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Hölder norms and the support theorem for diffusions

by

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ABSTRACT. – We show that the Stroock-Varadhan [S-V] support theorem is valid in α-Hölder norm (Theorem 4). The central tool is an estimate (stated in Theorem 1 and Theorem 2) of the probability that the Brownian motion has a large Hölder norm but a small uniform norm.

Key words : Diffusion processes, support theorem, Brownian motion, Hölder norm, gaussian measures.

RÉSUMÉ. – Nous montrons dans cette note que le théorème du support de Stroock-Varadhan [S-V] est valide en norme α-hölderienne, (*cf.* Théorème 4). L'outil principal est une majoration (énoncée au Théorème 1 et Théorème 2) de la probabilité pour qu'un mouvement brownien ait une grande norme hölderienne et une petite norme uniforme.

1. INTRODUCTION

What is the probability that the Brownian motion oscillates rapidly conditionally on the fact that it is small in uniform norm? More precisely, what is the probability that the α-Hölder norm of the Brownian motion is

Classification A.M.S. : 60 J 60, 60 H 10, 60 J 65, 46 E 15, 60 G 15.

large conditionally on the fact that its uniform norm (or more generally its β-Hölder norm with $\beta < \alpha$) is small?

This is the kind of question that naturally appears if one wants to extend Stroock-Varadhan characterization of the support of the law of diffusion processes [S-V] to sharper topologies than the one induced by the uniform norm.

We deal with this question in section 2 and show that those tails are much smaller than the gaussian tails one would get without the conditioning. This gives a family of examples where the conjecture (stated in [DG-E-...]) that two convex symmetric bodies are positively correlated (for gaussian measures) is true.

Our proofs are based on Ciesielski isomorphism [C] (*see* [B-R] for other applications of this theorem) and on correlation inequality. We give in an appendix a proof which avoids these tools.

This enables us to control in section 4 the probability that a Brownian stochastic integral oscillates rapidly conditionally on the fact that the Brownian motion is small in uniform norm. This is the tool to extend Stroock-Varadhan support theorem to α-Hölder norms.

2. CONDITIONAL TAILS FOR OSCILLATIONS OF THE BROWNIAN MOTION

If x is a continuous real function on [0, 1], vanishing at zero, one defines the sequence $(\xi_m(x))_{m \geqq 1}$ by the formula

$$\xi_{2^n+k}(x) = 2^{n/2}\left(2\,x\left(\frac{2k-1}{2^{n+1}}\right) - x\left(\frac{k}{2^n}\right) - x\left(\frac{k-1}{2^n}\right)\right),$$

for $n \geqq 0$ and $k = 1, \cdots, 2^n$ and the norms

$$\|x\|_0 = \sup_{0 \leqq t \leqq 1} |x_t|, \tag{1}$$

$$\|x\|_\alpha = \sup_{0 \leqq s < t \leqq 1} \frac{|x_t - x_s|}{|t-s|^\alpha}, \qquad \alpha \in]0, 1], \tag{2}$$

$$\|x\|'_\alpha = \sup_{m \geqq 1} |m^{\alpha-(1/2)}\,\xi_m(x)|, \qquad \alpha \in [0, 1]. \tag{3}$$

It is now classical that, for $\alpha \in]0, 1[$ the norms $\|\cdot\|_\alpha$ and $\|\cdot\|'_\alpha$ are equivalent (*see* [C]):

$$2^{\alpha-1}\|x\|'_\alpha \leqq \|x\|_\alpha \leqq 2^{-(1/2)}\,k_\alpha \|x\|'_\alpha, \qquad \alpha \in]0, 1[, \tag{4}$$

where

$$k_\alpha = \frac{2}{(2^\alpha - 1)(2^{1-\alpha} - 1)},$$

and

$$2^{-2}\|x\|'_0 \leqq \|x\|_0. \tag{5}$$

We want to estimate the probability that $\|w.\|_\alpha$ is large conditionally on the fact that $\|w.\|_\beta$ is small. We will first tackle the same problem with the norms $\|\cdot\|'$.

THEOREM 1. – *Let* (r, R) *be a couple of real positive numbers and* $v = \left(\frac{\mathrm{R}^b}{r^a}\right)^{1/(b-a)}$. *Let us denote*

$$\mathrm{H}_{\alpha,\beta}(r, \mathrm{R}) = \frac{\varphi(v)}{v} + \frac{1}{a}\mathrm{R}^{-(1/a)} \int_v^\infty \varphi(t)\, t^{(1/a)-2}\, dt. \tag{6}$$

where $\varphi(t) = \frac{e^{-(t^2/2)}}{\sqrt{2\pi}}$, $a = \frac{1}{2} - \alpha$, $b = \frac{1}{2} - \beta$. *Then*

$$\mathrm{P}(\|w.\|'_\alpha > \mathrm{R} \mid \|w.\|'_\beta < r) \leqq \frac{1}{\int_0^v \varphi(t)\, dt}\mathrm{H}_{\alpha,\beta}(r, \mathrm{R}), \tag{7}$$

$$\mathrm{P}(\|w.\|'_\alpha > \mathrm{R} \mid \|w.\|'_\beta < r) \leqq \mathrm{H}_{\alpha,\beta}(p_\beta\, r, \mathrm{R}), \tag{8}$$

where $p_\beta = 2^{1-\beta}$, *if* $\beta > 0$ *and* $p_0 = 4$;

$$\mathrm{P}(\|w.\|_\alpha > \mathrm{R} \mid \|w.\|_\beta < r) \leqq \mathrm{H}_{\alpha,\beta}(p_\beta\, r, 2^{1/2} k_\alpha^{-1}\mathrm{R}). \tag{9}$$

For the proof of the theorem we need the following lemma.

LEMMA 1. – *Let us denote* $n_0 = \left[\left(\frac{\mathrm{R}}{r}\right)^{1/(b-a)}\right]$. *Then*

$$\sum_{n \geqq n_0+1} \int_{\mathrm{R}n^a}^\infty \varphi(t)\, dt \leqq \mathrm{H}_{\alpha,\beta}(r, \mathrm{R}). \tag{10}$$

Proof. – By the classical bound:

$$\int_t^\infty \varphi(s)\, ds \leqq \frac{\varphi(t)}{t} \equiv \psi(t), \quad \text{for every } t > 0,$$

and the fact that ψ is decreasing, we have:

$$\sum_{n \geqq n_0+1} \int_{\mathrm{R}n^a}^{\infty} \varphi(t)\,dt \leqq \sum_{n \geqq n_0+1} \psi(\mathrm{R}\,n^a) = \psi(\mathrm{R}\,(n_0+1)^a)$$

$$+ \sum_{n \geqq n_0+2} \psi(R\,n^a) \leqq \psi(v) + \int_{n_0+1}^{\infty} \psi(\mathrm{R}\,t^a)\,dt$$

$$= \frac{\varphi(v)}{v} + \frac{1}{a} \mathrm{R}^{-(1/a)} \int_{\mathrm{R}\,(n_0+1)^a}^{\infty} \psi(t)\, t^{(1/a)-1}\,dt.$$

From this the conclusion follows.

Q.E.D.

Proof of theorem 1.

Proof of (7). – We remark that $g_n = \xi_n(w)$ is a sequence of independent identically distributed $\mathrm{N}(0, 1)$ random variables. Then

$$\mathrm{P}(\|w.\|'_\alpha > \mathrm{R} \mid \|w.\|'_\beta < r)$$

$$= \mathrm{P}(\sup_{n \geqq 1} |n^{-a}\, g_n| > \mathrm{R} \mid \sup_{m \geqq 1} |m^{-b}\, g_m| < r)$$

$$= \mathrm{P}(\bigcap_{m \geqq 1} |g_m| < rm^b)^{-1}$$

$$\times\ \mathrm{P}(\bigcup_{n \geqq 1} (|g_n| > \mathrm{R}\,n^a) \bigcap \bigcap_{m \geqq 1} (|g_m| < rm^b))$$

$$\leqq \prod_{m \geqq 1} \mathrm{P}(|g_m| < rm^b)^{-1}$$

$$\times\ \{\sum_{n \geqq 1} \mathrm{P}((\mathrm{R}\,n^a < |g_n| < rn^b) \bigcap \bigcap_{m \geqq 1,\, m \neq n} (|g_m| < rm^b))\}$$

$$= \sum_{n \geqq 1} \frac{\mathrm{P}(\mathrm{R}\,n^a < |g_n| < rn^b)}{\mathrm{P}(|g_n| < rn^b)} \cdot 1_{(\mathrm{R}n^a < rn^b)}$$

$$= \sum_{n \geqq 1} 2 \int_{\mathrm{R}n^a}^{rn^b} \frac{e^{-(s^2/2)}}{\sqrt{2\pi}}\,ds \cdot \frac{1}{2 \int_0^{rn^b} \frac{e^{-(s^2/2)}}{\sqrt{2\pi}}\,ds} \cdot 1_{(Rn^a < rn^b)}$$

$$= \sum_{n \geqq n_0+1} \frac{\int_{\mathrm{R}n^a}^{rn^b} \varphi(t)\,dt}{\int_0^{rn^b} \varphi(t)\,dt} \leqq \frac{1}{\int_0^v \varphi(t)\,dt} \cdot \sum_{n \geqq n_0+1} \int_{\mathrm{R}n^a}^{\infty} \varphi(t)\,dt.$$

Clearly we have $rn^b \geqq r(n_0+1)^b \geqq v$ so the last inequality is true. Then (7) is a consequence of Lemma 1.

Proof of (8). – We can write again

$$\begin{aligned} &\mathrm{P}(\|w.\|'_\alpha > \mathrm{R} \mid \|w.\|_\beta \leqq r) \\ &\quad = \mathrm{P}(\sup_{n \geqq 1} |n^{-a} g_n| > \mathrm{R} \mid \|w.\|_\beta \leqq r) \\ &\quad = \mathrm{P}(\bigcup_{n \geqq 1} (|g_n| > \mathrm{R}\, n^a) \mid \|w.\|_\beta \leqq r) \\ &\quad \leqq \sum_{n \geqq 1} \mathrm{P}(|g_n| > \mathrm{R}\, n^a) \mid \|w.\|_\beta \leqq r). \end{aligned}$$

But when $\|w.\|_\beta \leqq r$, by (4) or (5) we have

$$|g_n| \leqq 2^{1-\beta} rn^b, \qquad \text{if} \quad \beta > 0$$

or

$$|g_n| \leqq 4\, rn^{1/2}, \qquad \text{if} \quad \beta = 0.$$

So the preceeding sum is taken over all integer $n \geqq 1$ such that $2^{1-\beta} rn^b \geqq \mathrm{R}\, n^a$, if $\beta > 0$, or $4\, rn^{1/2} \geqq \mathrm{R}\, n^a$, if $\beta = 0$, *i.e.*

$$n \geqq 2^{(\beta-1)/(\alpha-\beta)} \left(\frac{\mathrm{R}}{r}\right)^{1/(b-a)} \left(\text{or } n \geqq 2^{-(2/\alpha)} \left(\frac{\mathrm{R}}{r}\right)^{1/\alpha}\right).$$

On the other hand $g_n = \xi_n(w)$ is linear form on the Wiener space so, by the correlation inequality in [DG-E-...] (*see* also [S-Z]), which is clearly true here, we obtain

$$\mathrm{P}(|g_n| > \mathrm{R}\, n^a,\ \|w.\|_\beta \leqq r) \geqq \mathrm{P}(|g_n| > \mathrm{R}\, n^a)\, \mathrm{P}(\|w.\|_\beta \leqq r).$$

So

$$\mathrm{P}(|g_n| > \mathrm{R}\, n^a | \ \|w.\|_\beta \leqq r) \leqq \mathrm{P}(|g_n| > \mathrm{R}\, n^a)$$

and therefore

$$\begin{aligned} \mathrm{P}(\|w.\|'_\alpha > \mathrm{R} \mid \|w.\|_\beta \leqq r) &\leqq \sum_{n \geqq n_1+1} \mathrm{P}(|g_n| > \mathrm{R}\, n^a) \\ &= \sum_{n \geqq n_1+1} \int_{\mathrm{R}\, n^a}^{\infty} \varphi(t)\, dt, \end{aligned}$$

where $n_1 = \left[p_\beta^{1/(b-a)} \left(\frac{\mathrm{R}}{r}\right)^{1/(b-a)} \right]$. By Lemma 1 we obtain (8).

Proof of (9). – It is a consequence of (4) [or (5)] and (8).

Q.E.D.

LEMMA 2. – *With the notations of Theorem* 1 *there exists a polynomial function* Q_a *increasing on* $]0, \infty[$ *such that*

$$H_{\alpha,\beta}(r, R) \leqq \frac{\varphi(v)}{v}\left(1 + \frac{1}{a} R^{-(1/a)} v^{(1/a)-2} Q_a\left(\frac{1}{v}\right)\right). \tag{11}$$

Proof. – We will simply give an upper bound for $\int_v^\infty \varphi(t)\, t^{(1/a)-2}\, dt$. Noting that $\varphi'(t) = -t\,\varphi(t)$ and integrating by parts one gets

$$\int_v^\infty \varphi(t)\, t^{(1/a)-2}\, dt = -\int_v^\infty \varphi'(t)\, t^{(1/a)-3}\, dt = \varphi(v)\, v^{(1/a)-3} + \left(\frac{1}{a} - 3\right)\int_v^\infty \varphi(t)\, t^{(1/a)-4}\, dt.$$

If $a \geqq \frac{1}{3}$,

$$\int_v^\infty \varphi(t)\, t^{(1/a)-2}\, dt \leqq \varphi(v)\, v^{(1/a)-3},$$

which gives (11) with $Q_a(x) \equiv 1$. If $a < \frac{1}{3}$, similarly,

$$\int_v^\infty \varphi(t)\, t^{(1/a)-4}\, dt = \varphi(v)\, v^{(1/a)-5} + \left(\frac{1}{a} - 5\right)\int_v^\infty \varphi(t)\, t^{(1/a)-6}\, dt.$$

So, if $\frac{1}{5} \leqq a < \frac{1}{3}$,

$$\int_v^\infty \varphi(t)\, t^{(1/a)-2}\, dt \leqq \varphi(v)\, v^{(1/a)-3} + \left(\frac{1}{a} - 3\right)\varphi(v)\, v^{(1/a)-5},$$

which is exactly (11) with $p = 1$ in the following expression:

$$Q_a(x) = 1 + \left(\frac{1}{a} - 3\right) x^2 + \cdots + \left(\frac{1}{a} - 3\right) \times \left(\frac{1}{a} - 5\right) \cdots \left(\frac{1}{a} - 2p - 1\right) x^{2p}.$$

Repeating the same reasoning the result is easily obtained for any p and any a such that: $\frac{1}{2p+3} \leqq a < \frac{1}{2p+1}$. Q_a has positive coefficients, it is therefore increasing on $]0, \infty[$.

Q.E.D.

COROLLARY 1. – *Let* (R, r) *such that* $v \geqq \epsilon > 0$. *Then*

$$P(\|w.\|'_\alpha > R \,|\, \|w.\|'_\beta < r) \leqq c(\epsilon) \frac{\varphi(v)}{\epsilon} \left(1 + Q_a\left(\frac{1}{\epsilon}\right)\left(\frac{R^\beta}{r^\alpha}\right)^{2/(\alpha-\beta)}\right); \tag{12}$$

$$P(\|w.\|'_\alpha > R \,|\, \|w.\|_\beta < r) \leqq \frac{\varphi(q_\beta v)}{q_\beta \epsilon} \left(1 + q_\beta^{(1/a)-2} Q_a\left(\frac{1}{\epsilon}\right)\left(\frac{R^\beta}{r^\alpha}\right)^{2/(\alpha-\beta)}\right); \tag{13}$$

$$P(\|w.\|_\alpha > R \,|\, \|w.\|_\beta < r) \leqq \frac{\varphi(c_{\alpha,\beta} v)}{c_{\alpha,\beta}\epsilon} \Bigg(1 + (2^{1/2} k_\alpha^{-1})^{-(1/a)} c_{\alpha,\beta}^{(1/a)-2} \times Q_a\left(\frac{1}{\epsilon}\right)\left(\frac{R^\beta}{r^\alpha}\right)^{2/(\alpha-\beta)}\Bigg) \tag{14}$$

Here $q_\beta = p_\beta^{-a/(b-a)}$, $c_{\alpha,\beta} = q_\beta (2^{1/2} k_\alpha^{-1})^{b/(b-a)}$ *and*

$$c(\epsilon) = \frac{1}{\int_0^\epsilon \varphi(t)\, dt}.$$

We note that if $\epsilon \to \infty$ *then* $c(\epsilon) \to 2$, $Q_a\left(\frac{1}{\epsilon}\right) \to 1$.

Proof. – Trivial by Lemma 2 and Theorem 1.

Q.E.D.

We need now a stronger result.

THEOREM 2. – *Let* α, β *two real numbers such that* $0 \leqq \beta < \alpha < \frac{1}{2}$. *There exist a positive number* $u_{\alpha,\beta} = \frac{1-2\alpha}{1-2\beta}$ *such that for every* $u \in [0, u_{\alpha,\beta}[$ *there exists* $M_0(\alpha, \beta, u)$ *and positive constants* $k_i(\alpha, \beta, u)$, $i = 1, 2$, *such that, for every* $M \geqq M_0$

$$\sup_{0<\delta\leqq 1} \mathrm{P}(\|w.\|_\alpha > \mathrm{M}\,\delta^u \mid \|w.\|_\beta < \delta) \leqq k_1\,\mathrm{M}^{2\beta/(\alpha-\beta)} \exp(-k_2\,\mathrm{M}^{(1-2\beta)/(\alpha-\beta)}). \tag{15}$$

Proof. – First of all we take in Corollary 1, $\mathrm{R} = \mathrm{M}\,\delta^u$ and $r = \delta$. So, for every $\delta \in]0, 1]$,

$$\mathrm{P}(\|w.\|_\alpha > \mathrm{M}\,\delta^u \mid \|w.\|_\beta < \delta) \leqq c_{\alpha,\beta}\,\mathrm{M}^{(1-2b)/(b-a)}\,\delta^{u(1-2b)-(1-2a)/(b-a)} \times \exp(-c'_{\alpha,\beta}\,\mathrm{M}^{2b/(b-a)}\,\delta^{2(ub-a)/(b-a)}).$$

It is clear then that when

$$\mathrm{M} \geqq \left(\frac{2}{c'_{\alpha,\beta}} \cdot \frac{u(1-2b)-(1-2a)}{b-a} \cdot \frac{b-a}{a-ub}\right)^{(b-a)/2b}$$

the right hand side of the last inequality is an increasing function of δ, when $\delta \in]0, 1]$, so that

$$\sup_{0<\delta\leqq 1} \mathrm{P}(\|w.\|_\alpha > \mathrm{M}\,\delta^u \mid \|w.\|_\beta < \delta) \leqq c_{\alpha,\beta}\,\mathrm{M}^{(1-2b)/(b-a)} \exp(-c'_{\alpha,\beta} \cdot \mathrm{M}^{2b/(b-a)}),$$

namely the conclusion.

3. HÖLDER BALLS OF DIFFERENT EXPONENT ARE POSITIVELY CORRELATED

If A, B are two symmetric convex sets a general conjecture stated in [DG-E-...] predicts that they are positively correlated for gaussian measures, *i. e.*

$$\mathrm{P}(\mathrm{A}|\mathrm{B}) \geqq \mathrm{P}(\mathrm{A}).$$

We here see that the conjecture is true for Hölder balls. Precisely, let us denote $\mathrm{B}_\alpha(\rho) = \{\|w.\|_\alpha \leqq \rho\}$ and $\mathrm{B}'_\alpha(\rho) = \{\|w.\|'_\alpha \leqq \rho\}$.

THEOREM 3. – *If* R *is sufficient large and if* r *is fixed then* $\mathrm{B}_\alpha(\mathrm{R})$ *and* $\mathrm{B}_\beta(r)$ *are positively correlated. This is also true for* $\mathrm{B}'_\alpha(\mathrm{R})$, $\mathrm{B}'_\beta(r)$ *and* $\mathrm{B}'_\alpha(\mathrm{R})$, $\mathrm{B}_\beta(r)$.

Proof. – We proved in Corollary 1, when $r = 1$ for example, that

$$P(B_\alpha(R)^C | B_\beta(1)) \leqq c_{\alpha,\beta} \exp(-c'_{\alpha,\beta} R^{(1-2\beta)/(\alpha-\beta)}), \tag{16}$$

for every $0 \leqq \beta < \alpha < \frac{1}{2}$. We can compare this estimate with the classical gaussian estimate, for R large,

$$P(\|w.\|_\alpha > R) \leqq \exp(-c_\alpha R^2) \tag{17}$$

(*see* [BA-L] or [B-BA-K] for other consequences of this inequality).

By large deviations principle one obtains in fact,

$$P(B_\alpha(R)^C) \sim e^{-c_\alpha R^2},$$

provided R is sufficient large. Therefore, by (16)

$$P(B_\alpha(R) | B_\beta(1)) \geqq P(B_\alpha(R)).$$

So, in this particular case, the general conjecture is valid: the two symmetric convex sets $B_\alpha(R)$ and $B_\beta(1)$ are positively correlated.

4. CONDITIONAL TAILS FOR OSCILLATIONS OF STOCHASTIC INTEGRALS

We shall estimate the Hölder norm of some stochastic integrals.

Let $\sigma_k(t, x)$, $k = 1, \cdots, m$, $b(t, x)$ be smooth vector fields on $\mathbb{R}^{d+1}$ and let us denote $(w^1, \cdots, w^m)$ a m-dimensional Brownian motion. Let P_x be the law of the diffusion (x_t), the solution of the Stratonovich stochastic differential equation

$$dx_t = \sum_{k=1}^{m} \sigma_k(t, x_t) \circ dw_t^k + b(t, x_t)\, dt, \qquad \text{with} \quad x_0 = x. \tag{18}$$

Also, we use the following class of stochastic processes:

DEFINITION 1. – *For* $\alpha, \beta \in \left[0, \frac{1}{2}\right[$ *and* $u \in [0, 1]$ *we will denote* $\mathcal{M}_u^{\alpha,\beta}$ *the set of stochastic processes* Y *such that*

$$\lim_{M \uparrow \infty} \sup_{0 < \delta \leqq 1} P(\|Y.\|_\alpha > M\delta^u \,|\, \|w.\|_\beta < \delta) = 0. \tag{19}$$

Here and elsewhere $\|w.\|_\alpha = \max_{1 \leqq i \leqq m} \|w.^i\|_\alpha$. We collect our results in the following lemma:

LEMMA 3. – *Let* $f : \mathbb{R}^d \to \mathbb{R}$ *be a smooth function and for* i, $j \in \{1, \cdots, r\}$ *we denote*

$$\eta_t^{ij} = \frac{1}{2}\int_0^t (w_s^i\, dw_s^j - w_s^j\, dw_s^i), \qquad \xi_t^{ij} = \int_0^t w_s^i \circ dw_s^j. \tag{20}$$

Then

(i) $w_\cdot^i \in \mathcal{M}_u^{\alpha,\beta}$, *for* $0 \leqq \beta < \alpha < \frac{1}{2}$ *and* $u \in \left[0, \frac{1-2\alpha}{1-2\beta}\right[$.

(ii) $\eta_\cdot^{ij} \in \mathcal{M}_u^{\alpha,0}$, *for* $\alpha \in \left[0, \frac{1}{2}\right[$ *and* $u \in [0, 1]$.

(iii) $\xi_\cdot^{ij} \in \mathcal{M}_u^{\alpha,0}$, *for* $\alpha \in \left[0, \frac{1}{2}\right[$ *and* $u \in [0, 1]$.

(iv) $\int_0^\cdot f(x_s)\, d\xi_s^{ij} \in \mathcal{M}_u^{\alpha,0}$, *for* $\alpha \in \left[0, \frac{1}{2}\right[$ *and* $u \in [0, 1]$.

(v) $\int_0^\cdot f(x_s) \circ dw_s^i \in \mathcal{M}_u^{\alpha,0}$, *for* $\alpha \in \left[0, \frac{1}{2}\right[$ *and* $u \in [0, 1-2\alpha[$.

Proof. – Clearly (i) is proved in Theorem 2.

(ii) We proceed as in [S-V]. There exists a one dimensional Brownian motion B such that when $i \neq j$

$$\eta_t^{ij} = \mathrm{B}(a(t)), \qquad a(t) = \frac{1}{4}\int_0^t ((w_s^i)^2 + (w_s^j)^2)\, ds,$$

where B is independent of the process $(w_t^i)^2 + (w_t^j)^2$ and so independent of $\|w.\|_0$. There exists a positive constant c such that $\|a\|_0$, $\|a\|_1$ are bounded by $c\|w.\|_0$. Then we can write

$$\begin{aligned}&\mathrm{P}(\|\eta_\cdot^{ij}\|_\alpha > \mathrm{M}\,\delta^u, \mid \|w.\|_0 < \delta)\\ &\quad = \mathrm{P}(\|w.\|_0 < \delta)^{-1} \cdot \mathrm{P}(\|\mathrm{B}(a(\cdot))\|_\alpha > \mathrm{M}\,\delta^u, \|w.\|_0 < \delta).\end{aligned}$$

If z is α-Hölder, $\tilde{z}$ is β-Hölder then $z \circ \tilde{z}$ is $\alpha\beta$-Hölder and

$$\|z \circ \tilde{z}\|_{\alpha\beta} \leqq \|z\|_{\alpha, \|\tilde{z}\|_0} \cdot \|\tilde{z}\|_\beta^\alpha,$$

so that

$$\|\mathrm{B}(a(\cdot))\|_\alpha \leqq \|\mathrm{B}\|_{\alpha, \|a\|_0} \cdot \|a\|_1^\alpha.$$

(Here and elsewhere $\|\cdot\|_{\alpha,\mathrm{T}}$ denotes the Hölder norm on $[0, \mathrm{T}]$.)

Therefore

$$\begin{aligned}&\mathrm{P}(\|\mathrm{B}(a(\cdot))\|_\alpha > \mathrm{M}\,\delta^u, \|w.\|_0 < \delta)\\ &\quad \leqq \mathrm{P}(\|\mathrm{B}\|_{\alpha, c\|w.\|_0^2}\, c\|w.\|_0^{2\alpha} > \mathrm{M}\,\delta^u, \|w.\|_0 < \delta).\end{aligned}$$

A scaling in Hölder norm shows that $\|B\|_{\alpha,\tau^2}$ and $\tau^{1-2\alpha}\|B\|_{\alpha,1}$ have the same law. Then we can write

$$\begin{aligned}&P(\|\eta^{ij}_{\cdot}\|_\alpha > M\,\delta^u \mid \|w_{\cdot}\|_0 < \delta)\\ &\leqq P(\|w_{\cdot}\|_0 < \delta)^{-1}\\ &\quad\cdot P(\|B_{\cdot}\|_\alpha\, c\|w_{\cdot}\|_0^{1-2\alpha}\|w_{\cdot}\|_0^{2\alpha} > M\,\delta^u,\ \|w_{\cdot}\|_0 < \delta).\end{aligned}$$

Finally,

$$\begin{aligned}&P(\|\eta^{ij}_{\cdot}\|_\alpha > M\,\delta^u \mid \|w_{\cdot}\|_0 < \delta)\\ &\leqq P(\|B_{\cdot}\|_\alpha c\,\delta > M\,\delta^u) \leqq \exp\left(-\frac{c_\alpha M^2}{\delta^{2(1-u)}}\right),\end{aligned}$$

by the independence of B and $\|w_{\cdot}\|_0$, and the gaussian inequality (17).

(iii) We note another trivial inequality, if z, $\tilde z$ are α-Hölder then $z\,\tilde z$ is α-Hölder and

$$\|z\,\tilde z\|_\alpha \leqq \|z\|_\alpha\,\|\tilde z\|_0 + \|z\|_0\,\|\tilde z\|_\alpha .$$

In particular

$$\|w^i_{\cdot}\,w^j_{\cdot}\|_\alpha \leqq 2\|w_{\cdot}\|_0\,\|w_{\cdot}\|_\alpha .$$

But

$$P(\|w_{\cdot}\|_0\,\|w_{\cdot}\|_\alpha > M\,\delta^u \mid \|w_{\cdot}\|_0 < \delta) \leqq P(\|w_{\cdot}\|_\alpha > M\,\delta^{u-1} \mid \|w_{\cdot}\|_0 < \delta),$$

so the conclusion follows at once from (i), (ii) and

$$\|\xi^{ij}_{\cdot}\|_\alpha \leqq \|\eta^{ij}_{\cdot}\|_\alpha + \frac{1}{2}\|w^i_{\cdot}\,w^j_{\cdot}\|_\alpha \leqq \|\eta^{ij}_{\cdot}\|_\alpha + \|w_{\cdot}\|_0\,\|w_{\cdot}\|_\alpha .$$

(iv) We apply Ito's formula several times (using the usual convention that repeated indices are summed) :

$$\begin{aligned}\int_0^t f(x_s)\,d\xi^{ij}_s &= f(x_t)\,\xi^{ij}_t - \int_0^t f_l(x_s)\,\sigma^l_k(x_s)\,\xi^{ij}_s\,dw^k_s\\ &\quad - \int_0^t (A_s f)(x_s)\,\xi^{ij}_s\,ds - \int_0^t f_l(x_s)\,\sigma^l_j(x_s)\,w^i_s\,ds\\ &= I_1 + I_2 + I_3 + I_4\end{aligned}$$

and it is sufficient to verify (iv) for each I_i, $i = 1, 2, 3, 4$ [here A_t is the generator of the diffusion (x_t)]. We readily see that

$$(a)\qquad I_3,\ I_4 \in \mathcal{M}^{\alpha,0}_u,$$

because

$$\|\mathrm{I}_4\| \leqq c\|w.\|_0 \quad \text{and} \quad \|\mathrm{I}_3\| \leqq c\|\xi^{ij}_{.}\|_0,$$

so we consider only I_1 and I_2. We first study I_1:

$$\begin{aligned}\mathrm{I}_1 &= f(x)\,\xi^{ij}_t + \left(\int_0^t (\mathrm{A}_s f)(x_s)\,ds\right)\xi^{ij}_t \\ &\quad + \left(\int_0^t f_l(x_s)\,\sigma^l_k(x_s)\,\xi^{ij}_s\,dw^k_s\right)\xi^{ij}_t \\ &= \mathrm{I}_{10} + \mathrm{I}_{11} + \mathrm{I}_{12}.\end{aligned}$$

We have again

$$(b) \qquad \mathrm{I}_{10},\ \mathrm{I}_{11} \in \mathcal{M}^{\alpha,0}_u,$$

because

$$\|\mathrm{I}_{10}\|_\alpha = c\|\xi^{ij}_{.}\|_\alpha, \qquad \|\mathrm{I}_{11}\|_\alpha \leqq c\|\xi^{ij}_{.}\|_\alpha$$

and setting $\alpha_k(x) \equiv -f_l(x)\,\sigma^l_k(x)$, $\alpha_{k,m} = \dfrac{\partial \alpha_k}{\partial x^m}$ we can write

$$\begin{aligned}\mathrm{I}_{12} &= -\alpha_k(x_t)\,w^k_t\,\xi^{ij}_t - \left(\int_0^t w^k_s\,(\mathrm{A}_s\,\alpha_k)(x_s)\,ds\right)\xi^{ij}_t \\ &\quad - \left(\int_0^t w^k_s\,\alpha_{k,m}(x_s)\,\sigma^m_n(x_s)\,dw^n_s\right)\xi^{ij}_t \\ &\quad + \left(\int_0^t (\alpha_{k,p})^2(x_s)\,(\sigma^p_k)^2(x_s)\,ds\right)\xi^{ij}_t \\ &= \mathrm{I}_{121} + \mathrm{I}_{122} + \mathrm{I}_{123} + \mathrm{I}_{124}.\end{aligned}$$

There is no problem to see that

$$\|\mathrm{I}_{122}\|_\alpha \leqq c\|w.\|_0\,\|\xi^{ij}_{.}\|_\alpha \quad \text{and} \quad \|\mathrm{I}_{124}\|_\alpha \leqq c\|\xi^{ij}_{.}\|_\alpha$$

and so

$$(c) \qquad \mathrm{I}_{122},\ \mathrm{I}_{124} \in \mathcal{M}^{\alpha,0}_u.$$

There exists a one-dimensional Brownian motion B such that

$$\mathrm{I}_{123} = \mathrm{B}(a(t))\,\xi^{ij}_t, \qquad a(t) = \int_0^t (\alpha_{k,m}\,\alpha_{k',m'}\,a^{mm'})(x_s)\,w^k_s\,w^{k'}_s\,ds$$

$\left(\text{here } a^{ij} = \sum_{k=1}^{m} \sigma_k^i \sigma_k^j\right)$. We can write

$$\begin{aligned} &\mathrm{P}(\|\mathrm{I}_{123}\|_\alpha > \mathrm{M}\,\delta^u \mid \|w_.\|_0 < \delta) \leqq \mathrm{P}(\|\xi_.^{ij}\|_\alpha > \mathrm{M}^{1/2}\,\delta \mid \|w_.\|_0 < \delta) \\ &\quad + \mathrm{P}(\|\mathrm{B}_.\|_{\alpha,\, c\|w_.\|_0^2}\, c\|w_.\|_0^{2\alpha}\|\xi_.^{ij}\|_\alpha \\ &\quad > \mathrm{M}\delta^u,\ \|\xi_.^{ij}\|_\alpha \leqq \mathrm{M}^{1/2}\,\delta \mid \|w_.\|_0 < \delta). \end{aligned}$$

By (iii) we must consider only the second term:

$$\begin{aligned} &\mathrm{P}(\|\mathrm{B}_.\|_\alpha > c\,\mathrm{M}^{1/2}\,\delta^{u-2}) \cdot \mathrm{P}(\|w_.\|_0 < \delta)^{-1} \\ &\leqq \exp\left(-\frac{c_\alpha\,\mathrm{M}}{\delta^{2\,(2-u)}} + \frac{c}{\delta^2}\right). \end{aligned}$$

This yields

$$(d) \qquad \mathrm{I}_{123} \in \mathcal{M}_u^{\alpha,\,0}.$$

Next we consider I_2:

$$\begin{aligned} \mathrm{I}_2 &= \int_0^t \alpha_k(x_s)\,\xi_s^{ij}\,dw_s^k \\ &= \alpha_k(x_t)\,\xi_t^{ij}\,w_t^k - \int_0^t \alpha_{k,\,l}(x_s)\,(\sigma_m^l)(x_s)\,\xi_s^{ij}\,w_s^k\,dw_s^m \\ &\quad - \int_0^t (\mathrm{A}_s\,\alpha_k)(x_s)\,\xi_s^{ij}\,w_s^k\,ds - \int_0^t \alpha_k(x_s)\,w_s^k\,d\xi_s^{ij} \\ &\quad - \int_0^t \alpha_j(x_s)\,w_s^i\,ds - \int_0^t \xi_s^{ij}\,\alpha_{k,\,l}(x_s)\,\sigma_m^l(x_s)\,\delta^{km}\,ds \\ &\quad - \int_0^t w_s^k\,\alpha_{k,\,l}(x_s)\,\sigma_j^l(x_s)\,w_s^i\,ds \\ &= \mathrm{J}_1 + \cdots + \mathrm{J}_7. \end{aligned}$$

Clearly,

$$(e) \qquad \mathrm{I}_{121} + \mathrm{J}_1 = 0$$

and

$$\|\mathrm{J}_3\|_\alpha \leqq c\|w_.\|_0\,\|\xi_.^{ij}\|_0, \qquad \|\mathrm{J}_5\|_\alpha \leqq c\|w_.\|_0,$$
$$\|\mathrm{J}_6\|_\alpha \leqq c\|\xi_.^{ij}\|_0, \qquad \|\mathrm{J}_7\|_\alpha \leqq c\|w_.\|_0^2,$$

So

$$(f) \qquad \mathrm{J}_3,\ \mathrm{J}_5,\ \mathrm{J}_6,\ \mathrm{J}_7 \in \mathcal{M}_u^{\alpha,0}.$$

By the same reasoning

$$\mathrm{J}_2 = \mathrm{B}(a(t)), \qquad a(t) = \int_0^t (\xi_s^{ij})^2 (\alpha_{k,l}\,\alpha_{k',l'}\,a^{ll'})(x_s)\, w_s^k\, w_s^{k'}\, ds,$$

so it suffices to estimate

$$\begin{aligned}
&\mathrm{P}(\|\mathrm{J}_2\|_\alpha > \mathrm{M}\,\delta^u,\ \|\xi^{ij}\|_0 \leqq \mathrm{M}^{1/2}\,\delta \mid \|w.\|_0 < \delta)\\
&\quad \leqq \mathrm{P}(\|\mathrm{B}.\|_{\alpha,\, c\|\xi_.^{ij}\|_0^2 \|w.\|_0^2}\, c\|\xi_.^{ij}\|_0^{2\alpha} \|w.\|_0^{2\alpha} > \mathrm{M}\delta^u,\\
&\qquad\qquad \|\xi_.^{ij}\|_0 < \mathrm{M}^{1/2}\,\delta \mid \|w.\|_0 < \delta)\\
&\quad \leqq \mathrm{P}(\|\mathrm{B}.\|_\alpha > c\,\mathrm{M}^{1/2}\,\delta^{u-2}) \cdot \mathrm{P}(\|w.\|_0 < \delta)^{-1}\\
&\quad \leqq \exp\left(-\frac{c_\alpha\,\mathrm{M}}{\delta^{2(2-u)}} + \frac{c}{\delta^2}\right).
\end{aligned}$$

Again

$$(g) \qquad \mathrm{J}_2 \in \mathcal{M}_u^{\alpha,0}.$$

Finally we have to study the martingale part of J_4 (the bounded variation being obviously controlled). We can write as above

$$\int_0^t \alpha_k(x_s)\, w_s^k\, w_s^i\, dw_s^j = \mathrm{B}(a(t)), \qquad a(t) = \int_0^t \alpha_k^2(x_s)\,(w_s^k\, w_s^i)^2\, ds.$$

Obviously

$$\begin{aligned}
&\mathrm{P}\left(\left\| \int_0^{\cdot} \alpha_k(x_s)\, w_s^k\, w_s^i\, dw_s^j \right\|_\alpha > \mathrm{M}\,\delta^u \,\Big|\, \|w.\|_0 < \delta\right)\\
&\quad \leqq \mathrm{P}(\|\mathrm{B}.\|_\alpha > c\,\mathrm{M}\,\delta^{u-2}) \cdot \mathrm{P}(\|w.\|_0 < \delta)^{-1}\\
&\quad \leqq \exp\left(-\frac{c_\alpha\,\mathrm{M}^2}{\delta^{2(2-u)}} + \frac{c}{\delta^2}\right).
\end{aligned}$$

So that

$$(h) \qquad \mathrm{J}_4 \in \mathcal{M}_u^{\alpha,0}.$$

Using formulas $(a) - (h)$ we can conclude that $\int_0^{\cdot} f(x_s)\, d\xi_s^{ij} \in \mathcal{M}_u^{\alpha,0}$.

(v) We use the same idea, namely we shall apply Ito's formula several times. Firstly, denoting $\dfrac{\partial f}{\partial x^l} = f_l$:

$$\int_0^t f(x_s)\, dw_s^i = f(x)\, w_t^i + \int_0^t dw_s^i \int_0^s (\mathrm{A}_u f)(x_u)\, du$$
$$+ \int_0^t dw_s^i \int_0^s f_l(x_u)\, \sigma_j^l(x_u)\, dw_u^j$$
$$= \mathrm{S}_1 + \mathrm{S}_2 + \mathrm{S}_3.$$

We have $\|\mathrm{S}_1\|_\alpha \leqq c\|w.\|_\alpha$ and

$$\mathrm{S}_2 = w_t^i \int_0^t (\mathrm{A}_s f)(x_s)\, ds - \int_0^t w_s^i (\mathrm{A}_s f)(x_s)\, ds = \mathrm{S}_{21} + \mathrm{S}_{22},$$

where $\|\mathrm{S}_{21}\|_\alpha \leqq c\|w.\|_\alpha$ and $\|\mathrm{S}_{22}\|_\alpha \leqq c\|w.\|_0$. Clearly we have

$$\mathrm{S}_1,\, \mathrm{S}_{21},\, \mathrm{S}_{22} \in \mathcal{M}_u^{\alpha,0}.$$

Next, with the same notation as in (iv),

$$\mathrm{S}_3 = -w_t^i \int_0^t \alpha_j(x_s)\, dw_s^j + \int_0^t w_s^i\, \alpha_j(x_s)\, dw_s^j + \int_0^t \alpha_j(x_s)\, ds$$
$$= \mathrm{S}_{31} + \mathrm{S}_{32} + \mathrm{S}_{33}.$$

By (iv) it is clear that

$$\mathrm{S}_{32} = \int_0^t \alpha_j(x_s)\, d\xi_s^{ij} \in \mathcal{M}_u^{\alpha,0}, \qquad \text{if} \quad i \neq j.$$

For $i = j$ we get a term with the same form as S_{33}, terms which are bounded in Hölder norm by a constant. To prove (v) it is sufficient to prove that $\mathrm{S}_{31} \in \mathcal{M}_u^{\alpha,0}$. But:

$$\mathrm{S}_{31} = -w_t^i\, w_t^j\, \alpha_j(x) - w_t^i \int_0^t dw_s^j \int_0^s (\mathrm{A}_u\, \alpha_j)(x_u)\, du$$
$$- w_t^i \int_0^t dw_s^j \int_0^s \alpha_{j,\,l}(x_u)\, \sigma_k^l(x_u)\, dw_u^k$$
$$= \mathrm{S}_{311} + \mathrm{S}_{312} + \mathrm{S}_{313}.$$

We have $\|\mathrm{S}_{311}\|_\alpha \leqq c\|w.\|_0\, \|w.\|_\alpha$ and

$$\mathrm{S}_{312} = w_t^i\, w_t^j \int_0^t (\mathrm{A}_s\, \alpha_j)(x_s)\, ds$$
$$-w_t^i \int_0^t w_s^j\, (\mathrm{A}_s\, \alpha_j)(x_s)\, ds = \mathrm{S}_{3121} + \mathrm{S}_{3122},$$

where

$$\|S_{3121}\|_\alpha \leqq c\|w.\|_\alpha \|w.\|_0 \quad \text{and} \quad \|S_{3122}\|_\alpha \leqq c\|w.\|_\alpha \|w.\|_0.$$

Again we have

$$S_{311},\ S_{3121},\ S_{3122} \in \mathcal{M}_u^{\alpha,0}.$$

We note $\beta_k(x) = -\alpha_{j,l}(x)\,\sigma_k^l(x)$ and then

$$\begin{aligned} S_{313} &= w_t^i\, w_t^j \int_0^t \beta_k(x_s)\,dw_s^k - w_t^i \int_0^t w_s^j\,\beta_k(x_s)\,dw_s^k - w_t^i \int_0^t \beta_k(x_s)\,ds \\ &= S_{3131} + S_{3132} + S_{3133}. \end{aligned}$$

Arguing as for S_{32}, S_{33} we see that $S_{3132} = -w_t^i \int_0^t \beta_k(x_s)\,d\xi_s^{jk}$, $j \neq k$ and S_{3133} are in $\mathcal{M}_u^{\alpha,0}$. We repeat with S_{3131} the computations we already performed for S_{31} and we easily see that the terms (with analogous notations)

$$S_{31311},\ S_{313121},\ S_{313122},\ S_{313133} \in \mathcal{M}_u^{\alpha,0}.$$

Then $S_{313132} = w_t^i\, w_t^j \int_0^t \gamma_l(x_s)\,d\xi_s^{kl}$, $l \neq k$, where $\gamma_l = \beta_m(x)\,\sigma_l^m(x)$, so S_{313132} satisfies (v) as above. To control the Hölder norm of S_{313131} we can write

$$S_{313131} = w_t^i\, w_t^j\, w_t^k \int_0^t \gamma_l(x_s)\,dw_s^l = w_t^i\, w_t^j\, w_t^k\, B(a(t)),$$

$$a(t) = \int_0^t \gamma_l^2(x_s)\,ds,$$

where B is a one-dimensional Brownian motion. We have

$$\begin{aligned} & P(\|S_{313131}\|_\alpha > M\,\delta^u \mid \|w.\|_0 < \delta) \\ &\quad \leqq P(\|w.\|_\alpha > M^{1/2}\,\delta^{u-(1/2)} \mid \|w.\|_0 < \delta) \\ &\quad + P(\|B.\|_\alpha c\|w.\|_\alpha \|w.\|_0^2 > M\,\delta^u,\ \|w.\|_\alpha \\ &\qquad \leqq M^{1/2}\,\delta^{u-(1/2)} \mid \|w.\|_0 < \delta) \\ &\quad \leqq P(\|w.\|_\alpha > M^{1/2}\,\delta^{u-(1/2)} \mid \|w.\|_0 < \delta) \\ &\quad + \exp\left(-\frac{c_\alpha M}{\delta^3} + \frac{c}{\delta^2}\right). \end{aligned}$$

From this we can easily conclude that S_{313131} satisfies (v) and the proof of the lemma is complete.

Q.E.D.

5. THE SUPPORT THEOREM IN HÖLDER NORM

Now we are able to extend the support theorem of Stroock-Varadhan for α-Hölder topology. Let us denote by Φ_x the mapping wich associates to $h \in \mathrm{L}^2 = \mathrm{L}^2([0, 1], \mathbb{R}^m)$ the solution of the differential equation

$$dy_t = \sum_{k=1}^{m} \sigma_k(t, y_t)\, h_t^k\, dt + b(t, y_t)\, dt, \qquad \text{with} \quad y_0 = x. \tag{21}$$

THEOREM 4. – *Let* $\alpha \in \left[0, \frac{1}{2}\right[$. *The support of the probability* P_x *for the norm* $\|\cdot\|_\alpha$ *coincide with the closure of* $\Phi_x(\mathrm{L}^2)$, i.e.

$$\operatorname{supp}_\alpha(\mathrm{P}_x) = \overline{\Phi_x(\mathrm{L}^2)}^{\alpha}. \tag{22}$$

Proof. – To begin with, we note that for every $\varepsilon > 0$ and $\delta = \left(\frac{\varepsilon}{2^n}\right)^{1/u}$, $u \in]0, 1 - 2\alpha[$, $n > 0$ integer, we have

$$\begin{aligned}
&\mathrm{P}\left(\left\| \int_0^{\cdot} \sigma_k(s, x_s) \circ dw_s^k \right\|_\alpha > \varepsilon \,|\, \|w.\|_0 < \delta\right) \\
&= \mathrm{P}\left(\left\| \int_0^{\cdot} \sigma_k(s, x_s) \circ dw_s^k \right\|_\alpha > 2^n \delta^u |\, \|w.\|_0 < \delta\right). \\
&\leqq \sup_{0<\eta\leqq 1} \mathrm{P}\left(\left\| \int_0^{\cdot} \sigma_k(s, x_s) \circ dw_s^k \right\|_\alpha > 2^n \eta^u |\, \|w.\|_0 < \eta\right).
\end{aligned}$$

Letting $\eta \uparrow \infty$, by (v) of Lemma 3 we obtain, for every $\varepsilon > 0$

$$\lim_{\delta \downarrow 0} \mathrm{P}\left(\left\| \int_0^{\cdot} \sigma_k(s, x_s) \circ dw_s^k \right\|_\alpha > \varepsilon \,|\, \|w.\|_0 < \delta\right) = 0. \tag{23}$$

Then we prove that, for every $\varepsilon > 0$

$$\lim_{\delta \downarrow 0} \mathrm{P}(\|x. - \Phi_x(0)\|_\alpha < \varepsilon \,|\, \|w.\|_0 < \delta) = 1, \tag{24}$$

using (23) and the following variant of Gronwall's lemma:

LEMMA 4. – *Let*

$$z_t = z + m(t) + \int_0^t l(z_s)\, ds, \qquad \tilde{z}_t = z + \int_0^t l(\tilde{z}_s)\, ds,$$

where $\|m\|_\alpha \leqq \epsilon$, $m(0) = 0$ *and* l *is a Lipschitz continuous function with Lipschitz constant* L. *Then*

$$\|z - \tilde{z}\|_\alpha \leqq (1 + \mathrm{L})\, e^{\mathrm{L}}\, \varepsilon.$$

Proof. – By Gronwall's lemma we can immediately write

$$\|z - \tilde{z}\|_0 \leqq \varepsilon\, e^{\mathrm{L}}.$$

Next we have

$$\begin{aligned}
\|z - \tilde{z}\|_{\alpha, t} &\leqq \epsilon + \left\| \int_0^{\cdot} (l(z_u) - l(\tilde{z}_u))\, du \right\|_{\alpha, t} \\
&\leqq \epsilon + \max_{0 \leqq p < q \leqq t} \frac{\mathrm{L}}{|p - q|^\alpha} \left| \int_q^p |z_u - \tilde{z}_u| du \right| \\
&\leqq \epsilon + \max_{0 \leqq p < q \leqq t} \frac{\mathrm{L}}{|p - q|^\alpha} \\
&\quad \times \left| \int_q^p (|z_q - \tilde{z}_q| + |u - q|^\alpha \, \|z - \tilde{z}\|_{\alpha, u})\, du \right| \\
&\leqq \epsilon + \mathrm{L}\, \|z - \tilde{z}\|_0 + \mathrm{L} \int_0^t \|z - \tilde{z}\|_{\alpha, u}\, du.
\end{aligned}$$

Gronwall's lemma ends up the proof of Lemma 4.

We apply this lemma with $z = \Phi_x(w.)$, $\tilde{z} = \Phi_x(0)$, $m(t) = \int_0^t \sigma_k(s, x_s) \circ dw_s^k$ and $l(x_s) = b(s, x_s)$. So, there exists a positive constant K such that

$$\|\Phi_x(w.) - \Phi_x(0)\|_\alpha < \mathrm{K}\,\varepsilon,$$

provided,

$$\left\| \int_0^{\cdot} \sigma_k(s, x) \circ dw_s^k \right\|_\alpha \leqq \varepsilon.$$

Thus we can write

$$\begin{aligned}
&\mathrm{P}(\|x. - \Phi_x(0)\|_\alpha > \varepsilon | \, \|w.\|_0 < \delta) \\
&= \mathrm{P}\Big((\|x. - \Phi_x(0)\|_\alpha > \varepsilon) \\
&\quad \cap \Big(\Big\| \int_0^{\cdot} \sigma_k(s, x_s) \circ dw_s^k \Big\|_\alpha > \frac{\varepsilon}{\mathrm{K}} \Big) \,|\, \|w.\|_0 < \delta \Big) \\
&\leqq \mathrm{P}\Big(\Big(\Big\| \int_0^{\cdot} \sigma_k(s, x_s) \circ dw_s^k \Big\|_\alpha > \frac{\varepsilon}{\mathrm{K}} \Big) \,|\, \|w.\|_0 < \delta \Big).
\end{aligned}$$

(24) is now a clear consequence of (23).

Finally Girsanov's formula gives for any $h \in \mathrm{L}^2$ and $\varepsilon > 0$

$$\lim_{\delta \downarrow 0} \mathrm{P}(\|\Phi_x(w.) - \Phi_x(h.)\|_\alpha < \varepsilon \,|\, \|w. - h.\|_0 < \delta) = 1 \tag{25}$$

(as in [S-V], Th. 5.1, p. 353). But, (25) implies

$$\mathrm{P}\left(\|\Phi_x(w_.) - \Phi_x(h_.)\|_\alpha < \varepsilon\right) > 0, \quad \text{for every } \varepsilon > 0 \tag{26}$$

and, consequently, we obtain the inclusion

$$\operatorname{supp}_\alpha (\mathrm{P}_x) \supseteqq \overline{\Phi_x(\mathrm{L}^2)}^{\alpha}. \tag{27}$$

The converse inclusion is easily obtained using the polygonal approximation of the Brownian motion. For each $n \geqq 0$ and $t \geqq 0$ we consider

$$t_n = \frac{[2^n]}{2^n}, \qquad t_n^+ = \frac{[2^n]+1}{2^n}, \qquad \dot{w}_t^{(n)} = 2^n\,(w_{t_n^+} - w_{t_n})$$

and let $(x_t^{(n)})$ be the solution of the equation (21) with $\dot{w}_t^{(n)\,k}$ instead h_t^k. If one denotes $\mathrm{P}_x^{(n)}$ the law of this solution it is obvious that

$$x_.^{(n)} \in \Phi_x(\mathrm{L}^2) \qquad \text{and} \qquad \mathrm{P}_x^{(n)}\left(\overline{\Phi_x(\mathrm{L}^2)}^{\alpha}\right) = 1.$$

It suffices to show that P_x is the weak limit of $(\mathrm{P}_x^{(n)})$ or, that is relatively weakly compact with respect to $\|\cdot\|_\alpha$-topology. By classical estimates, for every $p \geqq 0$ there exists a positive constant c_p such that for every positive integer n and for every $s, t \in [0, 1]$,

$$\mathrm{E}|x_t^{(n)} - x_s^{(n)}|^{2p} \leqq c_p |t-s|^p.$$

(*see* for instance [Bi], Chap. I, Prop. 1.3). It is easy to see that

$$\sup_n \mathrm{E}\left(\|x_.^{(n)}\|_{\alpha'}^{2p}\right) < c, \qquad \text{if} \quad \alpha' < \frac{p-1}{2p}.$$

If one chooses p large enough so that $\alpha < \dfrac{p-1}{2p}$, and if $\alpha' \in \left]\alpha, \dfrac{p-1}{2p}\right[$, it is clear that the set $\mathrm{K}(c) = \{z : \|z\|_{\alpha'} < c\}$ is compact in $\|\cdot\|_\alpha$-topology, and that for every $\epsilon > 0$ there exists a positive constant c_ε such that

$$\sup_n \mathrm{P}_x^{(n)}(\mathrm{K}(c_\varepsilon)) < \varepsilon.$$

So $(\mathrm{P}_x^{(n)})$ is tight. The proof of Theorem 4 is now complete.

Q.E.D.

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APPENDIX

We give now another proof of a variant of (9) [or (14)] when $\beta = 0$ which does not require the use of Ciesielski's theorem [*i.e.* (4) and (5)] nor the correlation inequality.

THEOREM 5. – *Let* (r, R) *be a couple of real positive numbers. For every* $a' < a$ *and* $b' > b$ *there exists a constant* c *such that, if* $\frac{\mathrm{R}^{a'}}{r^{b'}} > c$, *then*

$$\mathrm{P}\left((\|w\|_\alpha > \mathrm{R}) \cap (\|w\|_\beta < r)\right) \leqq \exp\left(-\frac{1}{2}\,\frac{\mathrm{R}^{(1-2\beta)/(\alpha-\beta)}}{r^{(1-2\alpha)/(\alpha-\beta)}}\right), \tag{28}$$

for $0 \leqq \beta < \alpha < \frac{1}{2}$.

Proof. – Let us consider

$$\eta = \left(\frac{r}{\mathrm{R}}\right)^{1/(\alpha-\beta)}$$

Then if $\|w\|_\beta < r$ we have

$$\sup_{s<t,\, t-s>\eta} \frac{|w_t - w_s|}{|t-s|^\alpha} \leqq \mathrm{R}.$$

Thus we can write

$$\begin{aligned} &\left((\|w\|_\alpha > \mathrm{R}) \cap (\|w\|_\beta < r)\right) \\ &\subset \left(\left(\sup_{s<t\leqq s+\eta} \frac{|w_t - w_s|}{|t-s|^\alpha} \geqq \mathrm{R}\right) \cap \left(\sup_t |w_t| < r\right)\right) \\ &\subset \left(\sup_{s<t\leqq s+\eta} \frac{|w_t - w_s|}{|t-s|^\alpha} \geqq \mathrm{R}\right) = \left(\sup_{v\in \mathrm{D}} |\mathrm{X}_v^\alpha| \geqq \mathrm{R}\right) \end{aligned}$$

where $v = (s, t)$, $\mathrm{D} = \{v : s < t \leqq s + \eta\}$ and $\mathrm{X}_v^\alpha = \frac{w_t - w_s}{|t-s|^\alpha}$ is a two-parameter gaussian variable. Now we can estimate

$$\begin{aligned} &\mathrm{P}\left((\|w\|_\alpha > \mathrm{R}) \cap (\|w\|_\beta < r)\right) \\ &\leqq \mathrm{P}\left(\sup_{v\in \mathrm{D}} |\mathrm{X}_v^\alpha| \geqq \mathrm{R}\right) \leqq \exp\left(-\frac{(\mathrm{R} - \mathrm{M}_\alpha)^2}{2\,\sigma_\alpha^2}\right), \end{aligned}$$

where the last inequality is valid when $\mathrm{R} \geqq \mathrm{M}_\alpha$ (*see* [L-T] (L.3.1., Sec. 3.1, p. 57)). Here

$$0 < \mathrm{M}_\alpha = \mathrm{E}\left(\sup_{v\in \mathrm{D}} |\mathrm{X}_v^\alpha|\right) \leqq \mathrm{E}\left(\|w\|_\alpha\right) < \infty$$

and

$$\sigma_\alpha^2 = \sup_{v \in D} E((X_v^\alpha)^2) = \eta^{1-2\alpha}.$$

So we obtain

$$\begin{aligned} &P((\|w\|_\alpha > R) \cap (\|w\|_\beta < r)) \\ &\leqq \exp\left(-\frac{R^2}{2\,\eta^{1-2\alpha}}\right) = \exp\left(-\frac{1}{2}\,\frac{R^{(1-2\beta)/(\alpha-\beta)}}{r^{(1-2\alpha)/(\alpha-\beta)}}\right), \qquad \beta \geqq 0. \end{aligned}$$

The restriction $R \geqq M_\alpha$ may be weakened as follows. Let $\alpha' > \alpha$ and we can write

$$\begin{aligned} \left(\sup_{s<t\leqq s+\eta} \frac{|w_t - w_s|}{|t-s|^\alpha} \geqq R\right) &= \left(\sup_{s<t\leqq s+\eta} \frac{|w_t - w_s|}{|t-s|^{\alpha'}} \cdot |t-s|^{\alpha'-\alpha} \geqq R\right) \\ &\subset \left(\sup_{s<t\leqq s+\eta} \frac{|w_t - w_s|}{|t-s|^{\alpha'}} \geqq R\,\eta^{\alpha-\alpha'}\right) \\ &= \left(\sup_{s<t\leqq s+\eta} \frac{|w_t - w_s|}{|t-s|^{\alpha'}} \geqq \frac{R^{(\alpha'-\beta)/(\alpha-\beta)}}{r^{(\alpha'-\beta)/(\alpha-\beta)}}\right). \end{aligned}$$

Now we need only

$$\frac{R^{(\alpha'-\beta)/(\alpha-\beta)}}{r^{(\alpha'-\alpha)/(\alpha-\beta)}} > E(\|w\|_{\alpha'}) = M_{\alpha'},$$

and so the proof of the theorem is complete.

Q.E.D.

Clearly, Theorem 5 implies that

$$\begin{aligned} P(\|w.\|_\alpha > R \,|\, \|w.\|_0 < r) &= \frac{P((\|w.\|_\alpha > R) \cap (\|w\|_0 < r))}{P(\|w\|_0 < r)} \\ &\leqq \exp\left(-\frac{1}{2}\,\frac{R^{1/\alpha}}{r^{(1/\alpha)-2}}\right) \exp\left(\frac{\pi^2}{8} \cdot \frac{1}{r^2}\right) \end{aligned}$$

and we need the condition $\alpha < \dfrac{1}{4}$ for an interesting estimate, if r is small.

At the end of this work we learned that a similar result was obtained independently by Millet-Sanz-Solé [M-S].

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