(0) = l$, the Lebesgue measure; *see* Lemma 4.6.1 below).

($\star\star\star$) For *supercritical dimensions* $d > \alpha\,(=\gamma)$ *a law of large numbers* (LLN) is true: For fixed $t \geqq 0$,

$$X^K(t) \xrightarrow[K\to\infty]{\mathcal{P}r} \mathcal{T}_t^\kappa \mu \qquad \text{if} \quad X^K(0) \xrightarrow[K\to\infty]{\mathcal{P}r} \mu, \qquad \mu \in \mathcal{M}_a,$$

where $\mathcal{T}_t^\kappa \mu$ is the measure which results if the α-stable flow with "diffusion" constant $\kappa \geqq 0$ acts on μ over a time period of length t; *see* Lemma 4.5.2 below. In this case (if $\rho > 0$) also the *Gaussian fluctuations* around the α-stable flow $\mathcal{T}^\kappa \mu$ can be computed, leading to Ornstein-Uhlenbeck processes; *see* e.g. Dawson, Fleischmann, and Gorostiza (1989) (specialized to a constant medium and to branching with finite variance).

So far we discussed the situation under the critical scaling $\gamma = \alpha \wedge d$. In the case of a *subcritical scaling* $\gamma < \alpha \wedge d$ (*i. e.* if the microscopic time grows only "moderately"), always a LLN holds; *see* Remark 4.6.5 below. On the other hand, for a *supercritical scaling* $\gamma > \alpha \wedge d$, under reasonable initial conditions one expects a *local extinction* $X^K(t) \xrightarrow[K\to\infty]{\mathcal{P}r} 0, t > 0$, provided that $d \leqq \alpha$, whereas in supercritical dimensions $d > \alpha$ again a LLN should hold.

1.5. Main Results

In this note we fix our attention to *large deviations* related to the law of large numbers ($\star\star\star$) above, *i. e.* with the most interesting LLN since in this case the scaling is critical.

For convenience, similarly to (1.3.4), we introduce a notation μ^K for a scaling of measures μ :

$$(\mu^K, \varphi) := (\mu, \varphi^K), \qquad \mu \in \mathcal{M}_a, \qquad K > 0, \qquad \varphi \in \Phi, \tag{1.5.1}$$

[with φ^K defined in (1.3.4)]. Write

$$\left.\begin{aligned} \Lambda_{\mu,t}(\varphi) := \log \mathbb{E}_{0,\mu}^{\kappa,\rho} \exp(X(t), \varphi) \in (-\infty, +\infty], \\ \mu \in \mathcal{M}_a, \qquad t > 0, \qquad \varphi \in \Phi, \end{aligned}\right\} \tag{1.5.2}$$

for the *log-Laplace functionals* related to the unscaled *X*.

THEOREM 1.5.3 (large deviation principle). – *Assume that $d > \alpha = \gamma$. Fix κ, $\rho \geqq 0$, a measure $\mu \in \mathcal{M}_a$, $\mu \neq 0$, and a (macroscopic) time point $t > 0$. For $K > 0$, let μ_K denote the measure in $\mathcal{M}_a$ which satisfies $(\mu_K)^K = \mu$ (for instance $\mu = l \equiv \mu_K$). Then the following large deviation principle (LDP) holds: There is a lower semi-continuous* convex *functional $S_{\mu,t} : \mathcal{M}_a \mapsto [0, +\infty]$ with $S_{\mu,t}(\mathcal{T}_t^\kappa \mu) = 0$ such that,*

(i) *for each open subset G of $\mathcal{M}_a$,*

$$\liminf_{K\to\infty} K^{-(d-\alpha)} \log \mathbb{P}_{0,\mu_K}^{\kappa,\rho} (X^K(t) \in G) \geqq - \inf_{\nu\in G} S_{\mu,t}(\nu),$$

(ii) *for each closed subset F of $\mathcal{M}_a$,*

$$\limsup_{K\to\infty} K^{-(d-\alpha)} \log \mathbb{P}_{0,\mu_K}^{\kappa,\rho} (X^K(t) \in F) \leqq - \inf_{\nu\in F} S_{\mu,t}(\nu),$$

(iii) $S_{\mu,t}$ *is a* "good" rate functional: *all sets $\{\nu \in \mathcal{M}_a;\ S_{\mu,t}(\nu) \leqq N\}$, $N > 0$, are compact.*

(iv) $S_{\mu,t}$ *is given by the variational formula*

$$S_{\mu,t}(\nu) = \sup\{(\nu, \varphi) - \Lambda_{\mu,t}(\varphi);\ \varphi \in \Phi\} =: \Lambda_{\mu,t}^*(\nu), \qquad \nu \in \mathcal{M}_a.$$

That is, roughly speaking, $\mathcal{P}r(X^K(t) = d\nu) \approx \exp[-K^{d-\alpha} S_{\mu,t}(\nu)]$, as $K \to \infty$, in the sense of logarithmic equivalence.

The point is that for (i) we do not need any smallness condition, *i. e.* a restriction to some small (open) neighborhoods G of $\mathcal{T}_t^\kappa \mu$.

Although the representation (iv) is rather implicit, nevertheless it is very useful since the log-Laplace functional $\Lambda_{\mu,t}$ can be characterized in terms of unique solutions of equation (1.3.2), see Theorem 3.3.1 and Corollary 3.3.4 below.

In the special case of a *vanishing "diffusion"* constant $\kappa = 0$ (*i. e.* if there is no motion in the model) equation (1.3.2) can be solved explicitly. Then we are able also to solve the variational problem (iv). To describe this, we need some notation. Within $\mathcal{M}_a$, each $\nu \in \mathcal{M}_a$ may be uniquely decomposed as $\nu = \nu_{ac} + \nu_\partial + \nu_\infty$. Here $\nu_{ac}(dy) =: g_{ac}(y)\,\mu(dy)$ is absolutely continuous with respect to (the fixed) measure μ, whereas ν_∂ and ν_∞ are singular with respect to μ. By definition, ν_∂ is concentrated on the (uniquely determined) closed support $\mathcal{S}$ of μ whereas $\nu_\infty(\mathcal{S}) = 0$.

THEOREM 1.5.4 (solution of the variational problem). – *For* $\mu \in \mathcal{M}_a$, $t, \rho > 0$, *but* $\kappa = 0$,

$$\left.\begin{aligned} S_{\mu,t}(\nu) = (\rho t)^{-1} \int \mu(dy)\,(\sqrt{g_{ac}(y)} - 1)^2 \\ +\nu_\partial(\mathbb{R}^d)/\rho t + \nu_\infty(\mathbb{R}^d).(+\infty), \qquad \nu \in \mathcal{M}_a, \end{aligned}\right\} \tag{1.5.5}$$

(*using the convention* $0.(+\infty) = 0$).

An interesting fact is that by (1.5.5) the rate functional $S_{\mu,t}$ is typically *not strongly convex,* since it is positively homogeneous along ν_∂. Hence its "conjugate" $\Lambda_{\mu,t}$ is not "steep". (Recall that steepness is often used as a starting point to get the lower bound (i) in terms of some Legendre transform.) We mention also that normally the rate functional $S_{\mu,t}$ is *not continuous*; *see* Example 5.2.4 below.

1.6. Reformulation and Methodology

By scaling properties of the stable semi-group and of the critical continuous-state Galton-Watson process, and by our assumed parameter relations, the time-space-mass scaling X^K of X as $K \to \infty$ can be reformulated as a limit in law of X under $\rho \to 0$ (see Lemma 4.6.1 below).

For the sake of a heuristic argument, let us restrict our attention for the moment to the case of the special branching rates $\rho = 1/N$, $N \to \infty$. Then by the branching property and again by scaling arguments, $X(t)$ with respect to $\mathbb{P}^{\kappa,\,1/N}_{0,\,\mu}$ has the same law as $N^{-1}\sum_{i=1}^{N} X^i(t)$, where the $X^i(t)$ are independent and distributed according to $\mathbb{P}^{\kappa,\,1}_{0,\,\mu}$. Now apply an *infinite dimensional version of Cramér's Theorem.* Here, of course, one has to be careful since the exponential moments of the $(X^i(t), \varphi)$, $\varphi \in \Phi_+$, are *infinite* as a rule. But they are finite for "small" $\varphi \in \Phi_+$ which is actually sufficient; *see* Corollary 5.1.3 below.

To be more precise, our approach is to investigate the large deviation probabilities

$$R^{-1} \log \mathbb{P}^{\kappa,\,\rho}_{0,\,R\mu}(R^{-1} X(t) \in .) \quad \text{as } R \to \infty,$$

which exist *without any dimension restriction* and may be expressed by means of some rate functional $S_{\mu, t}$ (*see* Theorem 4.1.1 below). We derive this via a general methodology for large deviation probabilities as presented in Chapters II and III of Deuschel and Stroock (1989), in conjunction with some results on superprocess log-Laplace functionals which we develop for this purpose. By scaling Theorem 1.5.3 above then follows with the same rate functional $S_{\mu, t}$.

Concerning technical details, a necessary step in the development is to deal with equation (1.3.2) for initial functions φ which admit also *positive* values. Here one has to take into account that, for given φ and a fixed time interval, solutions u may *not exist* (think of the explosive behavior of the ordinary equation $\frac{d}{dt} u(t) = \rho u^2(t)$, for $\rho > 0$ and positive initial values). Perhaps we should add at this place, that (1.3.2) will be handled by transfering it to the corresponding integral equation [*mild* solutions of (1.3.2)]. A rather detailed picture is given in the Theorem 2.4.3 below, which in particular covers known results due to Fujita (1966) or Nagasawa and Sirao (1969).

We mention that the methods in this note are useful also for dealing with functional deviations *in time* (and not only in space), *see* Fleischmann *et al.* (1993), and for large deviations related to other variants of the law of large numbers (subcritical scaling).

1.7. Outline

The relevant tools concerning equation (1.3.2) for φ with possibly changing sign are compiled in Section 2 in a more general set-up than needed for the present particular application (for the sake of later reference). In Section 3, by analytic continuation methods, the connection to the log-Laplace functionals is given. The large deviation estimates follow in Section 4, whereas the final section is devoted to the identification of the rate functional.

2. ON THE CUMULANT EQUATION

The main content of this section is Theorem 2.4.3 below which provides a rather detailed picture concerning the explosive reaction diffusion equation (1.3.2) when we drop the assumption $\varphi \leqq 0$. This is a variation of a type of result which has appeared in many forms and it may be regarded as essentially known. Nevertheless we include it here for the sake of being self-contained. We also stress the fact that most results in the literature only refer to the Brownian case $\alpha = 2$. In general Δ_α are integral operators outside the area of classical partial differential equations and this motivated our approach here which is based on functional analytic methods.

2.1. Further Preliminaries

In this subsection we introduce the function space Φ^I in which solutions of the equation (1.3.2) "live". Recall the parameters d, α and $a = [a_1, a_2]$ where we first go back to our earlier assumption $a_2 > 0$. Fix a finite closed time interval $I := [L, T]$, $L \leqq T$. Let Φ^I denote the linear space of all *continuous* curves u defined on I and with values in Φ. Equip Φ^I with the supremum norm, denoted by

$$\|u\|_I := \sup\{\|u(t)\|;\, t \in I\}, \qquad u \in \Phi^I.$$

By setting $u(t, y) := u(t)(y)$, $t \in I$, $y \in \mathbb{R}^d$, we also regard u as a function on $I \times \mathbb{R}^d$, and we get a continuous embedding $\Phi^I \subset \mathcal{C}_0[I \times \mathbb{R}^d]$ since $\|u\|_\infty \leqq \|u\|_I$, $u \in \Phi^I$. Moreover, we immediately obtain:

LEMMA 2.1.1. – *The spaces Φ and Φ^I are* Banach algebras *with respect to the pointwise product of functions.*

From now on we again include the boundary case $a_2 = 0$ in which $\Phi = \mathcal{C}_1[\mathbb{R}^d]$ and $\Phi^I = \mathcal{C}_1[I \times \mathbb{R}^d]$ where $\mathcal{C}_1$ refers to spaces of continuous functions with a finite limit at infinity.

2.2. On the Stable Flow

Recall that $\kappa \geqq 0$ is a fixed ("diffusion") constant. If $\kappa > 0$, then the *stable semigroup* $\{\mathcal{T}_t^\kappa;\, t \geqq 0\}$ with generator $\kappa\Delta_\alpha = -\kappa(-\Delta)^{\alpha/2}$ possesses continuous transition density functions

$$p^\kappa(s, t, x, y) = p^\kappa(t - s, y - x), \qquad s < t, \qquad x, y \in \mathbb{R}^d,$$

with characteristic functions

$$\int dz\, p^\kappa(r, z)\, e^{i\,\theta . z} = \exp[-\kappa\, r\, |\theta|^\alpha], \qquad r > 0, \quad \theta \in \mathbb{R}^d. \tag{2.2.1}$$

(Note again that with $\alpha = 2$ the *heat flow* is included.)

For $\varphi \in \Phi$ we set $\mathcal{T}^{0,I}\varphi := \varphi$ and for $\kappa > 0$ define $\mathcal{T}^{\kappa,I}\varphi := \{\mathcal{T}^{\kappa}_{T-s}\varphi;\ s \in I\}$ where by definition

$$\mathcal{T}^{\kappa}_t \varphi(x) = \int dy\, p^{\kappa}(t, y-x)\,\varphi(y), \qquad t > 0, \qquad x \in \mathbb{R}^d.$$

The following lemma can be found, for instance, in Dawson and Fleischmann (1988), Lemma 4.1. (Note that $\mathcal{T}^{\kappa}_t = \mathcal{T}^{1}_{\kappa t}$.)

LEMMA 2.2.2. – *$[\kappa, \varphi] \mapsto \mathcal{T}^{\kappa,I}\varphi$ is a* continuous *mapping of* $\mathbb{R}_+ \times \Phi$ *into* Φ^I.

As a simple consequence we get [*see* also Dawson and Fleischmann (1992), formula line (3.4)]:

LEMMA 2.2.3. – *The linear operators $\mathcal{T}^{\kappa}_t$ acting in Φ are* uniformly bounded *for bounded t and κ.*

Proof. – In fact, for $0 \leqq t, \kappa \leqq c$,

$$\|\mathcal{T}^{\kappa}_t \varphi\| \leqq \|\varphi\|\, \|\mathcal{T}^{1}_{\kappa t}\varphi_a\| \leqq \|\varphi\|\, \|\mathcal{T}^{1,J}\varphi_a\|_J = \text{const.}\, \|\varphi\|,$$

where for the moment we set $J := [0, c^2]$, and const. always denotes a finite constant. ■

2.3. Another Convolution Map

For $u \in \Phi^I$ we introduce $W^{\kappa,I}u$ by setting

$$(W^{\kappa,I}u)(s) = \int_s^T dr\, \mathcal{T}^{\kappa}_{r-s}u(r), \qquad s \in I = [L, T].$$

LEMMA 2.3.1. – *$[\kappa, u] \mapsto W^{\kappa,I}u$ is a* continuous *mapping of* $\mathbb{R}_+ \times \Phi^I$ *into* Φ^I.

Proof. – According to Lemma 2.2.2, $\mathcal{T}^{\kappa}_{r-s}u(r)$ belongs to Φ, for each pair r, s satisfying $r \geqq s$. In view of Lemma 2.2.3,

$$\|\mathcal{T}^{\kappa}_s\varphi - \mathcal{T}^{\kappa}_r\psi\| \leqq \|\mathcal{T}^{\kappa}_s(\varphi-\psi)\| + \|\mathcal{T}^{\kappa}_s\psi - \mathcal{T}^{\kappa}_r\psi\|$$
$$\leqq \text{const.}\,\|\varphi-\psi\| + \text{const.}\,\|\mathcal{T}^{\kappa}_{|s-r|}\psi - \psi\|, \qquad s, r, \kappa \geqq 0, \quad \varphi, \psi \in \Phi.$$

Assume $\kappa_n \to \kappa$ and $u_n \to u$ as $n \to \infty$. For $s_n \in I$, by the previous estimates,

$$\|W^{\kappa_n,I}u_n(s_n) - W^{\kappa,I}u(s_n)\| \leqq \int_I dr\, \|\mathcal{T}^{1}_{\kappa_n|r-s_n|}u_n(r) - \mathcal{T}^{1}_{\kappa|r-s_n|}u(r)\|$$
$$\leqq \int_I dr\,(\text{const.}\,\|u_n(r) - u(r)\| + \text{const.}\,\|\mathcal{J}^{1}_{|\kappa_n-\kappa||r-s_n|}u(r) - u(r)\|).$$

In virtue of Lemma 2.2.2, the latter norm expression converges to 0 as $n \to \infty$, for each r. Moreover, by Lemma 2.2.3, it is bounded above by const. $\|u(r)\| \leqq$ const. $\|u\|_I =$ const. Hence, by dominated convergence, the integral over the second norm expression converges to 0 as $n \to \infty$. But for the first term we get $\leqq$ const. $\|u_n - u\|_I$ which converges to 0, too. Summarizing,

$$\|W^{\kappa_n, I} u_n - W^{\kappa, I} u\|_I = \sup_{s \in I} \|W^{\kappa_n, \mathrm{I}} u_n(s) - W^{\kappa, I} u(s)\| \to 0$$

as $n \to \infty$, and we are done. ■

2.4. Implicit Function Theorem Setting

Recall that $I = [L, T]$. Now we are prepared to introduce the functional

$$\mathbf{F}(\kappa, \rho, \varphi, \psi, u) := u - \mathcal{T}^{\kappa, I} \varphi - W^{\kappa, I} \psi - \rho W^{\kappa, I} (u^2) \qquad (2.4.1)$$

defined for $[\kappa, \rho, \varphi, \psi, u] \in \mathbb{R}_+ \times \mathbb{R}_+ \times \Phi \times \Phi^I \times \Phi^I$. We will study the equation

$$\mathbf{F}(\kappa, \rho, \varphi, \psi, u) = 0 \qquad (2.4.2)$$

which covers (1.3.2). In fact, in more details it can be written as

$$\left. \begin{array}{c} u(s) = \mathcal{T}^{\kappa}_{T-s} \varphi + \int_s^T dr\, \mathcal{T}^{\kappa}_{r-s} \psi(r) + \rho \int_s^T dr\, \mathcal{T}^{\kappa}_{r-s} (u^2(r)), \\ s \in I, \end{array} \right\} \qquad (2.4.2')$$

and a formal differentiation to the time variable s yields

$$-\frac{\partial}{\partial s} u = \kappa \Delta_\alpha u + \psi + \rho u^2, \qquad u|_{s=T-} = \varphi. \qquad (2.4.2'')$$

(To rebuild (1.3.2), set $L = 0$, $\psi = 0$, and reverse the time: $s \mapsto T - t$; later the *backward formulation* is needed to express some functionals of the occupation time process related to X.) Our purpose will be to solve (2.4.2) with the help of the implicit function theorem, for adequate $[\kappa, \rho, \varphi, \psi]$.

THEOREM 2.4.3 (cumulant equation). – *Recall that* $0 < \alpha \leqq 2$, $1 \leqq d < a_1 \leqq d + \alpha$, $a_2 \geqq 0$ *and* $I = [L, T]$ *are fixed.*

(i) (uniqueness). – *To each* $[\kappa, \rho, \varphi, \psi] \in \mathbb{R}_+^2 \times \Phi \times \Phi^I$ *there exists at most one element* $u \in \Phi^I$ *which solves* $\mathbf{F}(\kappa, \rho, \varphi, \psi, \mu) = 0$.

(ii) (existence). – *The set* $\mathcal{U}$ *of all those* $[\kappa, \rho, \varphi, \psi] \in \mathbb{R}_+^2 \times \Phi \times \Phi^I$ *such that there exists an element* $u =: u_{[\kappa, \rho, \varphi, \psi]}$ *in* Φ^I *for which* $\mathbf{F}(\kappa, \rho, \varphi, \psi, u) = 0$ *is* open *and includes* $\mathbb{R}_+ \times \{0\} \times \Phi \times \Phi^I$ *as well as* $\mathbb{R}_+^2 \times \Phi_- \times \Phi_-^I$. *In particular,* $u_{[\bullet, \bullet, 0, 0]} = 0$.

(iii) (continuity, convexity and analyticity). – *The map* $[\kappa, \rho, \varphi, \psi] \mapsto u_{[\kappa, \rho, \varphi, \psi]} \in \Phi^I$ *defined on* $\mathcal{U}$ *is continuous, and for fixed* $[\kappa, \rho]$, *the map* $[\varphi, \psi] \mapsto u_{[\kappa, \rho, \varphi, \psi]}$ *is convex and analytic (with* $[\kappa, \rho, \varphi, \psi]$ *ranging in* $\mathcal{U}$*).*

(iv) (blow-up)[1]. *If* $L < T$ *then* $\mathcal{U}$ *is different from* $\mathbb{R}_+^2 \times \Phi \times \Phi^I$, *and* $\sup\{u_{[\kappa, \rho, \varphi, \psi]}(s, y);\ [s, y] \in I \times \mathbb{R}^d\} \to +\infty$ *as* $[\kappa, \rho, \varphi, \psi] \to [\overline{\kappa}, \overline{\rho}, \overline{\varphi}, \overline{\psi}] \in \partial\mathcal{U}$, *the boundary of* $\mathcal{U}$.

(v) (maximum principle). $u_{[\kappa, \rho, \varphi, \psi]} \leqq 0\ (\geqq 0)$ *provided that* $\varphi, \psi \leqq 0$ $(\geqq 0$, resp.).

(vi) (global solutions). *Fix* $[\kappa, \rho] \in \mathbb{R}_+^2$. *If* $\varphi, \psi \leqq 0$, *then even a global solution exists, that is the solution can be extended from* $I = [L, T]$ *to all of* $(-\infty, T]$. *On the other hand, if* $d > \alpha$ (supercritical dimension) *and* $[\varphi_+, \psi_+]$ *is sufficiently small in norm, then again a global solution exists.*

[Of course, in our real Banach space setting, *analyticity* at a point means that the power series expansion converges absolutely in a neighborhood of that point; *see* e.g. Zeidler (1986), Section 8.2.]

The reader who is more interested in the direction of the paper as a whole or is willing to accept this theorem as it stands might want to skip the proof in the next subsection and proceed directly to Section 3.

2.5. Proof of Theorem 2.4.3

To prepare for the proof, first note that $\mathbf{F}$ maps $\mathbb{R}_+^2 \times \Phi \times \Phi^I \times \Phi^I$ continuously into Φ^I, *see* the Lemmas 2.2.2, 2.3.1 and 2.1.1. Furthermore, at each point $[\kappa, \rho, \varphi, \psi, u] \in \mathbb{R}_+^2 \times \Phi \times \Phi^I \times \Phi^I$, we get the following first *partial (Fréchet) derivative of* $\mathbf{F}$ with respect to u :

$$D_u^1 \mathbf{F}(\kappa, \rho, \varphi, \psi, u)\, v = v - 2\rho W^{\kappa, I}(uv), \qquad v \in \Phi^I. \tag{2.5.1}$$

Consequently, this partial derivative is linear in u and continuous in $[\kappa, \rho, \varphi, \psi, u]$ (again by the Lemmas 2.3.1 and 2.1.1).

---

(1) For some specific blow-up properties in the case $\alpha = 2$, we refer to Mueller and Weissler (1985).

LEMMA 2.5.2. – *For each* $[\kappa, \rho, \varphi, \psi, u] \in \mathbb{R}_+^2 \times \Phi \times \Phi^I \times \Phi^I$, *the (bounded linear) operator* $D_u^1 \mathbf{F}(\kappa, \rho, \varphi, \psi, u) : \Phi^I \mapsto \Phi^I$ *is* bijective.

Proof. – Suppose $L < T$ (otherwise $W^{\kappa, I} = 0$ and $D_u^1 \mathbf{F}(\kappa, \rho, \varphi, \psi, u)$ is the identity). Fix $[\kappa, \rho, \varphi, \psi, u]$ and let v belong to Φ^I with $D_u^1 \mathbf{F}(\kappa, \rho, \varphi, \psi, u)\, v = 0$. By (2.5.1) and boundedness of the operator $W^{\kappa, I}$ according to Lemma 2.3.1,

$$\|v(s)\| \leqq \text{const.}\, \|u\|_I \int_s^T dr\, \|v(r)\|, \qquad s \in I.$$

Then Grönwall's Lemma (pass to $\|v(T-s)\|$) implies $\|v(s)\| \equiv 0$, *i. e.* $v = 0$. Consequently, the first partial derivative under consideration is a *one-to-one* operator.

Let $w \in \Phi^I$. We want to show that there is a $v \in \Phi^I$ with $v - 2\rho W^{\kappa, I}(uv) = w$, *i. e.* that v solves the linear equation

$$v(s) = 2\rho \int_s^T dr\, \mathcal{T}_{r-s}^{\kappa}\, u(r)\, v(r) + w(s), \qquad s \in I. \tag{2.5.3}$$

To this purpose we will decompose the interval I into sufficiently small pieces in order to replace the integral operator in (2.5.3) by an operator with norm strictly smaller than 1, which then will allow us to apply the so-called *main theorem for linear operator equations* in Banach spaces.

Fix $w \in \Phi^I$, let $N > 1$ be a natural number (to be specified later), set $\tau := (T-L)/N$, and introduce the intervals $I(i) := [T-(i+1)\tau, T - i\tau]$, $J(i) := [T - i\tau, T]$, $0 \leqq i < N$. Fix i. For $s \in I(i)$, instead of (2.5.3) we get

$$\begin{aligned} v(s) =& 2\rho\, (W^{\kappa, I(i)}(uv))(s) \\ &+ \mathcal{T}_{(T-i\tau)-s}^{\kappa}\, (W^{\kappa, J(i)}(uv))(T - i\tau) + w(s), \end{aligned} \tag{2.5.4}$$

Now

$$\|W^{\kappa, I(i)}(uv)\|_{I(i)} \leqq C\|u\|_I\, \|v\|_{I(i)}\, \tau, \qquad u \in \Phi^I,$$

where the constant C can be chosen independently of i and τ. Fix N so large that $2\rho C\, \|u\|_I\, \tau < 1$, in order to ensure that the bounded linear operator $W^{\kappa, I(i)}(u\bullet)$ acting in $\Phi^{I(i)}$ has a norm smaller than 1.

First assume that $i = 0$. Then the middle expression at the r.h.s. of equation (2.5.4) disappears, and (2.5.4) has a (unique) solution v on $I(0)$; *see,* for instance, Zeidler (1986), Theorem 1.B.

For a proof by induction on i suppose that v is already constructed on $J(i)$ for some i, $0 \leqq i < N-1$. Then apply the same theorem to extend v

continuously to $I(i) \cup J(i)$. Summarizing, the operator under consideration maps *onto* Φ^I, and the proof is finished. ■

Now we are ready to complete the *Proof of Theorem* 2.4.3.

1° *(uniqueness)*. Take $[\kappa, \rho, \varphi, \psi] \in \mathbb{R}_+^2 \times \Phi \times \Phi^I$ and assume that $\mathbf{F}(\kappa, \rho, \varphi, \psi, u) = 0 = \mathbf{F}(\kappa, \rho, \varphi, \psi, v)$ for some $u, v \in \Phi^I$. From (2.4.1),

$$\|u(s) - v(s)\| \leqq \rho \, \|W^{\kappa, I} u^2(s) - W^{\kappa, I} v^2(s)\|, \qquad s \in I.$$

Using the Lemmas 2.1.1 and 2.2.3, we can continue with

$$\leqq \text{const.} \, \|u + v\|_I \int_s^T dr \, \|u(r) - v(r)\|,$$

and again Grönwall's Lemma yields $\|u(s) - v(s)\| \equiv 0$. This proves the claim (i).

2° *(open domain of existence)*. Fix a point $[\kappa_0, \rho_0, \varphi_0, \psi_0, u_0] \in \mathbb{R}_+^2 \times \Phi \times \Phi^I \times \Phi^I$, and assume that $\mathbf{F}(\kappa_0, \rho_0, \varphi_0, \psi_0, u_0) = 0$ (as is the case for $\varphi_0 = \psi_0 = u_0 = 0$). Based on the formula (2.5.1), Lemma 2.5.2 and 1°, from the *implicit function theorem* we conclude the existence of an (open) neighborhood $\mathcal{U}_0$ of $[\kappa_0, \rho_0, \varphi_0, \psi_0]$ in $\mathbb{R}_+^2 \times \Phi \times \Phi^I \times \Phi^I$ such that there is a unique map $[\kappa, \rho, \varphi, \psi] \mapsto u_{[\kappa, \rho, \varphi, \psi]}$ defined on $\mathcal{U}_0$ with $\mathbf{F}(\kappa, \rho, \varphi, \psi, u_{[\kappa, \rho, \varphi, \psi]}) = 0$; see for instance Theorem 4.B in Zeidler (1986). (Here we have to mention that in applying the implicit function theorem we could replace $\mathbb{R}^2$ by $\mathbb{R}_+^2$ where the neighborhoods of points at the half axis $\kappa = 0$ or $\rho = 0$ are defined in a one-sided way). This shows that the non-empty set $\mathcal{U}$ defined in (ii) is *open*. For the remaining claims of (ii) we refer to the last step of proof below.

3° *(continuity and analyticity)*. The *continuous* dependence of $u_{[\kappa, \rho, \varphi, \psi]}$ on $[\kappa, \rho, \varphi, \psi]$ directly follows by the implicit function theorem from the continuity of $D_u^1 \mathbf{F}(\kappa, \rho, \varphi, \psi, u)$ in $[\kappa, \rho, \varphi, \psi]$.

For fixed $\kappa, \rho \geqq 0$, the directional derivative of $\mathbf{F}$ in the direction of the vector $[\varphi, \psi]$ is given by

$$D_{[\varphi, \psi]}^1 \mathbf{F}(\kappa, \rho, \varphi, \psi, u)[\xi, \zeta] = -\mathcal{T}^{\kappa, I} \xi - W^{\kappa, I} \zeta, \tag{2.5.5}$$

$[\xi, \zeta] \in \Phi \times \Phi^I$, hence is independent of $[\varphi, \psi]$. Combining this with (2.5.1), we obtain that the first partial derivative $D_{[\varphi, \psi, u]}^1 \mathbf{F}(\kappa, \rho, \varphi, \psi, u)$ exists and is even continuous in $[\varphi, \psi, u]$. Next,

$$D_u^2 \mathbf{F}(\kappa, \rho, \varphi, \psi, u) \, vw = -2 \rho \, W^{\kappa, I}(vw), \qquad v, w \in \Phi^I,$$

i. e. $D_u^2 \mathbf{F}(\kappa, \rho, \varphi, \psi, u)$ is independent of $[\varphi, \psi, u]$. Consequently, all higher partial derivatives of $\mathbf{F}$ with respect to $[\varphi, \psi, u]$ will disappear (in other words, $\mathbf{F}$ is a polynomial in $[\varphi, \psi, u]$). Therefore $\mathbf{F}(\kappa, \rho, \varphi, \psi, u)$ is *analytic* in $[\varphi, \psi, u]$, for each fixed $[\kappa, \rho]$. Then the *analyticity* property in the statement (iii) follows; *see* Zeidler (1986), Corollary 4.23.

4° *(blow-up)*. Assume that $[\kappa_n, \rho_n, \varphi_n, \psi_n] \to [\overline{\kappa}, \overline{\rho}, \overline{\varphi}, \overline{\psi}] \in \partial\mathcal{U}$ and that the corresponding solutions $u_n := u_{[\kappa_n, \rho_n, \varphi_n, \psi_n]}$ satisfy $\|u_n\|_\infty \leqq C$, $n \geqq 1$, for some finite constant C. From (2.4.1.) and (2.4.2), for $s \in I$,

$$\begin{aligned}\|u_{n+m}(s) - u_n(s)\| \leqq &\|\mathcal{T}_{T-s}^{k_{n+m}} \varphi_{n+m} - \mathcal{T}_{T-s}^{\kappa_n} \varphi_n\| \\ &+ \|W^{\kappa_{n+m}, I} \psi_{n+m}(s) - W^{\kappa_n, I} \psi_n(s)\| \\ &+ \|\rho_{n+m} W^{\kappa_{n+m}, I} u_{n+m}^2(s) - \rho_n W^{\kappa_n, I} u_n^2(s)\|.\end{aligned}$$

From the Lemmas 2.2.2 and 2.3.1, the first two terms on the r.h.s. are of the order $o(1)$ as $n, m \to \infty$, uniformly in s. Since the sequence ρ_n is bounded, the remaining term can be estimated from above by

$$\leqq o(1) + \text{const.}\,(\|u_{n+m}\|_\infty + \|u_n\|_\infty) \int_s^T dr\, \|u_{n+m}(r) - u_n(r)\|$$

[the $o(1)$ is again uniform in s]. Using the boundedness of the sequence $\|u_n\|_I$ and Grönwall's inequality we get $\|u_{n+m} - u_n\|_I = o(1)$ as $n, m \to \infty$. Hence the u_n form a Cauchy sequence in the Banach space Φ^I. Let $\overline{u}$ denote its limit. From the Lemmas 2.1.1, 2.2.2 and 2.3.1 we conclude that $\mathbf{F}(\overline{\kappa}, \overline{\rho}, \overline{\varphi}, \overline{\psi}, \overline{u}) = 0$. However, this contradicts the statement in 2° since by assumption $[\overline{\kappa}, \overline{\rho}, \overline{\varphi}, \overline{\psi}]$ does not belong to the maximal open set $\mathcal{U}$ of existence. Therefore $\|u_n\|_\infty$ is unbounded. From (2.4.2′) as well as the Lemmas 2.2.2 and 2.3.1,

$$u_n \geqq \mathcal{T}^{\kappa_n, I} \varphi_n + W^{\kappa_n, I} \psi_n \underset{n\to\infty}{\longrightarrow} \mathcal{T}^{\overline{\kappa}, I} \overline{\varphi} + W^{\overline{\kappa}, I} \overline{\psi} \in \Phi^I,$$

i. e. the u_n are bounded below, which yields the (one-sided) blow-up property claimed in (iv).

5° *(points of non-existence)*. Take $[\kappa, \rho, \varphi, \psi] \in \mathbb{R}_+^2 \times \Phi_+ \times \Phi_+^I$ with $\rho > 0$ and $\varphi \neq 0$. Let $\theta > 0$. From 2° we know that $[\kappa, \rho, \theta\varphi, \theta\psi]$ belongs to $\mathcal{U}$ for θ sufficiently small. Assume that it belongs to $\mathcal{U}$ for all $\theta > 0$. Applying the operator $\mathcal{T}_{s-L}^\kappa$ on the solution $u_\theta := u_{[\kappa, \rho, \theta\varphi, \theta\psi]}$ at time $s \in I = [L, T]$, we get

$$\mathcal{T}_{s-L}^\kappa u_\theta(s) = \theta\, \mathcal{T}_{T-L}^\kappa \varphi + \theta \int_s^T dr\, \mathcal{T}_{r-L}^\kappa \psi(r) + \int_s^T dr\, \mathcal{T}_{r-L}^\kappa (u_\theta^2(r)).$$

Setting $\mathcal{T}_{s-L}^\kappa u_\theta(s)(y) =: f_\theta(s)$, $s \in I$, for a fixed $y \in \mathbb{R}$, from Jensen's inequality we obtain

$$f_\theta(s) \geqq f_\theta(T) + \int_s^T dr\, f_\theta^2(r), \qquad s \in I.$$

Therefore f_θ dominates the solution of the equation

$$g(s) = f_\theta(T) + \int_s^T dr\, g^2(r), \qquad s \in I,$$

for all $\theta > 0$. But the latter equation is solvable only for $(T-L)\, f_\theta(T) < 1$ and its solution $g(s) = f_\theta(T)/(1 - (T-s)\, f_\theta(T))$, $s \in I$, explodes as $(T - L)\, f_\theta(T) \uparrow 1$. On the other hand, $L < T$ by assumption, and $f_\theta(T) = \theta\, \mathcal{T}^\kappa_{T-L}\, \varphi(y)$ ranges continuously from 0+ to $+\infty$ on $\{\theta > 0\}$ for an appropriate y by our assumption on φ. This is certainly a contradiction. Consequently, $[\kappa, \rho, \theta\varphi, \theta\psi]$ does not belong to $\mathcal{U}$ for θ sufficiently large. This completes the proof of (iv).

6° (*convexity*). At this stage we use the standard iteration scheme, which we recall here without going into any details. (For this technique, *see* for instance Dawson and Fleischmann (1988), Proposition 4.6, or also Dawson and Fleischamnn (1992).) Fix $\kappa, \rho \geqq 0$. Let $[\kappa, \rho, \varphi, \psi]$ belong to $\mathcal{U}$. Set

$$u_0 := 0, \qquad u_{n+1} = \mathcal{T}^{\kappa, I}\varphi + W^{\kappa, I}\psi + \rho\, W^{\kappa, I}(u_n^2), \qquad n \geqq 0. \tag{2.5.6}$$

We may assume that I is sufficiently small (otherwise decompose I as in the proof of Lemma 2.5.2.). Then we get $u_n \underset{n\to\infty}{\longrightarrow} u_{[\kappa, \rho, \varphi, \psi]} =: u$ in Φ^I. Take additionally $[\kappa, \rho, \varphi', \psi'] \in \mathcal{U}$ and consider the corresponding approximating functions u'_n of the solution $u_{[\kappa, \rho, \varphi', \psi']} =: u'$. For a constant $0 < \beta < 1$, we want to show that $u_{[\kappa, \rho, \varphi_\beta, \psi_\beta]} =: u_\beta$ exists, where $[\varphi_\beta, \psi_\beta] := \beta\,[\varphi, \psi] + (1 - \beta)\,[\varphi', \psi']$, and that

$$u_\beta \leqq \beta\, u + (1 - \beta)\, u'. \tag{2.5.7}$$

To this end, by using (2.5.6), show by induction that

$$u_{n,\beta} \leqq \beta\, u_n + (1 - \beta)\, u'_n$$

holds. On the other hand,

$$\|u_{n+m,\beta}(s) - u_{n,\beta}(s)\| \leqq \text{const.} \int_s^T dr\, \|u_{n+m-1,\beta}(r) - u_{n-1,\beta}(r)\|,$$

because from (2.5.6) and (2.5.8),

$$\|u_{n,\beta}\|_I \leqq (\|u_n\|_I + \|u'_n\|_I) \vee (\|\mathcal{T}^{\kappa, I}\varphi\|_I + \|W^{\kappa, I}\psi\|_I) \leqq \text{const.}$$

Consequently, $u_{n,\beta}$ converges in Φ^I to the desired solution u_β as $n \to \infty$, and the inequality (2.5.7) is obvious. Summarizing, u has the desired convexity property. This completes the proof of (iii).

7° (*special cases*). If $\rho = 0$ then $u = \mathcal{T}^{\kappa, I}\varphi + W^{\kappa, I}$ hence $\{\rho = 0\} \subseteq \mathcal{U}$. If $\varphi, \psi \geqq 0$, then obviously $u_{[\kappa, \rho, \varphi, \psi]} \geqq 0$, (if they exist). On the other hand, if $\varphi, \psi \leqq 0$ then non-positive solutions $u_{[\kappa, \rho, \varphi, \psi]}$ can always be constructed by the iteration scheme. Since I is arbitrary, we can easily extend the solutions to all of $(-\infty, T]$. Such global solutions exist also if, for $[\kappa, \rho] \in \mathbb{R}_+^2$ fixed, $[\varphi_+, \psi_+]$ is sufficiently small in norm, provided that we are in supercritical dimensons $d > \alpha$; we refer to Fujita (1966) or Nagasawa and Sirao (1969). This completes the proof of (ii), (v) and (vi) and finishes the proof of Theorem 2.4.3 at all. ■

3. LOG-LAPLACE FUNCTIONALS

In this section we shall introduce the super-α-stable motion X. The main result will be a characterization of its exponential moments in terms of the equation (1.3.2); *see* Theorem 3.3.1 and Corollary 3.3.4 below.

3.1. Preliminaries: The *a*-Vague Topology

Let $[\Phi^*, \|\,.\,\|_*]$ denote the dual Banach space to $[\Phi, \|\,.\,\|]$. Then $\mathcal{M}_a$ can be considered as a *convex subset* of Φ^* equipped with the *weak* topology* (*i. e.* the a-vague topology in $\mathcal{M}_a$ is nothing else than the topology induced in $\mathcal{M}_a$ by the weak* topology in Φ^*). Note that

$$|(\mu, \varphi)| \leqq \|\varphi\| \, (\mu, \varphi_a), \qquad \varphi \in \Phi, \quad \mu \in \mathcal{M}_a, \tag{3.1.1.}$$

from which in particular follows that the "duality" relation $(\bullet, \bullet)$ between $\mathcal{M}_a$ and Φ is *continuous* in both "components", and that

$$\|\mu\|_* = (\mu, \varphi_a), \qquad \mu \in \mathcal{M}_a. \tag{3.1.2}$$

There exists a sequence $\{f_n;\, n \geqq 1\}$ of functions in $\mathcal{C}_+^{\text{comp}}$ such that, with $f_0 := \varphi_a$,

$$\rho_a(\mu, \nu) := \sum_{n=0}^{\infty} 2^{-n-1} \left(1 - \exp\left[-|(\mu, f_n) - (\nu, f_n)|/\|f_n\|\right]\right),$$
$$\mu, \nu \in \mathcal{M}_a,$$

is a *translation-invariant metric* on $\mathcal{M}_a$ which generates the a-vague topology; *cf.* Kallenberg (1983), Appendix A.7. $\mathcal{M}_a$ is a *separable* metric space.

LEMMA 3.1.3. – *Each open ball*

$$B(\nu, r) := \{\mu \in \mathcal{M}_a;\ \rho_a(\mu, \nu) < r\}, \qquad \nu \in \mathcal{M}_a, \qquad r > 0,$$

is a convex *subset of* $\mathcal{M}_a$.

Proof. – This can be concluded from the inequality

$$\left.\begin{array}{c}\rho_a(\theta\mu_1 + (1-\theta)\mu_2, \nu) \leqq \rho_a(\mu_1, \nu) \vee \rho_a(\mu_2, \nu),\\ 0 \leqq \theta \leqq 1, \quad \mu_1, \mu_2, \nu \in \mathcal{M}_a,\end{array}\right\} \tag{3.1.4}$$

which follows from the corresponding property of the Euclidean metric entering into the exponents in the definition of ρ_a, combined with the fact that the function $1 - e^{-r}$, $r \geqq 0$, is monotonically increasing. ■

Set $\dot{R}^d := R^d \cup \{\infty_a\}$ with ∞_a an isolated point. Denote by $\dot{\varphi}_a$ the extension of φ_a to $\dot{R}^d$ by setting $\dot{\varphi}_a(\infty_a) := 1$. Write $\dot{\mathcal{M}}_a$ for the set of all measures μ on $\dot{R}^d$ satisfying $(\mu, \dot{\varphi}_a) < \infty$. Define the a-vague topology in $\dot{\mathcal{M}}_a$ as we did in the case $\mathcal{M}_a$ but now with φ_a replaced by $\dot{\varphi}_a$. The following criterion is taken from Iscoe (1986) [*see* also Dawson (1993), § 3.1.5].

LEMMA 3.1.4. – *A subset A of* $\dot{\mathcal{M}}_a$ *is* relatively compact *if and only if there is a natural number k such that* $A \subseteq \{\mu \in \dot{\mathcal{M}}_a;\ (\mu, \dot{\varphi}_a) \leqq k\}$ *holds.*

Finally, from the definition of ρ_a we conclude that

$$\rho_a(\mu, \nu) \leqq \|\mu - \nu\|_*, \qquad \mu, \nu \in \mathcal{M}_a. \tag{3.1.6}$$

3.2. Superstable Motion in $\mathbb{R}^d$

Recall that $0 < \alpha \leqq 2$, $d < a_1 \leqq d + \alpha$, $a_2, \kappa, \rho \geqq 0$ and $a = [a_1, a_2]$. A critical *superstable motion* X in $\mathbb{R}^d$ with motion index α, "diffusion" constant $\kappa \geqq 0$, and (constant) branching rate $\rho \geqq 0$ can be defined as a time-homogeneous Markov process $[X, \mathbb{P}^{\kappa, \rho}_{s, \mu};\ s \in \mathbb{R}, \mu \in \mathcal{M}_a]$ with *continuous* trajectories in $\mathcal{M}_a$ and with Laplace transition functionals

$$\left.\begin{array}{c}\mathbb{E}^{\kappa, \rho}_{s, \mu} \exp(X(t), \varphi) = \exp(\mu, u_{[\kappa, \rho, \varphi, 0]}(-(t-s))),\\ s \leqq t, \quad \mu \in \mathcal{M}_a, \quad \varphi \in \Phi_-,\end{array}\right\} \tag{3.2.1}$$

where $u_{[\kappa, \rho, \varphi, 0]} = u$ solves

$$u(s) = \mathcal{T}^{\kappa}_{-s}\varphi + \rho \int_s^0 dr\, \mathcal{T}^{\kappa}_{r-s}(u^2(r)), \qquad s \leqq 0, \tag{3.2.2}$$

or as a short-hand

$$-\frac{\partial}{\partial s} u = \kappa \Delta_\alpha u + \rho u^2, \qquad u|_{s=0-} = \varphi;$$

that is, $u_{[\kappa,\,\rho,\,\varphi,\,0]}$, $\varphi \in \Phi_-$, is the unique extension from $I \subset \mathbb{R}_-$ to $\mathbb{R}_-$ of the solution according to Theorem 2.4.3.

Note that by the continuity properties of solutions and by (3.1.1) the Laplace functional expression (3.2.1) is *continuous in all its variables* $s,\, t,\, \mu,\, \varphi$ as described. Note also that if $\rho = 0$ then X reduces to the stable flow $\{\mathcal{T}_t^\kappa \mu;\ t \geqq 0\}$ in $\mathcal{M}_a$ defined by $(\mathcal{T}_t^\kappa \mu,\, \varphi) := (\mu,\, \mathcal{T}_t^\kappa \varphi)$, $\varphi \in \Phi_+$.

3.3. Exponential Moments

The (weighted) *occupation time process* Y related to X is defined by $Y(t) := \int_0^t ds\, X(s)$, $t > 0$. Now we want to describe the exponential moments of $[X(t),\, Y(t)]$, $t \geqq 0$, with the help of solutions to the equation (2.4.2′).

THEOREM 3.3.1 (log-Laplace functional). – *Fix* $I = [L,\, T]$, $L < T$, $\kappa,\, \rho \geqq 0$, *and let* $\mathfrak{V} := \mathfrak{V}[\kappa,\, \rho]$ *denote the set of all those* $[\varphi,\, \psi] \in \Phi \times \Phi^I$ *such that* $V[\varphi,\, \psi] := v$ *defined by*

$$\left.\begin{aligned} v(s,\, y) := \log \mathbb{E}_{s,\,\delta_y}^{\kappa,\,\rho} \exp \Big[(X(T),\, \varphi) + \textstyle\int_s^T dr\, (X(r),\, \psi(r))\Big], \\ [s,\, y] \in I \times \mathbb{R}^d, \end{aligned}\right\} \quad (3.3.2)$$

satisfies $\sup\{\nu_{[\kappa,\,\rho,\varphi,\,\psi]}(s,\, y);\ [s,\, y] \in I \times \mathbb{R}^d\} < +\infty$. *Then* $\mathfrak{V}$ *is an* open convex *set which covers* $\Phi_- \times \Phi_-^I$. *Moreover,* $[\varphi,\, \psi] \in \mathfrak{V}[\kappa,\, \rho]$ *if and only if* $[\kappa,\, \rho,\, \varphi,\, \psi] \in \mathcal{U}$ *with* $\mathcal{U}$ *defined in Theorem 2.4.3* (ii). *In this case* $V[\varphi,\, \psi] = u_{[\kappa,\,\rho,\,\varphi,\psi]}$, *the (unique) solution to (2.4.2′).*

Note that this theorem provides a *probabilistic representation* of the solutions to (2.4.2′).

Proof of Theorem 3.3.1. – 1° Fix $[\kappa,\, \rho,\, \varphi,\, \psi] \in \mathbb{R}_+^2 \times \Phi \times \Phi^I$, and assume for the moment that $\varphi,\, \psi \geqq 0$. Then, for all $\theta \leqq 0$, the functions $V[\theta\varphi,\, \theta\psi]$ defined in (3.3.2) belong to Φ^I and solve (2.4.2′), hence coincide with $u_\theta := u_{[\kappa,\,\rho,\,\theta\varphi,\,\theta\psi]} \in \Phi^I$. In fact, if $\psi = 0$ then this is a version of the log-Laplace functional in (3.2.1), and the formula can be extended to (3.3.2) by approximating ψ by appropriate step functions using that X is a Markov process; *see* Iscoe (1986).

2° Now drop the additional assumption $\varphi,\, \psi \geqq 0$. Let $\varphi = \varphi_+ - \varphi_-$, $\psi = \psi_+ - \psi_-$ denote the minimal decomposition with $\varphi_+,\, \psi_+ \geqq 0, \varphi_-$, $\psi_- \geqq 0$. Consider $\tilde{\theta} := [\theta_1, \ldots, \theta_4] \leqq 0$. Then from 1° we know that $v_{\tilde{\theta}} := V[\theta_1\, \varphi_+ + \theta_2\, \varphi_-,\, \theta_3\, \psi_+ + \theta_4\, \psi_-]$ belongs to Φ^I and satisfies

(2.4.2′) with $[\varphi, \psi]$ replaced by $A(\tilde{\theta}) := [\theta_1 \varphi_+ + \theta_2 \varphi_-, \theta_3 \psi_+ + \theta_4 \psi_-]$, that is, $v_{\tilde{\theta}} = u_{[\kappa, \rho, \theta_1 \varphi_+ + \theta_2 \varphi_-, \theta_3 \psi_+ + \theta_4 \psi_-]} =: u_{\tilde{\theta}}$ for $\tilde{\theta} \leqq 0$.

3° Keeping the notations from the previous step of proof, set

$$\left.\begin{array}{r}\widetilde{\Theta} := \{\tilde{\theta} \in \mathbb{R}^4;\ A(\tilde{\theta}) \in \mathfrak{V}\},\\ \Theta := \{\tilde{\theta} \in \mathbb{R}^4;\ [\kappa, \rho, \theta_1 \varphi_+ + \theta_2 \varphi_-, \theta_3 \psi_+ + \theta_4 \psi_-] \in \mathcal{U}\}\end{array}\right\} \tag{3.3.3}$$

with $\mathcal{U}$ defined in Theorem 2.4.3 (ii). Note that $\mathbb{R}^4_- \subseteq \widetilde{\Theta} \cap \Theta$. By Hölder's inequality, $\widetilde{\Theta}$ is a convex subset of $\mathbb{R}^4$. On the other hand, Θ i open by Theorem 2.4.3. (ii), and also convex by the convexity of $\tilde{\theta} \mapsto u_{\tilde{\theta}}(s, y)$ which follows from Theorem 2.4.3 (iii).

Fix $[s, y] \in I \times \mathbb{R}^d$ for the moment. Well-known properties of bilateral Laplace functions imply that $\tilde{\theta} \mapsto v_{\tilde{\theta}}(s, y)$ is an *analytic* function on the interior $\widetilde{\Theta}^\circ$ of $\widetilde{\Theta}$. On the other hand, $\tilde{\theta} \mapsto u_{\tilde{\theta}}(s, y)$ is an analytic function on Θ by Theorem 2.4.3 (iii). But by 2° both coincide on $\{\tilde{\theta};\ \tilde{\theta} \leqq 0\}$, and by uniqueness of analytic continuation we conclude that $\nu_{\tilde{\theta}}(s, y) = u_{\tilde{\theta}}(s, y)$ on $\widetilde{\Theta}^\circ \cap \Theta$, and that both $v_\bullet(s, y)$ and $u_\bullet(s, y)$ are branches of a unique analytic function defined on $\widetilde{\Theta}^\circ \cup \Theta$. Since $[s, y]$ is arbitrary, the $(I \times \mathbb{R}^d)^{\mathbb{R}}$-valued mappings $v_\bullet$ and $u_\bullet$ coincide on $\widetilde{\Theta} = \widetilde{\Theta}^\circ = \Theta$. In fact, both $\widetilde{\Theta}$ and Θ are maximal by their definition: $v_{\tilde{\theta}}$ has a finite supremum on $I \times \mathbb{R}^d$ if and only if $\tilde{\theta} \in \widetilde{\Theta}$ whereas $u_{\tilde{\theta}}$ blows up at the boundary of Θ as described in Theorem 2.4.3 (iv). Passing to $\theta_1 = \theta_3 = \theta$ and $\theta_2 = \theta_4 = -\theta$, we get that $[\theta\varphi, \theta\psi] \in \mathfrak{V}$ if and only if $[\kappa, \rho, \theta\varphi, \theta\psi] \in \mathcal{U}$, and in this case $V[\theta\varphi, \theta\psi] = u_{[\kappa, \rho, \theta\varphi, \theta\psi]}$. Specialize to $\theta = 1$ to finish the proof. ■

From δ-initial measures we may pass to any initial measure:

COROLLARY 3.3.4. (exponential moments). – *Fix* $I = [L, T]$, $L < T$ *and* $\mu \in \mathcal{M}_a$. *If* $[\kappa, \rho, \varphi, \psi]$ *belongs to* $\mathcal{U}$, *then*

$$\mathbb{E}^{\kappa, \rho}_{L, \mu} \exp\left[(X(T), \varphi) + \int_L^T dr\,(X(r), \psi(r))\right] = \exp(\mu, u_{[\kappa, \rho, \varphi, \psi]}(L)) < +\infty$$

with $u_{[\kappa, \rho, \varphi, \psi]}$ *the solution to* (2.4.2′) *as defined in Theorem* 2.4.3.

Proof. – This follows from Theorem 3.3.1 if we approximate μ by discrete measures with a finite set of atoms and use the branching property and obvious continuities. ■

4. LARGE DEVIATION ESTIMATES

Here we shall derive the announced large deviation principle (LDP), actually in a reformulated setting; later we turn back to the original statement.

4.1. Reformulation of the LDP Theorem 1.5.3

Note that here only our basic parameter assumptions $0 < \alpha \leqq 2$, $1 \leqq d < a_1 \leqq d + \alpha$, $a_2, \kappa, \rho \geqq 0$ are enforced.

THEOREM 4.1.1 (dimension independent version of the LDP). – *Fix $t > 0$ and $\mu \in \mathcal{M}_a$, $\mu \neq 0$. There exists a lower semi-continuous* convex *"good" rate functional $S_{\mu, t} : \mathcal{M}_a \mapsto [0, +\infty]$ with $S_{\mu, t} (\mathcal{T}_t^\kappa \mu) = 0$ such that,*

(i) *for each open subset G of $\mathcal{M}_a$,*

$$\liminf_{R \to \infty} R^{-1} \log \mathbb{P}^{\kappa, \rho}_{0, R\mu} (R^{-1} X (t) \in G) \geqq - \inf_{\nu \in G} S_{\mu, t} (\nu),$$

(ii) *for each closed subset F of $\mathcal{M}_a$,*

$$\limsup_{R \to \infty} R^{-1} \log \mathbb{P}^{\kappa, \rho}_{0, R\mu} (R^{-1} X (t) \in F) \leqq - \inf_{\nu \in F} S_{\mu, t} (\nu).$$

The proof of this theorem is provided within the next four subsections.

4.2. Supermultiplicativity

As an immediate preparation for the proof of the previous theorem, we formulate the following simple lemma.

LEMMA 4.2.1. – *Fix $t > 0$, $\mu \in \mathcal{M}_a$ and a convex Borel subset A of $\mathcal{M}_a$. The function*

$$f (R) := \mathbb{P}^{\kappa, \rho}_{0, R\mu} (R^{-1} X (t) \in A), \qquad R > 0, \tag{4.2.2}$$

is supermultiplicative: $f (R + S) \geqq f (R) f (S)$, $\quad R, S > 0$.

Proof. – Fix $R, S > 0$. Let $[X', X'']$ be distributed according to the product measure $\mathbb{P}^{\kappa, \rho}_{0, R\mu} \times \mathbb{P}^{\kappa, \rho}_{0, S\mu}$. Then

$$f (R) f (S) = \mathbb{P}^{\kappa, \rho}_{0, R\mu} \times \mathbb{P}^{\kappa, \rho}_{0, S\mu} (R^{-1} X' (t) \in A, S^{-1} X'' (t) \in A).$$

However, if both $R^{-1} X'(t)$ and $S^{-1} X''(t)$ belong to A then also its convex combination $(R+S)^{-1}(X'(t)+X''(t))$ is in A. Consequently

$$f(R)\, f(S) \leqq \mathbb{P}^{\kappa,\rho}_{0,R\mu} \times \mathbb{P}^{\kappa,\rho}_{0,S\mu}((R+S)^{-1}(X'(t)+X''(t)) \in A).$$

But by the *branching property*, which follows directly from the form of the Laplace functional (3.2.1), the sum $X'(t)+X''(t)$ has the law $\mathbb{P}^{\kappa,\rho}_{0,(\mathrm{R+S})\mu}$. Hence

$$f(R)\, f(S) \leqq \mathbb{P}^{\kappa,\rho}_{0,(R+S)\mu}((R+S)^{-1} X(t) \in A) = f(R+S). \quad \blacksquare$$

LEMMA 4.2.3. – *In addition to the assumptions in the previous lemma, suppose that $A \subseteq \mathcal{M}_a$ is open. If now $f(R) > 0$ for some $R > 0$ then f is bounded away from 0 on some non-empty open interval.*

Proof. – Assume that $f(R) > 0$ for a fixed $R > 0$. Since $\mathcal{M}_a$ is separable there exists a $\nu \in \mathcal{M}_a$ and an $\varepsilon_0 > 0$ such that for the open ball $B(\nu, \varepsilon_0)$ contained in A we have

$$f(R) \geqq \mathbb{P}^{\kappa,\rho}_{0,R\mu}(R^{-1} X(t) \in B(\nu, \varepsilon_0)) > 0. \tag{4.2.4}$$

By the continuity of finite measures, this even holds with ε_0 replaced by some $\varepsilon \in (0, \varepsilon_0)$. Hence we may fix a $\delta > 0$ with $2\delta < \rho_a(B(\nu, \varepsilon), \mathcal{M}_a \setminus A)$, and a natural number s_0 with $\varepsilon + \|\nu\|_* < s_0\,\delta$. For $0 \leqq r < R$ and a natural number s, let $[X', X'']$ be distributed according to $\mathbb{P}^{\kappa,\rho}_{0,sR\mu} \times \mathbb{P}^{\kappa,\rho}_{0,r\mu}$. Then, by the branching property,

$$f(S) = \mathbb{P}^{\kappa,\rho}_{0,sR\mu} \times \mathbb{P}^{\kappa,\rho}_{0,r\mu}(S^{-1}(X'(t)+X''(t)) \in A) \quad \text{with} \quad S := s\,R+r.$$

But a sum belongs to A certainly if the first summand belongs to $B(\nu, \varepsilon+\delta)$ and the second summand has a $\|\bullet\|_*$-norm smaller than δ [recall (3.16)]:

$$\left.\begin{aligned} f(S) \geqq \mathbb{P}^{\kappa,\rho}_{0,sR\mu}(S^{-1} X'(t) \in B(\nu, \varepsilon+\delta)) \\ \mathbb{P}^{\kappa,\rho}_{0,r\mu}(\|S^{-1} X''(t)\|_* < \delta). \end{aligned}\right\} \tag{4.2.5}$$

The first factor on the right hand side can be estimated further in a similar way: $(s\,R+r)^{-1} X'(t) \in B(\nu, \varepsilon+\delta)$ is certainly fulfilled if $(s\,R)^{-1} X'(t) \in B(\nu, \varepsilon)$ and if the ρ_a-distance of the difference of both "vectors" is smaller than δ. But this is actually true under $(s\,R)^{-1} X'(t) \in B(\nu, \varepsilon)$ and $s \geqq s_0$. In fact, by the translation-invariance of the metric ρ_a,

$$\begin{aligned} &\rho_a((s\,R+r)^{-1} X'(t), (s\,R)^{-1} X'(t)) \\ &\quad = \rho_a(r\,(s\,R+r)^{-1}(s\,R)^{-1} X'(t), 0) \end{aligned}$$

which by $r\,(s\,R + r)^{-1} \leqq s^{-1}$ can be continued with

$$\leqq s^{-1}\,\rho_a\,((s\,R)^{-1}\,X'\,(t),\,0) \leqq s^{-1}\,(\rho_a\,((s\,R)^{-1}\,X'\,(t),\,\nu) + \rho_a\,(\nu,\,0))$$
$$\leqq s^{-1}\,(\varepsilon + \|\nu\|_*) < \delta$$

by the triangular inequality, (3.1.6), and our choice of s_0. Thus the first factor at the r.h.s. of (4.2.5) can be estimated from below by

$$\geqq \mathbb{P}^{\kappa,\,\rho}_{0,\,s\,R\,\mu}\,((s\,R)^{-1}\,X'\,(t) \in B\,(\nu,\,\varepsilon))$$
$$\geqq [\mathbb{P}^{\kappa,\,\rho}_{0,\,R\,\mu}\,(R^{-1}\,X'\,(t) \in B\,(\nu,\,\varepsilon))]^s > 0, \qquad s \geqq s_0, \quad (4.2.6)$$

where we applied the supermultiplicativity Lemma 4.2.1 to the convex (by Lemma 3.1.3) set $B\,(\nu,\,\varepsilon)$. Concerning the second factor at the r.h.s. of (4.2.5) pass to the complement and proceed for $\theta > 0$ as follows by using (3.1.2):

$$\mathbb{P}^{\kappa,\,\rho}_{0,\,r\,\mu}\,(\|S^{-1}\,X''\,(t)\|_* \geqq \delta) = \mathbb{P}^{\kappa,\,\rho}_{0,\,r\,\mu}\,((X''\,(t),\,\theta\varphi_a) \geqq S\,\delta\theta)$$
$$\leqq e^{-S\,\delta\theta}\,\mathbb{E}^{\kappa,\,\rho}_{0,\,r\,\mu}\exp\,(X''\,(t),\,\theta\varphi_a).$$

By Corollary 3.3.4 with $I = [-t,\,0]$ and applying time-homogeneity, this exponential moment is finite for a sufficiently small $\theta > 0$ and equals

$$\exp\,[(r\,\mu,\,u_{[\kappa,\,\rho,\,\theta\varphi_a,\,0]}\,(-t))],$$

hence is bounded in $r \leqq R$. On the other hand, $e^{-S\,\delta\theta} \to 0$ as $S \to \infty$. Consequently, the second factor on the r.h.s. of (4.2.5) is bounded away from 0 for sufficiently large $S = s\,R + r$. Combined with (4.2.6) we conclude that $f\,(S)$ is bounded away from 0 on some non-empty open interval. This finishes the proof. ■

4.3. Weak Large Deviation Principle

Let $\mathfrak{A}$ denote the system of all those non-empty subsets of $\mathcal{M}_a$ which are open and convex . Fix $\mu \in \mathcal{M}_a$, $t > 0$ and, for the moment, $A \in \mathfrak{A}$. In Lemma 4.2.1 go over to $-\log f$ to conclude that the function

$$\sigma\,(\mathrm{R}) := -\log^{\kappa,\,\rho}_{0,\,R\,\mu}\,(R^{-1}\,X\,(t) \in A) \in [0,\,+\infty], \qquad R > 0,$$

is *subadditive, i. e.* $\sigma(R+S) \leqq \sigma(R) + \sigma(S)$, $R, S > 0$. Moreover, Lemma 4.2.3 yields that σ is either bounded on some non-empty open interval, or identically $+\infty$. Hence, the subadditivity of σ implies that all the limits

$$\left. \begin{array}{c} \mathbf{S}_{\mu, t}(A) := -\lim\limits_{R\to\infty} R^{-1} \log \mathbb{P}^{\kappa, \rho}_{0, R\mu}(R^{-1} X(t) \in A) \in [0, +\infty], \\ A \in \mathfrak{A}, \end{array} \right\} \quad (4.3.1)$$

exist; *see*, for instance Lemma 4.2.5 in [9]. Recall that by Lemma 3.1.3 all open balls $B(\nu, r)$, $r > 0$, $\nu \in \mathcal{M}_a$, belong to $\mathfrak{A}$. By monotonicity, set

$$\left. \begin{array}{c} \mathbf{S}_{\mu, t}(\nu) := \lim\limits_{r\downarrow 0} \mathbf{S}_{\mu, t}(B(\nu, r)) = \sup\{\mathbf{S}_{\mu, t}(A);\ \nu \in A \in \mathfrak{A}\}, \\ \nu \in \mathcal{M}_a. \end{array} \right\} \quad (4.3.2)$$

Obviously, $\mathbf{S}_{\mu, t} : \mathcal{M}_a \mapsto [0, +\infty]$ is a *lower semi-continuous* functional. For *convexity*, it is enough to show that

$$\mathbf{S}_{\mu, t}((\nu_1 + \nu_2)/2) \leqq (\mathbf{S}_{\mu, t}(\nu_1) + \mathbf{S}_{\mu, t}(\nu_2))/2, \qquad \nu_1, \nu_2 \in \mathcal{M}_a. \quad (4.3.3)$$

Set $(\nu_1 + \nu_2)/2 =: \nu$, take any $A \in \mathfrak{A}$ with $\nu \in A$, and choose $A_i \in \mathfrak{A}$ such that $\nu_i \in A_i$ and $A \supseteq (A_1 + A_2)/2$. Then, by (4.3.1) and the branching property,

$$\begin{aligned} \mathbf{S}_{\mu, t}(A) &= -\lim_{R\to\infty} (2R)^{-1} \log \mathbb{P}^{\kappa, \rho}_{0, 2R\mu}((2R)^{-1} X(t) \in A) \\ &\leqq (\mathbf{S}_{\mu, t}(A_1) + \mathbf{S}_{\mu, t}(A_2))/2 \leqq (\mathbf{S}_{\mu, t}(\nu_1) + \mathbf{S}_{\mu, t}(\nu_2))/2, \end{aligned}$$

and (4.3.2) implies (4.3.3).

Immediately from (4.3.1) and (4.3.2) we get

$$\left. \begin{array}{c} \liminf\limits_{R\to\infty} R^{-1} \log \mathbb{P}^{\kappa, \rho}_{0, R\mu}(R^{-1} X(t) \in G) \geqq -\inf\limits_{\nu\in G} \mathbf{S}_{\mu, t}(\nu), \\ \text{open} \quad G \subseteq \mathcal{M}_a. \end{array} \right\} \quad (4.3.4)$$

On the other hand, if C is a compact subset of $\mathcal{M}_a$ and $i := \inf\limits_{\nu\in C} \mathbf{S}_{\mu, t}(\nu)$ is positive, then for $0 < \varepsilon < i$ we find finitely many open balls $B_1, \ldots, B_M$ which cover C and satisfy $\mathbf{S}_{\mu, t}(B_m) \geqq i - \varepsilon$, $1 \leqq m \leq M$. Then again with (4.3.1) and (4.3.2), we obtain (*cf.* [9], p. 62)

$$\begin{aligned} &\limsup_{R\to\infty} R^{-1} \log \mathbb{P}^{\kappa, \rho}_{0, R\mu}(R^{-1} X(t) \in C) \\ &\quad \leqq -\inf_{\nu\in C} \mathbf{S}_{\mu, t}(\nu), \quad \text{compact } C \subseteq \mathcal{M}_a. \end{aligned} \quad (4.3.5)$$

Summarizing, with the estimates (4.3.4) and (4.3.5) we proved that the family $R^{-1} \log \mathbb{P}^{\kappa, \rho}_{0, R\mu} (R^{-1} X(t) \in \bullet)$, $R > 0$, satisfies a *weak large deviation principle* with the convex rate functional $\mathbf{S}_{\mu, t}$ defined in (4.3.1).

4.4. Full Large Deviation Principle

For convenience, for $\mu \in \mathcal{M}_a$, $t > 0$ we introduce the largest *open* set $\Phi_{\mu, t}$ of all those functions $\varphi \in \Phi$ such that $\Lambda_{\mu, t}(\varphi) < +\infty$, with $\Lambda_{\mu, t}$ defined in (1.5.2).

LEMMA 4.4.1. – *Fix $\mu \in \mathcal{M}_a$ and $t > 0$. For all $\varphi \in \Phi_{\mu, t}$,*

$$\lim_{N \to \infty} \lim_{R \to \infty} R^{-1} \log \mathbb{E}^{\kappa, \rho}_{0, R\mu} \{\exp(X(t), \varphi); (R^{-1} X(t), \varphi) \geqq N\} = -\infty.$$

Proof. – Fix μ, t, φ as in the lemma. Since $\Phi_{\mu, t}$ is open by definition, we find a $\theta > 0$ such that also $(1 + \theta)\varphi$ belongs to $\Phi_{\mu, t}$. Writing $\varphi = \varphi(1 + \theta) - \theta\varphi$ in the exponent and using the condition $(X(t), \theta\varphi) > RN\theta$, we immediately get

$$\begin{aligned} R^{-1} \log \mathbb{E}^{\kappa, \rho}_{0, R\mu} \{\exp(X(t), \varphi); (R^{-1} X(t), \varphi) > N\} \\ \leqq -\theta N + R^{-1} \Lambda_{R\mu, t}((1 + \theta)\varphi), \end{aligned} \tag{4.4.2}$$

$R, N > 0$. But by the branching property,

$$\Lambda_{R\mu, t}(\varphi) = \int R\mu(dy) \log \mathbb{E}^{\kappa, \rho}_{0, \delta_y} \exp(X(t), \varphi) = R\Lambda_{\mu, t}(\varphi), \tag{4.4.3}$$

$R > 0$. Hence, the r.h.s. in (4.4.2) is finite, and letting first $R \to \infty$ and then $N \to \infty$, the claim follows. ■

By Lemma 4.4.1 with $\varphi = \theta\varphi_a$ and $\theta > 0$ sufficiently small,

$$\lim_{N \to \infty} \lim_{R \to \infty} R^{-1} \log \mathbb{P}^{\kappa, \rho}_{0, R\mu} ((R^{-1} X(t), \theta\varphi_a) \geqq N) = -\infty.$$

From the compactness Lemma 3.1.5 and (3.1.2) we learn that to each $M > 0$ we find a compact set $C_M \subseteq \mathcal{M}_a$ such that (interpreting measures on R^d as measures on $\dot{R}^d$, and distributions on $\mathcal{M}_a$ as distributions on $\dot{\mathcal{M}}_a$)

$$\limsup_{R \to \infty} R^{-1} \log \mathbb{P}^{\kappa, \rho}_{0, R\mu} (R^{-1} X(t) \notin C_M) \leqq -M.$$

In other words, we have *exponential tightness* in $\dot{\mathcal{M}}_a$. Since the superprocess X lives in $\mathcal{M}_a$, together with the results of the previous subsection we get for $R^{-1}\log \mathbb{P}^{\kappa,\rho}_{0,R\mu}(R^{-1}X(t)\in\bullet)$, $R>0$, a *full LDP with the convex "good" rate functional* $\mathbf{S}_{\mu,t}$ [of (4.3.1)]; *see* [9], Lemma 2.1.5.

4.5. Law of Large Numbers

We need the following simple *scaling property* of the superprocess:

LEMMA 4.5.1. – *Fix $\mu\in\mathcal{M}_a$ and a constant $c\geqq 0$. If X is distributed according to $\mathbb{P}^{\kappa,\rho}_{0,\mu}$ then cX has the law $\mathbb{P}^{\kappa,c\rho}_{0,c\mu}$.*

Proof. – By the Markov property, this directly follows from the identity $u_{[\kappa,\rho,c\varphi,0]}=cu_{[\kappa,c\rho,\varphi,0]}$, $\varphi\in\Phi_-$, via (3.2.1) and (3.2.2). ■

As a complement to Theorem 4.1.1 we add here the following

LEMMA 4.5.2 (law of large numbers). – *Fix $t>0$, $\mu\in\mathcal{M}_a$. For all neighborhoods $\mathcal{U}(\mathcal{T}^\kappa_t\mu)$ of $\mathcal{T}^\kappa_t\mu$,*

$$\mathbb{P}^{\kappa,\rho}_{0,R\mu}(R^{-1}X(t)\in\mathcal{U}(\mathcal{T}^\kappa_t\mu))\xrightarrow[\mathrm{K}\to\infty]{}1,$$

Proof. – By Lemma 4.5.1,

$$\begin{aligned}&\mathbb{E}^{\kappa,\rho}_{0,R\mu}\exp(R^{-1}X(t),\varphi)\\&=\mathbb{E}^{\kappa,\rho/R}_{0,\mu}\exp(X(t),\varphi)\\&=\exp(\mu,u_{[\kappa,\rho/R,\varphi,0]}(-t)),\qquad\varphi\in\Phi_-.\end{aligned}$$

The claim then follows from continuity properties, since

$$(\mu,u_{[\kappa,0,\varphi,0]}(t))=(\mu,\mathcal{T}^K_t\varphi)=(\mathcal{T}^\kappa_t\mu,\varphi).\quad ■$$

Completion of Proof of Theorem 4.1.1. – By the LLN Lemma 4.5.2, Lemma 3.1.3 and (4.3.1), we have $\mathbf{S}_{\mu,t}(B(\mathcal{T}^\kappa_t\mu,r))=0$ for all $r>0$, and (4.3.2) implies that $\mathbf{S}_{\mu,t}(\mathcal{T}^\kappa_t\mu)=0$. ■

4.6. Proof of the LDP Theorem 1.5.3

Here we come back to our scaled processes X^K defined in (1.3.4). By some scaling arguments, the LDP of Theorem 1.5.3 is in fact a consequence of Theorem 4.1.1. First of all, X^K coicides in law with the original process X but with other parameters κ, ρ [recall (1.5.1)]:

LEMMA 4.6.1 (space-time-mass scaling). – *For* $K \geqq 1$, *let* μ_K *belong to* $\mathcal{M}_a$, *and set* $\kappa_K := \kappa K^{\gamma-\alpha}$ *as well as* $\rho_K := \rho K^{\gamma-d}$. *Then*

$$\mathbb{P}^{\kappa,\rho}_{0,\mu_K}(X^K(t) \in \bullet) = \mathbb{P}^{\kappa_K,\rho_K}_{0,(\mu_K)^K}(X(t) \in \bullet), \qquad K \geqq 1, \quad t > 0.$$

Proof. – Fix $K \geqq 1$. By the *self-similarity* of the stable transition density functions $p^\kappa(t) := p^\kappa(t, \bullet)$, $t > 0$, introduced in Subsection 2.2, we have

$$p^\kappa(K^\gamma t) = (p^{\kappa_K}(t))^K, \qquad t > 0, \tag{4.6.2}$$

[which directly follows from (2.2.1)]. This implies

$$\mathcal{T}^\kappa_{K^\gamma t}(\varphi^K) = (\mathcal{T}^{\kappa_K}_t \varphi)^K, \qquad t \geqq 0, \quad \varphi \in \Phi_-. \tag{4.6.3}$$

But $K^{-d}(\psi^2)^K = (\psi^K)^2$, $\psi \in \Phi_-$, and the uniqueness of solutions $u = u_{[\kappa,\rho,\varphi,0]}$ to equation (3.2.2) yields

$$u_{[\kappa,\rho,\varphi^K,0]}(-K^\gamma t) = (u_{[\kappa_K,\rho_K,\varphi,0]}(-t))^K, \qquad t \geqq 0, \quad \varphi \in \Phi_-.$$

Then from (3.2.1) for $t \geqq 0$, $\varphi \in \Phi_-$,

$$\begin{aligned}\mathbb{E}^{\kappa,\rho}_{0,\mu_K} \exp(X^k(t), \varphi) &= \mathbb{E}^{\kappa,\rho}_{0,\mu_K} \exp(X(K^\gamma t), \varphi^K) \\ &= \exp(\mu_K, u_{[\kappa,\rho,\varphi^K,0]}(-K^\gamma t)).\end{aligned}$$

By the previous identity and again by (3.2.1) we can continue with

$$\begin{aligned}&= \exp(\mu_K, (u_{[\kappa_K,\rho_K,\varphi,0]}(-t))^K) = \exp((\mu_K)^K, u_{[\kappa_K,\rho_K,\varphi,0]}(-t))) \\ &= \mathbb{E}^{\kappa_K,\rho_K}_{0,(\mu_K)^K} \exp(X(t), \varphi).\end{aligned}$$

This coincidence of Laplace functionals implies the claim. ■

Proof of Theorem 1.5.3 [except (iv)]. – The scaling Lemmas 4.6.1 and 4.5.1 are now the essential steps in order to see that Theorem 1.5.3 [except (iv)] follows from Theorem 4.1.1. In fact, for $\gamma = \alpha$,

$$\begin{aligned}\mathbb{P}^{\kappa,\rho}_{0,\mu_K}(X^K(t) \in \bullet) &= \mathbb{P}^{\kappa,\rho_K}_{0,\mu}(X(t) \in \bullet) \\ &= \mathbb{P}^{\kappa,\rho}_{0,K^{d-\alpha}\mu}(K^{-(d-\alpha)} X(t) \in \bullet), \qquad K \geqq 1. \end{aligned} \tag{4.6.4}$$

We have only to set $K^{d-\alpha} =: R$ and to take into account that $d > \alpha$ by assumption. ■

Remark 4.6.5. – Under *subcritical scaling,* that is if $\gamma < \alpha \wedge d$, the LLN

$$X^K(t) \underset{K\to\infty}{\overset{\mathcal{P}r}{\to}} \mu \qquad \text{if} \qquad X^K(0) \underset{K\to\infty}{\overset{\mathcal{P}r}{\to}} \mu,$$

mentioned in the end of Subsection 1.4 above, follows similarly as in the proof of Lemma 4.5.2, since here $\kappa_K \to 0$ in view of Lemma 4.6.1 and $\mathcal{T}_t^0$ equals the identity operator. □

5. IDENTIFICATION OF THE RATE FUNCTIONAL

This section is devoted to the representation of the rate functional formulated in assertion (iv) of Theorem 1.5.3, and the solution of this variational problem in the $\kappa = 0$ case. Again only our basic parameter assumptions $0 < \alpha \leqq 2$, $1 \leqq d < a_1 \leqq d + \alpha$, and $a_2, \kappa, p \geqq 0$ are enforced.

5.1. Proof of the Representation (iv) of Theorem 1.5.3

Fix again $\mu \in \mathcal{M}_a$, $\mu \neq 0$ and $t > 0$. First note that the LDP according to Theorem 4.1.1. with $R \to \infty$ can be weakened as a LDP along $N = 1, 2, ...$ *(parameter discretization)*. Thus by uniqueness of the rate functional (*see* e.g. Lemma 2.1.1 in [9]), $\mathbf{S}_{\mu,t} = \Lambda^*_{\mu,t}$ will be true if we can show that Theorem 4.1.1. holds with $R \to \infty$ and $\mathbf{S}_{\mu,t}$ replaced by $N \to \infty$ and $\Lambda^*_{\mu,t}$, respectively.

Recall the definition (1.5.2) of $\Lambda_{\mu,t}$. By the identity (4.4.3),

$$N^{-1} \log \mathbb{E}^{\kappa,\rho}_{0,N\mu} \exp(X(t), \varphi) \equiv \Lambda_{\mu,t}(\varphi), \qquad \varphi \in \Phi, \tag{5.1.1}$$

(reflecting the i.i.d. structure). Hence, the discrete version of the LDP Theorem 4.1.1 implies

$$\Lambda_{\mu,t}(\varphi) \geqq (\nu, \varphi) - \mathbf{S}_{\mu,t}(\nu), \qquad \varphi \in \Phi, \quad \nu \in \mathcal{M}_a;$$

see, for instance, [9], Lemma 2.1.7. Therefore, $-\mathbf{S}_{\mu,t} \leqq -\Lambda^*_{\mu,t}$, which gives the desired large deviation upper bound. It remains to show that for fixed non-empty open $G \subseteq \mathcal{M}_a$

$$\liminf_{N\to\infty} N^{-1} \log \mathbb{P}^{\kappa,\rho}_{0,N\mu}(N^{-1} X(t) \in G) \geqq -\Lambda^*_{\mu,t}(\nu), \quad \nu \in G, \tag{5.1.2}$$

holds. By the branching property, the random variable $N^{-1} X(t)$ with respect to the law $\mathbb{P}^{\kappa, \rho}_{0, N\mu}$ can be read as $N^{-1} \sum_{i=1}^{N} X^i(t)$ with i.i.d. $X^1, \dots, X^N$ with respect to $\mathbb{P}^{\kappa, \rho}_{0, \mu}$, and we will do this in the following.

In order to get (5.1.2), we want to apply the lower large deviation estimate of Cramér's Theorem in finite dimensions. To this end, fix $\nu_0 \in G$. Then there exists a finite sequence $\varphi_1, \dots, \varphi_m \in \Phi$ and an $\varepsilon > 0$ such that

$$\mathcal{U} := \{\nu \in \mathcal{M}_a;\ [(\nu, \varphi_1), \dots, (\nu, \varphi_m)] \in B\} \subseteq G$$

with

$$B := \{z = [z_1, \dots, z_m] \in \mathbb{R}^m;\ |z_i - (\nu_0, \varphi_i)| < \varepsilon,\ 1 \leqq i \leqq m\}.$$

By the branching property and Cramér's large deviation lower bound in $\mathbb{R}^m$, which holds without the strong assumption on everywhere finite exponential moments, see, for instance, de Acosta, Ney and Nummelin (1991),

$$\begin{aligned} &\liminf_{N \to \infty} N^{-1} \log \mathbb{P}^{\kappa, \rho}_{0, N\mu} (N^{-1} [(X(t), \varphi_1), \dots, (X(t), \varphi_m)] \in B) \\ &\quad \geqq - \inf_{z \in B} \lambda^*_{\mu, t}(z), \end{aligned}$$

where

$$\lambda^*_{\mu, t}(z) := \sup \{\sum_i z_i \theta_i - \lambda_{\mu, t}(\theta);\ \theta \in \mathbb{R}^m\}, \qquad z \in \mathbb{R}^m,$$

with

$$\lambda_{\mu, t}(\theta) := \Lambda_{\mu, t}(\sum_i \theta_i \varphi_i), \qquad \theta \in \mathbb{R}^m.$$

Thus,

$$\liminf_{N \to \infty} N^{-1} \log \mathbb{P}^{\kappa, \rho}_{0, N\mu} (N^{-1} X(t) \in \mathcal{U}) \geqq -\lambda^*_{\mu, t}([(\nu, \varphi_1), \dots, (\nu, \varphi_m)]),$$

$\nu \in \mathcal{U}$, and therefore, since $\nu_0 \in \mathcal{U}$,

$$\liminf_{N \to \infty} N^{-1} \log \mathbb{P}^{\kappa, \rho}_{0, N\mu} (N^{-1} X(t) \in G) \geqq -\lambda^*_{\mu, t}([(\nu, \varphi_1), \dots, (\nu, \varphi_m)]),$$

$\nu_0 \in G$, But

$$\begin{aligned} &\lambda^*_{\mu, t}[(\nu_0, \varphi_1), \dots, (\nu_0, \varphi_m)]) \\ &\quad = \sup \{(\nu_0, \sum_i \varphi_i \theta_i) - \Lambda_{\mu, t}(\sum_i \theta_i \varphi_i);\ \theta \in \mathbb{R}^m\} \\ &\quad \leqq \Lambda^*_{\mu, t}(\nu_0), \end{aligned}$$

and (5.1.2) follows. This finishes the proof. ■

At this place we mention that as a consequence of the proofs in the Subsections 4.2-4.4, combined with the previous proof, we get the following *infinite dimensional version of Cramér's Theorem,* written in terms of random measures. (In the case of everywhere *finite* exponential moments *see* also Dawson and Gärtner (1987), Theorem 3.4.)

COROLLARY 5.1.3 (infinite dimensional Cramér's Theorem). – *Let* $\mathcal{X}^1, \mathcal{X}^2, \ldots$ *be an i.i.d. sequence of random elements in* $\mathcal{M}_a$. *Set* $\Lambda(\varphi) := \log \mathcal{E} \exp(\mathcal{X}^1, \varphi)$, $\varphi \in \Phi$, *and assume that there is an* $\varepsilon > 0$ *such that* $\Lambda(\varphi) < +\infty$ *if* $\|\varphi\| < \varepsilon$. *Then the sequence* $N^{-1} \sum_{i=1}^{N} \mathcal{X}^i$, $N = 1, 2, \ldots$, *satisfies the LDP with "good" convex rate functional* Λ^* *defined by* $\Lambda^*(\nu) := \sup\{(\nu, \varphi) - \Lambda(\varphi);\ \varphi \in \Phi\}$, $\nu \in \mathcal{M}_a$.

5.2. Solution of the Variational Problem

The purpose of this subsection is to compute the *Legendre transform* of the log-Laplace functional $\Lambda_{\mu, t}$ of $X(t)$ with respect to $\mathbb{P}^{0, \rho}_{0, \mu}$, which implies Theorem 1.5.4. To this end, recall that Φ^* is the dual to the Banach space Φ equipped with the weak* topology, and that we use the convention $0.(+\infty) = 0$.

THEOREM 5.2.1 (Legendre transform). – *Fix* $\rho, t > 0$, $\mu \in \mathcal{M}_a$. *In the case* $\kappa = 0$, *the Legendre transform*

$$\Lambda^*_{\mu, t}(\varphi^*) := \sup_{\varphi \in \Phi} \{(\varphi^*, \varphi) - \Lambda_{\mu, t}(\varphi)\}, \qquad \varphi^* \in \Phi^*, \tag{5.2.2}$$

of $\Lambda_{\mu, t}$ *[defined in* (1.5.2)*] has the following form: For* $\nu \in \mathcal{M}_a$,

$$\Lambda^*_{\mu, t}(\nu) = (\rho t)^{-1} \int \mu(dy) \left(\sqrt{g_{ac}(y)} - 1\right)^2 + \nu_\partial(\mathbb{R}^d)/\rho t + \nu_\infty(\mathbb{R}^d) . (+\infty) \tag{5.2.3}$$

with $\nu_{ac}, \nu_\partial, \nu_\infty, g_{ac}$ *introduced before Theorem 1.5.4, whereas* $\Lambda^*_{\mu, t}(\varphi^*) = +\infty$ *for the remaining* $\varphi^* \in \Phi^*$.

Note that $\Lambda^*_{\mu, t}$ is *positively homogeneous along* ν_∂, hence it is *not strongly convex* at $\nu = \nu_\partial \neq 0$. Roughly speaking, strong convexity is violated at measures ν which spatially "deviate" inside the closed support of the starting measure μ in a singular way. Moreover, the following example shows that as a rule $\Lambda^*_{\mu, t}$ is *not continuous*.

Example 5.2.4. – Let μ be the uniform distribution on the cube $[-1, 1]^d$ and ν_ε the mean zero Gaussian distribution on $\mathbb{R}^d$ with variance $\varepsilon > 0$, but restricted to $[-1, 1]^d$. Then $\nu_\varepsilon = (\nu_\varepsilon)_{ac}$ converges (even weakly) to $\delta_0 = (\delta_0)_\partial$ as $\varepsilon \to 0$ however $\Lambda^*_{\mu, t}(\nu_\varepsilon) \underset{\varepsilon \to 0}{\to} 2/\rho t$ whereas $\Lambda^*_{\mu, t}(\delta_0) = 1/\rho t$. ■

Note also that the integral in (5.2.3) appears in a criterion for absolute continuity respectively singularity of *Poisson point processes* in $\mathbb{R}^d$ with intensity measures μ and ν; *see* e.g. Theorem 1.12.3 in Matthes *et al.* (1978).

Proof of Theorem 5.2.1. – 1° First we want to compute $\Lambda_{\mu, t}$. For the moment, fix $\varphi \in \Phi$. Then equation (2.4.2′) (with $\psi = 0$) degenerates to the ordinary equation

$$u(s, y) = \varphi(y) + \rho \int_s^0 dr\, u^2(r, y), \qquad s < 0,$$

which has the unique (pointwise) solution

$$u(s, y) = \begin{cases} \varphi(y)/(1 + \rho s \varphi(y)) & \text{if} \quad \rho s \varphi(y) > -1, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.2.5}$$

$s < 0,\ y \in \mathbb{R}^d$. By analytic continuation as in the proof of Theorem 3.3.1 we conclude that

$$\log \mathbb{E}^{0, \rho}_{0, \delta_y} \exp(X(t), \varphi) = u(-t, y) \in (-\infty, +\infty], \qquad t > 0.$$

In addition, fix $t > 0$ and $\mu \in \mathcal{M}_a$. Then

$$\begin{aligned} \Lambda_{\mu, t}(\varphi) &= \log \mathbb{E}^{\kappa, \rho}_{0, \mu} \exp(X(t), \varphi) \\ &= \int \mu(dy)\, u(-t, y) \in (-\infty, +\infty] \end{aligned} \tag{5.2.6}$$

with u given in (5.2.5).

2° Without loss of generality we may assume that $\rho = 1$ (otherwise make a time change). Also, by the special form (5.2.6) of $\Lambda_{\mu, t}$, in the definition of $\Lambda^*_{\mu, t}$ the supremum can be restricted to those $\varphi \in \Phi$ such that $t\varphi(y) < 1$ μ-a.e. [since (φ^*, φ) is always finite].

3° To prove that $\Lambda^*_{\mu, t} = +\infty$ outside $\mathcal{M}_a$, we fix φ^* in Φ^* and assume that $\sup \{(\varphi^*, \varphi) - \Lambda_{\mu, t}(\varphi)\} < +\infty$ where φ runs through the set just described. Then we have to show that φ^* can be generated by a measure in $\mathcal{M}_a$. To this purpose we want to apply the *Daniell-Stone Theorem*; *see*, for instance, Bauer (1974), Satz 39.4. Indeed, Φ is a Stone lattice, and we will show that φ^* is non-negative and that $(\varphi^*, \varphi_n) \to 0$ as φ_n pointwise monotonously decreases to 0 (as $n \to \infty$). Assume that there exists a non-negative $\varphi \in \Phi$ such that $(\varphi^*, \varphi) < 0$. Then $\theta\varphi \leqq 0$ for $\theta < 0$, and the supremum in the definition of the Legendre transform can be estimated below by taking into account only $\theta\varphi$:

$$\Lambda^*_{\mu, t}(\varphi^*) \geqq \theta(\varphi^*, \varphi) \quad \text{since} \quad -\Lambda_{\mu, t}(\theta\varphi) \geqq 0.$$

Letting $\theta \to -\infty$ we get a contradiction to the assumed finiteness. Hence, φ^* is non-negative. Suppose that in Φ there exists a sequence $\varphi_n \downarrow 0$ pointwise as $n \to \infty$ and such that $(\varphi^*, \varphi_n) \geqq \varepsilon$, $n \geqq 1$, for some $\varepsilon > 0$. All φ_n are continuous and wil vanish as $|y| \to \infty$. Hence, $\|\varphi_n\|_\infty \to 0$ as $n \to \infty$. Thus, for each $\theta > 0$, $0 \leqq \theta\varphi_n \leqq \|\theta\varphi_n\|_\infty < 1/t$ for all sufficiently large n. Therefore, by (5.2.6) and (5.2.5),

$$\Lambda^*_{\mu, t}(\varphi^*) \geqq \theta(\varphi^*, \varphi_n) - \Lambda_{\mu, t}(\theta\varphi_n) \geqq \theta\varepsilon - \int d\mu\, \theta\varphi_n/(1 - t\,\theta\varphi_n)$$

for sufficiently large n. But even

$$1 - t\,\theta\varphi_n(y) \geqq 1 - t\,\theta\|\varphi_n\|_\infty \geqq 1/2$$

as $n \to \infty$. Hence,

$$\Lambda^*_{\mu, t}(\varphi^*) \geqq \theta\varepsilon - 2\int \mu(dy)\,\theta\varphi_n(y).$$

However, the latter integral is finite and tends to 0 by monotone convergence as $n \to \infty$. Thus $\Lambda^*_{\mu, t}(\varphi^*) \geqq \theta\varepsilon$, for all $\theta > 0$. Letting $\theta \to \infty$ we arrive at the desired contradiction. Summarizing, φ^* is an abstract integral and can then be represented by some measure ν. Here ν is defined on the smallest σ-field making all $\varphi \in \Phi$ measurable, which is nothing else than the usual Borel σ-field on $\mathbb{R}^d$. Of course, ν has the necessary finiteness property, *i. e.* it belongs to $\mathcal{M}_a$. It remains to calculate $\Lambda^*_{\mu, t}$ on $\mathcal{M}_a$.

4° By calculus methods one can easily handle the "*zero-dimensional*" case:

$$\sup_{\theta < 1/t} (x\,\theta - \theta/(1 - t\,\theta)) = t^{-1}(\sqrt{x} - 1)^2, \qquad x \geqq 0,$$

where the supremum is uniquely "realized" at $\theta = (1 - 1/\sqrt{x})/t$ (read $\theta = -\infty$ if $x = 0$).

5° Next we will deal with the case $\nu_\infty \neq 0$. Here we have to show that $\Lambda^*_{\mu, t}(\nu) = +\infty$. Now there is a bounded Borel set $B \subseteq \mathbb{R}^d \setminus \mathcal{S}$ with $\nu_\infty(B) > 0$. By regularity, there is even a compact set $C \subseteq B$

with $\nu_\infty(C) > 0$. Since the closed sets C and $\mathcal{S}$ are apart by a positive (Euclidean) distance, for all sufficiently small $\varepsilon > 0$ the open ε-neighborhood $\mathcal{U}_\varepsilon(C) =: \mathcal{U}$ of C is also disjoint to $\mathcal{S}$. For such ε we may choose some $\psi_\varepsilon \in \Phi$ with the property that $\varepsilon^{-1} 1_C \leqq \psi_\varepsilon \leqq \varepsilon^{-1} 1_{\mathcal{U}}$. Then $\Lambda_{\mu,t}(\psi_\varepsilon) = 0$ and

$$\Lambda^*_{\mu,t}(\nu) \geqq (\nu, \psi_\varepsilon) \geqq \varepsilon^{-1} \nu_\infty(C) \to +\infty \quad \text{as } \varepsilon \to 0.$$

6° For the remaining proof we can assume that $\nu_\infty = 0$. Then

$$\Lambda^*_{\mu,t}(\nu) = \sup\{(\nu_\partial + \nu_{\mathrm{ac}}, \varphi) - \Lambda_{\mu,t}(\varphi)\}$$

where the supremum is taken over those $\varphi \in \Phi$ such that $t\varphi(y) < 1$ μ-a.e. We can estimate from above as follows (recall that $g_{ac} =: g$ is the density of ν with respect to μ) :

$$\Lambda^*_{\mu,t}(\nu) \leqq \nu_\partial(\mathbb{R}^d)/t + \int \mu(dy) \sup_{\theta < 1/t} [g(y)\theta - \theta/(1 - t\theta)].$$

Here for the first term we used $\varphi \leqq 1/t$ on $\mathcal{S}$ by the continuity of φ, whereas for the remaining term we passed to pointwise suprema. Together with 4° we get the desired expression as an upper estimate for $\Lambda^*_{\mu,t}(\nu)$. It remains to deal with estimates from below. Here the *key idea of proof* consists in choosing a $\psi \in \Phi$ such that approximately $\psi(y) \approx 1/t$ for those y where ν_∂ has its mass, whereas $\psi(y) \approx (1 - 1/\sqrt{g(y)})/t$ on the "support" of ν_{ac}. Here of course some technical work has to be done.

7° We start with the case $\mu(\{g < 1 - \delta\}) = +\infty$ for some $\delta > 0$. Then also $\mu(\{(\sqrt{g} - 1)^2 > \delta^2\}) \geqq \mu(\{\sqrt{g} < 1 - \delta\}) = +\infty$ for some $\delta > 0$ we fix in the following. We have to show that $\Lambda^*_{\mu,t}(\nu) \geqq +\infty$. Let $A \subseteq \mathcal{S}$ be a supporting Borel set of μ with the property that $\nu_\partial(A) = 0$. By our assumption, to each $K > 0$ there is a compact set $C := C_K \subseteq \{\sqrt{g} < 1 - \delta\} \cap A$ with $\mu(C) > K$. By regularity, we find a bounded open neighborhood $\mathcal{U}(C_K) =: \mathcal{U}$ such that $(\nu + \mu)(\mathcal{U} \setminus C) < 1$. Choose $\psi \in \Phi$ with

$$-t^{-1}\delta(1 - \delta)^{-1} 1_{\mathcal{U}} \leqq \psi \leqq -t^{-1}\delta(1 - \delta)^{-1} 1_C. \tag{5.2.7}$$

Now

$$\Lambda^*_{\mu,t}(\nu) \geqq (\nu_\partial + \nu_{ac}, \psi) - \Lambda_{\mu,t}(\psi),$$

and using both estimates of (5.2.7) and since $\nu_\partial(C) = 0$ and $r \mapsto r/(1-r)$ is monotone, we can continue with

$$\geqq -(\nu_\partial + \nu_{ac})(\mathcal{U} \setminus C)\, t^{-1}\, \delta\, (1-\delta)^{-1}$$

$$- \int_C \mu(dy)\, [g(y)\, t^{-1}\, \delta\, (1-\delta)^{-1} - t^{-1}\, \delta]$$

$$\geqq -t^{-1}\, \delta\, (1-\delta)^{-1} + \mu(C)\, t^{-1}\, \delta^2$$

$$\geqq -t^{-1}\, \delta\, (1-\delta)^{-1} + K\, t^{-1}\, \delta^2 \underset{K \to \infty}{\longrightarrow} +\infty$$

8° By the previous step of proof, from now on we can assume that $\mu\{g < 1-\delta\}) < +\infty$ for all $\delta > 0$. Let E denote the halfopen unit cube $[0, 1)^d$ in $\mathbb{R}^d$, and let z_i, $i = 1, 2, \ldots$, run through all points of the lattice $\mathbb{Z}^d$. Each Borel set $B \subseteq \mathbb{R}^d$ can be decomposed into disjoint bounded sets by setting $B_i := B \cap (E + z_i)$, $i \geqq 1$. We will apply this construction (and reserve the index i for it) to the sets $\mathcal{S} \setminus A$ and $A \cap \{g \geqq 1-\delta\}, \delta > 0$, which have possibly infinite mass with respect to ν_∂ and μ. (Although we could also deal separately with the cases $\nu_\partial(\mathcal{S} \setminus A) = +\infty$ and $\mu(\{g \geqq 1+\delta\}) = +\infty$ similarly as in 7°, since then $\Lambda^*_{\mu, t}(\nu) = +\infty$.)

9° For the next steps of proof we fix a number $\delta := 2^{-m}$, $m > 1$, and set $\varepsilon := \varepsilon_{\delta, n} = \sqrt{\delta 2^{-n}}$, $n \geqq 1$. For $i \geqq 1$ choose compact sets $C_{\varepsilon, i} \subseteq (\mathcal{S} \setminus A)_i$ such that

$$\nu_\partial((\mathcal{S} \setminus A)_i \setminus C_{\varepsilon, i}) < \varepsilon^2\, 2^{-i}. \tag{5.2.8}$$

For $1 \leqq j \leqq (1-\delta)/\varepsilon^2$ we introduce the Borel sets

$$B_{\varepsilon, j} := \{(j-1)\, \varepsilon^2 \leqq g < j\, \varepsilon^2\} \cap A.$$

Select compact sets $K_{\varepsilon, j} \subseteq B_{\varepsilon, j}$ satisfying $\mu(B_{\varepsilon, j} \setminus K_{\varepsilon, j}) < \varepsilon^4$. For $i \geqq 1$ take compact subsets $L_{\varepsilon, i}$ of $(\{1-\delta \leqq g < 1\} \cap A)_i$ with the property that

$$\mu((\{1-\delta \leqq g < 1\} \cap A)_i \setminus L_{\varepsilon, i}) < \varepsilon^2\, 2^{-i}.$$

Finally, for $i \geqq 1$ and $0 \leqq k < (1-\varepsilon^2)/\varepsilon^4$ set

$$B_{\varepsilon, i, k} := (\{1 + k\, \varepsilon^2 \leqq g < 1 + (k+1)\, \varepsilon^2\} \cap A)_i$$

and take compact sets $C_{\varepsilon, i, k} \subseteq B_{\varepsilon, i, k}$ such that

$$(\nu_{ac} + \mu)\, (B_{\varepsilon, i, k} \setminus C_{\varepsilon, i, k}) < \varepsilon^6\, 2^{-i}. \tag{5.2.9}$$

Note that all these compact sets $C_{\varepsilon, i}$, $K_{\varepsilon, j}$, $L_{\varepsilon, i}$ and $C_{\varepsilon, i, k}$ (where i, j, k are running as above) have pairwise a positive distance. Now choose $\psi_\varepsilon \in \Phi$ with the property that

$$\psi_\varepsilon(y) = \begin{cases} (1-\varepsilon)/t & \text{on } C_{\varepsilon, i} \\ (1 - 1/\sqrt{j\varepsilon^2})/t & \text{on } K_{\varepsilon, j} \\ 0 & \text{on } L_{\varepsilon, i} \\ (1 - 1/\sqrt{1 + (k-1)\, \varepsilon^2})/t & \text{on } C_{\varepsilon, i, k} \end{cases}$$

where $i \geqq 1$, $1 \leqq j \leqq (1-\delta)/\varepsilon^2$ and $0 \leqq k < (1-\varepsilon^2)/\varepsilon^4$. Moreover, we impose

$$-1/\varepsilon t \leqq \psi_\varepsilon(y) \leqq (1-\varepsilon)/t \tag{5.2.10}$$

for the remaining y. This choice of ψ_ε is actually possible since ψ_ε has these bounds also on all the compact sets above (for the fixed ε).

10° Now we are ready to provide the estimates from below. In fact, $\Lambda^*_{\mu,t}(\nu) \geqq (\nu_\partial, \psi_\varepsilon) + I_1 + I_2 + I_3$ where the last three terms refer to the integral $\int d\mu\,(g\,\psi_\varepsilon - \psi_\varepsilon/(1-t\,\psi_\varepsilon))$ restricted to $\{g < 1-\delta\}$, to $\{1-\delta \leqq g < 1\}$ and to $\{g \geqq 1\}$, respectively. First of all,

$$(\nu_\partial, \psi_\varepsilon) \geqq \nu_\partial\Big(\bigcup_i C_{\varepsilon,i}\Big)(1-\varepsilon)/t - \nu_\partial\Big((\mathcal{S}\setminus A)\setminus \bigcup_i C_{\varepsilon,i}\Big)/\varepsilon t$$

where the first term converges to the desired expression $\nu_\partial(\mathbb{R}^d)/t$ as $\varepsilon \to 0$, whereas, using (5.2.8), the second term can be estimated further from below by $\geqq -\varepsilon/t$, converging to zero as $\varepsilon \to 0$.

11° Turning to I_1 we proceed as follows. On each set $K_{\varepsilon,j}$, for the integrand we have

$$g\,\psi_\varepsilon - \psi_\varepsilon/(1-t\,\psi_\varepsilon) \geqq (\sqrt{j\,\varepsilon^2}-1)^2/t$$

since $g < j\,\varepsilon^2$ and noting that ψ_ε is non-positive because of $j\,\varepsilon^2 < 1$. Further, on $B_{\varepsilon,j}\setminus K_{\varepsilon,j}$ use $0 \leqq g \leqq 1$ and (5.2.10) to get $g\,\psi_\varepsilon - \psi_\varepsilon/(1-t\,\psi_\varepsilon) \geqq -2/t\,\varepsilon$. Decomposing $\{g < 1-\delta\} = \bigcup_j (K_{\varepsilon,j} \cup (B_{\varepsilon,j}\setminus K_{\varepsilon,j}))$ we obtain

$$I_1 \geqq \sum_j [\mu(K_{\varepsilon,j})(\sqrt{j\,\varepsilon^2}-1)^2/t - \mu(B_{\varepsilon,j}\setminus K_{\varepsilon,j})\,2/t\,\varepsilon].$$

Since $\mu(B_{\varepsilon,j}\setminus K_{\varepsilon,j}) < \varepsilon^4$ and taking into account that there are at most $1/\varepsilon^2$ indices j, further

$$I_1 \geqq \sum_j \mu(K_{\varepsilon,j})(\sqrt{j\,\varepsilon^2}-1)^2/t - 2\,\varepsilon/t.$$

Now set $f_\varepsilon(y) := \sum_j j\,\varepsilon^2\,1_{B_{\varepsilon,j}}(y)$ to get

$$I_1 \geqq t^{-1}\int \mu(dy)\,1_{\bigcup_j K_{\varepsilon,j}}(y)\,(\sqrt{f_\varepsilon(y)}-1)^2 - 2\,\varepsilon/t.$$

Recall that $\varepsilon^2 = \delta\,2^{-n}$, and let $n \to \infty$. Then on $\{g < 1-\delta\}\cap A$ we have $f_\varepsilon \to g$ and $1_{K_{\varepsilon,j}} \to 1$ pointwise. But f_ε is bounded by 1 and $\mu(\{g < 1-\delta\}) < +\infty$, thus by bounded convergence we get $I_1 \geqq t^{-1} \int_{\{g<1-\delta\}} d\mu\ (\sqrt{g}-1)^2$ where the latter expression finally converges to $t^{-1} \int_{\{g<1\}} d\mu\ (\sqrt{g}-1)^2$ as $\delta \to 0$.

12° Since $\psi_\varepsilon(y) = 0$ in the main term of I_2, its estimation results into the error term

$$I_2 \geqq \sum_i \mu((\{1-\delta \leqq g < 1\} \cap A)_i \setminus L_{\varepsilon,i})\, 2\varepsilon/t \geqq -2\varepsilon/t \underset{\varepsilon\to 0}{\to} 0.$$

13° Finally, ψ_ε is non-negative on each $\bigcup_i C_{\varepsilon,i,k}$. Hence, on these sets $g \geqq 1 + k\varepsilon^2 =: \eta_k$, and then

$$I_3 \geqq \sum_k [\mu(\bigcup_i C_{\varepsilon,i,k})\,(\eta_k(1-1/\sqrt{\eta_k})/t - (\sqrt{\eta_k}-1)/t)$$
$$-\nu_{ac}(\bigcup_i B_{\varepsilon,i,k} \setminus \bigcup_i C_{\varepsilon,i,k})/\varepsilon t - \mu(\bigcup_i B_{\varepsilon,i,k} \setminus \bigcup_i C_{\varepsilon,i,k})/\varepsilon t].$$

By (5.2.9) we can continue with

$$\geqq \sum_k [\mu(\bigcup_i C_{\varepsilon,i,k})\,(\sqrt{\eta_k}-1)^2/t - 2\varepsilon^5/t].$$

Using the notation $h_\varepsilon(y) := \sum_k \eta_k 1_{\bigcup_i B_{\varepsilon,i,k}}(y)$ and taking into account that we have at most $1/\varepsilon^4$ indices k, the latter expressions can be written as and estimated from below by

$$\geqq t^{-1} \int d\mu\, 1_{\bigcup_{i,k} C_{\varepsilon,i,k}} (\sqrt{h_\varepsilon}-1)^2 - 2\varepsilon/t.$$

Here we can additionally assume that in 9° the construction of the sets $C_{\varepsilon,i,k}$ had been done in such a way that the union $\bigcup_{i,k} C_{\varepsilon,i,k}$ *monotonously* increases to $\{g \geqq 1\}$ as $n \to \infty$ (via $\varepsilon = \sqrt{\delta\, 2^{-n}}$). But h_ε converges monotonously to $g\, 1_{\{g \geqq 1\}}$ and then by monotone convergence as $n \to \infty$ we arrive at the estimate $I_3 \geqq t^{-1} \int_{\{g\geqq 1\}} d\mu\,(\sqrt{g}-1)^2$.

14° Combining the estimates in 10°-13°, we get the desired lower bound, and the proof of Theorem 5.2.1 is complete. ■

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