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On symmetric stable random variables and matrix transposition

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ABSTRACT. – It is shown that the symmetric stable distribution of index α $(0 < \alpha \leq 2)$ may be characterized by an invariance property relative to the transposition of square matrices of all dimensions. This property gives an explanation of the Ciesielski-Taylor identities in law.

Key words : stable variables, matrix transposition.

RÉSUMÉ. – On montre que la loi stable symétrique d'indice α $(0 < \alpha \leq 2)$ peut être caractérisée par une propriété d'invariance relative à la transposition des matrices carrées de toutes dimensions. Cette propriété permet d'expliquer les identités en loi de Ciesielski-Taylor.

Classification A.M.S. : Primary 60 J 65; Secondary 60 J 60.

0. INTRODUCTION

(0.1) The present work takes its origin in the simple proofs given by two of the authors of certain identities in law between some functionals of Brownian motion or Bessel processes (*see* [3] and [6]).

Precisely : (i) if $(B_t, t \geq 0)$ denotes a one-dimensional Brownian motion starting from 0, and $(\tilde{B}_t \,;\, 0 \leq t \leq 1)$ a standard Brownian bridge, then :

$$\int_0^1 ds\,(B_s - G)^2 \overset{\text{(law)}}{=} \int_0^1 ds\,\tilde{B}_s^2, \qquad \text{where} \quad G \overset{\text{def}}{=} \int_0^1 ds\,B_s \tag{0.a}$$

(*see* [3], where this identity in law is obtained, together with several extensions).

(ii) if, for $\delta > 0$, $(R_\delta(t), t \geq 0)$ denotes a δ-dimensional Bessel process starting from 0, then :

$$\int_0^\infty ds\,1_{(R_{\delta+2}(s) \leq 1)} \overset{\text{(law)}}{=} T_1(R_\delta), \tag{0.b}$$

where $T_1(X) \overset{\text{def}}{=} \inf\{t \,:\, X_t = 1\}$.

The identity in law (0.b) is due to Ciesielski-Taylor (1962) for integer dimensions; for a detailed discussion and further extensions, *see* [6] and [7]. The proofs of both identities in law (0.a) and (0.b) rely essentially upon the following (Fubini type) identity in law:

$$\int_0^\infty ds\left(\int_0^\infty \varphi(s, u)\,dB_u\right)^2 \overset{\text{(law)}}{=} \int_0^\infty ds\left(\int_0^\infty \varphi(u, s)\,dB_u\right)^2, \tag{0.c}$$

where $\varphi \in L^2(\mathbb{R}_+^2, ds\,du)$.

(0.2) In the first section of the present work, we show that some discrete analogue of the identity in law (0.c) holds for a sequence of i.i.d. Gaussian variables, namely: if $\underline{G}_n = (G_1, \ldots, G_n)$ is a random vector which consists of n independent N (0, 1) random variables, then the identity in law:

$$l_2(A\underline{G}_n) \overset{\text{(law)}}{=} l_2(A^*\,\underline{G}_n) \tag{1.a$_2$}$$

holds, where A is any $n \times n$ real matrix, and $l_2(x) = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ denotes the euclidean norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

In fact, more generally, we show that, for any $0 < \alpha \leq 2$, we have:

$$l_\alpha \left(\mathrm{A}\underline{\mathrm{C}}_n^{(\alpha)}\right) \overset{(\mathrm{law})}{=} l_\alpha \left(\mathrm{A}^* \underline{\mathrm{C}}_n^{(\alpha)}\right), \qquad (1.\mathrm{a})_\alpha$$

where $\underline{\mathrm{C}}_n^{(\alpha)} = (\mathrm{C}_1^{(\alpha)}, \ldots, \mathrm{C}_n^{(\alpha)})$ now denotes a random vector, the components of which are n standard independent symmetric stable r.v's, with parameter α, and

$$l_\alpha (x) = \left(\sum_{i=1}^{n} |x_i|^\alpha \right)^{1/\alpha}.$$

(0.3) In the second section, we are interested in the study of a converse to the property $(1.\mathrm{a})_\alpha$, namely: if $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n, \ldots$ is a sequence of i.i.d. random variables which satisfies, for any $n \in \mathbb{N}$, and any $n \times n$ matrix A:

$$l_\alpha (\mathrm{A}\underline{\mathrm{X}}_n) \overset{(\mathrm{law})}{=} l_\alpha (\mathrm{A}^* \underline{\mathrm{X}}_n), \qquad (1.\mathrm{a})'_\alpha$$

where $\underline{\mathrm{X}}_n \overset{\mathrm{def}}{=} (\mathrm{X}_1, \ldots, \mathrm{X}_n)$, then we show that X_1 is a symmetric stable random variable of index α.

Hence, in this sense, the property $(1.\mathrm{a})_\alpha$ characterizes the symmetric stable law of index α.

(0.4) In section 3, we consider a fixed finite dimension n, and we try to characterize the laws of n-dimensional random variables $\underline{\mathrm{X}}_n = (\mathrm{X}_1, \ldots, \mathrm{X}_n)$ such that $(1.\mathrm{a})'_\alpha$ is satisfied, but we do not assume any other property on the vector $\underline{\mathrm{X}}_n$.

We obtain a complete description of such vectors for $\alpha = 2$, but our description remains incomplete for $\alpha \neq 2$.

(0.5) In section 4, we go back to the infinite dimensional case; if $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n, \ldots$ is a sequence of random variables which satisfies, for any $n \in \mathbb{N}$, and any $n \times n$ matrix A, the identity in law $(1.\mathrm{a})'_\alpha$ and, if moreover, $\underline{\mathrm{X}}_\infty \overset{(\mathrm{law})}{=} -\underline{\mathrm{X}}_\infty$, then, we prove that:

$$\underline{\mathrm{X}}_\infty \overset{(\mathrm{law})}{=} (\mathrm{H}\mathrm{C}_n^{(\alpha)}\ ;\ n \in \mathbb{N}) \qquad (0.\mathrm{d})$$

where $(\mathrm{C}_n^{(\alpha)}; n \in \mathbb{N})$ is a sequence of independent symmetric standard stable variables of index α, and H is a non negative r.v. which is independent of the sequence $\underline{\mathrm{C}}_\infty^\alpha$. In fact, we may even get rid of the assumption $\underline{\mathrm{X}}_\infty \overset{(\mathrm{law})}{=} -\underline{\mathrm{X}}_\infty$, and the general result is that (0.d) holds up to the introduction of a Bernoulli, ± 1 valued, random variable (*see* Theorem 3 below for a precise statement).

(0.6) As a conclusion of this introduction, we describe how this paper relates to its companions [3], [6] and [7]: whereas in [3] and [6], the authors presented some applications, mainly (0.a) and (0.b), of the Fubini identity

(0.c), the aim of this paper, together with [7], is to understand in a deeper way the role of the identity (0.c):

in the present paper, we restrict ourselves to the case of a (possibly finite) sequence of variables, and, therefore, we discuss identities in law such as $(1.a)_\alpha$ and $(3.a)_\alpha$, whilst, in [7], we consider continuous time processes and, in particular, we characterize all processes $(X_t,\, t \geq 0)$ which satisfy:

$$\int_0^\infty ds \left(\int_0^\infty \varphi(s,\, u)\, dX_u \right)^2 \stackrel{\text{(law)}}{=} \int_0^\infty ds \left(\int_0^\infty \varphi(u,\, s)\, dX_u \right)^2, \qquad (0.c)'$$

for all simple functions $\varphi \,:\, \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$.

1. THE MAIN IDENTITY IN LAW

Let $0 < \alpha \leq 2$, and $n \in \mathbb{N}\backslash\{0\}$. We consider the application $l_\alpha \,:\, \mathbb{R}^n \to \mathbb{R}_+$ defined by :

$$l_\alpha(a) = \left(\sum_{i=1}^n |a_i|^\alpha \right)^{1/\alpha}.$$

We also consider an n-dimensional random vector

$$\underline{C}_n^{(\alpha)} = (C_1^{(\alpha)}, \ldots, C_n^{(\alpha)}),$$

the components of which are n independent, standard, symmetric variables, which are stable with exponent α, that is:

$$E[\exp(i\lambda C_j^{(\alpha)})] = \exp(-|\lambda|^\alpha) \qquad (\lambda \in \mathbb{R}).$$

Then, we have the:

THEOREM 1. – *For any* $n \times n$ *real matrix* A, *we have:*

$$l_\alpha(A\underline{C}_n^{(\alpha)}) \stackrel{\text{(law)}}{=} l_\alpha(A^* \underline{C}_n^{(\alpha)}), \qquad (1.a)_\alpha$$

where A^* *is the transpose of* A.

Proof. – We introduce $\underline{\tilde{C}}_n^{(\alpha)}$ an independent copy of $\underline{C}_n^{(\alpha)}$, and we write:

$$(A\underline{C}_n^{(\alpha)},\, \underline{\tilde{C}}_n^{(\alpha)}) = (\underline{C}_n^{(\alpha)},\, A^*\underline{\tilde{C}}_n^{(\alpha)}).$$

We then compute, in two different manners, the characteristic function of the above random variable; we obtain thus:

$$\mathrm{E}\,[\exp -|\lambda|^{\alpha}(l_{\alpha}\,(\mathrm{A}\underline{\mathrm{C}}_{n}^{(\alpha)})^{\alpha})] = \mathrm{E}\,[\exp -|\lambda|^{\alpha}(l_{\alpha}\,(\mathrm{A}^{*}\tilde{\underline{\mathrm{C}}}_{n}^{(\alpha)})^{\alpha})]$$

for every $\lambda \in \mathbb{R}$.

Using the fact that: $\underline{\mathrm{C}}_{n}^{(\alpha)} \overset{(\text{law})}{=} \tilde{\underline{\mathrm{C}}}_{n}^{(\alpha)}$ and the injectivity of the Laplace transform, we obtain $(1.\mathrm{a})_{\alpha}$. □

Remark 1. – In the case $\alpha = 2$, there is also the following alternative proof: if $\underline{\mathrm{G}}_{n} = (\mathrm{G}_{1}, \ldots, \mathrm{G}_{n})$ is an n-dimensional random vector, with the $\mathrm{G}_{i}'s$ independent, centered, each with variance equal to 1, then we have:

$$(l_{2}\,(\mathrm{A}\underline{\mathrm{G}}_{n}))^{2} = (\mathrm{A}\underline{\mathrm{G}}_{n},\, \mathrm{A}\underline{\mathrm{G}}_{n}) = (\underline{\mathrm{G}}_{n},\, \mathrm{A}^{*}\,\mathrm{A}\underline{\mathrm{G}}_{n}) \overset{(\text{law})}{=} (\underline{\mathrm{G}}_{n},\, \mathrm{A}\mathrm{A}^{*}\,\underline{\mathrm{G}}_{n}),$$

since AA* and A*A have the same eigenvalues, with the same order of multiplicity, and the law of $\underline{\mathrm{G}}_{n}$ is invariant by orthogonal transforms.

Then, the proof is ended by remarking that:

$$(\underline{\mathrm{G}}_{n},\, \mathrm{A}\mathrm{A}^{*}\,\underline{\mathrm{G}}_{n}) = (l_{2}\,(\mathrm{A}^{*}\,\underline{\mathrm{G}}_{n}))^{2}.$$

Obviously, these arguments cannot be used for $\alpha \neq 2$.

2. A CHARACTERIZATION OF THE SYMMETRIC STABLE LAWS

We now consider a given application $l\,:\,\mathrm{FS} \longrightarrow \mathbb{R}_{+}$, where FS is the set of finite sequences $a = (a_{1}, \ldots, a_{n},\, 0,\, 0,\, \ldots)$ for some n, and $a_{i} \in \mathbb{R}$, such that the following hypotheses are satisfied:

$$l\,(a) > 0, \quad \text{for every } a \neq 0; \tag{2.a}$$

$$l\,(\lambda a) = |\lambda|\, l\,(a), \quad \text{for every } a, \text{ and } \lambda \in \mathbb{R}. \tag{2.b}$$

We also consider a real-valued random variable X and a sequence of i.i.d. random variables $\mathrm{X}_{1}, \mathrm{X}_{2}, \ldots, \mathrm{X}_{n}, \ldots$, with the same distribution as X, and we write $\underline{\mathrm{X}}_{n}$ for the truncated sequence $(\mathrm{X}_{1}, \ldots, \mathrm{X}_{n}, 0, 0, \ldots)$, which we sometimes identify with the $\mathbb{R}^{n}$-valued r.v.: $(\mathrm{X}_{1}, \ldots, \mathrm{X}_{n})$.

We can now state and prove our main result.

THEOREM 2. – *The following properties are equivalent:*

1) X *is a symmetric stable random variable, with parameter* α;

2) *there exists* $l : \mathrm{FS} \to \mathbb{R}_+$, *which satisfies the properties* (2.a) *and* (2.b) *and such that, for every* $n \in \mathbb{N}^*$, *and every* $n \times n$ *real matrix* A, *we have:*

$$l\,(\mathrm{A}\underline{\mathrm{X}}_n) \overset{(\text{law})}{=} l\,(\mathrm{A}^*\,\underline{\mathrm{X}}_n); \tag{2.c}$$

3) *there exists an application* $\tilde{l} : \mathrm{FS} \to \mathbb{R}_+$ *such that:*

$$\tilde{l}\,(a) > 0, \quad \textit{for every } a \neq 0, \tag{2.ã}$$

and:

$$\textit{for every } a = (a_1, \ldots, a_n, 0, 0, \ldots), \left|\sum_{i=1}^{n} a_i\,\mathrm{X}_i\right| \overset{(\text{law})}{=} \tilde{l}\,(a)|\mathrm{X}_1|. \tag{2.d}$$

Moreover, when 1) *is satisfied, the applications* l *and* $\tilde{l}$ *are given by:*

$$l\,(a) = c\,l_\alpha\,(a), \quad \textit{for some } c > 0, \quad \textit{and} \quad \tilde{l}\,(a) = l_\alpha\,(a).$$

Remark 2. – In the statement of Theorem 2, we have tried to make some minimal hypothesis about the application $l : \mathrm{FS} \to \mathbb{R}_+$, namely (2.a) and (2.b).

However, even these hypotheses may be superfluous, as the following seems to suggest: if X is a symmetric stable random variable with parameter α, and $f : \mathbb{R}_+ \to \mathbb{R}$ is a Borel function, then: $l = f o l_\alpha$ obviously satisfies (2.c).

Such applications l may well be the largest class of applications from FS to $\mathbb{R}$ which satisfy (2.c). □

Notation. – In the proof of Theorem 2, it will be convenient to associate to $x \in \mathbb{R}$ the element $\hat{x}$ of FS defined by $\hat{x} = (x, 0, 0, \ldots)$.

Proof of Theorem 2. – *a*) From Theorem 1, we already know that 1) $\Longrightarrow$ 2), with $l = l_\alpha$.

To prove that 2) $\Longrightarrow$ 3), we remark that, if we take $a = (a_1, a_2, \ldots, a_n, 0, 0, \ldots)$, and the $n \times n$ matrix

$$\mathrm{A} = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ a_n & 0 & \ldots & 0 \end{pmatrix}, \quad \text{so that: } \mathrm{A}^* = \begin{pmatrix} a_1 & \ldots & a_n \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 \end{pmatrix},$$

we obtain, from our hypothesis

$$l\,(a\mathrm{X}_1) \overset{(\text{law})}{=} l\,((\Sigma a_i\,\mathrm{X}_i)\hat{}\,) \tag{2.c}$$

and now, using (2.b), we have:

$$|X_1|\, l\,(a) \overset{(\text{law})}{=} \left|\sum_{i=1}^{n} a_i\, X_i\right| l\,(\hat{1}). \tag{2.e}$$

Therefore, 3) is satisfied with: $\tilde{l}\,(a) = l\,(a)/l\,(\hat{1})$.

b) It now remains to prove that 3) $\Longrightarrow$ 1), and that, when 1) is satisfied, l and $\tilde{l}$ are determined as announced in the last statement of the Theorem. Indeed, let us assume for one moment that we have proved the implication: 3) $\Longrightarrow$ 1), so that X_1 is symmetric, stable, with exponent α.

Then, we deduce from (2.d) that: $\tilde{l}\,(a)\,|X_1| \overset{(\text{law})}{=} l_\alpha\,(a)|X_1|$, so that: $\tilde{l} = l_\alpha$, and we deduce from (2.e) that:

$$l\,(a) = c l_\alpha\,(a), \qquad \text{with} \quad c = l\,(\hat{1}).$$

c) We now prove that 3) $\Longrightarrow$ 1).

To help the reader with the sequel of the proof, we first assume that X_1 is symmetric; then, we deduce from (2.d) that we have, by taking $a \equiv (1,\, 1,\, \ldots,\, 1,\, 0,\, 0,\, \ldots)$ (1 is featured here n times):

$$X_1 \overset{(\text{law})}{=} \frac{1}{\lambda_n}\left(\sum_{i=1}^{n} X_i\right).$$

This implies (*see* Feller [4], p. 166) that $\lambda_n \sim n^{1/\alpha}$, for some $0 < \alpha \leq 2$, and that X_1 is symmetric, stable, with exponent α.

d) Now, we give the complete proof without assuming *a priori* that X_1 is symmetric.

By taking $a = (1,\, \ldots,\, 1,\, -1,\, \ldots,\, -1,\, 0,\, 0,\, \ldots)$ (the n first components are equal to 1, and the n next ones are equal to –1), we obtain, from our hypothesis (2.d), that:

$$|X_1| \overset{(\text{law})}{=} \frac{1}{\lambda_n}\left|\sum_{i=1}^{n} (X_i - X_i')\right|$$

and, consequently, the (symmetric) law of $\frac{1}{\lambda_n}\sum_{i=1}^{n}(X_i - X_i')$ does not depend on n. Consequently, just as above in *c*), we obtain that $X_1 - X_1'$ is (symmetric) stable with some exponent α.

e) It now remains to show that X_1 is symmetric.

To do this, we shall use the hypothesis (2.d), with $a_1 = (1,\, 1)$, and $a_2 = (1,\, -1)$. Thus, we deduce from (2.d) that:

$$|X_1 + X_2| \overset{(\text{law})}{=} \mu |X_1|, \qquad \text{and} \qquad |X_1 - X_2| \overset{(\text{law})}{=} \nu |X_1|. \tag{2.f}$$

This identity (2.f) is equivalent to:

$$\varepsilon\,(X_1 + X_2) \overset{(\text{law})}{=} \mu\varepsilon X_1 \qquad \text{and} \qquad X_1 - X_2 \overset{(\text{law})}{=} \nu\varepsilon X_1, \tag{2.f'}$$

where ε is a symmetric Bernoulli variable, which is independent of the pair (X_1, X_2).

We define $\phi\,(t) \equiv E\,(\exp i\,(t\,X_1))$, and we remark that, since we now know $X_1 - X_2$ to be (symmetric) stable, with exponent α, we have:

$$|\phi\,(t)| = \exp\,(-c|t|^\alpha), \quad \text{for some } c. \tag{2.g}$$

Hence, the identity (2.f′) may be written as the following pair of identities:

$$\frac{1}{2}\,(\phi\,(t) + \overline{\phi\,(t)}) = \frac{1}{2}\left(\left(\phi\left(\frac{t}{\mu}\right)\right)^2 + \left(\overline{\phi\left(\frac{t}{\mu}\right)}\right)^2\right), \tag{2.h}$$

and

$$\frac{1}{2}\,(\phi\,(t) + \overline{\phi\,(t)}) = \left|\phi\left(\frac{t}{\nu}\right)\right|^2 = \exp\left(-2c\left|\frac{t}{\nu}\right|^\alpha\right), \quad \text{[from (2.g)]}. \tag{2.i}$$

Now, we remark that the right-hand side of (2.h) is (obviously) equal to:

$$\frac{1}{2}\left(\phi\left(\frac{t}{\mu}\right) + \overline{\phi\left(\frac{t}{\mu}\right)}\right)^2 - \left|\phi\left(\frac{t}{\mu}\right)\right|^2,$$

and then, using (2.g) and (2.i), the identity (2.h) may now be written as:

$$\exp\left(-2c\left|\frac{t}{\nu}\right|^\alpha\right) = \frac{1}{2}\left\{2\,\exp -2c\left|\frac{t}{\mu\nu}\right|^\alpha\right\}^2 - \exp\left(-2c\left|\frac{t}{\mu}\right|^\alpha\right),$$

which, if we write $s = 2c\,|t|^\alpha$, $m = \dfrac{1}{\mu^\alpha}$, $n = \dfrac{1}{\nu^\alpha}$, is equivalent to:

$$2\,\exp\,(-2\,mns) = \exp\,(-ms) + \exp\,(-ns), \quad \text{for all } s \geq 0.$$

From the injectivity of Laplace transforms (for instance!), we now deduce that: $2mn = m = n$, so that: $\mu = \nu$, and we now deduce from (2.f) that:

$$|X_1 - X_2| \overset{(\text{law})}{=} |X_1 + X_2|. \tag{2.j}$$

This relation (2.j) implies, by Lemma 1 below, that X_1 is symmetric, and the proof of our Theorem 2 is finished. □

It may be helpful to isolate the following characterization of a symmetric random variable.

LEMMA 1. – *A real-valued random variable* X *is symmetric if, and only if:*

$$|X + X'| \overset{\text{(law)}}{=} |X - X'|, \tag{2.k}$$

where X′ *is an independent copy of* X.

Proof. – All we need to show is that, if (2.k) is satisfied, then X is symmetric.

Consider a symmetric Bernoulli random variable ε, which is independent of the pair (X, X′). Then, (2.k) is equivalent to:

$$X - X' \overset{\text{(law)}}{=} \varepsilon\,(X + X'), \tag{2.l}$$

and, if we note: $z = E[e^{itX}]$, we have, from (2.l):

$$|z|^2 = \frac{1}{2}\{z^2 + \bar{z}^2\},$$

which is equivalent to: $\text{Im}(z) = 0$; hence, $E[e^{itX}]$ is real, and X is symmetric. □

3. THE FINITE DIMENSIONAL STUDY

Let $n \in \mathbb{N}$, $n \geq 1$, and $0 < \alpha \leq 2$. In this section, we should like to characterize the n-dimensional random variables $\underline{X}_n = (X_1, \ldots, X_n)$ which satisfy:

$$l_\alpha(A\underline{X}_n) \overset{\text{(law)}}{=} l_\alpha(A^*\,\underline{X}_n). \tag{1.a)$_\alpha$}$$

The difference with the study made in the previous sections is that we do not assume here that the components $X_1, X_2, \ldots, X_n$ are independent, nor that they are identically distributed.

Our first result in this study is the following

PROPOSITION 1. – *The n-dimensional r.v.* $\underline{X}_n = (X_1, \ldots, X_n)$ *satisfies* $(1.a)_\alpha$ *if, and only if, for any* $(a_i)_{1 \leq i \leq n} \in \mathbb{R}^n$, *we have*

$$\left|\sum_{i=1}^{n} a_i\, X_i\right| \overset{\text{(law)}}{=} \left(\sum_{i=1}^{n} |a_i|^\alpha\right)^{1/\alpha} |X_1|. \tag{3.a)$_\alpha$}$$

Proof. – 1) Using arguments similar to those in the proof of Theorem 1, it is easily seen that $\underline{X}_n$ satisfies $(1.a)_\alpha$ if, and only if, for every $n \times n$ matrix A, we have:

$$(\underline{C}_n^{(\alpha)}, A\underline{X}_n) \overset{(\text{law})}{=} (\underline{C}_n^{(\alpha)}, A^*\underline{X}_n), \tag{3.b}$$

where the vector $\underline{C}_n^{(\alpha)}$ is assumed to be independent of $\underline{X}_n$.

Letting A vary among all $n \times n$ matrices, we obtain that (3.b) is equivalent to:

$$(C_i^{(\alpha)} X_j;\ 1 \le i \le j \le n) \overset{(\text{law})}{=} (X_i\, C_j^{(\alpha)};\ 1 \le i,\ j \le n). \tag{3.c}$$

Now, if two vectors (a_i) and (a'_i) satisfy: $\sum_{i=1}^n |a_i|^\alpha = \sum_{i=1}^n |a'_i|^\alpha$, then, the variables:

$$\sum_{i=1}^n a_i\, C_i^{(\alpha)} \quad \text{and} \quad \sum_{i=1}^n a'_i\, C_i^{(\alpha)}$$

have the same law. We then deduce from (3.c) that:

$$\left\{ \left(\sum_{i=1}^n a_i\, X_i \right) C_j;\ j \le n \right\} \overset{(\text{law})}{=} \left\{ \left(\sum_{i=1}^n a'_i\, X_i \right) C_j;\ j \le n \right\}$$

which is equivalent to: $\left| \sum_{i=1}^n a_i\, X_i \right| \overset{(\text{law})}{=} \left| \sum_{i=1}^n a'_i\, X_i \right|$.

Consequently, we have obtained that $(3.a)_\alpha$ is satisfied.

2) Conversely, we aim to show that if $(3.a)_\alpha$ is satisfied, then so is $(1.a)_\alpha$. We remark the following equivalences, with the help of our above notations for (3.b):

$$\begin{aligned}
(1.a)_\alpha &\iff (\underline{C}_n^{(\alpha)}, A\underline{X}_n) \overset{(\text{law})}{=} (\underline{C}_n^{(\alpha)}, A^*\underline{X}_n), \quad \text{for every } n \times n \text{ matrix A},\\
&\iff (A^*\underline{C}_n^{(\alpha)}, \underline{X}_n) \overset{(\text{law})}{=} (A\underline{C}_n^{(\alpha)}, \underline{X}_n), \quad \text{for every matrix A.}\\
&\iff (3.d) : (A^*\underline{C}_n^{(\alpha)}, \tilde{\underline{X}}_n) \overset{(\text{law})}{=} (A\underline{C}_n^{(\alpha)}, \tilde{\underline{X}}_n), \quad \text{for every A,}
\end{aligned}$$

where we have denoted: $\tilde{\underline{X}}_n = \tilde{\varepsilon}\underline{X}_n$, with $\tilde{\varepsilon}$ a symmetric Bernoulli variable which is independent of the pair of n-dimensional variables $\underline{C}_n^{(\alpha)}$ and $\underline{X}_n$. Now, the property $(3.a)_\alpha$ is equivalent to:

$$\sum_{i=1}^n a_i\, \tilde{X}_i \overset{(\text{law})}{=} \left(\sum_{i=1}^n |a_i|^\alpha \right)^{1/\alpha} \tilde{X}_1, \tag{$\widetilde{(3.a)}_\alpha$}$$

and we shall deduce (3.d), hence $(1.a)_\alpha$, from $\widetilde{(3.a)}_\alpha$.

Indeed, we have:

$$\begin{aligned}(A^* \underline{C}_n^{(\alpha)}, \underline{\tilde{X}}_n) &\overset{(\text{law})}{=} l_\alpha (A^* \underline{C}_n^{(\alpha)}) \tilde{X}_1 \qquad [\text{from } \widetilde{(3.a)}_\alpha]\\ &\overset{(\text{law})}{=} l_\alpha (A\underline{C}_n^{(\alpha)}) \tilde{X}_1 \qquad [\text{since } \underline{C}_n^{(\alpha)} \text{ satisfies } (1.a)_\alpha]\\ &\overset{(\text{law})}{=} (A\underline{C}_n^{(\alpha)}, \underline{\tilde{X}}_n) \qquad [\text{from } \widetilde{(3.a)}_\alpha].\end{aligned}$$

Hence, we have shown (3.d), and the proof is finished. □

In the case $\alpha = 2$, we have the following characterization of all vectors $\underline{X}_n$ which satisfy $(1.a)_2$.

PROPOSITION 2. – *An n-dimensional random variable* $\underline{X}_n = (X_1, \ldots, X_n)$ *satisfies* $(1.a)_2$ *if, and only if, it may be represented (possibly on a larger probability space than the original one) as:*

$$\underline{X}_n = \varepsilon \rho \, \underline{U}_n, \tag{3.e}$$

where ρ *is a r.v. which takes its values in* $\mathbb{R}_+$, $\underline{U}_n$ *is uniformly distributed on the unit sphere* S_{n-1}, ε *takes only the values* $+1$ *and* -1, *and* ρ *and* $\underline{U}_n$ *are independent (but no stochastic relationship between* ε *and the couple* $(\rho, \underline{U}_n)$ *is assumed).*

Proof. – As we have already seen, $\underline{X}_n$ satisfies $(1.a)_\alpha$ if, and only if, it satisfies:

$$\sum_{i=1}^n a_i \tilde{X}_i \overset{(\text{law})}{=} \left(\sum_{i=1}^n |a_i|^\alpha\right)^{1/\alpha} \tilde{X}_1, \tag{$\widetilde{3.a}_\alpha$}$$

where: $\underline{\tilde{X}}_n = \varepsilon \underline{X}_n$, with ε a Bernoulli random variable which is independent of $\underline{X}_n$.

Thus, $\underline{\tilde{X}}_n$ satisfies $\widetilde{(3.a)}_2$ if, and only if, its law is rotationnally invariant. Hence, we can write:

$$\underline{\tilde{X}}_n = \rho \underline{U}_n, \tag{3.f}$$

where ρ and $\underline{U}_n$ satisfy the properties stated in the Proposition.

Finally, since we have: $\underline{\tilde{X}}_n = \varepsilon \underline{X}_n$, we deduce from (3.c) that:

$$\underline{\tilde{X}}_n = \varepsilon \rho \, \underline{U}_n. \tag{3.e}$$

Remark 3. – As a complement to Proposition 2, we should like to mention a result of Letac [8], who proved that if $n \geq 3$, and $X_1, \ldots, X_n$

are independent, and satisfy $P(X_j = 0) = 0$, for every j, then, the hypothesis: (3.e′) $\underline{X}_n = \rho\, \underline{U}_n$ (with the same notation as in the statement of Proposition 2) implies that the X_j's are normal.

Remark 4. – Two other (infinite dimensional) variants of the previous result have been obtained by Letac-Milhaud [9] and Berman [10]:

– in [9], it is proven that if $\underline{X}_\infty = (X_1, X_2, \dots, X_n, \dots)$ is a stationary sequence of r.v's, such that, for every $n \in \mathbb{N}$, $P(\|\underline{X}_n\| = 0) = 0$, and $\frac{1}{\|\underline{X}_n\|} \underline{X}_n$ is uniformly distributed on the unit (Euclidean) sphere of $\mathbb{R}^n$, then: $\underline{X}_\infty \overset{\text{(law)}}{=} H\, \underline{G}_\infty$, where $\underline{G}_\infty = (G_1, G_2, \dots, G_n, \dots,)$ is a sequence of independent centered, Gaussian variables, independent of an $\mathbb{R}_+$-valued r.v. H.

– in [10], Berman replaces the (Euclidean) l_2-norms by l_p-norms and obtains thus an extension of the result of [9], but with $\underline{G}_\infty$ now changed into $\underline{G}^p_\infty = (G_1^{(p)}, G_2^{(p)}, \dots, G_n^{(p)}, \dots)$ a sequence of independent random variables such that:

$$P(G_1^{(p)} \in dx) = c_p \exp(-|x|^p)\, dx.$$

In the case: $0 < \alpha < 2$, we can prove, in contrast with Proposition 2, that not every n-dimensional variable $\underline{X}_n$ which satisfies $(3.a)_\alpha$ can be written in the form:

$$\underline{X}_n = \varepsilon \rho\, \underline{U}_n^{(\alpha)}, \qquad (3.e)_\alpha$$

where ρ is ≥ 0, independent of $\underline{U}_n^{(\alpha)}$, a vector which is assumed to have a "universal" distribution depending only on n and α.

However, we are able to exhibit a number of examples of variables $\underline{X}_n$ which satisfy $(1.a)_\alpha$ [or, equivalently, $(3.a)_\alpha$].

In order to do this, it is of interest to introduce the class of variables $\underline{T}_n = (T_1, \dots, T_n)$, all components of which take their values in $\mathbb{R}_+$, and which satisfy:

$$\sum_{i=1}^n a_i T_i \overset{\text{(law)}}{=} \left(\sum_{i=1}^n a_i^{\alpha/2}\right)^{2/\alpha} T_1, \qquad (3.a)^+_{\alpha/2}$$

for all $a_i \geq 0$, $1 \leq i \leq n$.

We can now state and prove the following

PROPOSITION 3. – *Consider two independent vectors* $\underline{\xi}_n = (\xi_1, \dots, \xi_n)$ *and* $\underline{T}_n = (T_1, \dots, T_n)$ *which satisfy respectively* $(3.a)_2$ *and* $(3.a)^+_{\alpha/2}$.

Then, the random vector:

$$\underline{X}_n = (\sqrt{T_j}\,\xi_j;\ 1 \le j \le n)$$

satisfies $(3.a)_\alpha$.

Proof. – We remark that, by conditioning first with respect to $\underline{T}_n$, we have for all $a_j \in \mathbb{R}$, $1 \le j \le n$:

$$\left|\sum_{j=1}^n a_j \sqrt{T_j}\,\xi_j\right| \overset{\text{(law)}}{=} \left(\sum_{j=1}^n a_j^2\, T_j\right)^{1/2} |\xi_1| \qquad [\text{since } \underline{\xi}_n \text{ satisfies } (3.a)_2]$$

$$\overset{\text{(law)}}{=} \left(\sum_{j=1}^n |a_j|^\alpha\right)^{2/\alpha} T_1^{1/2}\,|\xi_1| \qquad [\text{since } \underline{T}_n \text{ satisfies } (3.a)^+_{\alpha/2}]$$

$$\overset{\text{(law)}}{=} \left(\sum_{j=1}^n |a_j|^\alpha\right)^{2/\alpha} |X_1|.$$

Hence, $\underline{X}_n$ satisfies $(3.a)_\alpha$. □

In fact, the same arguments allow us to obtain the following generalization of Proposition 3.

Proposition 3′. – *Let* $0 < \alpha \le \gamma \le 2$, *and consider two independent vectors* $\underline{\xi}_n = (\xi_1, \dots, \xi_n)$ *and* $\underline{T}_n = (T_1, \dots, T_n)$ *which satisfy respectively* $(3.a)_\gamma$ *and* $(3.a)^+_{\alpha/\gamma}$.

Then, the random vector $\underline{X}_n = ({T_j}^{1/\gamma}\,\xi_j;\ 1 \le j \le n)$ *satisfies* $(3.a)_\alpha$.

In order to obtain a better understanding of the class of vectors $\underline{X}_n$ which satisfy either $(3.a)_\alpha$, for $0 < \alpha \le 2$, or $(3.a)^+_\alpha$, for $0 < \alpha \le 1$, we find it interesting to introduce the following.

Definition. – *An* $\mathbb{R}_+$*-valued random variable* ρ *is called a simplifiable r.v. if the identity in law:*

$$\rho X \overset{\text{(law)}}{=} \rho Y,$$

where X, *resp:* Y, *is an* $\mathbb{R}_+$ *valued random variable which is assumed to be independent of* ρ, *implies:* $X \overset{\text{(law)}}{=} Y$.

The interest of this definition in our study shows up in the following

Lemma 2. – 1) *If* $\underline{T}_n = \rho\,\underline{S}_n$ *satisfies* $(3.a)^+_\alpha$, *for some* $\alpha \le 1$, *and if* ρ *is a simplifiable random variable which is independent of* $\underline{S}_n$, *then* $\underline{S}_n$ *satisfies* $(3.a)^+_\alpha$;

2) *A similar statement holds with* $\underline{X}_n = \rho\, \underline{Y}_n$ *which is assumed to satisfy* $(3.a)_\alpha$, *for some* $\alpha \leq 2$.

The proof of this lemma is obvious from the definition of a simplifiable variable, and the properties $(3.a)_\alpha^+$ and $(3.a)_\alpha$.

As an application, we remark that, if $\underline{U}_n$ and $\underline{U}'_n$ are two independent n-dimensional random variables which are uniformly distributed on the unit sphere S_{n-1}, then:

$$\left.\begin{aligned} &U_n/\underline{U}'_n \equiv \left(\frac{U_i}{U'_i};\ 1 \leq i \leq n\right) \quad \text{satisfies } (3.a)_1, \\ &\text{and} \\ &1/(\underline{U}'_n)^2 \equiv \left(\frac{1}{(U'_i)^2};\ 1 \leq i \leq n\right) \quad \text{satisfies } (3.a)_{1/2}. \end{aligned}\right\} \tag{3.g}$$

The property (3.g) may be proven as follows:
if $\underline{G}_n = (G_1, \ldots, G_n)$ and $\underline{G}'_n = (G'_1, \ldots, G'_n)$ are two independent n-dimensional centered Gaussian vectors, each component of which has variance 1, then $|\underline{G}_n|$, $|\underline{G}'_n|$, $\underline{U}_n \equiv \underline{G}_n/|\underline{G}_n|$, $\underline{U}'_n \equiv \underline{G}'_n/|\underline{G}'_n|$ are independent, and $|\underline{G}_n|$, hence: $|\underline{G}'_n|$ is simplifiable; likewise, the second assertion in (3.g) is proven by remarking that:

$$\underline{T}_n \equiv \left(\frac{1}{G_i^2};\ 1 \leq i \leq n\right)$$

is an n-dimensional vector which consists of independent one-sided stable $\left(\frac{1}{2}\right)$ random variables, hence $\underline{T}_n$ satisfies $(3.a)_{1/2}^+$.

Consequently, since $\underline{T}_n = \frac{1}{|\underline{G}_n|^2}\left(\frac{1}{(\underline{U}_n)^2}\right)$, and $\frac{1}{|\underline{G}_n|^2}$ is simplifiable, then $\frac{1}{(\underline{U}_n)^2}$ satisfies $(3.a)_{1/2}$.

In the preceding discussion, we asserted that certain random variables are simplifiable; these assertions are justified by the

LEMMA 3. – 1) *If* ρ *is a simplifiable random variable, then:*

(i) $P(\rho > 0) = 1$;

(ii) *for any* $m \in \mathbb{R}$, ρ^m *is simplifiable.*

2) *A gamma distributed random variable is simplifiable.*

3) *A strictly positive random variable* ρ *is simplifiable if, and only if, the characteristic function of* $(\log \rho)$ *does not vanish on any interval of* $\mathbb{R}$.

Proof. – The proof of this lemma is elementary; hence, we leave it to the reader.

Now, we can state and prove the following converse of Proposition 3.

PROPOSITION 4. – *Consider two independent vectors* $\underline{\xi}_n = (\xi_1, \ldots, \xi_n)$ *and* $\underline{\mathrm{T}}_n = (\mathrm{T}_1, \ldots, \mathrm{T}_n)$ *such that:*

(i) $\underline{\xi}_n$ *satisfies* $(3.\mathrm{a})_2$, *and* (ii) $\underline{\mathrm{X}}_n \equiv (\sqrt{\mathrm{T}_j}\,\xi_j; 1 \leq j \leq n)$ *satisfies* $(3.\mathrm{a})_\alpha$.

Then, if moreover $|\xi_1|$ *is a simplifiable variable, the sequence* $\underline{\mathrm{T}}_n$ *satisfies* $(3.\mathrm{a})^+_{\alpha/2}$.

Proof. – From our hypothesis on $\underline{\mathrm{X}}_n$, we have, for any $(a_j)_{j\leq n} \in \mathbb{R}^n$:

$$\left|\sum_{j=1}^n a_j \sqrt{\mathrm{T}_j}\,\xi_j\right| \overset{(\mathrm{law})}{=} \left(\sum_{j=1}^n |a_j|^\alpha\right)^{2/\alpha} \sqrt{\mathrm{T}_1}\,|\xi_1|. \tag{3.h}$$

From our hypothesis on $\underline{\xi}_n$, the left-hand side of (3.h) is equal in law to:

$$\left(\sum_{j=1}^n a_j^2\,\mathrm{T}_j\right)^{1/2} |\xi_j|.$$

Hence, we have:

$$\left(\sum_{j=1}^n a_j^2\,\mathrm{T}_j\right)^{1/2} |\xi_j| \overset{(\mathrm{law})}{=} \left(\sum_{j=1}^n |a_j|^\alpha\right)^{2/\alpha} \sqrt{\mathrm{T}_1}\,|\xi_1| \tag{3.h'}$$

which, since $|\xi_1|$ is a simplifiable variable, implies that $\underline{\mathrm{T}}_n$ satisfies $(3.\mathrm{a})^+_{\alpha/2}$. □

4. THE GENERAL INFINITE DIMENSIONAL STUDY

The aim of this section is to bridge the gap which exists between section 2, where we consider a sequence $\mathrm{X}_1, \ldots, \mathrm{X}_n, \ldots$ of i.i.d. random variables, and section 3 where we consider a finite dimensional sequence $\mathrm{X}_1, \ldots, \mathrm{X}_n$, for which we make no *a priori* independence, nor distributional identity property assumption.

In this section, we consider an infinite sequence $\underline{\mathrm{X}}_\infty = (\mathrm{X}_1, \ldots, \mathrm{X}_n, \ldots)$ such that for any $n \in \mathbb{N}^*$, the finite sequence $\underline{\mathrm{X}}_n = (\mathrm{X}_1, \ldots, \mathrm{X}_n)$ satisfies $(1.\mathrm{a})_\alpha$, for some α, with $0 < \alpha \leq 2$.

Thanks to the infinite dimensionality of the sequence $\underline{\mathrm{X}}_\infty$, we obtain a characterization result which completes Theorem 2.

THEOREM 3. – *The following properties are equivalent:*
1) *for any* $n \in \mathbb{N}^*$, $\underline{X}_n$ *satisfies:*

$$l_\alpha(A\underline{X}_n) \overset{(\text{law})}{=} l_\alpha(A^*\underline{X}_n), \quad \text{for every matrix } A, \tag{1.a)$_\alpha$}$$

2) *for any* $n \in \mathbb{N}^*$, *and* $(a_i)_{i \le n} \in \mathbb{R}^n$, $\underline{X}_n$ *satisfies:*

$$\left|\sum_{i=1}^{n} a_i X_i\right| \overset{(\text{law})}{=} \left(\sum_{i=1}^{n} |a_i|^\alpha\right)^{1/\alpha} |X_1|, \qquad (3.a)_\alpha$$

3) *there exist* ε, H *and* $\underline{C}_\infty^{(\alpha)} \equiv (C_n^{(\alpha)};\ n \in \mathbb{N})$ *such that*

$$\underline{X}_\infty \overset{(\text{law})}{=} (\varepsilon H\, C_n^{(\alpha)};\ n \in \mathbb{N}) \qquad (4.a)$$

where ε *takes the values* ± 1, H *is an* $\mathbb{R}_+$*-valued random variable,* $\underline{C}_\infty^{(\alpha)}$ *is a sequence* $(\underline{C}_n^{(\alpha)},\ n \in \mathbb{N})$ *of independent symmetric standard stable* (α) *random variables, and* H *and* $\underline{C}_\infty^{(\alpha)}$ *are independent [but no distributional relationship is assumed about* ε *with respect to the pair* $(H,\ \underline{C}_\infty^{(\alpha)})$*].*

Proof. – *a*) Proposition 1 ensures the equivalence between properties 1) and 2); moreover, since $\underline{C}_n^{(\alpha)}$ satisfies $(3.a)_\alpha$ for any n, it is immediate that, if $\underline{X}_\infty$ satisfies (4.a), then it satisfies $(3.a)_\alpha$ for any $n \in \mathbb{N}$. Hence, it remains to show that 2) implies 3).

b) We first introduce (if necessary on an enlarged probability space) a symmetric Bernoulli random variable ε which is assumed to be independent of $\underline{X}_\infty$. Call $\underline{\tilde{X}}_\infty = \varepsilon\, \underline{X}_\infty \equiv (\varepsilon\, X_n;\ n \in \mathbb{N})$. Then, we deduce from 2) that $\underline{\tilde{X}}_n$ satisfies:

$$\sum_{i=1}^{n} a_i \tilde{X}_i \overset{(\text{law})}{=} l_\alpha(a)\, \tilde{X}_1, \qquad \widetilde{(3.a)}_n$$

for any $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$.

In particular, the sequence $\underline{\tilde{X}}_\infty$ is exchangeable. Consequently, from Neveu ([5], Exercice IV.5.2, p. 137), or Chow-Teicher ([2], Theorem 2, section 7.2), there exists a sub σ-field $\mathcal{G}$ such that, conditionnally on $\mathcal{G}$, the variables $(\tilde{X}_n;\ n \in \mathbb{N})$ are i.i.d; here, we may take for $\mathcal{G}$ the σ-field of symmetrical events in $\sigma\{\underline{\tilde{X}}_\infty\}$, or the asymptotic σ-field $\bigcap_n \sigma\{\tilde{X}_m;\ m \ge n\}$.

We now show that the conditional distribution of $\tilde{X}_1$ given $\mathcal{G}$ is the symmetric stable law of index α.

Indeed, from $\widetilde{(3.a)}_n$, we have:

$$\Phi_n (a_1, \ldots, a_n) \stackrel{\text{def}}{=} \mathrm{E}\left[\exp i \sum_{j=1}^{n} a_j \tilde{\mathrm{X}}_j\right] = \Phi_1 (l_\alpha (a)). \tag{4.b}$$

Now, if we denote by $\phi_\omega (a) = \mathrm{E}[\exp (ia\, \tilde{\mathrm{X}}_1)|\mathcal{G}] (\omega)$ the characteristic function of $\tilde{\mathrm{X}}_1$ given $\mathcal{G}$, then Bretagnolle, Dacunha-Castelle and Krivine ([1], p. 234-235) show that, as a consequence of (4.b), one has:

$$\phi_\omega (a) = \exp (-\mathrm{K} (\omega)|a|^\alpha),$$

for some $\mathcal{G}$-measurable $\mathbb{R}_+$-valued r.v. K.

The property (4.a) now follows easily. □

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