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CARLO BOLDRIGHINI
ROBERT A. MINLOS
ALESSANDRO PELLEGRINOTTI

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Interacting random walk in a dynamical random environment II. Environment from the point of view of the particle

by

Carlo BOLDRIGHINI (*)

Dipartimento di Matematica e Fisica, Università di Camerino,
Via Madonna delle Carceri, 62032 Camerino, Italy.

Robert A. MINLOS (*)

Moscow State University, Moscow, Russia.

and

Alessandro PELLEGRINOTTI (*)

Dipartimento di Matematica, Università di Roma La Sapienza,
P. A. Moro 2, 00187 Roma, Italy.

ABSTRACT. – We consider, as in I, a random walk $X_t \in \mathbb{Z}^\nu$, $t \in \mathbb{Z}_+$ and a dynamical random field $\xi_t(x)$, $x \in \mathbb{Z}^\nu$ in mutual interaction with each other. The model is a perturbation of un unperturbed model in which walk and field evolve independententy. Here we consider the environment process in a frame of reference that moves with the walk, *i.e.*, the "field from the point of view of the particle" $\eta_t(.) = \xi_t(X_t + .)$. We prove that its distribution tends, as $t \to \infty$, to a limiting distribution μ, which is absolutely continuous with respect to the unperturbed equilibrium distribution. We also prove that, for $\nu \geqq 3$, the time correlations of the field η_t decay as const $\frac{e^{-\alpha t}}{t^{\frac{\nu}{2}}}$.

Key words: Random walk in random environment, mutual influence, environment from the point of view of the particle.

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RÉSUMÉ. – On considère un chemin aléatoire $X_t \in \mathbb{Z}^\nu$, $t \in \mathbb{Z}_+$ et un milieu aléatoire dynamique $\xi_t(x)$, $x \in \mathbb{Z}^\nu$ en interaction mutuelle. Le modèle est une perturbation d'un modèle imperturbé, dans lequel le chemin aléatoire et le milieu évoluent d'une façon indépendante. On étudie ici le procès du milieu dans un repère qui se déplace avec le chemin aléatoire, c'est-à-dire le « milieu du point de vue de la particule » $\eta_t(.) = \xi_t(X_t + .)$. On montre que la distribution de η_t tend, pour $t \to \infty$ à une distribution limite μ, qui est absolument continue par rapport à la distribution d'équilibre du milieu imperturbé. On montre aussi que le premier terme du développement asymptotique des corrélations temporelles du champ η_t est donné par const $\frac{e^{-\alpha t}}{t^{\frac{\nu}{2}}}$.

1. INTRODUCTION

The present paper is second in a series of two papers. As in the preceding paper [1] (hereafter referred to as Part I), we study the time evolution of a random walk X_t on the ν-dimensional lattice $\mathbb{Z}^\nu$ and a field (environment) $\xi_t \equiv \{\xi_t(x) : x \in \mathbb{Z}^\nu\}$, subject to a mutual interaction of local character. Time is discrete, $t \in \mathbb{Z}_+$, and the field $\xi_t(x)$, $x \in \mathbb{Z}^\nu$ takes values in a finite set S.

We briefly recall the main features of the model, and refer the reader to Part I for more details. The assumptions of the present paper which differ from those of Part I are listed in Section 2.

As a starting point we consider an "unperturbed" model, in which the random walk and the environment evolve independently. The random walk is homogeneous with transition probabilities $\{P_0(y) : y \in \mathbb{Z}^\nu\}$, and the evolution of the environment at each site is an ergodic Markov chain, with a finite space state S, and stochastic operator $Q_0 = \{q_0(s, s') : s, s' \in S\}$, the same for all sites. π_0 will denote the unique stationary measure of the chain. The evolution at different sites is independent.

For the interacting model the random walk transition probabilities are written as

$$\mathbb{P}(X_{t+1} = x + y \mid X_t = x, \xi_t = \bar{\xi}) = P_0(y) + \varepsilon c(y, \bar{\xi}(x))$$

and the transition probabilities for the environment at the site $x \in \mathbb{Z}^\nu$ are

$$\mathbb{P}\left(\xi_{t+1}(x) = s \,|\, X_t = z,\, \xi_t = \bar{\xi}\right) = \begin{cases} q_0\left(\bar{\xi}(x),\, s\right) & \text{if } \ x \neq z \\ q_0\left(\bar{\xi}(x),\, s\right) + \varepsilon \bar{q}\left(\bar{\xi}(x),\, s\right) & \text{if } \ x = z. \end{cases}$$

$c(.\,,.)$ and $\bar{q}(.\,,.)$ satisfy some compatibility conditions, and ε is a small parameter.

Under some general assumptions we deduced in the paper [2] the local central limit theorem for the displacements of the particle. In Part I we studied the asymptotic (in time) decay of the correlations of the environment in a fixed frame of reference. The present paper is devoted to the investigation of the environment process in a frame of reference which moves with the particle, *i.e.*, of the process $\{\eta_t : t \in \mathbb{Z}_+\}$ defined as $\eta_t(x) = \xi_t(X_t + x)$ (sometimes called "field from the point of view of the particle"). η_t evolves as an infinite-dimensional Markov chain.

We prove for any dimension ν that, as $t \to \infty$, the distribution of η_t tends to a limiting invariant distribution μ, which is absolutely continuous with respect to the invariant distribution $\Pi_0 = \pi_0^{\mathbb{Z}^\nu}$ of the environment in the unperturbed case ($\varepsilon = 0$). We also study the time asymptotics of the (time) correlations of the field η_t, for $\nu \geqq 3$. We prove that the leading term is of the type const $e^{-\alpha t}\, t^{-\frac{\nu}{2}}$. The constant factor depends on the initial conditions, whereas $\alpha \in (0, \infty)$ depends only on the parameters of the model. This result should be compared with the long time tail of the correlations of the field in a fixed frame of reference, which was studied in Part I. The different behavior is explained by the fact that in a fixed frame of reference the environment process ξ_t by itself is not Markov.

The methods used in the proofs are, as in the previous papers [1] and [2], based on the spectral analysis of the stochastic operator (or transfer matrix) $\mathcal{T}$ of the Markov chain $\{\eta_t : t \in \mathbb{Z}_+\}$, acting on the space $\mathcal{H} = L_2(\Omega, \Pi_0)$. Here $\Omega = S^{\mathbb{Z}^\nu}$ is the state space of the field. The method of the proof is based, as in Part I, on the analysis of the leading spectral subspaces.

The paper is organized as follows. Section 2 is devoted to the definition of the model and to the statement of the results. In Section 3 we prove the main technical theorem (Theorem 3.1), which gives the decomposition of $\mathcal{H}$ in invariant (with respect to $\mathcal{T}$) subspaces. Section 4 contain the proofs of the theorems, which rely on the results of Section 3, and Section 5 is devoted to some concluding remarks. In the Appendix we prove a technical lemma which is needed in the proof of Theorem 3.1.

2. DEFINITIONS AND FORMULATION OF THE RESULTS

The model is described in detail in Section 2 of Part I, to which we refer. We state here only the assumption which differ from the ones of Part I. Throughout the paper we will write (I n.m) to denote formula (n.m) of Part I.

The basis $\{ \Psi_\Gamma : \Gamma \in \mathfrak{M} \}$ in $\mathcal{H}$, which plays an important role in what follows, is defined by (I 3.1a).

Throughout the paper we assume conditions I, II and III of Part I on the random walk transition probabilities, and conditions IV and V on the transition probabilities of the random field. Condition VI on the spectrum of Q_0 is replaced by the following one.

VI*. If $|S| > 2$ the spectrum of Q_0 is such that

$$1 = \mu_0 > |\mu_1| > |\mu_2|.$$

Condition VI* is the old condition VI plus the assumption that the eigenvalue μ_1 is nondegenerate (hence real, since complex eigenvalues occur only in conjugate pairs).

Further, we assume Condition VII of Part I for the function c, but replace Condition VIII by the following (stronger) condition.

VIII*. The Fourier coefficients

$$\hat{r}(u) = \int_{T^\nu} \frac{1}{\tilde{p}_0(\lambda)} e^{-i(\lambda, u)} \, d\lambda, \tag{2.1a}$$

of the inverse of the function $\tilde{p}_0(\lambda)$ [eq. (I 2.3 d)] satisfy the inequality

$$\frac{\mu_*}{|\mu_1|} \sum_{u \in \mathbb{Z}^\nu} |\hat{r}(u)| < 1, \qquad \mu_* = \max \{ |\mu_2|, \mu_1^2 \}. \tag{2.1 b}$$

Inequality (2.1b) implies the existence of a spectral gap

$$\min_{\lambda \in T^\nu} |\mu_1| \, |\tilde{p}_0(\lambda)| > \mu_*. \tag{2.1c}$$

We now define the random field which will be studied in this paper, the "environment from the point of view of the particle". This is the random field $\eta_t \in \Omega$, $t \in \mathbb{Z}_+$ given by the relation

$$\eta_t(y) = \xi_t(y + X_t). \tag{2.2a}$$

For any bounded functional F on Ω we have for the conditional average [with respect to the distribution (I 2.1)]

$$\langle F(\eta_t) \,|\, \eta_{t-1} = \bar\eta,\, X_{t-1} = \bar x \rangle = \sum_{y \in \mathbb{Z}^\nu} (P_0(y) + \varepsilon c(y, \bar\eta(0))) \times \int_\Omega dP_y(\eta_t \,|\, \bar\eta)\, F(\eta_t), \tag{2.2b}$$

where $P_y(. \,|\, \bar\eta)$ is a product measure

$$P_y(\eta \,|\, \bar\eta) = \prod_{x \neq 0} q_0(\bar\eta(x),\, \eta(x-y))\, q_1(\bar\eta(0),\, \eta(-y)), \tag{2.2c}$$

$q_1(s, s')$, $s, s' \in S$ being the matrix elements of the matrix Q_1 defined in eq. (I 2.5b). The conditional average (2.2b) does not depend on X_{t-1}, and we can consider it as an average over the Markov field $\{\eta_t(y)\}$, with transition probabilities

$$P(\eta_t \,|\, \eta_{t-1} = \bar\eta) = \sum_{y \in \mathbb{Z}^\nu} (P_0(y) + \varepsilon c(y, \bar\eta(0)))\, P_y(\eta_t \,|\, \bar\eta). \tag{2.2d}$$

The stochastic operator or transfer matrix $\mathcal{T}$ of this field is defined, as usual, by its action on the bounded functionals F on Ω :

$$(\mathcal{T} F)(\bar\eta) = \int_\Omega F(\eta)\, dP(\eta \,|\, \bar\eta). \tag{2.3}$$

We denote the average of a random variable f with respect to any measure ν by $\langle f \rangle_\nu$, and the correlations by $\langle f, g \rangle_\nu = \langle fg \rangle_\nu - \langle f \rangle_\nu \langle g \rangle_\nu$.

We fix an initial distribution Π of the field η, induced by some initial distribution $\tilde\Pi$ of the field ξ for a fixed initial position $x_0 = X_0$ of the random walk. Π is simply the shift of $\tilde\Pi$ by x_0. $\mathcal{P}_\Pi$ denotes the distribution on the space of trajectories of the Markov field $\{\eta_t : t \in \mathbb{Z}_+\}$ generated by the initial distribution Π. From the definition (2.3) we find, for F as before

$$\langle F(\eta_t) \rangle_{\mathcal{P}_\Pi} = \int_\Omega (\mathcal{T}^t F)(\eta)\, d\Pi(\eta).$$

We shall also consider condition IX of Part I (symmetry of the random walk). For the field η it implies the following statement, which may be considered as a new condition.

IX*.

$$P(\eta_t = \eta \mid \eta_{t-1} = \bar{\eta}) = P(\eta_t = \mathcal{V}\eta \mid \eta_{t-1} = \mathcal{V}\bar{\eta}).$$

Here, as in Part I, $\mathcal{V}$ is the space reflection: $(\mathcal{V}\eta)(x) = \eta(-x)$.

We need the following further assumption on the correlations of the initial measure Π.

X*. The spatial correlations with respect to the distribution Π of the vectors $\{\Psi_\Gamma : \Gamma \in \mathfrak{M}\}$ of the basis in $\mathcal{H}$ [*see* (I 3.1a)] satisfy the following cluster inequality

$$|\langle \Psi_\Gamma, \Psi_{\Gamma_j^{(x)}} \rangle_\Pi| < \text{const } \bar{M}^{|\operatorname{supp}\Gamma|}\, \bar{q}^{\hat{d}(x, \operatorname{supp}\Gamma)},$$

$$\hat{d}(x, A) = \max_{y \in A} |x - y|, \qquad A \subset \mathbb{Z}^\nu.$$

Here $\Gamma_j^{(x)}$ denotes the multi-index with $\operatorname{supp}\Gamma = \{x\}\ \gamma(x) = j$, and $\bar{M}$, $\bar{q}$ are two constants such that $\bar{M} > 1$, and $\bar{q} \in (0, 1)$.

As in Part I, by d_A, $A \subset \mathbb{Z}^\nu$, we denote the minimal length of the connected graphs which join all points of the set A.

We now formulate the main results of our paper. We understand throughout that condition I-IX are the ones of Part I. Conditions introduced in the present paper are denoted by a star.

THEOREM 2.1. – *Let* $\Pi^{(t)}$, $t \in \mathbb{Z}_+$ *denote the family of measures generated by the Markov process* η_t *with* $\Pi^{(0)} = \Pi$, *and assume conditions* I-V, VI*, VII *and* VIII*. *Then, as* $t \to \infty$, *the measures* $\Pi^{(t)}$ *tend weakly to an invariant measure* μ, *which is absolutely continuous with respect to the independent measure* Π_0.

Moreover there are positive constants c_1 *and* $\bar{q} \in (0, 1)$ *such that*

$$|\langle \Psi_\Gamma \rangle_\mu| < \textit{const}\, (c_1\, \varepsilon)^{|\operatorname{supp}\Gamma|}\, \bar{q}^{d_{\operatorname{supp}\Gamma \cup \{0\}}}, \qquad \Gamma \in \mathfrak{M}. \tag{2.4}$$

Remark 2.1. – Inequality (2.4) implies that Condition X* above holds for the invariant measure μ as well.

In Theorem 2.1 Condition VI* could be replaced by the weaker Condition VI, at the cost of a longer proof. We shall indicate below how it can be done.

The other result concerns the behavior of the time correlations of the field, for which we write down the precise asymptotics. Consider the correlation

$$\langle f_1(\eta_t(x_1)),\ f_2(\eta_0(x_2))\rangle_{\mathcal{P}_\Pi}, \tag{2.5}$$

where $x_1,\ x_2$ are two fixed points.

THEOREM 2.2. – *Let* $\nu \geqq 3$, *and add conditions* IX* *and* X* *to the hypotheses of Theorem* 2.1. *Then, as* $t \to \infty$,

$$\langle f_1(\eta_t(x_1)),\ f_2(\eta_0(x_2))\rangle = \mathrm{C}\left(\frac{\mu_1}{|\mu_1|}\right)^t \frac{e^{-\alpha t}}{t^{\frac{\nu}{2}}}(1+o(1)) \tag{2.6}$$

where $\alpha > 0$ *is a constant depending only on the parameters of the model and the constant C depends in addition on* $f_1,\ f_2,\ x_1,\ x_2$, *and* Π.

By Remark 2.1 the invariant measure μ satisfies the conditions of Theorem 2.2, so that the equilibrium time correlations for the measure μ are also of the type (2.6).

3. EXISTENCE AND PROPERTIES OF THE INVARIANT SUBSPACES

3.1. More notation and preliminary results

In what follows we denote by const any absolute constant, independent of ε.

The action of $\mathcal{T}$ on the basis functions $\{\Psi_\Gamma : \Gamma \in \mathfrak{M}\}$ is given by

$$(\mathcal{T}\,\Psi_\Gamma)(\eta) = (\mathcal{T}^0\,\Psi_\Gamma)(\eta) + \varepsilon \sum_y \sum_j \frac{\mu_\Gamma}{\mu_{\gamma(-y)}} \times L(\gamma(-y),\ j\,;\ y)\,\Psi_{A_j(0)(\Gamma+y)}(\eta). \tag{3.1a}$$

Here $A_j(x)\Gamma$ is the multi-index obtained by replacing the value $\gamma(x)$ of Γ at x with j, leaving all the other values $\gamma(z),\ z \neq x$ unchanged. $\mathcal{T}^0$ is the unperturbed operator

$$\mathcal{T}^0\,\Psi_\Gamma = \mu_\Gamma \sum_y P_0(y)\,\Psi_{\Gamma+y}, \tag{3.1b}$$

and the coefficients $L(\dots)$ are given by

$$L(m, j; y) = q_j^m P_0(y) + \mu_m \sum_{l=0}^{|S|-1} b_{m,l}^j c_l(y) + \sum_{l,k=0}^{|S|-1} q_l^m b_{k,l}^j c_k(y). \tag{3.1c}$$

Here we have used the equality $A_j(-y)\Gamma + y = A_j(0)(\Gamma + y)$, and the coefficients $c_j(y)$, $q_j^{j'}$, and $b_{m,m'}^j$ are the coefficients of the expansions in the basis $\{e_j\}$ of the functions $c(u, s)$, $\sum_{s'} \bar{q}(s, s') e_j(s')$ and $e_k(s) e_m(s)$, respectively [*see* (I 2.6f)]. Note that $c_0(u) = 0$ and $q_k^0 = 0$, which implies $L(0, j; u) = c_j(u)$.

We shall write $\mathcal{T} = \mathcal{T}^0 + \varepsilon\Delta$. $\mathcal{T}^0$ is translation invariant whereas $\mathcal{T}$ is not.

LEMMA 3.1. – *$\mathcal{T}$ and Δ are bounded linear operators on $\mathcal{H}$.*

Proof. – By eqs. (3.1a, b, c) we have

$$\mathcal{T}_{\Gamma\Gamma'} = \sum_{y \in \mathbb{Z}^\nu} \hat{\mu}_\Gamma(-y)\, \delta_{\Gamma'_0, (\Gamma+y)_0}\, \hat{L}(\gamma(-y), \gamma'(0); y),$$

where we use the notation

$$\hat{\mu}_\Gamma(y) = \frac{\mu_\Gamma}{\mu_{\gamma(y)}}, \qquad \Gamma_z = A_0(z)\Gamma,$$

$$\hat{L}(j, j'; y) = \mu_j\, \delta_{j,j'}\, P_0(y) + \varepsilon L(j, j'; y).$$

Recalling the expression (I 3.1b) of the scalar product in $\mathcal{H}$, we have

$$\begin{aligned}(\mathcal{T} f, g) &= \sum_{\Gamma, \Gamma', \Gamma''} f_{\Gamma''}\, a_{\Gamma,\Gamma'}\, \mathcal{T}_{\Gamma''\Gamma}\, \overline{g_{\Gamma'}} \\ &= \sum_y \sum_{\Gamma,\Gamma',\Gamma''} \hat{\mu}_{\Gamma''}\, f_{\Gamma''}\, a_{\Gamma,\Gamma'}\, \overline{g_{\Gamma'}} \\ &\quad \times \hat{L}(\gamma''(-y), \gamma(0); y)\, \delta_{\Gamma_0, (\Gamma''+y)_0}.\end{aligned}$$

Setting, for fixed y, $\Gamma^* = \Gamma'' + y$, $f_{\Gamma^*}^{(y)} = f_{\Gamma^* - y}$, we have

$$(\mathcal{T} f, g) = \sum_y (\mathcal{T}^{(y)} f^{(y)}, g),$$

where the operator $\mathcal{T}^{(y)}$ is defined by the position

$$(\mathcal{T}^{(y)} f, g) = \sum_{\Gamma, \Gamma', \Gamma^*} f_{\Gamma^*} a_{\Gamma\Gamma'} \mu_{\Gamma_0^*} \hat{L}(\gamma^*(0), \gamma(0); y) \overline{g_{\Gamma'}} \delta_{\Gamma_0, \Gamma_0^*}.$$

The proof that $\mathcal{T}^{(y)}$ is a bounded operator is done exactly as in Lemma A.1 of [2], and since $\| f^{(y)} \| = \| f \|$ by translation invariance of the norm, and the sum is over the finite set $|y| < D$, it follows that $\mathcal{T}$ is a bounded operator in $\mathcal{H}$. The proof that Δ is bounded is an easy consequence. Lemma 3.1 is proved. ■

As in Part I we consider, for $M > M^* \equiv \max\{\max_{j,s} |e_j(s)|, 2\}$, the dense subspace $\mathcal{H}_M \subset \mathcal{H}$, with norm $\| . \|_M$ defined by (I 3.5a).

By the definition (I 3.1a) of the basis $\{\Psi_\Gamma\}$ it is easy to see that the following inequalities between norms hold

$$\| f \|_{\mathcal{H}} \leqq \| f \|_\infty \leqq \| f \|_M,$$

where $\| f \|_\infty \equiv \sup_{\xi \in \Omega} | f(\xi) |$ denotes the supremum norm of f.

For the space $\mathcal{H}_M$ we state the analogue of Lemma 3.1.

LEMMA 3.2. – (i) *The operators $\mathcal{T}$ and Δ are bounded on $\mathcal{H}_M$.* (ii) *$\mathcal{H}_M$ is invariant under $\mathcal{T}$.*

Proof. – It is the same as the proof of the analogous Lemma 3.4 in [2]. ■

We denote by $\mathfrak{M}/\mathbb{Z}^\nu$ the set of the equivalence classes of multi-indices which differ only by a shift. Let $\zeta \in \mathfrak{M}/\mathbb{Z}^\nu$ and $\hat{\Gamma} \in \zeta$ be a representative of the class ζ. Since $\mathcal{T}^0$ is translation invariant, the subspace $\mathcal{H}_\zeta$, consisting of the vectors

$$\sum_{z \in \mathbb{Z}^\nu} c_z \Psi_{\hat{\Gamma}+z}, \tag{3.2}$$

is invariant under $\mathcal{T}^0$. It is easy to see that

$$\mathcal{T}^0 \left(\sum_{z \in \mathbb{Z}^\nu} c_z \Psi_{\hat{\Gamma}+z} \right) = \mu_{\hat{\Gamma}} \sum_{z \in \mathbb{Z}^\nu} \tilde{c}_z \Psi_{\hat{\Gamma}+z}, \qquad \tilde{c}_z = \sum_{u \in \mathbb{Z}^\nu} c_u P_0(z-u).$$

Hence the spectrum of the operator $\mathcal{T}^0|_{\mathcal{H}_\zeta}$ coincides with the range of the function $\mu_{\hat{\Gamma}} \tilde{p}_0(\lambda)$. Note that $\mu_{\hat{\Gamma}}$ does not depend on the choice of the representative $\hat{\Gamma} \in \zeta$. We denote by $\mathcal{H}_1^0$ the space $\mathcal{H}_\zeta$ when ζ is the class of equivalence of the multi-indices Γ which contains $\Gamma_1^{(0)}$ (defined in

Condition X* of § 2). Condition VIII* implies that the spectrum of $\mathcal{T}^0$ in the space $\mathcal{H}_1^0$ is separated from the spectrum of $\mathcal{T}^0$ in $\mathcal{H} \setminus \mathcal{H}_0^1$.

3.2. Construction of the invariant subspaces

The main result of this section is the following theorem.

THEOREM 3.1. – *Under the assumptions* I-V, VI*, VII, *and* VIII*, *and for* ε *small enough the following assertions hold.*

(i) *One can find two subspaces of* $\mathcal{H}$, $\mathcal{H}_1$ *and* $\mathcal{H}_2$, *invariant with respect to the operator* $\mathcal{T}$, *such that*

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2, \tag{3.3a}$$

$$\mathcal{H}_M = \mathcal{H}_0 + \mathcal{H}_M^{(1)} + \mathcal{H}_M^{(2)}, \qquad \mathcal{H}_M^{(i)} = \mathcal{H}_i \cap \mathcal{H}_M \quad i = 1,\, 2, \tag{3.3b}$$

$\mathcal{H}_0$ *being the space of the constants.*

(ii) *In the space* $\mathcal{H}_M^{(1)}$ *one can find a basis* $\{ h_z,\ z \in \mathbb{Z}^\nu \}$ *on which the operator* $\mathcal{T}_1 \equiv \mathcal{T}\,|_{\mathcal{H}_M^{(1)}}$ *acts according to the formula*

$$\mathcal{T}_1\, h_x = \sum_{y \in \mathbb{Z}^\nu} \bar{p}\,(x - y)\, h_y + \sum_{y \in \mathbb{Z}^\nu} S\,(x,\, y)\, h_y. \tag{3.4a}$$

Moreover $\bar{p}\,(y)$ *is real and given by the relation*

$$\bar{p}\,(y) = \mu_1\, P_0\,(y) + \hat{\delta}\,(y), \tag{3.4b}$$

and, for some $q \in (0,\, 1)$, *the following inequalities hold*

$$|\, S\,(x,\, y)\,| < const\, \varepsilon q^{|\,x\,| + |\,y\,|}, \qquad |\, \hat{\delta}\,(y)\,| < const\, \varepsilon q^{|\,y\,|}. \tag{3.4c}$$

(iii) *The norms of the restrictions of* $\mathcal{T}$ *to the subspaces* $\mathcal{H}_M^{(1)}$ *and* $\mathcal{H}_M^{(2)}$ *(endowed with the norm* $\|\,.\,\|_M$*) satisfy respectively the estimates*

$$\|\, \mathcal{T}\,|_{\mathcal{H}_M^{(1)}}\, \|_{\mathcal{H}_M} < |\, \mu_1\,| + const\, \varepsilon, \tag{3.5a}$$

$$\|\, \mathcal{T}\,|_{\mathcal{H}_M^{(2)}}\, \|_{\mathcal{H}_M} < \mu_* + const\, \varepsilon. \tag{3.5b}$$

iv) *Under the additional condition* IX* $\bar{p}\,(y)$ *is an even function of* $y \in \mathbb{Z}^\nu$.

Remark 3.1. – The decomposition (I 3.2) of the space $\mathcal{H}$ in a direct integral of the eigenspaces $\mathcal{H}_\lambda$ of the group of translations $\{ \mathcal{U}_v\, :\, v \in \mathbb{Z}^\nu \}$ does not reduce the operator $\mathcal{T}$, since for $\varepsilon > 0$ $\mathcal{T}$ does not commute with the space shifts. This is why an additional sum appears in (3.4a), and we had to assume inequality (2.1c) in Condition VIII*.

Remark 3.2. – If we have condition VI, instead of VI*, *i.e.*, the eigenvalue μ_1 has a multiplicity $s > 1$, Theorem 3.1 holds with some obvious changes. Namely the basis in the space $\mathcal{H}_M^{(1)}$ will be labeled by two indices: $\{ h_z^{(i)} \, : \, z \in \mathbb{Z}^\nu, \, i = 1, \ldots, s \}$. Moreover the operator $\mathcal{T} \,|_{\mathcal{H}_M^{(1)}} \equiv \mathcal{T}_1$ in this basis will act as follows

$$\mathcal{T}_1 \, h_z^{(i)} = \sum_{\substack{y \in \mathbb{Z}^\nu \\ j=1, \ldots, s}} \bar{p}_{ij} (z - y) \, h_y^{(j)} + \sum_{\substack{y \in \mathbb{Z}^\nu \\ j=1, \ldots, s}} S_{ij} (z, \, y) \, h_y^{(j)},$$
$$i = 1, \ldots, s, \quad z \in \mathbb{Z}^\nu,$$

where

$$\bar{p}_{ij} (u) = \mu_i \, P_0 (u) \, \delta_{ij} + \hat{\delta}_{ij} (u), \qquad i, j = 1, \ldots, s, \quad u \in \mathbb{Z}^\nu,$$

and the matrix elements $\{ \delta_{ij} (u) \}$, and $\{ S_{ij} (z, \, y) \}$ satisfy estimates analogous to (3.4c).

Proof of Theorem 3.1. – The proof makes use of the same ideas as the analogous proof in section 3 of [2]. It is based on several lemmas, which are stated and proved in the rest of the present section.

We shall first find the invariant subspace $\mathcal{H}_M^{(1)}$, and then obtain $\mathcal{H}_1$ as the closure of $\mathcal{H}_M^{(1)}$ in $\mathcal{H}$.

For $\varepsilon = 0$ we have clearly

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1^0 + \mathcal{H}_2^0,$$
$$\mathcal{H}_M = \mathcal{H}_0 + \mathcal{H}_{1,\,M}^0 + \mathcal{H}_{2,\,M}^0, \qquad \mathcal{H}_{i,\,M}^0 = \mathcal{H}_i^0 \cap \mathcal{H}_M \quad i = 1, \, 2.$$

Here $\mathcal{H}_0$ is the space of the constants, $\mathcal{H}_1^0$ was defined above and $\mathcal{H}_2^0$ is the (closed) subspace spanned by the functions $\{ \Psi_\Gamma \, : \, \Gamma \in \mathcal{G}_2 \}$, $\mathcal{G}_2$ being the set of the multi-indices Γ with either $|\operatorname{supp} \Gamma| > 1$ or $|\operatorname{supp} \Gamma| = 1$ but $\gamma (x) \neq 1$ for $x \in \operatorname{supp} \Gamma$.

It is convenient for the moment to set

$$\tilde{\mathcal{H}}_1^0 = \mathcal{H}_0 + \mathcal{H}_1^0, \qquad \tilde{\mathcal{H}}_{1,\,M}^0 = \mathcal{H}_0 + \mathcal{H}_{1,\,M}^0,$$

so that $\mathcal{H}_M = \tilde{\mathcal{H}}_{1,\,M}^0 + \mathcal{H}_{2,\,M}^0$, and $\mathcal{T}$ (as an operator on $\mathcal{H}_M$) is written as an operator matrix

$$\mathcal{T} = \begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{pmatrix}, \tag{3.6}$$

where the operators $\mathcal{T}_{ij}$ act as follows: $\mathcal{T}_{11} : \tilde{\mathcal{H}}^0_{1,M} \to \tilde{\mathcal{H}}^0_{1,M}$, $\mathcal{T}_{12} : \mathcal{H}^0_{2,M} \to \tilde{\mathcal{H}}^0_{1,M}$, $\mathcal{T}_{21} : \tilde{\mathcal{H}}^0_{1,M} \to \mathcal{H}^0_{2,M}$, $\mathcal{T}_{22} : \mathcal{H}^0_{2,M} \to \mathcal{H}^0_{2,M}$. The space $\tilde{\mathcal{H}}^0_{1,M}$ is spanned by the functions $\{ \Psi_{\Gamma_1^{(x)}} : x \in \mathbb{Z}^\nu \}$ and $\Psi_{\bar{0}}$.

We first study the inverse of the operator $\mathcal{T}_{11}$ in the space $\tilde{\mathcal{H}}^0_{1,M}$. ($\Gamma_1^{(z)}$ is defined in Condition X* of Section 2.) We introduce for brevity the notation $\varphi_z = \Psi_{\Gamma_1^{(z)}}$, and we will allow z to take the value $\bar{0} : \varphi_{\bar{0}} = \Psi_{\bar{0}}$. For the indices of the matrix elements of operators on $\tilde{\mathcal{H}}^0_1$ we shall also write z instead of $\Gamma_1^{(z)}$.

LEMMA 3.3. – *The operator $\mathcal{T}_{11} = \mathcal{T}\,|_{\tilde{\mathcal{H}}^0_{1,M}}$ is invertible in $\tilde{\mathcal{H}}^0_{1,M}$ and its inverse is given by*

$$(\mathcal{T}_{11})^{-1} = (\mathcal{T}^0_{11})^{-1} + \varepsilon\, \mathcal{D}, \tag{3.7a}$$

where the operator $\mathcal{D}$ has the following properties:

$$\left.\begin{array}{ll} \mathcal{D}_{\bar{0},z} = 0, \quad z \in \mathbb{Z}^\nu \cup \bar{0}, & |\mathcal{D}_{z,\bar{0}}| < const\, \theta^{|z|}, \\ |\mathcal{D}_{z,z'}| < const\, \theta^{|z|+|z'|}, & z,\ z' \in \mathbb{Z}^\nu, \end{array}\right\} \tag{3.7b}$$

for some $\theta \in (0, 1)$.

Proof. – The operator $\mathcal{T}_{11}$ can be written as

$$\mathcal{T}_{11} = \mathcal{T}^0_{11} + \varepsilon\, \hat{\mathcal{D}}$$

and, as it follows from formulas (3.1a, b, c) $\mathcal{T}_{11}$ can be written as a matrix (corresponding to the decomposition $\tilde{\mathcal{H}}^0_{1,M} = \mathcal{H}_0 + \mathcal{H}^0_{1,M}$)

$$\mathcal{T}_{11} = \begin{pmatrix} 1 & \hat{\mathcal{T}}_{01} \\ 0 & \hat{\mathcal{T}}_{11} \end{pmatrix}, \tag{3.8a}$$

where

$$\left.\begin{array}{c} (\hat{\mathcal{T}}_{11})_{x,y} = \mu_1\, P_0\,(y-x) + \varepsilon\, \hat{\mathcal{R}}_{x,y}, \quad (\hat{\mathcal{T}}_{01})_{x,\bar{0}} = \varepsilon \mu_1\, c_1\,(-x), \\ \hat{\mathcal{R}}_{x,y} = L\,(1,\ 1;\ -x)\, \delta_{y,0}. \end{array}\right\} \tag{3.8b}$$

Its inverse is represented by the matrix

$$(\mathcal{T}_{11})^{-1} = \begin{pmatrix} 1 & -\hat{\mathcal{T}}_{01}\,(\hat{\mathcal{T}}_{11})^{-1} \\ 0 & (\hat{\mathcal{T}}_{11})^{-1} \end{pmatrix}, \tag{3.8c}$$

with

$$\left.\begin{array}{c} (\hat{\mathcal{T}}_{11})^{-1} = (\hat{\mathcal{T}}^0_{11})^{-1} + \varepsilon\, \mathcal{R}, \\ (\hat{\mathcal{T}}^0_{11})^{-1}_{x,y} = \dfrac{1}{\mu_1} \displaystyle\int d\lambda\, \frac{e^{i\,(\lambda,\, x-y)}}{\tilde{p}_0\,(\lambda)} = \frac{1}{\mu_1}\, r\,(y-x) \end{array}\right\} \tag{3.8d}$$

and $\mathcal{R}$ is given by a series

$$\mathcal{R} = \sum_{k=1}^{\infty} \varepsilon^{k-1} (-1)^k (\hat{\mathcal{T}}_{11}^0)^{-1} \hat{\mathcal{R}} (\hat{\mathcal{T}}_{11}^0)^{-1} \ldots \hat{\mathcal{R}} (\hat{\mathcal{T}}_{11}^0)^{-1},$$

where in the running term of the sum the product $\hat{\mathcal{R}} (\hat{\mathcal{T}}_{11}^0)^{-1}$ is repeated k times. From eq. (3.8d) we get $| (\hat{\mathcal{T}}_{11}^0)^{-1}_{x,y} | \leqq \text{const } \theta^{|x-y|}$ for some $\theta \in (0, 1)$. Inserting the expression for $\hat{\mathcal{R}}$ given by (3.8b) one easily finds the estimate

$$| \mathcal{R}_{x,y} | < \text{const } \theta^{|x|+|y|}, \qquad x, y \in \mathbb{Z}^\nu.$$

One finds also that

$$| (\hat{\mathcal{T}}_{01} (\hat{\mathcal{T}}_{11})^{-1})_{z,\bar{0}} | < \text{const } \varepsilon\, \theta^{|z|}, \qquad z \in \mathbb{Z}^\nu.$$

This proves Lemma 3.3. ■

We look for an invariant subspace $\tilde{\mathcal{H}}_M^{(1)}$ of the form

$$\tilde{\mathcal{H}}_M^{(1)} = \{ u + \mathcal{S} u \,:\, u \in \tilde{\mathcal{H}}_{1,M}^0 \}, \tag{3.9a}$$

where $\mathcal{S} \,:\, \tilde{\mathcal{H}}_{1,M}^0 \to \mathcal{H}_{2,M}^0$ is an unknown operator. The invariance condition for $\tilde{\mathcal{H}}_M^{(1)}$ leads to the following equation for $\mathcal{S}$ [which is analogous to eq. (I 3.6c)] :

$$\mathcal{S} = \mathcal{K}(\mathcal{S}) \equiv \mathcal{T}_{21} (\mathcal{T}_{11})^{-1} + \mathcal{T}_{22}\, \mathcal{S} (\mathcal{T}_{11})^{-1} - \mathcal{S}\, \mathcal{T}_{12}\, \mathcal{S} (\mathcal{T}_{11})^{-1}. \tag{3.9b}$$

$\mathcal{K}$ is considered as acting on the space $\mathcal{A}(\tilde{\mathcal{H}}_{1,M}^0, \mathcal{H}_{2,M}^0)$ of the maps from $\tilde{\mathcal{H}}_{1,M}^0$ to $\mathcal{H}_{2,M}^0$, endowed with the operator norm. Equation (3.9b) has a unique solution, as stated by the following lemma.

LEMMA 3.4. – *If ε is small enough one can find a number $\kappa_0 > 0$ such that the map $\mathcal{K}$ is a contraction in the sphere of radius κ_0 centered at the origin of the space $\mathcal{A}(\tilde{\mathcal{H}}_{1,M}^0, \mathcal{H}_{2,M}^0)$.*

Proof. – We denote with $\mathcal{S}_{z,\Gamma}$ the matrix elements of the operator $\mathcal{S}$.

By equations (3.1a, b, c) we see that $(\mathcal{T}_{21})_{\bar{0},\Gamma} = 0$, and for $z \in \mathbb{Z}^\nu$

$$(\mathcal{T}_{21})_{z,\Gamma} = \left\{ \begin{array}{ll} \varepsilon L(1, \gamma(0)\,;\, -z) & \operatorname{supp}\Gamma = \{0\} \\ \varepsilon\, \mu_1\, c_{\gamma(0)}(u - z) & \operatorname{supp}\Gamma = \{0\} \cup \{u\} \\ 0 & \text{otherwise.} \end{array} \right\} \tag{3.10}$$

For $(\mathcal{T}_{12})_{\Gamma,z}$ we have, if $\operatorname{supp}\Gamma = \{u\}$,

$$(\mathcal{T}_{12})_{\Gamma,z} = \left\{ \begin{array}{ll} L(\gamma(u), 0\,;\, -u) & z = \bar{0} \\ L(\gamma(u), 1\,;\, -u)\, \delta_{z,0}, & z \in \mathbb{Z}^\nu, \end{array} \right\} \tag{3.11a}$$

and if $\operatorname{supp}\Gamma = \{u\} \cup \{v\}$, and $z \in \mathbb{Z}^\nu$

$$(\mathcal{T}_{12})_{\Gamma, z} = \mu_1 [L(\gamma(u), 0; -u)\, \delta_{z, v-u} (1 - \delta_{\gamma(u), 1}) \\ + L(\gamma(v), 0; -v)\, \delta_{z, u-v} (1 - \delta_{\gamma(v), 1})]. \tag{3.11b}$$

$(\mathcal{T}_{12})_{\Gamma, z} = 0$ in all other cases.

For the norms of the operators appearing in equation (3.9b) we have

$$\| (\mathcal{T}_{11})^{-1} \| \leqq \sup_{z \in \mathbb{Z}^\nu \cup \bar{0}} \sum_{u \in \mathbb{Z}^\nu \cup \bar{0}} | (\mathcal{T}_{11})^{-1}_{z, u} | \\ \leqq \frac{1}{|\mu_1|} \sum_u | \hat{r}(u) | + \varepsilon C_{11}, \tag{3.12a}$$

$$\| \mathcal{T}_{12} \| \leqq \sup_{\Gamma \in \mathcal{G}_2} \sum_{z \in \mathbb{Z}^\nu \cup \bar{0}} | (\mathcal{T}_{12})_{\Gamma, z} | < \varepsilon C_{12} \tag{3.12b}$$

$$\| \mathcal{T}_{21} \| \leqq \sup_{z \in \mathbb{Z}^\nu} \sum_{\Gamma \in \mathcal{G}_2} | (\mathcal{T}_{21})_{z, \Gamma} | M^{|\operatorname{supp}\Gamma|} \leqq \varepsilon C_{21} \tag{3.12c}$$

$$\| \mathcal{T}_{22} \| \leqq \mu_* + \varepsilon C_{22}, \tag{3.12d}$$

where the constants $C_{i,j}$, $i, j = 1, 2$ depend on the parameters of the model and on M. (3.12a) comes from Lemma 3.3, by observing that

$$1 \leqq \frac{1}{|\mu_1| \min_{\lambda \in T^\nu} | \tilde{p}_0(\lambda) |} \leqq \frac{1}{|\mu_1|} \sum_u | \hat{r}(u) |.$$

Inequalities (3.12b, c) are easily derived, using the explicit expressions (3.10), (3.11a, b) and (3.12d) follows by observing that for $\varepsilon = 0$, $\mathcal{T}_{22} = \mathcal{T}_{22}^0$ with $\| \mathcal{T}_{22}^0 \| = \mu_*$, and that $\varepsilon^{-1}(\mathcal{T}_{22} - \mathcal{T}_{22}^0)$ is a bounded operator (Lemma 3.2).

By inequalities (3.12a, b, c, d) and Condition VIII* [inequality (2.1b)], we see that the second term on the right-hand side of eq. (3.9b) is bounded, for small ε, by $\beta \| \mathcal{S} \|$, where $\beta \in (0, 1)$. Since the other two terms are of order ε and $\varepsilon \| \mathcal{S} \|^2$ respectively, $\mathcal{K}$ is a contraction in any sufficiently small sphere of $\mathcal{A}(\tilde{\mathcal{H}}_{1, M}^0, \mathcal{H}_{2, M}^0)$. Lemma 3.4 is proved. ■

Hence there is a unique $\mathcal{S}$, solution of eq. (3.9b) and the space $\tilde{\mathcal{H}}_M^{(1)}$ defined by eq. (3.9a) is invariant with respect to $\mathcal{T}$. Since $M > 1$, it is not difficult to see that

$$\| \mathcal{S} \|_{\mathcal{A}(\tilde{\mathcal{H}}_{1, M}^0, \mathcal{H}_{2, M}^0)} \leqq \sup_{z \in \mathbb{Z}^\nu \cup \bar{0}} \sum_{\Gamma \in \mathcal{G}_2} | \mathcal{S}_{z, \Gamma} | M^{|\operatorname{supp}\Gamma|} \equiv \| \mathcal{S} \|_M.$$

The Banach space of the elements of $\mathcal{A}(\tilde{\mathcal{H}}^0_{1,M}, \mathcal{H}^0_{2,M})$ with the norm $||\,.\,||_M$ is denoted by $\mathcal{A}_M$. The solution $\mathcal{S}$ of equation (3.9b) is actually in $\mathcal{A}_M$, as it follows from the following lemma.

LEMMA 3.5. – *The right-hand side of equation* (3.9b) *defines a map* $\mathcal{K}_M : \mathcal{A}_M \to \mathcal{A}_M$, *and for* ε *small enough one can find a positive number* $\bar{\kappa}$ *so small that* $\mathcal{K}_M$ *is a contraction in the sphere of radius* $\bar{\kappa}$ *centered at the origin of* $\mathcal{A}_M$.

Proof. – We have, by inequalities (3.12a, b, c, d)

$$||\,\mathcal{T}_{21}\,(\mathcal{T}_{11})^{-1}\,||_M \leqq ||\,\mathcal{T}_{21}\,||_M \sup_{z\,\in\,\mathbb{Z}^\nu\cup\bar{0}} \sum_{u\,\in\,\mathbb{Z}^\nu\cup\bar{0}} |\,(\mathcal{T}_{11})^{-1}_{z,u}\,| \leqq \text{const}\;\varepsilon,$$

$$\begin{aligned} ||\,\mathcal{S}\,\mathcal{T}_{12}\,\mathcal{S}\,(\mathcal{T}_{11})^{-1}\,||_M &\leqq ||\,\mathcal{S}\,||_M \sup_{\Gamma} \sum_{z\,\in\,\mathbb{Z}^\nu\cup\bar{0}} |\,(\mathcal{T}_{12})_{\Gamma,z}\,|\,||\,\mathcal{S}\,(\mathcal{T}_{11})^{-1}\,||_M \\ &\leqq ||\,\mathcal{S}\,||^2_M \sup_{\Gamma} \sum_{z\,\in\,\mathbb{Z}^\nu\cup\bar{0}} |\,(\mathcal{T}_{12})_{\Gamma,z}\,| \\ &\quad\times \sup_{z\,\in\,\mathbb{Z}^\nu\cup\bar{0}} \sum_{u\,\in\,\mathbb{Z}^\nu\cup\bar{0}} |\,(\mathcal{T}_{11})^{-1}_{zu}\,| \leqq \text{const}\;\varepsilon\,||\,\mathcal{S}\,||^2_M. \end{aligned} \tag{3.13a}$$

For the second term, which is the leading one, we have

$$\begin{aligned} ||\,\mathcal{T}_{22}\,\mathcal{S}\,(\mathcal{T}_{11})^{-1}\,||_M &\leqq ||\,\mathcal{T}_{22}\,||\;||\,\mathcal{S}\,||_M \sup_{z\,\in\,\mathbb{Z}^\nu\cup\bar{0}} \sum_{u\,\in\,\mathbb{Z}^\nu\cup\bar{0}} |\,(\mathcal{T}_{11})^{-1}_{u,z}\,| \\ &\leqq \left(\frac{1}{\mu_1}\sum_u |\,r\,(u)\,| + \varepsilon\,C_1\right) \\ &\quad\times (\mu_* + \varepsilon\,C_{22})\,||\,\mathcal{S}\,||_M \leqq \beta\,||\,\mathcal{S}\,||_M, \end{aligned} \tag{3.13b}$$

where $\beta \in (0,\,1)$. The conclusion follows as for the preceding lemma. ■

Hence there is a unique solution $\mathcal{S} \in \mathcal{A}_M$ of eq. (3.9b), which coincides with the previous one.

We introduce a basis in $\tilde{\mathcal{H}}^{(1)}_M$, by setting

$$u_z = \varphi_z + \mathcal{S}\,\varphi_z, \qquad z \in \mathbb{Z}^\nu \cup \bar{0}. \tag{3.14}$$

The action of $\mathcal{T}$ on the new basis is given by the following formulas

$$\mathcal{T}\,u_z = \sum_{z'\,\in\,\mathbb{Z}^\nu\cup\bar{0}} A_{z,z'}\,u_{z'}, \qquad z \in \mathbb{Z}^\nu \cup \bar{0}, \tag{3.15a}$$

$$A_{z,\,z'} = (\mathcal{T}_{11})_{z,\,z'} + (\mathcal{T}_{12}\,\mathcal{S})_{z,\,z'},\ z,\ z' \in \mathbb{Z}^\nu \cup \bar{0}. \tag{3.15b}$$

We need to prove a property of fast decay as $z,\ z' \to \infty$ of the matrix elements $A_{z,\,z'}$. To this aim, we use the fact that, as it was proved in [3], $\mathcal{S}$ can be written as a series

$$\mathcal{S} = \sum_{r=1}^{\infty} \sum_{\alpha_1,\,\dots,\,\alpha_r} x_{\alpha_1,\,\dots,\,\alpha_r}\,\mathcal{B}_{\alpha_1}\,\mathcal{T}_{12}\,\mathcal{B}_{\alpha_2}\,\mathcal{T}_{12}\dots\mathcal{T}_{12}\,\mathcal{B}_{\alpha_r}, \tag{3.16a}$$

where the summation goes over all sequences of pairs $\alpha_i = (s_i,\ q_i)$, $s_i,\ q_i \in \mathbb{Z}_+$,

$$\mathcal{B}_{s,\,q} = \mathcal{T}_{22}^{s}\,\mathcal{T}_{21}\,(\mathcal{T}_{11})^{-q-1}, \qquad s,\ q = 0,\ 1,\ \dots, \tag{3.16b}$$

and $x_{\alpha_1,\,\dots,\,\alpha_r}$ are real numbers (*see* [3]). We first prove the convergence of the series.

LEMMA 3.6. – *For ε small enough the series* (3.16a) *converges in the norm of $\mathcal{A}_M$.*

Proof. – By reasoning exactly as in the deduction of inequality (3.13a) we find

$$\|\,\mathcal{B}_{\alpha_1}\,\mathcal{T}_{12}\,\mathcal{B}_{\alpha_2}\,\mathcal{T}_{12}\dots\mathcal{T}_{12}\,\mathcal{B}_{\alpha_r}\,\|_M \leqq \Big(\sup_{\Gamma\in\mathcal{G}_2} \sum_{z\in\mathbb{Z}^\nu\cup\bar{0}} |\,(\mathcal{T}_{12})_{\Gamma,\,z}\,|\Big)^{r-1} \times \prod_{i=1}^{r} \|\,\mathcal{B}_{\alpha_i}\,\|_M,$$

and, in analogy with the deduction of inequality (3.13b), we find

$$\|\,\mathcal{B}_{s,\,q}\,\|_M \leqq \|\,\mathcal{T}_{22}^{r}\,\|\;\|\,\mathcal{T}_{21}\,\|_M \sup_{z\in\mathbb{Z}^\nu\cup\bar{0}} \sum_{u\in\mathbb{Z}^\nu\cup\bar{0}} |\,(\mathcal{T}_{11})_{z,\,u}^{-q-1}\,|, \tag{3.17}$$

where, as before, by $\|\,.\,\|$ we denote the operator norm of $\mathcal{T}_{22} : \mathcal{H}_{2,\,M}^{(0)} \to \mathcal{H}_{2,\,M}^{(0)}$.

Now, by Lemma 3.3, expanding the power $(\mathcal{T}_{11})^{-s} = ((\mathcal{T}_{11}^{0})^{-1} + \varepsilon\,\mathcal{D})^{s}$, we find

$$(\mathcal{T}_{11})^{-s} = (\mathcal{T}_{11}^{0})^{-s} + \mathcal{D}^{(s)}, \tag{3.18}$$

where

$$(\mathcal{T}_{11}^{0})_{z,\,z'}^{-s} = \int_{T^\nu} \frac{e^{i\,(\lambda,\,z'-z)}}{(\mu_1\,\tilde{p}_0\,(\lambda))^{s}}\,d\lambda, \qquad z,\ z' \in \mathbb{Z}^\nu.$$

and $\mathcal{D}^{(q)}$ is written as a sum of products. The function $\dfrac{1}{\mu_1\,\tilde{p}_0(\lambda)}$, which is the Fourier transform of $(\mathcal{T}_{11}^0)^{-1}$, is analytic in some complex neighborhood $W_d \equiv \{\lambda : |\operatorname{Im}\lambda_i| \leqq d,\, i = 1, \ldots, \nu\}$ of the torus T^ν, if d is sufficientely small, and satisfies in W_d the inequality

$$\left|\frac{1}{\mu_1\,\tilde{p}_0(\lambda)}\right| < \frac{a+\eta}{\mu_1}, \qquad a = \sum_{u \in \mathbb{Z}^\nu} |\hat{r}(u)|, \tag{3.19}$$

where $\hat{r}(u)$, $u \in \mathbb{Z}^\nu$ are the Fourier coefficients (2.1a), and the quantity η is small for small d. It is easy to see that the Fourier transforms

$$\left.\begin{aligned} \tilde{\mathcal{D}}^{(s)}(\lambda, \lambda') &= \sum_{z, z' \in \mathbb{Z}^\nu} \mathcal{D}^{(s)}_{z, z'}\, e^{i(\lambda, z) - i(\lambda', z')}, \\ \tilde{\mathcal{D}}^{(s)}(\lambda) &= \sum_{z} \mathcal{D}^{(s)}_{z, \bar{0}}\, e^{i(\lambda, z)} \end{aligned}\right\} \tag{3.20a}$$

are analytic for $\lambda,\ \lambda' \in W_d$. Using the inequality $(1+x)^s - 1 \leqq x^{\frac{1}{2}}(1 + x^{\frac{1}{2}})^s$, we find

$$\left.\begin{aligned} |\tilde{\mathcal{D}}^{(s)}(\lambda, \lambda')| &< c_1\,\varepsilon^{\frac{1}{2}}\left(\frac{a+\eta}{\mu_1} + c_2\,\varepsilon^{\frac{1}{2}}\right)^s, \\ |\tilde{\mathcal{D}}^{(s)}(\lambda)| &< c_1\,\varepsilon^{\frac{1}{2}}\left(\frac{a+\eta}{\mu_1} + c_2\,\varepsilon^{\frac{1}{2}}\right)^s \end{aligned}\right\} \tag{3.20b}$$

where $c_1,\ c_2$ are constants independent of s. [Note that, by formula (3.8c) we have $(\mathcal{T}_{11}^0)^{-s}_{\bar{0}\bar{0}} = 1$ and $\mathcal{D}^{(s)}_{\bar{0}, z} = 0$, $z \in \mathbb{Z}^\nu \cup \bar{0}$.] In computing the Fourier coefficients we can now shift the integration region in the complex region W_d, and we get from (3.19), (3.20a, b) (maybe by redefining η) the relations

$$\left.\begin{aligned} |(\mathcal{T}_{11}^0)^{-s}_{z, z'}| &< \left(\frac{a+\eta}{\mu_1}\right)^s \theta^{|z - z'|}, \\ |\mathcal{D}^{(s)}_{z, z'}| &< c_1\,\varepsilon^{\frac{1}{2}}\left(\frac{a+\eta}{\mu_1}\right)^s \theta^{|z| + |z'|}, \\ |\mathcal{D}^{(s)}_{z, \bar{0}}| &< c_1\,\varepsilon^{\frac{1}{2}}\left(\frac{a+\eta}{\mu_1}\right)^s \theta^{|z|}, \qquad z,\ z' \in \mathbb{Z}^\nu. \end{aligned}\right\} \tag{3.21}$$

Relations (3.21) imply

$$\sup_{z \neq \bar{0}} \sum_{u \in \mathbb{Z}^\nu} |(\mathcal{T}_{11})^{-s}_{z, u}| < (1 + c\,\varepsilon^{\frac{1}{2}})\left(\frac{a+\eta}{|\mu_1|}\right)^s, \tag{3.22}$$

where c is a constant independent of s. From (3.17), (3.12d), (3.22) we find

$$\|\mathcal{B}_{s, q}\|_M \leqq C_{21}\,\varepsilon\,(1 + c\,\varepsilon^{\frac{1}{2}})\,(\mu_* + \varepsilon\, C_{22})^s \left(\frac{a+\eta}{|\mu_1|}\right)^{q+1}, \tag{3.23a}$$

whence

$$\| \mathcal{B}_{\alpha_1} \mathcal{T}_{12} \mathcal{B}_{\alpha_2} \mathcal{T}_{12} \ldots \mathcal{T}_{12} \mathcal{B}_{\alpha_r} \|_M \leqq (\bar{C}\,\varepsilon)^{2r-1} (\mu_* + \varepsilon\, C_{22})^{S_r} \left(\frac{a+\eta}{\mu_1}\right)^{Q_r + r} \tag{3.23b}$$

where $S_r = \sum_1^r s_i$ and $Q_r = \sum_1^r q_i$.

From the analysis in paper [3] it follows that $x_{\alpha_i, \ldots, \alpha_r} = 0$ if $S_r \neq Q_r + r + 1$, and that

$$\sum_{\substack{\alpha_1, \ldots, \alpha_r \\ S_r = S}} | x_{\alpha_1, \ldots, \alpha_r} | \leqq w_{S, r}$$

where $w_{S, r}$ are the coefficients of the expansion

$$w(z, \zeta) = \sum_{S, r=0}^{\infty} w_{S, r}\, z^S \zeta^r \tag{3.24a}$$

of the solution $w(z, \zeta)$ of the equation

$$w = \zeta + zw + w^2, \qquad w(0, 0) = 1. \tag{3.24b}$$

By looking at the singularities in the complex z-plane of the solution of the quadratic equation (3.24b) it is not hard to see that the radius of convergence of the power series in z of $w(\zeta, z)$ tends to 1 as $\zeta \to 0$. Since in our case $|\zeta| \leqq (\bar{C}\,\varepsilon)^2$, and the number $\mu_* (1 + \varepsilon\, C_{22}) \left(\frac{a+\eta}{|\mu_1|}\right)$ is, by Condition VIII*, less than 1 for ε and d small enough, we conclude that there is a positive number ε_0 such that for, $\varepsilon < \varepsilon_0$, $\mu_* (1 + \varepsilon\, C_{22}) \left(\frac{a+\eta}{|\mu_1|}\right)$ will be smaller than the radius of convergence (in z) of the series (3.24a), which implies absolute convergence in $\mathcal{A}_M$ for the series (3.16a). Lemma 3.6 is proved. ■

Remark 3.3. – Note that since $(\mathcal{T}_{11})^{-1} \Psi_{\bar{0}} = \Psi_{\bar{0}}$ and $\mathcal{T}_{21} \Psi_{\bar{0}} = 0$, it follows that $S\, \Psi_{\bar{0}} = 0$, which implies $u_{\bar{0}} = \varphi_{\bar{0}} = \Psi_{\bar{0}}$.

Remark 3.4. – Inequality (3.23b) implies $\| \mathcal{S} \|_M <$ const ε.

For what follows we need a more accurate analysis of the matrix elements of the operator $\mathcal{T}_{12}\, \mathcal{S}$. What we need is provided by the following Lemma.

LEMMA 3.7. – *The following relations hold*

$$\left.\begin{array}{ll} (\mathcal{T}_{12}\, \mathcal{S})_{z, z'} = R(z - z') + V_{z, z'}, & z, z' \in \mathbb{Z}^\nu; \\ (\mathcal{T}_{12}\, \mathcal{S})_{\bar{0}, z} = 0, \quad (\mathcal{T}_{12}\, \mathcal{S})_{z, \bar{0}} = V_{z, \bar{0}}, & z \in \mathbb{Z}^\nu \cup \{\bar{0}\}, \end{array}\right\} \tag{3.25a}$$

where the functions R and V satisfy, for some $\theta \in (0, 1)$, *the estimates*

$$\begin{aligned} |R(u)| &< const\, \varepsilon^2\, \theta^{|u|}, \qquad u \in \mathbb{Z}^\nu \\ |V_{z,z'}| &< const\, \varepsilon^2\, \theta^{|z|+|z'|}, \qquad z, z' \in \mathbb{Z}^\nu, \\ |V_{z,\bar{0}}| &< const\, \varepsilon^2\, \theta^{|z|}. \end{aligned} \tag{3.25b}$$

Proof. – The proof is rather lengthy and requires additional constructions. It is deferred to the Appendix. ■

By Lemmas 3.5 and 3.7 we have proved that $\tilde{\mathcal{H}}_M^{(1)}$ is invariant with respect to $\mathcal{T}$, and that one can find in it a basis $\{u_z : z \in \mathbb{Z}^\nu \cup \bar{0}\}$ on which $\mathcal{T}$ acts as in (3.15a), and the coefficients $\{A_{z,z'} : z, z' \in \mathbb{Z}^\nu \cup \bar{0}\}$ are, by (3.8a, b), (3.15b) and (3.25a),

$$\left.\begin{aligned} A_{\bar{0},\bar{0}} = 1,\ A_{\bar{0},z} = 0,\ A_{z,\bar{0}} = \varepsilon\,\mu_1\, c_1(-z) + V_{z,\bar{0}},\ z \in \mathbb{Z}^\nu, \\ A_{z,z'} = \mu_1\, P_0(z-z') + \varepsilon\,\hat{\mathcal{R}}_{z,z'} + R(z-z') + V_{z,z'}, \\ z, z' \in \mathbb{Z}^\nu. \end{aligned}\right\} \tag{3.26}$$

To accomplish the proof of assertion (ii) of Theorem 3.1 we introduce in $\tilde{\mathcal{H}}_M^{(1)}$ a new basis

$$h_{\bar{0}} = u_{\bar{0}}, \qquad h_z = u_z + H_z\, u_{\bar{0}}, \tag{3.27a}$$

and we show that the sequence $H = \{H_z : z \in \mathbb{Z}^\nu\}$ can be determined in such a way that

$$\mathcal{T}\, h_z = \sum_{z' \in \mathbb{Z}^\nu} A_{z,z'}\, h_{z'}, \qquad \mathcal{T}\, h_{\bar{0}} = h_{\bar{0}}. \tag{3.27b}$$

In fact, by eqs. (3.15a) and (3.27a) we have

$$\mathcal{T}\, h_z = \sum_{z' \in \mathbb{Z}^\nu} A_{z,z'}\, h_{z'} - \Big(\sum_{z' \in \mathbb{Z}^\nu} A_{z,z'}\, H_{z'} - A_{z,\bar{0}} - H_z\Big)\, u_{\bar{0}},$$

and we can define H to be the solution of the system of equations

$$H_z - \sum_{z' \in \mathbb{Z}^\nu} A_{z,z'}\, H_{z'} = A_{z,\bar{0}}. \tag{3.28a}$$

Let B_q, $q \in (0, 1)$ be the space of the sequences H such that the norm

$$\|H\|_q = \sup_{z \in \mathbb{Z}^\nu} |H_z|\, q^{-|z|}$$

is finite. From formula (3.15b), Lemmas 3.3 and 3.7 it follows that for $q > \theta$ the matrix $\{A_{z,z'} : z, z' \in \mathbb{Z}^\nu\}$ defines a bounded operator $\mathcal{A}$ in B_q, with norm $\|\mathcal{A}\|_q < 1$, and, moreover $\{A_{z,\bar{0}} : z \in \mathbb{Z}^\nu\} \in B_q$.

Hence there is a unique solution in B_q of the system (3.28a). Moreover, by formula (3.26) it is clear that $\| A_{\cdot,\bar{0}} \|_q \leqq \text{const}\, \varepsilon$, whence it follows that the same inequality

$$\| H \|_q \leqq \text{const}\, \varepsilon \tag{3.28b}$$

holds for the solution of eq. (3.28a).

By (3.27b) the space $\tilde{\mathcal{H}}_M^{(1)}$ splits into two invariant subspaces: the subspace of the constants $\mathcal{H}_0 = \{ c\, h_{\bar{0}} \} = \{ c\, \Psi_{\bar{0}} \}$, and the subspace $\tilde{\mathcal{H}}_M^{(1)}$ which is spanned by the basis $\{ h_z \,:\, z \in \mathbb{Z}^\nu \}$. Setting now $\hat{\delta}(y) = R(y)$, and $S(x, y) = \varepsilon \hat{\mathcal{R}}(x, y) + V_{x,y}$, using the estimates (3.25b) we get the relations (3.4a, b, c) of Theorem 3.1.

We now prove inequality (3.5a) for the restriction of $\mathcal{T}$ to $\tilde{\mathcal{H}}_M^{(1)}$. Using the explicit expression (3.27a) for the functions of the basis $\{ h_z \}$, Remark 3.4 and inequality (3.28b) we see that

$$\| h_z \|_M \leqq \| H \|_q + M + \| \mathcal{S} \|_M \leqq M\,(1 + \text{const}\, \varepsilon), \tag{3.29}$$

where the constant is independent of ε and z. For any vector $v = \sum_z v_z h_z \in \mathcal{H}_M^{(1)}$, by susbstituting the explicit expression (3.27a) of the functions h_z, we find

$$\| v \|_M = | \sum_z v_z\, H_z | + M \sum_z | v_z | + \sum_{\Gamma \in \mathcal{G}_2} M^{|\operatorname{supp} \Gamma|} \, | \sum_z \mathcal{S}_{z,\Gamma}\, v_z |,$$

whence, reasoning as above, one deduces, for some $c_1 > 0$,

$$M\,(1 - c_1\, \varepsilon) \sum_z | v_z | \leqq \| v \|_M \leqq M\,(1 + c_1\, \varepsilon) \sum_z | v_z |. \tag{3.30}$$

Using (3.29), (3.30) and the inequality

$$\sup_{z \in \mathbb{Z}^\nu} \sum_{z' \in \mathbb{Z}^\nu} | A_{z,z'} | \leqq | \mu_1 | \,(1 + c_2\, \varepsilon), \tag{3.31}$$

which follows easily from equation (3.26), and Lemmas 3.3 and 3.7, we find, for ε small enough

$$\sup_{v \in \mathcal{H}_M^{(1)}} \frac{\| \mathcal{T}\, v \|_M}{\| v \|_M} \leqq \frac{\sum_{z'} | \sum_z v_z\, A_{z,z'} | \, \| h_{z'} \|_M}{M\,(1 - c_1\, \varepsilon) \sum_z | v_z |}$$

$$\leqq | \mu_1 | \frac{1 + c_2\, \varepsilon}{1 - c_1\, \varepsilon} < | \mu_1 | \,(1 + \text{const}\, \varepsilon).$$

Inequality (3.5a) is proved.

We are left with the construction of the space $\mathcal{H}_M^{(2)}$. We recall that in [2] we introduced a basis $\{\Psi_\Gamma^* : \Gamma \in \mathfrak{M}\}$ in the space $\mathcal{H}$

$$\Psi_\Gamma^*(\eta) = \prod_x e_{\gamma(x)}^*(\eta(x)), \tag{3.32}$$

where $\{e_j^* : j = 0, \ldots, |S| - 1\}$ is a basis in the space $l_2(S, \pi_0)$, given by the normalized eigenvectors of the operator Q_0^*, and bi-orthogonal to the basis $\{e_j : j = 0, \ldots, |S| - 1\}$. The basis $\{\Psi_\Gamma^*, \Gamma \in \mathfrak{M}\}$ is bi-orthogonal to the basis $\{\Psi_\Gamma : \Gamma \in \mathfrak{M}\}$ in $\mathcal{H}$. We observe furthermore that the matrix $\{\mathcal{T}_{\Gamma,\Gamma'}^*\}$ of the operator $\mathcal{T}^*$, adjoint of $\mathcal{T}$, in the basis $\{\Psi_\Gamma^*, \Gamma \in \mathfrak{M}\}$ is the adjoint of the matrix $\{\mathcal{T}_{\Gamma,\Gamma'}\}$ of the operator $\mathcal{T}$ in the basis $\{\Psi_\Gamma, \Gamma \in \mathfrak{M}\}$, *i.e.*,

$$\mathcal{T}_{\Gamma,\Gamma'}^* = \overline{\mathcal{T}_{\Gamma',\Gamma}}.$$

By applying to the operator $\mathcal{T}^*$ the same considerations as above we find a subspace $\tilde{\mathcal{H}}_M^{*(1)} \subset \mathcal{H}_M$, invariant with respect to $\mathcal{T}^*$, which, in analogy with (3.9a), is given by

$$\tilde{\mathcal{H}}_M^{*(1)} = \{u + \mathcal{S}^* u : u \in \tilde{\mathcal{H}}_{1,M}^{0*}\}.$$

Here $\tilde{\mathcal{H}}_{1,M}^{0*}$ is the space spanned by the vectors $\varphi_{\bar{0}}^*$ and $\{\varphi_z^* : z \in \mathbb{Z}^\nu\}$. $\varphi_z^* = \Psi_{\Gamma_1^{(z)}}^*$, is defined in analogy to φ_z, and the operator $\mathcal{S}^* : \tilde{\mathcal{H}}_{1,M}^{0*} \to \mathcal{H}_{2,M}^{0*}$ ($\mathcal{H}_{2,M}^{0*} \subset \mathcal{H}_M$ is the space spanned by the vectors $\{\Psi_\Gamma^* : \Gamma \in \mathcal{G}_2\}$) is defined in analogy to $\mathcal{S}$. In the space $\tilde{\mathcal{H}}_{1,M}^{0*}$ we can choose a basis $\{u_z^* : z \in \mathbb{Z}^\nu \cup \bar{0}\}$ of the form $u_z^* = \varphi_z^* + \mathcal{S}^* \varphi_z^*$. It is not hard to see that

$$(u_z, u_{z'}^*) = \delta_{z,z'} + \mathcal{C}_{z,z'}, \qquad \mathcal{C}_{z,z'} = \sum_{\Gamma \in \mathcal{G}_2} \mathcal{S}_{z,\Gamma} \overline{\mathcal{S}_{z',\Gamma}^*}. \tag{3.33}$$

$\mathcal{T}^*$ acts on the basis $\{u_z^*\}$ as follows

$$\mathcal{T}^* u_z^* = \sum_{z' \in \mathbb{Z}^\nu \cup \bar{0}} A_{z,z'}^* u_{z'}^*, \qquad z \in \mathbb{Z}^\nu \cup \bar{0},$$

and, in analogy with eq.s (3.26), $A_{\bar{0},\bar{0}}^* = 1$ and $A_{z,\bar{0}}^* = 0$ for all $z \in \mathbb{Z}^\nu$. As above we can go over to a new basis in $\tilde{\mathcal{H}}_M^{(1)*}$

$$h_z^* = u_z^*, \quad z \in \mathbb{Z}^\nu, \qquad h_{\bar{0}}^* = u_{\bar{0}}^* + \sum_{z \in \mathbb{Z}^\nu} H_z^* u_z^* \tag{3.34}$$

and the sequence $\{H_z^*, z \in \mathbb{Z}^\nu\}$ is the solution of the system

$$H_z^* - \sum_{z' \in \mathbb{Z}^\nu} A_{z',z}^* H_{z'}^* = A_{\bar{0},z}^*, \quad z \in \mathbb{Z}^\nu,$$

which, reasoning as for eq. (3.28a), is in the space B_q. We find, as above,

$$\mathcal{T}^* h_z^* = \sum_{z' \in \mathbb{Z}^\nu} A^*_{z,z'} h^*_{z'}, \quad z \in \mathbb{Z}^\nu, \qquad \mathcal{T}^* h^*_{\bar{0}} = h^*_{\bar{0}},$$

i.e., the space $\tilde{\mathcal{H}}_M^{(1)*}$ splits into two invariant (with respect to $\mathcal{T}^*$) subspaces:

$$\tilde{\mathcal{H}}_M^{(1)*} = \mathcal{H}_0^* + \mathcal{H}_M^{(1)*}, \tag{3.35}$$

$\mathcal{H}_0^*$ and $\mathcal{H}_M^{(1)*}$ being spanned by the vectors $h^*_{\bar{0}}$, and $\{ h_z^* : z \in \mathbb{Z}^\nu \}$.

We denote by $\tilde{\mathcal{H}}_1^*$ and $\mathcal{H}_1^* \subset \tilde{\mathcal{H}}_1^*$ the subspaces of $\mathcal{H}$ obtained by taking the closure of the spaces $\tilde{\mathcal{H}}_M^{(1)*}$ and $\mathcal{H}_M^{(1)*}$ in the norm of $\mathcal{H}$. They are invariant with respect to $\mathcal{T}^*$ and for them a decomposition analogous to (3.35) holds.

We now come to the proof of the reality of the function $\bar{p}(y)$. It follows from the reality of the operator $\mathcal{T}$ (*i.e.*, from the fact that $\overline{\mathcal{T} f} = \mathcal{T} \bar{f}$), and the reality of the functions $\varphi_z(\xi) = e_1(\xi(z)) : z \in \mathbb{Z}^\nu$, and φ_z^*, which, in its turn, follows from Condition VIII*. This implies the reality of the operators $\mathcal{T}_{11}$, $\mathcal{T}_{12}$, $\mathcal{T}_{21}$, and $\mathcal{T}_{22}$.

In fact, consider for instance a real function $f \in \mathcal{H}^0_{2,M}$. As $\mathcal{T}$ is real, $\mathcal{T} f = \mathcal{T}_{12} f + \mathcal{T}_{22} f$ is real, and so is $\mathcal{T}_{12} f = \sum_{z \in \mathbb{Z}^\nu \cup \bar{0}} (\mathcal{T} f, \varphi_z^*) \varphi_z$. Hence $\mathcal{T}_{12}$ and $\mathcal{T}_{22}$ are real operators. Analogous arguments show that $\mathcal{T}_{11}$ and $\mathcal{T}_{12}$ are also real. This implies, by equation (3.16a), the reality of $\mathcal{S}$, and hence the reality of the basis $\{ h_z : z \in \mathbb{Z}^\nu \}$ and of the matrix elements $A_{z,z'}$. Since, as it follows from estimate (3.4c) and formulas (3.26), $\bar{p}(y) = \lim_{z \to \infty} A_{z,z-y}$, $\bar{p}(y)$ is a real function.

If we now assume that the symmetry Condition IX* holds, it is not hard to see that $A_{z,z'} = A_{z',z}$. In fact, by the discussion of Condition IX in Part I, $P_0(y)$ and $c_j(y)$, $j = 1, \ldots |S| - 1$ are symmetric in $y \in \mathbb{Z}^\nu$. It follows that the operators Δ^j defined by formula (A.4) of the Appendix, are also symmetric in y_j. This gives the required property of the matrix $A_{z,z'}$, which, with the help of (3.26) leads to the symmetry of the function $\bar{p}(y)$. Assertions (ii) and (iv) of Theorem 3.1 are proved.

We now pass to the rest of the proof. We have

$$\langle h^*_{\bar{0}} \rangle_{\Pi_0} = (h^*_{\bar{0}}, h_{\bar{0}}) = 1.$$

As a consequence of the bi-orthogonality of the bases $\{ \Psi_\Gamma : \Gamma \in \mathfrak{M} \}$ and $\{ \Psi_\Gamma^* : \Gamma \in \mathfrak{M} \}$, $h_{\bar{0}}$ is orthogonal to the subspace $\mathcal{H}_1^*$. Moreover $h^*_{\bar{0}}$ is orthogonal to the subspace $\mathcal{H}_1$. In fact, by the invariance of $h^*_{\bar{0}}$ with respect to $\mathcal{T}^*$ we have $(h^*_{\bar{0}}, (\mathcal{E} - \mathcal{T}) v) = 0$ for any $v \in \mathcal{H}_M^{(1)}$, where $\mathcal{E}$ denotes the identity operator. But, by inequality (3.5a) $\mathcal{E} - \mathcal{T}$ is invertible in $\mathcal{H}_M^{(1)}$, which implies that $h_{\bar{0}}^* \perp v$ for any $v \in \mathcal{H}_M^{(1)}$. Since $\mathcal{H}_M^{(1)}$ is dense in $\mathcal{H}_1$, the assertion follows.

Next we construct the bi-orthogonal basis to $\{ h_z : z \in \mathbb{Z}^\nu \}$. By relations (3.14), (3.26), (3.31) and (3.32) we have

$$(h_z, h_{z'}^*) = (u_z, u_{z'}^*) = \delta_{z, z'} + \mathcal{C}_{z, z'}, \qquad z, z' \in \mathbb{Z}^\nu.$$

We introduce a new basis $\{ \hat{h}_z^*, z \in \mathbb{Z}^\nu \}$ in the space $\mathcal{H}_1^*$ of the form

$$\hat{h}_z^* = h_z^* + \sum_{z'} \mathcal{F}_{z, z'} h_{z'}^*. \tag{3.36}$$

The matrix $\mathcal{F} = \{ \mathcal{F}_{z, z'} \}$ is chosen in such a way that

$$(\mathcal{E} + \mathcal{C}) (\mathcal{E} + \mathcal{F}^*) = \mathcal{E}$$

where $\mathcal{F}^*$ is the adjoint matrix of $\mathcal{F}$.

In Lemma 3.8 below we prove that the operator $\mathcal{C}$ has a small norm in B_q. Hence $\mathcal{F}$, which can be written as

$$\mathcal{F} = -\mathcal{C}^* (\mathcal{C}^* + \mathcal{E})^{-1}$$

exists as an operator on B_q, and has also small norm.

An easy check shows that the basis $\{ \hat{h}_z^* : z \in \mathbb{Z}^\nu \}$ is bi-orthogonal to the basis $\{ h_z : z \in \mathbb{Z}^\nu \}$ in $\mathcal{H}_1$.

We now introduce the space

$$\mathcal{H}_2 = (\tilde{\mathcal{H}}_1^*)^\perp \subset \mathcal{H}$$

Clearly this space is invariant with respect to $\mathcal{T}$, and for any vector $f \in \mathcal{H}$ we have a unique decomposition

$$f = f^{(0)} + f^{(1)} + f^{(2)}, \qquad f^{(i)} \in \mathcal{H}_i, \quad i = 0, 1, 2. \tag{3.37a}$$

This last assertion can be proved as follows. We set

$$\begin{gathered} f^{(0)} = (f, h_{\bar{0}}^*) \in \mathcal{H}_0, \qquad f^{(1)} = \sum_y h_y (f, \hat{h}_y^*) \in \mathcal{H}_1 \\ f^{(2)} = f - f^{(0)} - f^{(1)}. \end{gathered} \tag{3.37b}$$

It is easy to see that $f^{(2)} \perp h_{\bar{0}}^*$ and $f^{(2)} \perp \mathcal{H}_1^*$, so that $f^{(2)} \in \mathcal{H}_2$. The uniqueness of the decomposition (3.37a) is evident.

This proves the decomposition of $\mathcal{H}$ (3.3a). To accomplish the proof we have to prove the analogous assertion (3.3b) for $\mathcal{H}_M$. We have to show that $f^{(1)}$ and $f^{(2)}$ are in $\mathcal{H}_M^{(1)}$ and $\mathcal{H}_M^{(2)}$, respectively, for any $f \in \mathcal{H}_M$. We first need an estimate for the matrix elements $\mathcal{S}_{z,\Gamma}$ and $\mathcal{S}^*_{z,\Gamma}$.

LEMMA 3.8. – *The following estimates hold*

$$|\mathcal{S}_{z,\Gamma}| < const\,(C_1\,\varepsilon)^{|\operatorname{supp}\Gamma|-1}\, q^{|z-u|}\, q^{d_{\{\operatorname{supp}\Gamma\setminus\{u\}\}\cup\{0\}}}, \qquad z \in \mathbb{Z}^\nu \tag{3.38a}$$

where $q \in (0, 1)$, C_1 *is an absolute constant, and u is the element of the set* supp Γ *which is closest to z. A similar estimate holds for* $\mathcal{S}^*_{z,\Gamma}$, $z \in \mathbb{Z}^\nu$, *and finally*

$$|\mathcal{S}^*_{\bar{0},\Gamma}| < \mathrm{const}\,(C_1\,\varepsilon)^{|\operatorname{supp}\Gamma|}\, q^{d_{\operatorname{supp}\Gamma\cup\{0\}}}. \tag{3.38b}$$

Proof. – It is not hard to deduce the estimate (3.38a) by analyzing the terms of the series (3.16a, b), much in the same way as it is was done in the proof of Lemma 3.7. As for $\mathcal{S}^*$, it can be represented in a series, which is simply obtained by replacing in the series (3.16a, b) the matrix elements of the operator $\mathcal{T}$ with the elements of the operator $\mathcal{T}^*$. The estimates that are needed are obtained as for $\mathcal{S}$. We omit the details. ■

To prove that the projection of any vector $f \in \mathcal{H}_M$ on $\mathcal{H}_2$ and $\mathcal{H}_1$ belongs respectively to $\mathcal{H}_M^{(1)}$ or $\mathcal{H}_M^{(2)}$, and inequality (3.5b) we introduce a basis in $\mathcal{H}_2$.

Consider the functions

$$U_\Gamma = \Psi_\Gamma - \sum_{z \in \mathbb{Z}^\nu \cup \bar{0}} \overline{\mathcal{S}^*_{z,\Gamma}}\, \varphi_z, \qquad \Gamma \in \mathcal{G}_2,$$

which, as it is easy to check, belong to the space $\mathcal{H}_2$. In addition we set $U_{\Gamma_1^{(z)}} = u_z$, $z \in \mathbb{Z}^\nu$ [*see* (3.14)], and $U_{\bar{0}} = u_{\bar{0}} = \Psi_{\bar{0}}$.

LEMMA 3.9. – *The following assertions hold.*

(i) *The functions* $\{U_\Gamma,\ \Gamma \in \mathcal{G}_2\}$ *are in* $\mathcal{H}_{2,M}$.

(ii) *Any vector* $f \in \mathcal{H}_M$ *is written as a series*

$$f = \sum_{\Gamma \in \mathfrak{M}} f'_\Gamma\, U_\Gamma \tag{3.39a}$$

which converges in the norm $\|\,.\,\|_M$, *and, for* ε *small enough, we have*

$$(1 - C\,\varepsilon) \sum_{\Gamma \in \mathfrak{M}} |f'_\Gamma|\, M^{|\operatorname{supp}\Gamma|} < \|f\|_M < (1 + C\,\varepsilon) \sum_{\Gamma \in \mathfrak{M}} |f'_\Gamma|\, M^{|\operatorname{supp}\Gamma|}. \tag{3.39b}$$

(iii) *The matrix elements of the operator $\mathcal{T}\,|_{\mathcal{H}_2}$ in the basis $\{U_\Gamma : \Gamma \in \mathcal{G}_2\}$ are*

$$\mathcal{T}_{\Gamma,\Gamma'} = (\mathcal{T}_{22})_{\Gamma,\Gamma'} - \sum_{z \in \mathbb{Z}^\nu \cup \bar{0}} (\mathcal{T}_{21})_{\Gamma',z}\, \overline{\mathcal{S}^*_{z,\Gamma}}. \tag{3.40}$$

Proof. – We set

$$U'_\Gamma = U_\Gamma + \overline{\mathcal{S}^*_{\bar{0},\Gamma}}\,\varphi_{\bar{0}} = \Psi_\Gamma - \sum_{z \neq \bar{0}} \overline{\mathcal{S}^*_{z,\Gamma}}\,\varphi_z, \qquad \Gamma \in \mathfrak{M} \setminus \{\bar{0}\}.$$

The transition matrix from the system $\{\Psi_\Gamma : \Gamma \in \mathfrak{M} \setminus \{\bar{0}\}\}$ to the system $\{U'_\Gamma : \Gamma \in \mathfrak{M} \setminus \{\bar{0}\}\}$ is

$$\mathcal{B} = \begin{pmatrix} \mathcal{E}_1 & \mathcal{S}_1 \\ \mathcal{S}_2 & \mathcal{E}_2 \end{pmatrix},$$

where $(\mathcal{E}_1)_{z,z'} = \delta_{z,z'}$, $(\mathcal{E}_2)_{\Gamma,\Gamma'} = \delta_{\Gamma,\Gamma'}$, $\mathcal{S}_1$ has matrix elements $\mathcal{S}_{z,\Gamma}$, and $\mathcal{S}_2$ has matrix elements $-\overline{\mathcal{S}^*_{z,\Gamma}}$. Here, as above, we write z for $\Gamma_1^{(z)}$. A simple computation shows that the inverse matrix $\mathcal{B}^{-1}$ has the form

$$\mathcal{B}^{-1} = \begin{pmatrix} \mathcal{E}_1 + \mathcal{K}_{11} & \mathcal{K}_{12} \\ \mathcal{K}_{21} & \mathcal{E}_2 + \mathcal{K}_{22} \end{pmatrix}$$

where

$$\begin{aligned} K_{11} &= \mathcal{S}_1\,(\mathcal{E}_2 - \mathcal{S}_2\,\mathcal{S}_1)^{-1}\,\mathcal{S}_2, & \mathcal{K}_{12} &= -(\mathcal{E}_1 - \mathcal{S}_1\,\mathcal{S}_2)^{-1}\,\mathcal{S}_1, \\ \mathcal{K}_{22} &= \mathcal{S}_2\,(\mathcal{E}_1 - \mathcal{S}_1\,\mathcal{S}_2)^{-1}\,\mathcal{S}_1, & \mathcal{K}_{21} &= -(\mathcal{E}_2 - \mathcal{S}_2\,\mathcal{S}_1)^{-1}\,\mathcal{S}_2. \end{aligned}$$

Hence it follows that

$$\left.\begin{aligned} &\Psi_{\bar{0}} = u_{\bar{0}} \\ &\varphi_z = u_z + \sum_{z'} (\mathcal{K}_{11})_{z,z'}\,u_{z'} + \sum_{\Gamma \in \mathcal{G}_2} (\mathcal{K}_{12})_{z,\Gamma}\,U_\Gamma \\ &\qquad + \left(\sum_{\Gamma} (\mathcal{K}_{12})_{z,\Gamma}\,\mathcal{S}^*_{\bar{0},\Gamma}\right) U_{\bar{0}} \\ &\Psi_\Gamma = U_\Gamma + \sum_{\Gamma' \in \mathcal{G}_2} (\mathcal{K}_{22})_{\Gamma,\Gamma'}\,U_{\Gamma'} + \sum_{z} (\mathcal{K}_{21})_{\Gamma,z}\,u_z \\ &\qquad + \left(\overline{\mathcal{S}^*_{\bar{0},\Gamma}} + \sum_{\Gamma' \in \mathcal{G}_2} (\mathcal{K}_{22})_{\Gamma,\Gamma'}\,\overline{\mathcal{S}^*_{\bar{0},\Gamma'}}\right) u_{\bar{0}}. \end{aligned}\right\} \tag{3.41}$$

Substituting this expression in the expansion $f = \sum_\Gamma f_\Gamma\,\Psi_\Gamma \in \mathcal{H}_M$ we get the expansion (3.39a), in which the coefficients f'_Γ are easily computed with the help of formula (3.41). Hence, using the estimates (3.38a) and (3.38b) for the matrix elements $\mathcal{S}_{z,\Gamma}$, $\mathcal{S}^*_{z,\Gamma}$, and the fact that $f \in \mathcal{H}_M$, we easily establish the convergence of the series (3.39a) and inequalities (3.39b).

Assertion (iii) follows from the expression of $\mathcal{T}_{21}$ [eq. (3.10)]. Finally from expression (3.40) for the matrix elements of the operator $\mathcal{T}\,|_{\mathcal{H}_2}$ in the basis $\{ U_\Gamma : \Gamma \in \mathcal{G}_2 \}$, and from the estimate (3.39b) it is easy to obtain the assertion (3.5b) of Theorem 3.1. Theorem 3.1 is proved. ■

4. PROOF OF THEOREMS 2.1 AND 2.2

4.1. Proof of Theorem 1

We first prove convergence of expectations of the type $\langle \mathcal{T}^t \, \Psi_\Gamma \rangle_{\mathcal{P}_\Pi}$. By Theorem 3.1 we can write

$$\Psi_\Gamma = \Psi_\Gamma^{(0)} + \Psi_\Gamma^{(1)} + \Psi_\Gamma^{(2)} \qquad \Psi_\Gamma^{(i)} \in \mathcal{H}_M^{(i)}, \qquad i = 0,\, 1,\, 2 \tag{4.1}$$

where

$$\Psi_\Gamma^{(0)} = (\Psi_\Gamma,\, h_{\bar{0}}^*) = \int_\Omega \Psi_\Gamma\,(\eta)\, h_{\bar{0}}^*\,(\eta)\, \Pi_0\,(d\eta)$$

$$\Psi_\Gamma^{(1)} = \sum_{z \in \mathbb{Z}^\nu} c_{\Gamma,\,z}\, h_z, \qquad c_{\Gamma,\,z} = (\Psi_\Gamma,\, \hat{h}_z^*),$$

$\hat{h}_z^*$ being defined in (3.36). Denoting by $b_\Gamma^{(y)}$ the coefficients of the expansion

$$\hat{h}_y^* = \sum b_\Gamma^{(y)}\, \Psi_\Gamma^* \tag{4.2a}$$

it is easy to see that

$$c_{\Gamma,\,z} = \overline{b_\Gamma^{(z)}}. \tag{4.2b}$$

By inequalities (3.5a, b) we have

$$|\langle \mathcal{T}^t\, \Psi_\Gamma^{(1)} \rangle_{\mathcal{P}_\Pi}| \leqq \|\mathcal{T}^t\, \Psi_\Gamma^{(1)}\|_M \leqq (\,|\mu_1| + \mathrm{const}\,\varepsilon)^t\, \|\Psi_\Gamma^{(1)}\|_M. \tag{4.3a}$$

$$|\langle \mathcal{T}^t\, \Psi_\Gamma^{(2)} \rangle_{\mathcal{P}_\Pi}| \leqq \|\mathcal{T}^t\, \Psi_\Gamma^{(2)}\|_M \leqq (\,|\mu_*| + \mathrm{const}\,\varepsilon)^t\, \|\Psi_\Gamma^{(2)}\|_M. \tag{4.3b}$$

Since $|\mu_*| < |\mu_1| < 1$, for ε small enough the estimates (4.3a, b) imply exponential convergence of the expectations $\langle \Psi_\Gamma \rangle_{\mathcal{P}_\Pi}$ to the corresponding expectations $\langle \Psi_\Gamma \rangle_\mu$, where μ is the measure on Ω which is absolutely continuous with respect to Π_0, with density $h_{\bar{0}}^*\,(\eta)$. Clearly the same holds for all cylinder functions depending only on the field in a bounded region.

This implies that the probabilities of the cylinder sets converge, and this is enough (*see* for example [4]) to ensure weak convergence of the measures $\Pi^{(t)}$ to μ, and the first assertion of Theorem 2.1 is proved.

As for the second assertion, note that, by (3.34), we have

$$\int_\Omega \Psi_\Gamma(\eta)\, d\mu = \int_\Omega \Psi_\Gamma(\eta)\, u_{\bar 0}^*(\eta)\, d\Pi_0(\eta) + \sum_z H_z^* \int_\Omega (\varphi_z^*(\eta) + \sum_{\Gamma' \in \mathcal{G}_2} \mathcal{S}_{z,\Gamma'}^* \Psi_{\Gamma'}^*)\, \Psi_\Gamma(\eta)\, d\Pi_0. \tag{4.4}$$

If $\Gamma = \Gamma_1^{(\bar z)}$ for some $\bar z \in \mathbb{Z}^\nu$, the right-hand side of eq. (4.4) is equal to $H_{\bar z}^*$. If on the other hand $\Gamma \in \mathcal{G}_2$ it is equal to

$$\mathcal{S}_{\bar 0,\Gamma}^* + \sum_{u \in \mathbb{Z}^\nu} \mathcal{S}_{u,\Gamma}^* H_u^*.$$

The second assertion of the theorem then follows with the help of Lemma 3.8, and the fact that $\{H_z^*\} \in B_q$.

Theorem 2.1 is proved. ■

4.2. Proof of Theorem 2.2

From now on we assume that all assumption I-V, VI*, VII, VIII*, IX* and X* hold. Moreover we will assume that μ_1 is positive. The changes that are needed for negative μ_1 are obvious.

By expanding the functions $f_{1,2}$ in the basis $\{\Psi_\Gamma : \Gamma \in \mathfrak{M}\}$, the asymptotic behavior of the correlation (2.6) is reduced to that of the quantities

$$\langle \mathcal{T}^t\, \Psi_{\Gamma_{j_1}^{(x_1)}},\, \Psi_{\Gamma_{j_2}^{(x_2)}} \rangle_{\mathcal{P}_\Pi} \qquad j_1, j_2 \in \{1, \ldots, |S| - 1\}.$$

For brevity we shall sometimes replace the index $\Gamma_{j_i}^{(x_i)}$ simply by $i = 1, 2$, and write $\mathcal{T}_j$ for $\mathcal{T}|_{\mathcal{H}_j}$, $j = 1, 2$. By (4.1) we have

$$\Psi_{\Gamma_{j_1}^{(x_1)}} = \Psi_1^{(0)} + \Psi_1^{(1)} + \Psi_1^{(2)}, \qquad \Psi_1^{(i)} \in \mathcal{H}_M^{(i)}, \qquad i = 0, 1, 2$$

and clearly

$$\langle \mathcal{T}^t\, \Psi_{\Gamma_{j_1}^{(x_1)}},\, \Psi_{\Gamma_{j_2}^{(x_2)}} \rangle_{\mathcal{P}_\Pi} = \langle \mathcal{T}_1^t\, \Psi_1^{(1)},\, \Psi_{\Gamma_{j_2}^{(x_2)}} \rangle_{\mathcal{P}_\Pi} + \langle \mathcal{T}_2^t\, \Psi_1^{(2)},\, \Psi_{\Gamma_{j_2}^{(x_2)}} \rangle_{\mathcal{P}_\Pi}. \tag{4.5a}$$

The second term on the right of eq. (4.5a) falls off, by ineq. (3.5b) as $(\mu_* + \mathrm{const}\,\varepsilon)^t$, which, as we shall see, is negligible with respect to the contribution of the first term in the asymptotic expansion. By (4.2a, b) we have

$$\Psi_1^{(1)} = \sum_y C_y^{(1)} h_y,$$

where $C_y^{(1)} = (\Psi_1^{(1)}, \hat{h}_y^*) = \overline{b^{(y)}_{\Gamma^{(x_1)}_{j_1}}}$. Whence we get that

$$\langle \mathcal{T}^t \Psi_1^{(1)}, \Psi_{\Gamma^{(x_2)}_{j_2}} \rangle_{\mathcal{P}_\Pi} = \sum_{y,\, z \in \mathbb{Z}^\nu} C_y^{(1)} A_{y,\, z}^t \langle h_z, \Psi_{\Gamma^{(x_2)}_{j_2}} \rangle_\Pi. \tag{4.5b}$$

Here $\{A_{y,\, z}^t\}$ denotes the matrix elements of the operator $(\mathcal{T}_1)^t$ in the basis $\{h_z : z \in \mathbb{Z}^\nu\}$.

LEMMA 4.1. – *The sequences $C^{(1)} \equiv \{C_y^{(1)} : y \in \mathbb{Z}^\nu\}$ and $\bar{D} \equiv \{\langle h_z, \Psi_{\Gamma^{(x_2)}_{j_2}} \rangle_\Pi : z \in \mathbb{Z}^\nu\}$ are in the space B_q.*

Proof. – The assertion regarding $C^{(1)}$ follows from Lemmas 3.7 and 3.8 and from the expression (3.36) for the vectors $\hat{h}_z^*$. Going over to the function $\bar{D}$, we can write it as

$$\bar{D}_z = \langle h_z, \Psi_{\Gamma^{(x_2)}_{j_2}} \rangle_\Pi = \langle \varphi_z, \Psi_{\Gamma^{(x_2)}_{j_2}} \rangle_\Pi + \sum \mathcal{S}_{z,\, \Gamma} \langle \Psi_\Gamma, \Psi_{\Gamma^{(x_2)}_{j_2}} \rangle_\Pi.$$

By using condition X^* and Lemma 3.8, we get the assertion that we need on $\bar{D}$. ■

We introduce the Fourier transforms

$$\phi(\lambda) = \sum_{y \in \mathbb{Z}^\nu} C_y^{(1)} e^{i(\lambda,\, y)}, \qquad \psi(\mu) = \sum_z \bar{D}_z e^{i(\mu,\, z)}.$$

Lemma 4.1 implies that $\phi(\lambda)$ and $\psi(\mu)$ are analytic in the complex neighborhood W_d of the torus T^ν for d small enough. Moreover for $t = 1$ we have

$$A(\lambda,\, \mu) = \sum_{y,\, z} A_{y,\, z} e^{i(\lambda,\, y) - i(\mu,\, z)} = \tilde{p}(\lambda)\, \delta(\lambda - \mu) + \tilde{S}(\lambda,\, \mu), \tag{4.6}$$

where $\tilde{p}(\lambda)$ and $\tilde{S}(\lambda,\, \mu)$ are the Fourier transforms of the functions $\bar{p}(y)$ and of $S(x,\, y)$ respectively, appearing in (3.4a). By the estimates (3.4c) they are both analytic for $\lambda,\, \mu \in W_d$. As $\mathcal{T}$ is a bounded operator in $\mathcal{H}$ we have for any integer t

$$\mathcal{T}^t = \frac{1}{2\pi i} \int_\gamma z^t \mathcal{R}(z)\, dz,$$

where $\mathcal{R}(z) = (\mathcal{T} - z\,\mathcal{E})^{-1}$ is the resolvent of $\mathcal{T}$, and the integral is over any closed contour γ in the complex plane, which contains the spectrum of $\mathcal{T}$. The kernel $\mathcal{R}_z(\lambda, \mu)$ of the resolvent $\mathcal{R}(z)$ for operators of the type (4.6) has the form

$$\mathcal{R}_z(\lambda, \mu) = \frac{\delta(\lambda - \mu)}{\tilde{p}(\lambda) - z} + \frac{\mathcal{D}(\lambda, \mu; z)}{\Delta(z)(\tilde{p}(\lambda) - z)(\tilde{p}(\mu) - z)},$$

where the function $\Delta(z)$ is a Fredholm determinant

$$\Delta(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(T^\nu)^n} \frac{K_n(\kappa_1, \ldots, \kappa_n)}{\prod_{i=1}^{n} (\tilde{p}(\kappa_i) - z)} d\kappa_1 \ldots d\kappa_n. \tag{4.7a}$$

Here

$$K_n(\kappa_1, \ldots, \kappa_n) = \det\{\tilde{S}(\kappa_i, \kappa_j)\}_{i,j=1,\ldots,n},$$

and the function $\mathcal{D}(\lambda, \mu; z)$ (Fredholm minor) is equal to

$$\begin{aligned}\mathcal{D}(\lambda, \mu; z) &= \tilde{S}(\lambda, \mu)\\ &\quad + \sum_{n=1}^{\infty} \frac{1}{n!} \int \frac{\tilde{K}_n(\lambda, \mu; \kappa_1, \ldots, \kappa_n)}{\prod_{i=1}^{n} (\tilde{p}(\kappa_i) - z)} d\kappa_1 \ldots d\kappa_n,\end{aligned} \tag{4.7b}$$

with

$$\tilde{K}_n(\lambda, \mu; \kappa_1, \ldots, \kappa_n) = \begin{vmatrix} \tilde{S}(\lambda, \mu) & \tilde{S}(\lambda, \kappa_1) & \cdots & \tilde{S}(\lambda, \kappa_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{S}(\kappa_n, \mu) & \tilde{S}(\kappa_n, \kappa_1) & \cdots & \tilde{S}(\kappa_n, \kappa_n) \end{vmatrix}.$$

Hence the correlation (4.5b), can be written as

$$\begin{aligned}\int (\mathcal{T}^t)(\lambda, \mu)\, \phi(\lambda)\, \overline{\psi(\mu)}\, d\lambda\, d\mu &= \int_{T^\nu} \tilde{p}^t(\lambda)\, \phi(\lambda)\, \overline{\psi(\lambda)}\\ &\quad + \frac{1}{2\pi i} \int_\gamma \frac{z^t}{\Delta(z)} dz \int_{(T^\nu)^2} \frac{\mathcal{D}(\lambda, \mu; z)\, \phi(\lambda)\, \overline{\psi(\mu)}}{(\tilde{p}(\lambda) - z)(\tilde{p}(\mu) - z)} d\lambda\, d\mu.\end{aligned} \tag{4.8}$$

Since $\tilde{p}(\lambda)$ is a real function we can apply to the first integral in (4.8) the Laplace method [7] and we obtain that

$$\int \tilde{p}^t(\lambda)\, \phi(\lambda)\, \overline{\psi(\lambda)}\, d\lambda\, d\mu = \frac{C\,\kappa^t}{t^{\nu/2}}(1 + o(1)) \tag{4.9}$$

where

$$\kappa = \max_{\lambda} \tilde{p}(\lambda) = \tilde{p}(0) \in (0, 1).$$

(The function $\tilde{p}(\lambda)$ for small ε has a unique maximum, as it follows from Condition II, and it is an even function, hence it attains its maximum at O.) Here

$$C = C(j_1, x_1, j_2, x_2, \Pi) = \frac{\kappa^{\frac{1}{2}} \phi(0) \overline{\psi(0)}}{\sqrt{2\pi} \det A},$$

and the elements of the matrix A are

$$a_{i,j} = \frac{\partial^2 \tilde{p}(\lambda)}{\partial \lambda_i \partial \lambda_j} \Big|_{\lambda=0}.$$

Note that for small ε κ is close to μ_1, so that

$$\kappa > (\mu_* + \varepsilon\delta), \tag{4.10}$$

where $\delta = \sum_y |\hat{\delta}(y)|$ (*see* relations 3.4b, c).

We now pass to the asymptotics of the second term in (4.8), and, to simplify the computations, assume from now on that $\nu = 3$. Note that $\Delta(z)$ and the integral

$$F(z) \equiv \int_{(T^3)^2} \frac{\mathcal{D}(\lambda, \mu; z) \phi(\lambda) \overline{\psi(\mu)}}{(\tilde{p}(\lambda) - z)(\tilde{p}(\mu) - z)} d\lambda\, d\mu$$

are both analytic functions of z outside the cut $\mathcal{C} = [\kappa', \kappa] \subset [-1, 1]$. Here $\kappa' = \min_{\lambda} \tilde{p}(\lambda)$, and for small ε, by Condition III, we have

$$\kappa' > -\kappa.$$

Consider the set

$$U_\beta = \{z \in \mathbb{C} : |z - \kappa| < \beta,\ z \notin \mathcal{C}\}, \qquad \beta > 0.$$

i.e., the β-neighborhood of κ outside the cut. We have

LEMMA 4.2. – *For β small enough the following limits exist*

$$\Delta_\pm(s) = \lim_{\varepsilon \to 0} \Delta(s \pm i\varepsilon), \qquad \kappa - \beta \leqq s \leqq \kappa,$$

$$F_\pm(s) = \lim_{\varepsilon \to 0} F(s \pm i\varepsilon), \qquad \kappa - \beta \leqq s \leqq \kappa,$$

Moreover for $z \in U_\beta$ the following representations hold

$$\left\{\begin{array}{l} \Delta(z) = C_0 + C_1 (z - \kappa)^{\frac{1}{2}} + C_2(z) \\ F(z) = B_0 + B_1 (z - \kappa)^{\frac{1}{2}} + B_2(z). \end{array}\right\} \tag{4.11}$$

Here C_0, C_1, B_0, B_1 are constants, and C_2, B_2 are analytic functions in z such that for some $K > 0$

$$\max\{|C_2(z)|, |B_2(z)|\} \leqq K|z-\kappa|. \tag{4.12}$$

$(z-\kappa)^{\frac{1}{2}}$ denotes the branch which takes positive values for z real and $z > \kappa$.

Proof. – Both assertions come from the following Lemma. ∎

LEMMA 4.3. – *Consider the integral*

$$H_n(z) = \int_{(T^3)^n} \frac{Q(\lambda_1, \ldots, \lambda_n)}{\prod_{i=1}^{n} (\tilde{p}(\lambda_i) - z)} \prod_{i=1}^{n} d\lambda_i, \tag{4.13}$$

where Q is analytic in each variable for $\lambda_i \in W_d$. Then for β small enough the following representation holds for $z \in U_\beta$

$$H_n(z) = C_0^{(n)} + C_1^{(n)}(z-\kappa)^{\frac{1}{2}} + C_2^{(n)}(z), \tag{4.14}$$

where

$$C_0^{(n)} = H_n(\kappa) = \int_{(T^3)^n} \frac{Q(\lambda_1, \ldots, \lambda_n)}{\prod_{i=1}^{n} (\tilde{p}(\lambda_i) - z)} \prod d\lambda_i$$

$$C_1^{(n)} = 2\pi^2 \sum_{q=1}^{n} \int_{(T^3)^{n-1}} \frac{Q(\lambda_1, \ldots, \lambda_q = 0, \ldots, \lambda_n)}{\prod_{i \neq q} (\tilde{p}(\lambda_i) - z)} \prod_{i \neq q} d\lambda_i,$$

and the function $C_2^{(n)}$ is analytic in U_β and satisfies there the estimate

$$|C_2^{(n)}| < MB^n \max_{\lambda_i \in W_d} |Q(\lambda_1, \ldots, \lambda_n)| \, |z-\kappa|, \tag{4.15}$$

where M and B are constants independent of n, z and of the function $Q(\lambda_1, \ldots, \lambda_n)$. Moreover the limiting values exist

$$H_\pm^{(n)}(s) = \lim_{\varepsilon \to 0} H_n(z \pm i\varepsilon), \qquad \kappa - \beta < s < \kappa. \tag{4.16}$$

Proof. – We consider first the case $n = 1$. We have

$$\begin{aligned} H_1(z) &= \int_{T^3} \frac{Q(\lambda)}{\tilde{p}(\lambda) - z} d\lambda \\ &= \int_{V_\delta} \frac{Q(\lambda)}{\tilde{p}(\lambda) - z} d\lambda + \int_{T^3 \setminus V_\delta} \frac{Q(\lambda)}{\tilde{p}(\lambda) - z} d\lambda \end{aligned} \tag{4.17}$$

Here V_δ is a neighborhood of the point 0:

$$V_\delta = \{\lambda \in T^3 : \tilde{p}(\lambda) > \kappa - \delta\}$$

δ is chosen so small that in V_δ there are no other critical points for the function $\tilde{p}(\lambda)$ except $\lambda = 0$. (From this it follows in particular that V_δ is connected.)

Clearly the second integral on the right in (4.17) is an analytic function of z in the circle $|z - \kappa| < \beta$ for $\beta < \delta$, and consequently can be represented in the form

$$\int_{T^3 \setminus V_\delta} \frac{Q(\lambda)}{\tilde{p}(\lambda) - z} \, d\lambda = \text{constant} + \tilde{C}_2(z) \tag{4.18}$$

where

$$|\tilde{C}_2(z)| < \tilde{K} \max_{\lambda \in W_d} |Q(\lambda)| \, \| z - \kappa | \tag{4.19}$$

and the constant $\tilde{K}$ is independent of z and Q.

We pass now to the first integral in (4.17), and perform in V_δ an analytic change of variables

$$\lambda' = \lambda + q(\lambda), \qquad |q(\lambda)| < c|\lambda|^2 \tag{4.20}$$

(where c is an absolute constant) such that in the new variable λ' the function $\tilde{p}(\lambda)$ for $\lambda \in V_\delta$ has the form

$$\tilde{p}(\lambda) = \kappa - b(\lambda')$$

Here $b(\lambda')$ is a positive definite quadratic form of λ' (which coincides with the second differential of $\tilde{p}(\lambda)$ at $\lambda = 0$ (*see* [5]). We introduce a new metric in V_δ

$$|\lambda'|^2 \equiv b(\lambda')$$

In the new variables the first integral in (4.17) is written as

$$\int_{|\lambda'| < \delta^{\frac{1}{2}}} \frac{Q'(\lambda')}{z - \kappa + |\lambda'|^2} \, d\lambda' \tag{4.21}$$

where $Q'(\lambda') = -Q(\lambda)\mathcal{I}(\lambda)$, and $\mathcal{I}(\lambda)$ is the jacobian of the change of variable (4.20). Further we represent $Q'(\lambda')$ in the form

$$Q'(\lambda') = Q'(0) + d_1(\lambda') + d_2(\lambda')$$

where $d_1(\lambda')$ is a linear function in λ' and

$$|d_2(\lambda')| < \text{const } |\lambda'|^2$$

The linear term d_1 gives no contribution, and the integral (4.21) is represented as

$$Q'(0) \int_{|\lambda'| < \delta^{\frac{1}{2}}} \frac{d\lambda'}{z - \kappa + |\lambda'|^2} + \int_{|\lambda'| < \delta^{\frac{1}{2}}} \frac{d_2(\lambda')}{z - \kappa + |\lambda'|^2} \, d\lambda'. \tag{4.22}$$

An easy computation shows that

$$\int_{|\lambda'|<\delta^{\frac{1}{2}}} \frac{d\lambda'}{z-\kappa+|\lambda'|^2} = \text{const} - 2\pi^2 (z-\kappa)^{\frac{1}{2}} + \bar{C}(z) \tag{4.23}$$

where the function $\bar{C}(z)$ is analytic in U_β and verifies the estimates

$$|\bar{C}(z)| < \bar{K}\,|z-\kappa| \tag{4.24}$$

for some constant $\bar{K}$ independent of z. One can show similarly that

$$\int_{|\lambda'|<\delta^{\frac{1}{2}}} \frac{d_2(\lambda')}{z-\kappa+|\lambda'|^2}\, d\lambda' = \text{const} + \bar{c}_2(z)$$

is, for small β an analytic function in U_β and satisfies there the estimate

$$|\bar{c}_2(z)| < K' \max_{\lambda\in W_d} |Q(\lambda)|\; |z-\kappa| \tag{4.25}$$

From formulas (4.17), (4.18), (4.22), (4.23), and estimate (4.19), (4.24), (4.25), taking into account the fact that $\mathcal{I}(0) = 1$ [by relations (4.20)], we get the representation

$$\int_{T^3} \frac{Q(\lambda)}{\tilde{p}(\lambda)-z}\, d\lambda = C_0 + C_1 (z-\kappa)^{\frac{1}{2}} + C_2(z), \qquad z\in U_\beta \tag{4.26}$$

where

$$C_0 = \int_{T^3} \frac{Q(\lambda)}{\tilde{p}(\lambda)-\kappa}\, d\lambda,$$

$$C_1 = 2\pi^2 Q(0),$$

and $C_2(z)$ is analytic in U_β and satisfies the estimate

$$|C_2(z)| < K'' \max_{\lambda\in W_d} |Q(\lambda)|\; |z-\kappa|$$

for some constant K'' independent of z, Q.

From (4.26) it follows that in the region U_β the following estimate holds

$$\left|\int \frac{Q(\lambda)}{\tilde{p}(\lambda)-z}\, d\lambda\right| < \hat{K} \max_{\lambda\in W_d} |Q(\lambda)|$$

where $\hat{K}$ is an absolute constants.

We now consider the general case of the integral (4.13) and we write it in the form

$$H_n(z) = \int_{T^3} \frac{d\lambda_n}{\tilde{p}(\lambda_n)-z} \cdots \int_{T^3} \frac{d\lambda_2}{\tilde{p}(\lambda_2)-z} \int \frac{Q(\lambda_1,\ldots,\lambda_n)}{\tilde{p}(\lambda_1)-z}\, d\lambda_1 \tag{4.27}$$

By applying the previous formulas and estimates to the internal integral we get

$$\int_{T^3} \frac{Q(\lambda_1, \ldots, \lambda_n)}{\tilde{p}(\lambda_1) - z} d\lambda_1 = C_0^{(1)}(\lambda_2, \ldots, \lambda_n) + C_1^{(1)}(\lambda_2, \ldots, \lambda_n)(z - \kappa)^{\frac{1}{2}} + C_2^{(1)}(\lambda_2, \ldots, \lambda_n; z)$$

where

$$C_0^{(1)}(\lambda_2, \ldots, \lambda_n) = \int_{T^3} \frac{Q(\lambda_1, \ldots, \lambda_n)}{\tilde{p}(\lambda_1) - \kappa} d\lambda_1$$

$$C_1^{(1)}(\lambda_2, \ldots, \lambda_n) = 2\pi^2 Q(\lambda_1 = 0, \lambda_2, \ldots, \lambda_n)$$

and the function $C_2^{(1)}(\lambda_2, \ldots, \lambda_n; z)$ is analytic in z for $z \in U_\beta$ and satisfies the estimate

$$|C_2^{(1)}(\lambda_2, \ldots, \lambda_n; z)| < K_2 \max_{\lambda_1, \ldots, \lambda_n \in W_d} |Q(\lambda_1, \ldots, \lambda_n)| \, |z - \kappa|.$$

By substituting the expression thus obtained in the integral (4.27) and integrating each term in λ_2 we obtain

$$\int_{T^3} \frac{d\lambda_2}{\tilde{p}(\lambda_2) - z} \int_{T^3} \frac{Q(\lambda_1, \ldots, \lambda_n)}{\tilde{p}(\lambda_1) - z} d\lambda_1 = C_0^{(2)}(\lambda_3, \ldots, \lambda_n) + C_1^{(2)}(\lambda_3, \ldots, \lambda_n)(z - \kappa)^{\frac{1}{2}} + C_0^{(2)}(\lambda_3, \ldots, \lambda_n; z)$$

where

$$C_0^{(2)}(\lambda_3, \ldots, \lambda_n) = \int_{(T^3)^2} \frac{Q(\lambda_1, \ldots, \lambda_n)}{(\tilde{p}(\lambda_1) - \kappa)(\tilde{p}(\lambda_2) - \kappa)} d\lambda_1 \, d\lambda_2$$

$$C_1^{(2)}(\lambda_3, \ldots, \lambda_n) = 2\pi^2 \left[\int_{T^3} \frac{Q(\lambda_1 = 0, \lambda_2, \ldots, \lambda_n)}{\tilde{p}(\lambda_2) - \kappa} d\lambda_2 + \int_{T^3} \frac{Q(\lambda_1, \lambda_2 = 0, \ldots, \lambda_n)}{\tilde{p}(\lambda_1) - \kappa} d\lambda_1 \right].$$

The function $C_2^{(2)}(\lambda_3, \ldots, \lambda_n; z)$ is analytic in U_β and verifies the estimate

$$|C_2^{(2)}(\lambda_3, \ldots, \lambda_n; z)| < K_2 \max_{\lambda_i \in W_d} |Q(\lambda_1, \ldots, \lambda_n)| \, |z - \kappa|$$

where

$$K_2 = K_1 (D + 2\pi^2 \beta + \hat{K}) + (2\pi^2)^2$$

and

$$D = \left| \int_{T^3} \frac{d\lambda}{\tilde{p}(\lambda) - \kappa} \right|$$

After $s < n$ integrations in (4.27) we get the expansion

$$C_0^{(s)} (\lambda_{s+1}, \ldots, \lambda_n) + C_1^{(s)} (\lambda_{s+1}, \ldots, \lambda_n) (z - \kappa)^{\frac{1}{2}} + C_2^{(s)} (\lambda_{s+1}, \ldots, \lambda_n; z), \qquad z \in U_\beta,$$

where

$$C_0^{(s)} (\lambda_{s+1}, \ldots, \lambda_n) = \int_{(T^3)^s} \frac{Q(\lambda_1, \ldots, \lambda_n)}{\prod_{i=1}^{s} (\tilde{p}(\lambda_i) - \kappa)} \prod_{i=1}^{s} d\lambda_i$$

$$C_1^{(s)} (\lambda_{s+1}, \ldots, \lambda_n) = 2\pi^2 \sum_{j=1}^{s} \int_{(T^3)^{s-1}} \frac{Q(\lambda_1, \ldots, \lambda_j = 0, \lambda_n)}{\prod_{\substack{i=1 \\ i \neq j}}^{s} (\tilde{p}(\lambda_i) - \kappa)} \prod_{\substack{i=1 \\ i \neq j}}^{s} d\lambda_i.$$

$C_2^{(s)} (\lambda_{s+1}, \ldots, \lambda_n; z)$ is an analytic function of z, for $z \in U_\beta$, and satisfies there the estimate

$$| C_2^{(s)} (\lambda_{s+1}, \ldots, \lambda_n; z) | < K_s \max_{\lambda_i \in W_d} | Q(\lambda_1, \ldots, \lambda_n) | \, | z - \kappa |$$

with K_s given by

$$K_s = K_{s-1} \hat{K} + K_1 (D^{s-1} + 2\pi^2 \beta D^{s-2} (s-1) + (s-1) (2\pi)^2 D^{s-2})$$

It is not hard to see that the recurrence formula leads to the estimate

$$K_s < sn \hat{D}^n \hat{K}^s L \tag{4.28}$$

where $\hat{D} = \max (D, 1)$, and $L > 0$ is some absolute constant. From (4.28) for $s = n$ we get the estimate (4.15). The expansion (4.14) is proved.

In order to prove (4.16) one has to represent the integral H_n in the form

$$H_n(z) = \int_C \cdots \int_C \frac{U(s_1, \ldots, s_n)}{(s_1 - z) \ldots (s_n - z)} ds_1 \ldots ds_n$$

where

$$\begin{aligned} U(s_1, \ldots, s_n) &= \int_{\tilde{p}(\lambda_n) = s_n} \cdots \int_{\tilde{p}(\lambda_1) = s_1} Q(\lambda_1, \ldots, \lambda_n) \, d\lambda_1 \ldots d\lambda_n \\ &= (-1)^n \frac{d^n}{ds_1 \ldots ds_n} \\ &\quad \times \int_{\tilde{p}(\lambda_n) > s_n} \cdots \int_{\tilde{p}(\lambda_1) > s_1} Q(\lambda_1, \ldots, \lambda_n) \, d\lambda_1 \ldots d\lambda_n \end{aligned}$$

Since we have only one critical point ($\lambda = 0$) of the function $\tilde{p}(\lambda)$ in V_δ the function $U(s_1, \ldots, s_n)$ is a smooth function of the variables $s_1, \ldots, s_n$ in the region $\kappa - \delta < s_1 < \kappa$, $i = 1, \ldots, n$ [6]. Hence by a repeated application of the formula of Sokhotskij [6] we get that for $\kappa - \delta < s < \kappa$

$$
\begin{aligned}
H_n^+(s) &= \lim_{\varepsilon \to 0} H_n(s + i\varepsilon) \\
&= \sum_{L \subseteq [1, \ldots, n]} (i\pi)^{|L|} \mathcal{P} \int_C \cdots \int_C \frac{U(s_1, \ldots, s_n)\,|_{s_i = s,\, i \in L}}{\prod_{m \notin L} (s_m - s)} \prod_{m \notin L} ds_m
\end{aligned}
$$

Here the summation goes over all subsets $L \subseteq \{1, \ldots, n\}$, the variables s_i are taken equal to s for $i \in L$ and the other variables are integrated (in the sense of the principal part). One can define similarly $H_n^-(s)$, which has a similar expression for $\kappa - \delta < s < \kappa$.

Lemma 4.3 is proved. ■

From Lemma 4.3 it is easy to deduce Lemma 4.2. In fact one has to apply it to reach integral in the expansions (4.7a) and (4.7b), and use the Hadamard inequality for the determinants $K_n(\kappa_1, \ldots, \kappa_n)$ and $\tilde{K}_n(\lambda, \mu, \kappa_1, \ldots, \kappa_n)$:

$$
\left\{
\begin{aligned}
&|K_n(\kappa_1, \ldots, \kappa_n)| < [\max_{\kappa_1, \kappa_2 \in W_d} |\tilde{S}(\kappa_1, \kappa_2)|]^n\, n^{\frac{n}{2}} \\
&|\tilde{K}_n(\lambda, \mu, \kappa_1, \ldots, \kappa_n)\, \phi(\lambda)\, \overline{\psi(\mu)}\,| \\
&\quad < \max_{\lambda \in W_d} |\phi(\lambda)| \max_{\mu \in W_d} |\psi(\mu)| \\
&\quad [\max_{\kappa_1, \kappa_2 \in W_d} |\tilde{S}(\kappa_1, \kappa_2)|]^{n+2} (n+2)^{\frac{n+2}{2}}.
\end{aligned}
\right\}
\tag{4.29}
$$

From these estimates it follows that the expansions of the type (4.14) for each of the terms of the series (4.7a) and (4.7b) can be summed term by term, so that one finally gets the relation (4.11). In the course of the proof we get also that

$$
\begin{aligned}
C_0 = \Delta(\kappa), \quad C_1 = 2\pi^2 \Bigg[\tilde{S}(0, 0) + \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{j=1}^{n} \int_{(T^3)^{n-1}} \\
\times \frac{K_n(\lambda_1, \ldots, \lambda_n)\,|_{\lambda_j = 0}}{\prod_{\substack{i=1 \\ i \neq j}}^{n} (\tilde{p}(\lambda_i) - \kappa)} \prod_{i \neq j} d\lambda_i \Bigg]
\end{aligned}
$$

and similar formulas for B_0, B_1. The assertion (4.12) is obtained in the same way. Lemma 4.2 is proved. ■

Some consequences of Lemma 4.2 are the following.

Remark 4.1. – Note that the expansion we obtained for $\Delta(z)$ implies that $\Delta(z) \neq 0$ for $z \in U_\beta$, for β small enough. This follows by observing that $\Delta(\kappa)$ is close to 1 for ε small enough. [It follows from the first estimate (3.4c) for the function S and from inequalities (4.29)]

Remark 4.2. – The zeroes of the determinant $\Delta(z)$ which lie outside the cut $\mathcal{C}$, are eigenvalues of the operator $\mathcal{T}$, and hence for small ε, lie in a small neighborhood V of the cut $\mathcal{C}$ (*see* [6]). The intersection of the external part of the circle $|z| < \kappa$ with this neighborhood lies inside U_β. Hence all zeroes of the determinant $\Delta(z)$ lie strictly inside the circle $|z| < \kappa$.

Remark 4.3. – From the expansion (4.11) and inequalities (4.29) it follows that the ratio $\dfrac{F(z)}{\Delta(z)}$ is analytic for $z \in U_\beta$, and can be represented as

$$\frac{F(z)}{\Delta(z)} = G_0 + G_1 (z-\kappa)^{\frac{1}{2}} + G_2(z) \tag{4.30}$$

where the function G_2 is analytic in U_β and is bounded by

$$|G_2(z)| < R|z-\kappa| \tag{4.31}$$

where R is some constant independent of z. Moreover, by (4.11),

$$G_0 = \frac{F(\kappa)}{\Delta(\kappa)}, \qquad G_1 = \frac{B_1 \Delta(\kappa) - C_1}{(\Delta(\kappa))^2}.$$

We are now able to compute the asymptotics as $t \to \infty$ of the second integral in (4.8), which can be written as

$$\frac{1}{2\pi i}\int_\gamma z^t \frac{F(z)}{\Delta(z)}\, dz$$

Since all singularities of $F(z)(\Delta(z))^{-1}$ are on the cut, except maybe for some zeroes of Δ which, as we said above, lie strictly inside the circle $|z| < \kappa$, we can choose for the integration contour γ the circle $|z| = \kappa - \beta$ (for β small enough) plus a small "deviation" which goes around the cut $\mathcal{C}$. Clearly the integral on the circle (more exactly on the arc on the circle) $|z| = \kappa - \beta$ gives a contribution which is bounded by

$$\text{constant}\,(\kappa-\beta)^t \tag{4.32}$$

The main contribution comes from the integral along the "deviation", which can be assumed to be as close to the cut as we want. Hence we are reduced to studying the integral

$$\frac{1}{2\pi i}\int_{\kappa-\beta}^{\kappa} s^t \left[\frac{F_+(s)}{\Delta_+(s)} - \frac{F_-(s)}{\Delta_-(s)}\right] ds. \tag{4.33}$$

From the expansion (4.30) it follows that

$$\frac{F_{+}(s)}{\Delta_{+}(s)} - \frac{F_{-}(s)}{\Delta_{-}(s)} = 2\,i\,G_1\,(\kappa - s)^{1/2} + \Delta G_2\,(s),$$
$$\Delta G_2\,(s) = G_2^{+}(\,s) - G_2^{-}\,(s),$$

Here $G_2^{\pm}\,(s)$ are the limiting values of $G_2\,(z)$ on the cut $\mathcal{C}$. The integral (4.33) takes the form

$$\frac{G_1}{\pi}\int_{\kappa-\beta}^{\kappa} (\kappa - s)^{\frac{1}{2}}\, s^t\, ds + \frac{1}{2\,\pi\,i}\int_{\kappa-\beta}^{\kappa} s^t\,\Delta G_2\,(s)\,ds$$

Introducing the new variable u by setting $s = \kappa\left(1 - \frac{u}{t}\right)$ we get the asymptotics of the first integral:

$$\frac{G_1\,\kappa^{t+\frac{3}{2}}}{\pi\,t^{\frac{3}{2}}}\int_0^{(\beta\,t)/\kappa} u^{1/2}\left(1 - \frac{u}{t}\right)^t du \asymp \frac{G_1\,\kappa^{\frac{3}{2}}\,\kappa^t}{2\sqrt{\pi}\,t^{\frac{3}{2}}}.$$

Using the estimate (4.31) for G_2, we get, after similar computations, that the second integral is bounded by

$$\text{const}\,\frac{\kappa^t}{t^2} \tag{4.34}$$

From (4.9), (4.32) and (4.34), and from the estimate (4.10) we get the final proof of Theorem 2.2 for the case $\nu = 3$. ■

For $\nu > 3$ the asymptotics is deduced along the same lines, only with more complicated calculations.

5. CONCLUDING REMARKS

If VI* is replaced by the weaker Condition VI (*i.e.*, if μ_1 has multiplicity $s > 1$) the proof of Theorem 2.2 is more complicated, and the asymptotics itself has a more complicated expression. One would have on the right-hand side of formula (2.6) an expression of the type

$$\frac{e^{-\alpha\,t}}{t^{\frac{\nu}{2}}}\,[C_0 + C_1\,\cos\,\beta_1\,t + \ldots + C_p\,\cos\,\beta_p\,t],$$

where $p \leqq s$, $\beta_i \neq 0$, $i = 1, \ldots, p$. The proof could be done following the same lines as above.

Another important remark concerns the connection between the spectral decomposition and the invariant (under $\mathcal{T}$) subspaces for the field ("from the point of view of the particle") η, and the corresponding constructions of Part I. In Part I we studied the full "particle + field model", with transfer matrix which we here denote, in order to avoid confusion, by $\mathcal{T}^{\text{tot}}$. $\mathcal{T}^{\text{tot}}$ and the operators $\{\mathcal{U}_v : v \in \mathbb{Z}^\nu\}$, which are a representation of the group of the space shifts, act on the space $\hat{\mathcal{H}} = L_2(S^{\mathbb{Z}^\nu}, \Pi_0) \times l_2(\mathbb{Z}^\nu)$. Since the operators $\mathcal{U}_v$ commute with $\mathcal{T}^{\text{tot}}$ (*see* Part I, Section 3), the space $\hat{\mathcal{H}}$ can be decomposed as a direct integral

$$\hat{\mathcal{H}} = \int d\lambda\, \mathcal{H}_\lambda \tag{5.1}$$

where the spaces $\mathcal{H}_\lambda$ are eigenspaces for the operators $\mathcal{U}_v$ with eigenvalues $e^{i(\lambda, v)}$, and (5.1) generates a decomposition for the operator $\mathcal{T}^{\text{tot}}$

$$\mathcal{T}^{\text{tot}} = \int \mathcal{T}^{\text{tot}}(\lambda)\, d\lambda$$

as a direct integral of the operators $\mathcal{T}^{\text{tot}}(\lambda)$, each of which acts on the space $\mathcal{H}_\lambda$.

A function $f(\xi, x) = g(\eta)$, depending only on η, is invariant with respect to the shifts, and it is not hard to see that the transfer matrix $\mathcal{T}$ which we study in the present paper coincides with the operator $\mathcal{T}^{\text{tot}}(0)$ acting on $\mathcal{H}_0$.

In Part I we constructed the leading subspace $\hat{\mathcal{H}}_1$, invariant for the operator $\mathcal{T}^{\text{tot}}$ (and the group $\{\mathcal{U}_v : v \in \mathbb{Z}^\nu\}$), as a perturbation of the subspace $\hat{\mathcal{H}}_1^0 = \Psi_{\bar{0}} \times l_2(\mathbb{Z}^\nu)$ (*see* Appendix A of Part I). In a similar way one can construct, under the assumptions of the present paper, an invariant subspace $\mathcal{H}_2$ as a perturbation of the subspace $\mathcal{H}_2^0 = \mathcal{H}_1^0 \times l_2(\mathbb{Z}^\nu)$, where $\mathcal{H}_1^0$ is the space spanned by the functions $\{\varphi_z : z \in \mathbb{Z}^\nu\}$, as explained in Section 3. $\hat{\mathcal{H}}_2$ is invariant with respect to the group $\{\mathcal{U}_v : v \in \mathbb{Z}^\nu\}$, and can be represented as a direct integral

$$\hat{\mathcal{H}}_2 = \int \hat{\mathcal{H}}_2(\lambda)\, d\lambda.$$

The decomposition leads to the representation of the operator $\mathcal{T}_2^{\text{tot}} \equiv \mathcal{T}^{\text{tot}}|_{\hat{\mathcal{H}}_2}$ as

$$\mathcal{T}_2^{\text{tot}} = \int \mathcal{T}_2^{\text{tot}}(\lambda)\, d\lambda.$$

Moreover the subspace $\hat{\mathcal{H}}_2(0) \subset \hat{\mathcal{H}}_0$ is isomorphic to the space $\mathcal{H}_1 \subset \mathcal{H}$ constructed in the present paper, and the operator $\mathcal{T}_2^{\text{tot}}(0)$ coincides with the operator $\mathcal{T}_1 = \mathcal{T}|_{\mathcal{H}_1}$.

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Appendix: Proof of Lemma 3.7

The representation (3.16a) of $\mathcal{S}$ as a series gives us that

$$(\mathcal{T}_{12}\,\mathcal{S})_{z,\,z'} = \sum_{r=1}^{\infty} \sum_{\alpha_1,\dots,\,\alpha_r} x_{\alpha_1,\dots,\,\alpha_r} \times \sum_{z_1,\dots,\,z_r} (\hat{\mathcal{B}}_{\alpha_r})_{z,\,z_r}\,(\hat{\mathcal{B}}_{\alpha_{r-1}})_{z_r,\,z_{r-1}} \cdots (\hat{\mathcal{B}}_{\alpha_1})_{z_1,\,z'},$$

where

$$\hat{\mathcal{B}}_{s,\,q} = \mathcal{T}_{12}\,\mathcal{B}_{s,\,q} = \mathcal{T}_{12}\,(\mathcal{T}_{22})^s\,\mathcal{T}_{21}\,(\mathcal{T}_{11})^{-q-1}. \tag{A.1}$$

We consider first the product $\mathcal{T}_{12}\,\mathcal{T}_{22}^s\,\mathcal{T}_{21}$. We write, in an obvious way $\mathcal{T}_{22} = \mathcal{T}_{22}^0 + \varepsilon\,\Delta_{22}$, so that the power $\mathcal{T}_{22}^s$ can be written as a sum of products, the elements of which can be either $\mathcal{T}_{22}^0$ or $\varepsilon\,\Delta_{22}$. Suppose that we have p factors $\varepsilon\,\Delta_{22}$ and $s-p$ factors $\mathcal{T}_{22}^0$:

$$\mathcal{T}_{i_1,\,i_2,\dots,\,i_p}^s = \varepsilon^p \prod_{j=1}^{s} \Delta^{(j)},$$

where $\Delta^{(j)} = \Delta_{22}$ for $j \in \{i_1, \dots, i_p\}$, and $\Delta^{(j)} = \mathcal{T}_{22}^0$ for $j \notin \{i_1, \dots, i_p\}$. The corresponding contribution can be written as

$$(\mathcal{T}_{12}\,\mathcal{T}_{i_1,\dots,\,i_p}^s\,\mathcal{T}_{21})_{z,\,z'} = \varepsilon^p \sum_{\Gamma^1,\dots,\,\Gamma^{s+1}} (\mathcal{T}_{21})_{z,\,\Gamma^1} \prod_{j=1}^{s} \Delta_{\Gamma^j\,\Gamma^{j+1}}^{(j)}\,(\mathcal{T}_{12})_{\Gamma^{s+1},\,z'}. \tag{A.2}$$

We can assume $z \neq \bar{0}$, since $(\mathcal{T}_{21})_{\bar{0},\,\Gamma} = 0$ for all $\Gamma \neq \bar{0}$. We take for the moment that $z' \in \mathbb{Z}^\nu$, and set $\Gamma^0 = \Gamma_1^{(z)}$, $\Gamma^{s+2} = \Gamma_1^{(z')}$. Each term on the right-hand side of eq. (A.2) can then be associated to a sequence

$$\Gamma_1^{(z)} \equiv \Gamma^0 \to \Gamma^1 \to \Gamma^2 \to \dots \to \Gamma^{s+1} \to \Gamma^{s+2} \equiv \Gamma_1^{(z')}. \tag{A.3}$$

If $j \notin \{i_1, \ldots, i_p\}$, Γ^{j+1} is obtained from Γ^j simply by a shift of a vector y_j, whereas if $j \in \{i_1, \ldots, i_p\}$, Γ^{j+1} is obtained by shifting and by changing the resulting multi-index $\Gamma^j + y_j$ at 0. We describe this process by assigning a "particle" to the points of the support of each multi-index Γ. The value $\gamma(x)$, $x \in \operatorname{supp}\Gamma$ will be the "species" of the particle. For $j \in \{i_1, \ldots, i_p\}$ an "interaction" takes place at the origin: a particle can be "created" (if $\gamma^{j+1}(0) > 0$ and $\gamma^j(-y_j) = 0$), can "disappear" (if $\gamma^{j+1}(0) = 0$, and $\gamma^j(-y_j) > 0$), or can change species (if $\gamma^j(-y_i) \neq \gamma^{j+1}(0) \neq 0$). It is convenient to assume that a particle disappears whenever it falls into the origin at one of the "times" $\{i_1, \ldots, i_p\}$, *i.e.*, a change of species is described by a destruction and a subsequent creation. We have, by (3.1a, b, c)

$$(\Delta^j)_{\Gamma^j \Gamma^{j+1}} = \left\{ \begin{array}{ll} \mu_{\Gamma^j} P_0(y_j)\, \delta_{\Gamma^j + y_j, \Gamma^{j+1}}, & j \notin \{i_1, \ldots, i_p\} \\ \hat{\mu}_{\Gamma^j}(-y_j)\, L(\gamma^j(-y_j), \gamma^{j+1}(0); -y_j) & \\ \quad \times \delta_{(\Gamma^j + y_j)_0, \Gamma_0^{j+1}}, & j \in \{i_1, \ldots, i_p\}. \end{array} \right\} \tag{A.4}$$

Let us suppose, for the moment, that $z' \in \mathbb{Z}^\nu$ and that z and z' are far from the origin, *i.e.*, $|z|$, $|z'| > D(s+2)$.

To help the intuition we interpret the sequence (A.3) as a time sequence, more precisely j will be the time at which the transition $\Gamma^j \to \Gamma^{j+1}$ takes place. At the time $j = i_0 = 0$ we have the first step of the sequence (A.3), which is described by the matrix element of $(\mathcal{T}_{21})_{z, \Gamma^1}$. By formula (3.10) we see that if z is far away the transition $\Gamma^0 \to \Gamma^1$ can be only a shift accompanied by a creation at 0. Formulas (3.11a, b) for $\mathcal{T}_{12}$ show also that the transition at the time $j = i_{p+1} = s+1$, $\Gamma^{s+1} \to \Gamma^{s+2}$ is necessarily a shift followed by the annihilation of a particle at 0. Since the particle originally at z cannot fall into the origin, it must clearly end up at z':

$$z + \sum_{j=0}^{s+1} y_j = z'.$$

All particles which are generated at the origin must fall back into the origin at some later time and disappear. Supposing that their number is n ($n \geq 1$, since one particle is certainly generated at $j = i_0 = 0$), we denote by $\{(j, j')\}_n = \{(j_0, j_0'), \ldots, (j_{n-1}, j_{n-1}')\}$ the sequence of pairs which gives the "lifetimes" of the particles. $\{j_0, \ldots, j_{n-1}\} \subset \{i_0, \ldots, i_p\}$ will be the times at which a particle is created, and $\{j_0', \ldots, j_{n-1}'\} \subset \{i_1, \ldots, i_{p+1}\}$ the times at which a particle disappears (the particle created at j_h disappearing at the time j_h'). We denote by $\{k\}_n = \{k_0, \ldots, k_{n-1}\}$ the species of the particles, k_h being the species of the particle created at the time j_h. Clearly if a particle is created at the time j and disappears at the time j', we must have $\sum_{l=j+1}^{j'} y_l = 0$.

One finds that the contribution to the expression (A.2) for a fixed choice of n, of the lifetimes $\{(j, j')\}_n$, and of the particle species $\{k\}_n$ can be written as

$$R_{i_1,\ldots,i_p}^{(s)}(\{(j, j')\}_n, \{k\}_n; z - z') = \sum_{\substack{Y_0,\ldots,Y_p \\ \{(j,j')\}_n}} \prod_{l=0}^{p} \mathcal{P}_l(Y_l; \{k\}_n) \prod_{j=0}^{s+1} \bar{\mu}_j \tag{A.5}$$

Here we have set

$$Y_l = \sum_{h=i_l+1}^{i_l+1} y_h, \qquad \bar{\mu}_j = \begin{cases} \hat{\mu}_{\Gamma^j}(-y_j) & j \in \{i_0, \ldots, i_{p+1}\} \\ \mu_{\Gamma^j} & j \notin \{i_0, \ldots, i_{p+1}\}. \end{cases}$$

Moreover the symbol $\{(j, j')\}_n$ under the summation sign means that the sum should be extended only to those Y_h's such that

$$\sum_{h=0}^{p+1} Y_h = z' - z, \qquad \text{and} \qquad \sum_{h=j_1+1}^{j_2} Y_h = 0,$$

the second relation holding for all $j_1, j_2 \in \{i_0, \ldots, i_{p+1}\}$ such that $j_1 = j_l$, $j_2 = j'_l$ for some $l = 1, \ldots, n$. (This is the condition for the particle to come back to the origin.) The function $\mathcal{P}_l(u; \{k\}_n)$ is the convolution of the function $L(k, k'; \cdot)$ with $i_{l+1} - i_l - 1$ factors $P_0(\cdot)$. Note that the species k and k', as well as the factors $\bar{\mu}_j$ are fixed by the choice of $\{(j, j')\}_n$ and of $\{k\}_n$. Summing the expression (A.5) over all choices of $\{(j, j')\}_n$ and of $\{k\}_n$ we get a quantity which is denoted by $R_{i_1,\ldots,i_p}^{(s)}(z' - z)$.

When z and z' are not far from the origin $R_{i_1,\ldots,i_p}^{(s)}(z' - z)$ makes sense, since it depends only on the difference $z' - z$, but does not correspond to the sum of the contributions of all possible sequences (A.3). More precisely, only the sequences (A.3) for which z does not fall into the origin at the times $\{i_0, \ldots, i_{p+1}\}$ give contributions corresponding to terms on the right side of equation (A.5), for some choice of $\{(j, j')\}_n$ and of $\{k\}_n$. The sum of the contribution of the other sequences is denoted by $\bar{V}_{i_0,\ldots,i_p}^{(s)}(z, z')$. To make up $R_{i_0,\ldots,i_p}^{(s)}$ we have to add and substract the terms appearing on the right side of eq. (A.5) corresponding to Y_h's for which the choice of z would imply that the particle originally at z falls into the origin at some time of the set $\{i_0, \dots, i_{p+1}\}$. Denoting the sum of such terms by $\tilde{V}^{(s)}_{i_0,\dots,i_p}(z, z')$, we can write

$$(\mathcal{T}_{21}\, \mathcal{T}^s_{i_1,\dots,i_p}\, \mathcal{T}_{12})_{z, z'} = R^{(s)}_{i_1,\dots,i_p}(z - z') + V^{(s)}_{i_0,\dots,i_p}(z, z').$$

where $V^{(s)}_{i_0,\dots,i_p}(z, z') = \bar{V}^{(s)}_{i_0,\dots,i_p}(z, z') - \tilde{V}^{(s)}_{i_0,\dots,i_p}(z, z')$. Clearly the quantity $V^{(s)}_{i_0,\dots,i_p}(z, z')$ can be written as a sum of terms of the type (A.5), where the shifts Y_h's are subject to some conditions expressing the fact that the particle originally at z falls into the origin.

To estimate the terms (A.5) we take absolute values, extend the sum to all values of y_j's such that $\sum_{j=0}^{s+1} y_j = z' - z$ and introduce the function $L^*(y) = \sum_{h, k} |L(h, k; -y)|$, so that finally

$$|R^{(s)}_{i_1,\dots,i_p}| \leqq \operatorname{const} \varepsilon^{2+p} (\mu_*)^s \times \sum_{y_0+\dots+y_{p+1}=z'-z} \prod_{l \in \{i_0,\dots,i_{p+1}\}} L^*(y_l) \prod_{j \notin \{i_0,\dots,i_{p+1}\}} P_0(y_j),$$

where the constant does not depend on s and p. Similar arguments lead to the inequality

$$|V^{(s)}_{i_0,\dots,i_p}(z, z')| \leqq \operatorname{const} \varepsilon^{2+p} (\mu_*)^s \sum_{\substack{j=1,\dots,p+1\\ h=0,\dots,p}} \sum_{\substack{y_0,\dots,y_p+1\\ \{j, h\}}} \times \prod_{l \in \{i_0,\dots,i_{p+1}\}} L^*(y_l) \prod_{j \notin \{i_0,\dots,i_{p+1}\}} P_0(y_j).$$

The notation $\{j, h\}$ under the summation sign stands for the conditions

$$\sum_{l=0}^{i_j} y_l = -z, \qquad \sum_{l=i_h+1}^{p+1} y_l = z'$$

which express the fact that i_j is the time at which the particle originally at z falls into the origin, and i_h is the time at which the particle ending up at z' is created. Summing up over all choices of $\{i_1, \dots, i_p\}$, for fixed p, and then over p, we get

$$\begin{aligned}(\mathcal{T}_{12}\,(\mathcal{T}_{22})^s\, \mathcal{T}_{21})_{z, z'} &= \sum_{\Gamma, \Gamma'} (\mathcal{T}_{21})_{z\,\Gamma}\, (\mathcal{T}_{22})^s_{\Gamma, \Gamma'}\, (\mathcal{T}_{12})_{\Gamma', z'} \\ &= R_s(z' - z) + V_s(z, z'), \qquad \text{(A.6a)}\end{aligned}$$

where $R_s(u)$ and $V_s(z, z')$ are 0 if $|u| > s\,D$ or $\max\{|z|, |z'|\} > s\,D$ respectively, and

$$\left\{\begin{aligned} &\sum_u |R_s(u)| \leqq \text{const}\,\varepsilon^2 (\mu_* (1+\varepsilon\, c))^s, \\ &\sum_{z, z'} |V_s(z, z')| \leqq \text{const}\,\varepsilon^2 (\mu_* (1+\varepsilon\, c))^s. \end{aligned}\right\} \tag{A.6b}$$

If $z' = \bar{0}$, then at the last but one step there is only one particle left (*i.e.* $|\operatorname{supp}\Gamma^{s+1}| = 1$), which ends up at the origin and disappears. In this case only the term $V^{(s)}_{i_1,\ldots,i_p}(z, \bar{0})$ is left, and summing up all such terms as before, we get a term $V_s(z, \bar{0})$ which is 0 if $|z| > D$, and satisfies the estimate

$$\sum_z |V_s(z, \bar{0})| \leqq \text{const}\,\varepsilon^3 (\mu_* (1+\varepsilon\, c))^s. \tag{A.6c}$$

From (A.6b, c) it follows that the Fourier transform of these functions

$$\tilde{R}_s(\lambda) = \sum_{u\in\mathbb{Z}^r} R_s(u)\, e^{i(\lambda, u)}$$

$$\tilde{V}_s(\lambda, \lambda') = \sum_{z, z'} V_s(z, z')\, e^{i(\lambda, z) - i(\lambda', z')}, \qquad \tilde{V}_s(\lambda) = \sum_z V_s(z, \bar{0})\, e^{i(\lambda, z)}$$

are analytic in the complex neighborhood $W_d \equiv \{\lambda : |\operatorname{Im}\lambda_i| \leqq d,\ i = 1, \ldots, \nu\}$, for d small, and in this neighborhood we have

$$|\tilde{R}_s(\lambda)| < \text{const}\,\varepsilon^2 (\mu_* (1+\varepsilon\, c))^s\, e^{d\,Ds}$$

$$|\tilde{V}_s(\lambda, \lambda')| < \text{const}\,\varepsilon^2 (\mu_* (1+\varepsilon\, c))^s\, e^{2\,d\,Ds},$$

$$|\tilde{V}_s(\lambda)| < \text{const}\,\varepsilon^2 (\mu_* (1+\varepsilon\, c))^s\, e^{d\,Ds}.$$

We now go back to the matrix elements of $\hat{\mathcal{B}}_{s,q}$, defined by eq. (A.1). Using relations (3.18) and (A.6a) we see that

$$(\hat{\mathcal{B}}_{s,q})_{z,z'} = \hat{R}_{s,q}(z - z') + \hat{V}_{s,q}(z, z'), \qquad z, z' \in \mathbb{Z}^\nu$$

$$(\hat{\mathcal{B}}_{s,q})_{z,\bar{0}} = \hat{V}_{s,q}(z, \bar{0})$$

where

$$\hat{R}_{s,q}(z - z') = \sum_u (\mathcal{T}^0_{11})^{-q-1}_{z,u} R_s(u - z'),$$

$$\hat{V}_{s,q}(z, z') = \sum_u [(\mathcal{T}^0_{11})^{-q-1}_{z,u} V_s(u, z') + \mathcal{D}^{(q+1)}_{z,u} R_s(z - z') + \mathcal{D}^{(q+1)}_{z,u} V_s(u, z')]$$

$$\hat{V}_{s,q}(z, \bar{0}) = \sum_u (\mathcal{T}^0_{11})^{-q-1}_{z,u} V_s(z, \bar{0}).$$

Consider now the Fourier transforms

$$\tilde{\hat{R}}_{s,q}(\lambda) = \sum_{u \in \mathbb{Z}^\nu} \hat{R}_{s,q}(u)\, e^{i(\lambda, u)},$$

$$\tilde{\hat{V}}_{s,q}(\lambda, \lambda') = \sum_{z, z'} \hat{V}_{s,q}(z, z')\, e^{i(\lambda, z) - i(\lambda', z')}$$

$$\tilde{\hat{V}}_{s,q}(\lambda) = \sum_{z} \hat{V}_{s,q}(z, \bar{0})\, e^{i(\lambda, z)}.$$

Using the estimate (3.23a) it is not hard to see that the functions $\tilde{\hat{R}}_{s,q}(\lambda)$, $\tilde{\hat{V}}_{s,q}(\lambda, \lambda')$ $\tilde{\hat{V}}_{s,q}(\lambda)$ are analytic for λ, λ' in the complex neighborood W_d and satisfy the estimates

$$\left\{\begin{array}{l} |\tilde{\hat{R}}_{s,q}(\lambda)| < \text{const}\, \varepsilon^2 (\mu_*(1+\varepsilon c))^s e^{dDs} \left(\dfrac{a+\eta}{\mu_1}\right)^{q+1}, \\ |\tilde{\hat{V}}_{s,q}(\lambda)| < \text{const}\, \varepsilon^2 (\mu_*(1+\varepsilon c))^s e^{dDs} \left(\dfrac{a+\eta}{\mu_1}\right)^{q+1} \\ \tilde{\hat{V}}_{s,q}(\lambda, \lambda')| < \text{const}\, \varepsilon^2 (\mu_*(1+\varepsilon c))^s e^{2dDs} \left(\dfrac{a+\eta}{\mu_1}\right)^{q+1}. \end{array}\right\} \tag{A.7}$$

where $\alpha_i = (s_i, q_i)$ $i = 1, \ldots, r$. We write

$$G_{\alpha_1, \ldots, \alpha_r}(z, z') = \hat{R}_{\alpha_1, \ldots, \alpha_r}(z - z') + \hat{V}_{\alpha_1, \ldots, \alpha_r}(z, z')$$

where $\hat{R}_{\alpha_1, \ldots, \alpha_r}$ is an operator product

$$\hat{R}_{\alpha_1, \ldots, \alpha_r} = \hat{R}_{\alpha_1} \hat{R}_{\alpha_2} \ldots \hat{R}_{\alpha_r},$$

and the operator $\hat{V}_{\alpha_1, \ldots, \alpha_r}$ satisfies the recurrent formula

$$\hat{V}_{\alpha_1, \ldots, \alpha_r} = \hat{R}_{\alpha_1} \hat{V}_{\alpha_2, \ldots, \alpha_r} + \hat{V}_{\alpha_1} \hat{R}_{\alpha_2, \ldots, \alpha_r} + \hat{V}_{\alpha_1} \hat{V}_{\alpha_2, \ldots, \alpha_r}. \tag{A.8}$$

From inequalities (A.7) we see that the Fourier transform

$$\tilde{\hat{R}}_{\alpha_1, \ldots, \alpha_r}(\lambda) = \sum_{z} \hat{R}_{\alpha_1, \ldots, \alpha_r}(u)\, e^{i(\lambda, u)}$$

is analytic in W_d and satisfies the estimate

$$|\tilde{\hat{R}}_{\alpha_1, \ldots, \alpha_r}(\lambda)| < (c\varepsilon^2)^r (\mu_*(1+\varepsilon c)\, e^{dD})^{S_r} \left(\frac{a+\eta}{\mu_1}\right)^{Q_r + r},$$

where $S_r = \sum_1^r s_i$, $Q_r = \sum_1^r q_i$. Using this estimate, (A.7) and (A.8) it is not hard to show that the Fourier transforms

$$\tilde{\hat{V}}\alpha_1, \ldots, \alpha_r (\lambda, \lambda') = \sum_{z, z'} \hat{V}_{\alpha_1, \ldots, \alpha_r} (z, z') \, e^{i (\lambda, z) - i (\lambda', z')}$$

$$\tilde{\hat{V}}\alpha_1, \ldots, \alpha_r (\lambda) = \sum_{z} \hat{V}_{\alpha_1, \ldots, \alpha_r} (z, \bar{0}) \, e^{i (\lambda, z)}$$

are analytic for λ, λ' in W_d and satisfy in W_d the following estimates

$$\left\{ \begin{aligned} & | \tilde{\hat{V}}_{\alpha_1, \ldots, \alpha_r} (\lambda, \lambda') < (c\,\varepsilon^2)^r \, (\mu_* \, (1 + \varepsilon\, c) \, e^{2\, d\, D})^{S_r} \left(\frac{\alpha + \eta}{\mu_1} \right)^{Q_r + r} \\ & | \tilde{\hat{V}}_{\alpha_1, \ldots, \alpha_r} (\lambda) < (c\,\varepsilon^2)^r \, (\mu_* \, (1 + \varepsilon\, c) \, e^{d\, D})^{S_r} \left(\frac{\alpha + \eta}{\mu_1} \right)^{Q_r + r}. \end{aligned} \right\} \tag{A.9}$$

The convergence of the series (3.16a) for $\mathcal{S}$ implies that the series

$$(\mathcal{T}_{12} \, \mathcal{S})_{z, z'} = \sum_{\alpha_1, \ldots, \alpha_r} x_{\alpha_1, \ldots, \alpha_r} \, G_{\alpha_1, \ldots, \alpha_r} (z, z') \tag{A.10}$$

is convergent and the sum is equal

$$(\mathcal{T}_{12} \, \mathcal{S})_{z, z'} = R (z - z') + V (z, z'), \qquad (\mathcal{T}_{12} \, \mathcal{S})_{z, \bar{0}} = V (z, \bar{0}),$$
$$z, \, z' \in \mathbb{Z}^\nu$$

where

$$R (z - z') = \sum_{\alpha_1, \ldots, \alpha_r} x_{\alpha_1, \ldots, \alpha_r} \, \hat{R}_{\alpha_1, \ldots, \alpha_r} (z - z'), \qquad z, \, z' \in \mathbb{Z}^\nu$$

and

$$V_{z, z'} = \sum_{\alpha_1, \ldots, \alpha_r} x_{\alpha_1, \ldots, \alpha_r} \, \hat{V}_{\alpha_1, \ldots, \alpha_r} (z, z'),$$

$$V_{z, \bar{0}} = \sum_{\alpha_1, \ldots, \alpha_r} x_{\alpha_1, \ldots, \alpha_r} \, \hat{V}_{\alpha_1, \ldots, \alpha_r} (z, \bar{0}), \qquad z, \, z' \in \mathbb{Z}^\nu.$$

From the convergence of the series (A.10) and the inequalities (A.9), one deduces easily, again by shifting the integration region into the complex W_d region, that the Fourier coefficients $R(u)$ and $V(z, z')$ verify, if ε is small enough, the estimates

$$| R (u) | < \text{const}\, \varepsilon^2 \, \theta^{| u |} \qquad u \in \mathbb{Z}^\nu$$

$$| V_{z, z'} | < \text{const}\, \varepsilon^2 \, \theta^{| z | + | z' |}, \qquad | V_{z, \bar{0}} | < \text{const}\, \varepsilon^2 \, \theta^{| z |}, \qquad z, \, z' \in \mathbb{Z}^\nu$$

for some $0 < \theta < 1$. Moreover $V_{\bar{0}, z} = V_{0, \bar{0}} = 0$.

Lemma 3.7 is proved. ■

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