ANNALES DE L'I. H. P., SECTION B

D. S. MINDLIN
B. A. RUBSHTEIN

Convolutional attractors of stationary sequences of random measures on compact groups

Annales de l'I. H. P., section B, tome 30, nº 2 (1994), p. 213-233
<http://www.numdam.org/item?id=AIHPB_1994__30_2_213_0>

Ann. Inst. Henri Poincaré,

Vol. 30, n° 2, 1994, p. 213-233. *Probabilités et Statistiques*

Convolutional attractors of stationary sequences of random measures on compact groups

by

D. S. MINDLIN

710 N. Genesee # 2,
Los Angeles, CA 90046, USA

and

B. A. RUBSHTEIN [1]

Dept. of Mathematics and Computer Science
Ben-Gurion University of the Negev,
P.O.B. 653, Beer-Sheva, Israel

ABSTRACT. – We consider a stationary ergodic sequence $\mu_n = \mu_n(\omega)$, $n \in \mathrm{N}$, of random probability measures on a compact group G and study the asymptotic behaviour of their convolutions

$$\nu_m^{(n)}(\omega) = \mu_{m+n-1}(\omega) * \ldots * \mu_m(\omega)$$

in the weak topology as $n \to \infty$.

Let $\mathscr{A}_m(\omega)$ be the set of all limit points of $\nu_m^{(n)}(\omega)$ as $n \to \infty$, $A_m(\omega) = \left(\bigcup_{n=1}^{\infty} \operatorname{supp} \nu_m^{(n)}(\omega) \right)^{-}$ and $\lambda_m(\omega) = \lim_{n \to \infty} \tilde{\nu}_m^{(n)}(\omega) * \nu_m^{(n)}(\omega)$. There exists a compact $\mathscr{A}_\infty$ such that a. s.

$$\mathscr{A}_\infty = \mathrm{A}_m(\omega)\, \lambda_m(\omega)\, \mathrm{A}_m(\omega)^{-1} = \overline{\lim_{m \to \infty}}\, \mathscr{A}_m(\omega) = \left(\bigcup_{m=1}^{\infty} \mathscr{A}_m(\omega) \right)^{-}$$

A.M.S. Classification: 60 J 15, 60 B 15.

[1] Research supported by the Israel Ministries of Science and of Absorption.

We call this set $\mathscr{A}_\infty$ the convolutional attractor of $\{\mu_m\}$, since also $\mathscr{A}_\infty = (\dot{\nu}_m^{(n)}(\omega), m \in \mathrm{N})^-$ a.s. where the sequence $\dot{\nu}_m^{(n)} = \nu_m^{(n)}(\omega) * \lambda_m(\omega)$ is asymptotically equivalent to $\nu_m^{(n)}(\omega)$ as $n \to \infty$ a.s. Describing properties of $\mathscr{A}_\infty$ we in particular find conditions under which $\lambda_m(\omega)$, $\mathrm{A}_m(\omega)$ and $\mathscr{A}_m(\omega)$ do not depend essentially on ω and $\mathscr{A}_\infty$ forms a group of measures as in the well known case of convolution powers $\mu^{(n)}$ of a single measure μ.

Key words : Random measures, convergence of convolutions, compact groups.

RÉSUMÉ. — Nous considérons une suite stationnaire et ergodique $\mu_n = \mu_n(\omega)$, $n \in \mathrm{N}$, de mesures de probabilités sur un groupe compact G et étudions le comportement asymptotique des produits de convolution $\nu_m^{(n)}(\omega) = \mu_{m+n-1}(\omega) * \ldots * \mu_m(\omega)$ dans la topologie faible lorsque $n \to \infty$.

Soit $\mathscr{A}_m(\omega)$ l'ensemble de tous les points d'adhérence de $\nu_m^{(n)}(\omega)$ lorsque $n \to \infty$, $\mathrm{A}_m(\omega) = \left(\bigcup_{n=1}^{\infty} \operatorname{supp} \nu_m^{(n)}(\omega)\right)^-$ et $\lambda_m(\omega) = \lim_{n \to \infty} \tilde{\nu}_m^{(n)}(\omega) * \nu_m^{(n)}(\omega)$.

Il existe un ensemble compact $\mathscr{A}_\infty$ tel que, p. p.,

$$\mathscr{A}_\infty = \mathrm{A}_m(\omega)\,\lambda_m(\omega)\,\mathrm{A}_m(\omega)^{-1} = \overline{\lim_{m \to \infty}}\, \mathscr{A}_m(\omega) = \left(\bigcup_{m=1}^{\infty} \mathscr{A}_m(\omega)\right)^-$$

Nous appelons l'attracteur convolutionnel de la suite $\{\mu_m\}$, puisque

$$\mathscr{A}_\infty = (\dot{\nu}_m^{(n)}(\omega), n, m \in N)^- \text{ p. p.}$$

où la suite $\dot{\nu}_m^{(n)} = \nu_m^{(n)}(\omega) * \lambda_m(\omega)$ est p. p. asymptotiquement équivalente à la suite $\nu_m^{(n)}(\omega)$ lorsque $n \to \infty$ p. p.

En décrivant les propriétés de $\mathscr{A}_\infty$ nous trouvons en particulier des conditions pour que $\lambda_m(\omega)$, $\mathrm{A}_m(\omega)$ et $\mathscr{A}_m(\omega)$ ne dépendent pas essentiellement de ω, et pour que $\mathscr{A}_\infty$ forme un groupe de mesures comme dans le cas bien connu des puissances de convolution $\mu^{(n)}$ d'une mesure unique μ est p. p.

---

1. INTRODUCTION

Let G be a compact Hausdorff group and $\mathscr{M}^1(\mathrm{G})$ be the convolution semigroup of Borel probability measures on G with the weak topology.

We consider a stationary random process $\mu_n = \mu_n(\omega)$, $n \in \mathrm{N}$, defined on the probability space $(\Omega, \mathscr{F}, \mathrm{P})$ with values in $\mathscr{M}^1(\mathrm{G})$ and study the limit behaviour of the random measures.

$$\nu_m^{(n)}(\omega) = \mu_{m+n-1}(\omega) * \ldots * \mu_m(\omega), \qquad m, n \in \mathrm{N}$$

for the typical realizations of the process $\mu_n(\omega)$ as $n \to \infty$.

The convergence of convolutions of probability measures on a compact group has been examined by many authors (e.g. *see* [1], [4], [6], [7], [8], [10], [11], [14]-[16] and references cited there).

Precisely, the asymptotic behaviour of the sequence of the convolution powers $\nu^{(n)} = \mu * \ldots * \mu$ (n-times), $n \in \mathrm{N}$ for a fixed $\mu \in \mathscr{M}_1(\mathrm{G})$ is described as follows (*see* [4], ch. II).

THEOREM 1.0. – *a) The set* $\mathscr{A} = \mathrm{L}m\mathrm{P}_{n \to \infty} \nu^{(n)}$ *of all limit points of the sequence* $\{\nu^{(n)}\}_{n=1}^{\infty}$ *has the form*

$$\mathscr{A} = \lambda \mathrm{H} = \{\lambda x, \, x \in \mathrm{H}\}$$

where $\lambda = \lambda_{\mathrm{K}}$ *is the normalized Haar measure of the subgroup*

$$\mathrm{K} = \left[\bigcup_{n=1}^{\infty} \mathrm{S}(\tilde{\nu}^{(n)} * \nu^{(n)}) \right]^{-}$$

K *is a normal subgroup of*

$$\mathrm{H} = [\mathrm{S}(\mu)]^{-} = \left[\bigcup_{n=1}^{\infty} \mathrm{S}(\nu^{(n)}) \right]^{-} = \overline{\lim_{n \to \infty}} \, \mathrm{S}(\nu^{n})$$

and furthermore

$$\lambda = \lim_{n \to \infty} \tilde{\nu}^{(n)} * v^{(n)} = \lim_{n \to \infty} \nu^{(n)} * \tilde{\nu}^{(n)}$$

b) The sequence $\nu^{(n)}$ *is asymptotically equivalent to the sequence* $\dot{\nu}^{(n)} = \dot{\mu} * \ldots * \dot{\mu}$ *of the convolution powers of the measure* $\dot{\mu} = \lambda * \mu$, i.e.

$$\lim_{n \to \infty} (\nu^{(n)} - \dot{\nu}^{(n)}) = 0$$

and

$$\mathscr{A} = \mathrm{L}m\mathrm{P}_{n \to \infty} \dot{\nu}^{(n)} = (\dot{\nu}^{(n)}, \, n \in \mathrm{N})^{-}$$

Here and elsewhere [A] denotes the group generated by the set A and A^{-} is its closure. $\mathrm{S}(\mu)$ denotes the support of the measure μ and we use the notation μx and $x\mu$ instead $\mu * \delta_x$ and $\delta_x * \mu$ where δ_x is a Dirac measure in a point. The measure $\tilde{\mu}$ is the image of μ by the involution $x \to x^{-1}$, $x \in \mathrm{G}$. The definition of $\underline{\lim}$ and $\overline{\lim}$ see in [4], ch. 2, or in [9], § 29, and $\mathrm{L}m\mathrm{P}_{n \to \infty}$ means the set of all limit (accumulation) points of the corresponding sequence as $n \to \infty$.

It's natural to call the set $\mathscr{A}$ in the above theorem 1.0 the *convolutional attractor* (CA) of the measure μ.

The main purpose of the paper is to construct the analogous (as it is possible) convolutional attractor for a stationary sequence of random measures (SSRM) $\mu_n = \mu_n(\omega)$, $n \in \mathrm{N}$. To this end we shall investigate the limit points of the corresponding convolutions $\nu_m^{(n)}(\omega)$ as $n \to \infty$.

For a given SSRM $\{\mu_n\}_{n=1}^{\infty}$ on G we introduce the following notation.

Denote by $\mathscr{A}^{(n)}$ the essential image of the random element $\nu_m^{(n)}$, *i.e.* the support of its distribution $P \circ (\nu_m^{(n)})^{-1}$ on $\mathscr{M}_1(G)$. Put also

$$\mathscr{A}^{(\infty)} = \overline{\lim_{n \to \infty}} \mathscr{A}^{(n)}, \qquad \mathscr{B}^{(\infty)} = \left(\bigcup_{n=1}^{\infty} \mathscr{A}^{(n)} \right)^{-},$$

$$H = [S(\nu), \nu \in \mathscr{B}^{(\infty)}]^{-}, \qquad K = [S(\tilde{\nu} * \nu), \nu \in \mathscr{B}^{(\infty)}]^{-}$$

We shall assume everywhere in the course of the paper that the following conditions hold.

A) The SSRM $\{\mu_n(\omega)\}_{n=1}^{\infty}$ is ergodic, *i.e.* every stationary event has the probability 0 or 1.

B) The compact set $\mathscr{B}^{(\infty)}$ (and therefore $\mathscr{A}^{(n)}$ for all n) has a countable base of its topology.

The condition B) is equivalent to the metrizability of the compact set $\mathscr{B}^{(\infty)}$ (*see* [9], § 41. II). But we do not assume any conditions of sepability or metrizability on G.

The main results of the paper are the theorems 1.1-1.4 stated below

THEOREM 1.1. — *For all m and a.a.ω the following statements hold.*

a) The set $\mathscr{A}_m(\omega) = \mathrm{L}m\mathrm{P}_{n \to \infty} \nu_m^{(n)}(\omega)$ *of all limit points of the sequence* $\nu_m^{(n)}(\omega)$ *as* $n \to \infty$ *has the form*

$$\mathscr{A}_m(\omega) = A_m(\omega)\lambda_m(\omega)$$

where

$$A_m(\omega) = \overline{\lim_{n \to \infty}} S(\nu_m^{(n)}(\omega)) = \left(\bigcup_{n=1}^{\infty} S(\nu_m^{(n)}(\omega)) \right)^{-}$$

and

$$\lambda_m(\omega) = \lim_{n \to \infty} \tilde{\nu}_m^{(n)}(\omega) * \nu_m^{(n)}(\omega)$$

are the Haar measures of the subgroups

$$K_m(\omega) = \left[\bigcup_{n=1}^{\infty} S(\tilde{\nu}_m^{(n)}(\omega) * \nu_m^{(n)}(\omega)) \right]^{-}$$

and

$$K = [K_m(\omega), m \in N]^{-}$$

Herewith the subgroups $K_m(\omega)$ *are conjugated in* H *and*

$$\mathrm{L}m\mathrm{P}_{n \to \infty} \nu_m^{(n)}(\omega) * \tilde{\nu}_m^{(n)}(\omega) = (\lambda_m(\omega), m \in N)^{-}$$

b) The equality

$$\dot{\mu}_n(\omega) = \mu_n(\omega) * \lambda_n(\omega)$$

defines a SSRM such that the sequence of corresponding convolutions

$$\dot{\nu}_m^{(n)}(\omega) = \dot{\mu}_{m+n-1}(\omega) * \ldots * \dot{\mu}_m(\omega)$$

is asymptotically equivalent to $\nu_m^{(n)}(\omega)$ *as* $n \to \infty$

$$\lim_{n \to \infty} (\nu_m^{(n)}(\omega) - \dot{\nu}_m^{(n)}(\omega)) = 0$$

for all m and a.a.ω.

c) There exists a compact subset $\mathscr{A}_\infty$ *of* $\mathscr{A}^{(\infty)} \subset \mathscr{B}^{(\infty)}$ *such that*

$$\mathscr{A}_\infty = A_m(\omega)\,\lambda_m(\omega)\,A_m(\omega)^{-1} = \overline{\lim_{m \to \infty}}\ \mathscr{A}_m(\omega)$$

$$= \left(\bigcup_{m=1}^{\infty} \mathscr{A}_m(\omega)\right)^- = (\dot{\nu}_m^{(n)}(\omega), n, m \in \mathrm{N})^-$$

for a.a.ω.

We shall call the above set $\mathscr{A}_\infty$ the *convolutional attractor* of the SSRM $\{\mu_n\}_{n=1}^\infty$

The asymptotic behavior of the convolutions $\nu_m^{(n)}$ as $n \to \infty$ is completely defined by the convolutions $\dot{\nu}_m^{(n)}(\omega)$ of the limiting SSRM $\{\dot{\mu}_n\}_{n=1}^\infty$. The correspondence

$$\{\mu_n\}_{n=1}^\infty \to \{\dot{\mu}_n\}_{n=1}^\infty$$

is retractive *i.e.* the limiting SSRM of $\{\dot{\mu}_n\}_{n=1}^\infty$ is $\{\{\dot{\mu}_n\}$ itself

It should be mentioned that the sets $K_m(\omega)$, $A_m(\omega)$ and $\mathscr{A}_m(\omega)$ (unlike K, H, $\mathscr{A}_\infty$, $\mathscr{A}^\infty$ and $\mathscr{B}^\infty$) can essentially depend on ω and m. The main new phenomenon arising here is that *CA need not to be a group of measures.* In particular it can contain the Haar measures of a family of distinct conjugated subgroups $K_m(\omega)$ of the group K.

Such phenomenon appears even in the case when $\{\mu_n\}$ forms a Markov chain with a finite state space (sec. 6). But it disappears for independent random measures μ_n.

THEOREM 1.2. — *The following conditions are related by* 8) $\Rightarrow$ 7) $\Leftrightarrow$ 6) $\Rightarrow$ 5) *and* 1)-5) *are equivalent among themeselves.*

1) *the mapping* $\omega \to \lambda_m(\omega)$ *is constant a.e.;*
2) $\lambda_m(\omega) = \lambda_K$ *a.e., where* λ_K *is the Haar measure of* K*;*
3) *there exists* $\lim_{n \to \infty} \nu_m^{(n)}(\omega) * \tilde{\nu}_m^{(n)}(\omega)$ *a.e.;*
4) $\lim_{n \to \infty} \nu_m^{(n)}(\omega) * \tilde{\nu}_m^{(n)}(\omega) = \lambda_K$ *a.e.;*
5) $\lambda_K \in \mathscr{B}^{(\infty)}$
6) $\mathscr{A}_\infty$ *is a subgroup of the semigroup* $\mathscr{M}_1(G)$.
7) $\mathscr{A}_\infty = \lambda_K H$*;*
8) $\mathscr{A}^{(n)} = \mathscr{A}^{(1)} * \ldots * \mathscr{A}^{(1)}$ *(n-times),* $n \in \mathrm{N}$.

COROLLARY 1.3. — *If* $\{\mu_n\}_{n=1}^\infty$ *is a sequence of independent identically distributed* (i.i.d.) *random measures, then the condition* 8) *and hence the other conditions of the Theorem* 1.2 *hold.*

In fact the i. i. d. sequence $\{\mu_n\}$ satisfies the following condition:

$$S(P_n) = S(P_1) \times \ldots \times S(P_1)\,(n\text{-times}), \qquad n \in N,$$

where P_n be the distributions of the random vectors $(\mu_1, \ldots, \mu_n)$. Thus 8) holds too.

Thus the CA of a sequence of i. i. d. random measures always has a quite similar form and properties as in the case of convolution powers $\{\mu^n\}_{n=1}^{\infty}$ (theorem 1.0).

As a consequence we obtain the convergence conditions for $v_m^{(n)}(\omega)$.

THEOREM 1.4. — *The following properties are equivalent.*

1) *One of the limits* $\lim\limits_{n \to \infty} v_m^{(n)}(\omega)$ *exists a. e.;*

2) $\lim\limits_{n \to \infty} v_m^{(n)}(\omega) = \lambda_H$, *a. e.* $\forall\, m \in N$;

3) $K_m(\omega) = H$ *a. e. for some (or for all)* $m \in N$;

4) $A_m(\omega) = \overline{\lim\limits_{n \to \infty}}\, S(v_m^{(n)}(\omega))$ *with positive probability;*

5) $\underline{\lim\limits_{n \to \infty}}\, S(v_m^{(n)}(\omega)) \neq \emptyset$ *with positive probability.*

6) $\lambda_H \in \mathscr{B}^{(\infty)}$.

This theorem generalizes the familar Ito-Kawada theorem (*see* [6], [7], [8], [15] and [4], ch. 2). It is an easy consequence of the above results. The condition 2) in the above theorem means the compositional convergence of the sequence $\{\mu_n(\omega)\}_{n=1}^{\infty}$ in the sence of Maksimov [11].

Our method of the study of the CA is based on the notion of a normal sequence, which is introduced in sec. 2. These are sequence with a block recurrence property in the topological sense. Every Borel normal sequence (*see* [16]) is a normal in our sence but not conversely.

It is easily verified (*see* ass. 5.1) that almost all realizations of a SSRM $\{\mu_n\}$ satisfying A) and B) are normal sequences. Therefore we can consider the CA of an arbitrary normal sequence of measures and obtain the above results as a consequence of the corresponding theorems for normal sequences in the sections 2-4. Some of the results about normal sequences (th. 3.1, th. 4.1 and others) are of independent interest.

A part of the results of this paper was announced in [12], [13].

We would like to thank Prof. M. Lin for useful and stimulating discussions.

2. NORMAL SEQUENCES

Recall that a sequence $\{a_n\}_{n=1}^{\infty}$ is said to be Borel normal (*see* e. q. [16]) if for every $l \geqq 1$ there exist infinitely many numbers n such that

$$a_{n+i} = a_i, \qquad i = 1, 2, \ldots, l$$

DEFINITION 2.1. – *A sequence* $\{a_n\}_{n=1}^{\infty}$ *of elements of a topological space* E *will be called normal if for every* $l \geqq 1$ *and for any collection of neighborhoods* $V_1, \ldots, V_l$ *of the points* $a_1, \ldots a_l$ *there exist infinitely many numbers n such that*

$$a_{n+i} \in V_i, \qquad i=1,2,\ldots,l \tag{2.1}$$

Every Borel normal sequence is obviously normal and these two notions coincide when E has the discrete topology.

The strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ which consists all n satisfying (2.1) will be called the recurrence sequence of the block $(a_1, \ldots, a_l)$ into the neighborhood $V_1 \times \ldots \times V_l$.

The next theorem plays an important part in the sequel.

Let now E be a compact semigroup and for an arbitrary sequence $\{a_n\}_{n=1}^{\infty}$ in E consider its partial products

$$b_n = a_n \ldots a_1, \qquad n \in \mathrm{N}.$$

THEOREM 2.2. – *Let* $\{a_n\}_{n=1}^{\infty}$ *be a normal sequence in a compact semigroup* E *and* $\mathscr{L}$ *denotes the set of all limit points of the corresponding sequence* $\{b_n\}_{n=1}^{\infty}$. *Then* $\mathscr{L}$ *contains at least one idempotent.*

Proof. – Let $\mathscr{U}$ be the totality of all sequences $\{U_n\}_{n=1}^{\infty}$, where U_n is an neighborhood of a_n for each n. We shall fix one such sequence $u=\{U_n\}_{n=1}^{\infty} \in \mathscr{U}$ and for every $l \geqq 1$ consider the recurrence sequence $n_k = n_k(u, l)$, $k \geqq 1$, of the block $(a_1 \ldots a_l)$ into $U_1 \times \ldots \times U_l$.

Let now $\mathscr{L}(u, l)$ be the set of all limit points of the sequence $\{b_{n_k}\}_{k=1}^{\infty}$ where $n_k = n_k(u, l)$.

The set $\mathscr{L}(u, l)$ is closed as the totality of all limits of the convergent subnets of the sequence $\{b_{n_k}\}_{k=1}^{\infty}$ and $\mathscr{L}(u, l) \neq \varnothing$ on account of the normality of $\{a_n\}$.

Since

$$\{n_k(u, l), k \leqq 1\} \supset \{n_k(u, l+1), k \geqq 1\}$$

we have a decreasing sequence of non-empty closed subsets $\{\mathscr{L}(u, l)\}_{e=1}^{\infty}$, which has the non-empty intersection $\mathscr{L}(u) = \bigcap_{l=1}^{\infty} \mathscr{L}(u, l)$.

We may define the intersection of a finite subset $\{u_i, i=1, \ldots, s\}$ of $\mathscr{U}$ by

$$\bigcap_{i=1}^{s} u_i = \left\{ \bigcap_{i=1}^{s} U_{n,i} \right\}_{n=1}^{\infty} \in \mathscr{U}$$

where $u_i=\{U_{n,i}\}_{n=1}^{\infty}\in\mathcal{U}$. Since $\{a_n\}$ is normal

$$\bigcap_{i=1}^{s}\mathcal{L}(u_i)=\bigcap_{l=1}^{\infty}\left(\bigcap_{i=1}^{s}\mathcal{L}(u_i,l)\right)\supset\bigcap_{l=1}^{s}\left(\mathcal{L}\left(\bigcap_{i=1}^{s}u_i,l\right)\right)=\mathcal{L}\left(\bigcap_{i=1}^{s}u_i\right)\neq\varnothing \tag{2.2}$$

We obtain the system $\{\mathcal{L}(u), u\in\mathcal{U}\}$ of nonempty closed subsets of $\mathcal{L}$. It is a centered system by (2.2), *i.e.* it has the finite intersection property. Thus its intersection $\mathcal{L}_0=\bigcap_{u\in\mathcal{U}}\mathcal{L}(u)$ is a non-empty closed susbset of $\mathcal{L}$.

We shall show now that

$$b_n\mathcal{L}_0\subset\mathcal{L},\qquad n\in\mathrm{N} \tag{2.3}$$

If this inclusion is false there exist $l\in\mathrm{N}$ and $b\in\mathcal{L}_0$ such that $b_l b\notin\mathcal{L}$. One can choose $u=\{U_n\}_{n=1}^{\infty}\in\mathcal{U}$, which satisfies

$$(U_l\cdot\ \ldots\ \cdot U_1 b)^-\cap\mathcal{L}=\varnothing \tag{2.4}$$

and $U_n=E$ for $n>l$. Since $b\in\mathcal{L}_0\subset\mathcal{L}(u,l)$, it is a limit point of the sequence $\{b_{n_k}\}_{k=1}^{\infty}$, where $n_k=n_k(u,l)$ is the recurrence sequence of the block $(a_1\ldots a_l)$ into $U_1\times\ldots\times U_l$. Taken a convergent net $b_{n_{k(\alpha)}}\to b$ we deduce from

$$b_{n_k+l}\in U_l\cdot\ \ldots\ \cdot U_1 b_{n_k}$$

that the set $(U_l\cdot\ \ldots\ \cdot U_1 b)^-$ contains limit points of the net $b_{n_{k(\alpha)}+l}$ and then limit points of b_n. This contradicts (2.4).

Thus (2.3) holds and hence $\mathcal{L}\ \mathcal{L}_0\subset\mathcal{L}$.

By construction we have $\mathcal{L}_0\subset\mathcal{L}$ and then $\mathcal{L}$ contains the compact semigroup $\left(\bigcup_{n=1}^{\infty}\mathcal{L}_0^n\right)^-$ generated by $\mathcal{L}_0$. Any compact semigroup contains an idempotent ([5], 9.18). Employing this assertion to the semigroup $\left(\bigcup_{n=1}^{\infty}\mathcal{L}_0^n\right)^-$ we complete the proof. ■

3. CENTERED CONVERGENCE AND ITS CONSEQUENCES

In the course of the sections 3 and 4 we shall consider a fixed normal sequence $\{\mu_n\}_{n=1}^{\infty}$ in $\mathcal{M}_1(G)$ and its convolutions

$$\nu_m^{(n)}=\mu_{m+n-1}*\ldots*\mu_m,\qquad m,n\in\mathrm{N} \tag{3.1}$$

Introduce the compact groups

$$K_m=\left[\bigcup_{n=1}^{\infty}S(\tilde{\nu}_m^{(n)})S(\nu_m^{(n)})\right]^-,\qquad m\in\mathrm{N}$$

of the group $H=[S(\mu_n), n\in N]^-$ and denote by λ_m the probability Haar measure of K_m.

The next theorem on centered convergence will be the main tool to describe limit points of $\nu_m^{(n)}$ as $n\to\infty$

THEOREM 3.1. – *For a normal sequence* $\{\mu_n\}_{n=1}^{\infty}$ *in* $\mathcal{M}_1(G)$ *there exist the following limits*

$$\lim_{n\to\infty} \tilde{x}_m^{(n)} \nu_m^{(n)} = \lambda_m, \qquad m\in N$$

where λ_m is the Haar measure of the subgroup K_m and $\{\tilde{x}_m^{(n)}\}_{n=1}^{\infty}$ is an arbitrary sequence of elements $\tilde{x}_m^{(n)}\in S(\tilde{\nu}_m^{(n)})$.

To prove this theorem we make use the left regular representation of G and $\mathcal{M}_1(G)$ in the Hilbert space $\mathcal{H}=L_2(G, \lambda_G)$, which are defined by

$$L(g)f=\delta_g * f, \qquad L(\mu)f=\mu * f$$

for $f\in\mathcal{H}$, $g\in G$ and $\mu\in\mathcal{M}_1(G)$. The mapping L is in fact a unitary representation of G and a *-representation of the convolutional semigroup $\mathcal{M}_1(G)$; $L(\tilde{\mu})=L(\mu)^*$ and $\|L(\mu)\|\leqq 1$ (*see* [5], § 27). Herewith, $L:\mu\to L(\mu)$ is a topological isomorphism of $\mathcal{M}_1(G)$ onto $L(\mathcal{M}_1(G))$ with the strong operator (so)-topology or with the weak operator (wo)-topology on $L(\mathcal{M}_1(G))$ on account of the compactness of $\mathcal{M}_1(G)$.

Proof of theorem 3.1. – It is enough to consider the case $m=1$.

Denote $T_n=L(\nu_1^{(n)})$, $n\in N$. We will use the order on $L(\mathcal{M}_1(G))$ which is induced by the cone of all non-negative defined operators on $\mathcal{H}$, *i.e.*

$$T\leqq T' \Leftrightarrow ((T'-T)f, f)\geqq 0, \qquad \forall f\in\mathcal{H}$$

Then $0\leqq T_n^* T_n\leqq I$, where $I=id_{\mathcal{H}}$, and

$$0\leqq L(\mu_n)^* L(\mu_n)\leqq I$$

implies

$$0\leqq T_n^* T_n = T_{n-1}^* L(\mu_n)^* L(\mu_n) T_{n-1} \leqq T_{n-1}^* T_{n-1} \leqq I \qquad (3.2)$$

i.e. the sequence $\{T_n^* T_n\}_{n=1}^{\infty}$ is a decreasing one and it is bounded below. Hence there exists the limit

$$(\text{wo})-\lim_{n\to\infty} T_n^* T_n = E, \qquad 0\leqq E\leqq I, \quad E\in L(\mathcal{M}_1(G))$$

(*see* [3], prob. 94).

On the other hand, there is an idempotent in the set $\mathcal{A}_1$ of all limit points of $\nu_1^{(n)}$ as $n\to\infty$ by the Theorem 2.2. Then λ is a limit point of the sequence $\tilde{\nu}_1^{(n)} * \nu_1^{(n)}$. Since $L:\mu\to L(\mu)$ is a homeomorphism, there exists the limit $\lim_{n\to\infty} \tilde{\nu}_1^{(n)} * \nu_1^{(n)}=\lambda$, where $L(\lambda)=E$.

The operator $L(\lambda)$ is an orthogonal projector on $\mathcal{H}$ and it gives the orthogonal decomposition $\mathcal{H}=X_1\oplus Y_1$ where $X_1=\operatorname{Im} E$ and $Y_1=\operatorname{Ker} E$.

We have by (3.2)

$$f \in X_1 \Leftrightarrow T_n^* T_n f \to f \Rightarrow (T_n^* T_n f, f) \to (f, f)$$
$$\Leftrightarrow \|T_n f\| \to \|f\| \Leftrightarrow \|T_n f\| = \|f\| \forall n$$
$$\Rightarrow (T_n^* T_n f, f) = (f, f) \forall n \Leftrightarrow T_n^* T_n f = f \forall n$$
$$\Rightarrow Ef = f \Leftrightarrow f \in X_1$$

and

$$f \in Y_1 \Leftrightarrow T_n^* T_n f \to 0 \Rightarrow (T_n^* T_n f, f) = \|T_n f\|^2 \to 0$$
$$\Rightarrow (Ef, f) = 0 \Rightarrow Ef = f \Leftrightarrow f \in Y_1$$

Thus

$$X_1 = \{ f \in \mathcal{H} : \|T_n f\| = \|f\| \forall n \} = \{ f \in \mathcal{H} : T_n^* T_n f = f \forall n \} \quad (3.3)$$
$$Y_1 = \{ f \in \mathcal{H} : \|T_n f\| \to 0,\ n \to \infty \} \quad (3.4)$$

We want to show now that $\lambda = \lambda_1$.

We have $\lambda * \lambda_1 = \lambda_1$ by $\tilde{\nu}_1^{(n)} * \nu_1^{(n)} \to \lambda$ and $S(\tilde{\nu}_1^{(n)} * \nu_1^{(n)}) \subset K_1$. Conversely, if $\lambda * f = f$, $f \in \mathcal{H}$ (*i.e.* $f \in X_1$) then $\tilde{\nu}_1^{(n)} * \nu_1^{(n)} * f = f$ for all n by (3.3) and hence $\delta_x * f = f$ a.e. for all $x \in S(\tilde{\nu}_1^{(n)} * \nu_1^{(n)})$, $n \in N$. Therefore $\delta_x * f = f$ a.e. for all $x \in K_1$ and $\lambda_1 * f = f$. Thus $\lambda_1 * \lambda = \lambda$ and hence $\lambda_1 = \lambda$. (It was used, that $\mu * f = f \Leftrightarrow \delta_x * f = f$ a.e. for all $x \in S(\mu)$, (*See* [4], 1.2.7).) Let now $\{ \tilde{x}_1^{(n)} \}_{n=1}^{\infty}$ with $\tilde{x}_1^{(n)} \in S(\tilde{\nu}_1^{(n)})$. For $f \in X_1$ we have $\tilde{x}_1^{(n)} \nu_1^{(n)} * f = f = \lambda_1 * f$ a.e. by (3.3) since $S(\tilde{x}_1^{(n)} \nu_1^{(n)}) \in K_1$. For $f \in Y_1$ we have

$$\| \tilde{x}_1^{(n)} \nu_!^{(n)} * f \| = \| \nu_1^{(n)} * f \| \to 0, \qquad n \to \infty$$

by (3.4). Taking into account the decomposition $\mathcal{H} = X_1 \oplus Y_1$ and $X_1 = L(\lambda_1) \mathcal{H}$ we obtain

$$\| \tilde{x}_1^{(n)} \nu_!^{(n)} * f - \lambda_1 * f \| \to 0, \qquad n \to 0$$

for all $f \in \mathcal{H}$ and hence $\tilde{x}_1^{(n)} * \nu_1^{(n)} \to \lambda_1$. ■

COROLLARY 3.2. — *For all* $m \in N$ *the following limits exist*

a) $\lim_{n \to \infty} \tilde{\nu}_m^{(n)} * \nu_m^{(n)} = \lambda_m$

b) $\lim_{n \to \infty} (\nu_m^{(n)} - \nu_m^{(n)} * \lambda_m) = 0$

c) $\lim_{n \to \infty} (\nu_m^{(n)} * \tilde{\nu}_m^{(n)} - x_m^{(n)} \lambda_m \tilde{x}_m^{(n)}) = 0$

for all $x_m^{(n)} \in S(\nu_m^{(n)})$ *and* $\tilde{x}_m^{(n)} \in S(\tilde{\nu}_m^{(n)})$.

Remark 3.3. — The choice of a centering sequence $\tilde{x}_m^{(n)}$ on the left side of $\nu_m^{(n)}$ is essentially connected with the order of the factors $\mu_{m+n-1}, \ldots, \mu_m$ in $\nu_m^{(n)}$. The following simple example shows that the sequence $\nu_m^{(n)} * \tilde{\nu}_m^{(n)}$ need not converge as $n \to \infty$. In this case $\nu_m^{(n)} x^{(n)}$ does not converge under any choice of $x^{(n)}$.

Example 3.4. — Let L_1 and L_2 be a pair of conjugate subgroups of G and $L_2 = x L_1 x^{-1}$, $L_1 \neq L_2$. Consider a periodic sequence $\{\mu_n\}$, supposing

$$\mu_{3k} = \lambda_{L_1}, \qquad \mu_{3k+1} = \delta_x, \qquad \mu_{3k+2} = \delta_{x^{-1}}, \qquad k = 0, 1, 2 \ldots$$

For $n \geqq 3$ we have $\tilde{\nu}_1^{(n)} * \nu_1^{(n)} = \lambda_{L_1}$, but $\nu_1^{(n)} * \tilde{\nu}_1^{(n)} = \lambda_{L_2}$ for $n = 3k+1$ and $\nu_1^{(n)} * \tilde{\nu}_!^{(n)} = \lambda_{L_1}$ otherwise. Then $\nu_m^{(n)} * \tilde{\nu}_m^{(n)}$ has exactly two limit points λ_{L_1} and λ_{L_2}.

Remark 3.5 If the Second Axiom of Countability holds on G the centering sequence always exists for every (even non-normal) sequence in $\mathscr{M}_1(G)$ (*see* [8]). In the case of a normal sequence we need not SAC-condition and the limit of the centered sequence of measures always has the form $x\lambda$, where $x \in H$ and λ is an idempotent.

We are able to describe now the limits points of $\nu_m^{(n)}$ as $n \to \infty$

Introduce the following notation.

$$B_m = \left(\bigcup_{n=1}^{\infty} S(\nu_m^{(n)}) \right)^-, \qquad A_m = \overline{\lim_{n \to \infty}}\, S(\nu_m^{(n)})$$

and C_m be the set of all limit points of all possible sequences $\{x_m^{(n)}\}_{n=1}^{\infty}$ as $n \to \infty$ where $x_m^{(n)} \in S(\nu_m^{(n)})$. At last let, $\mathscr{A}_m$ be the set of all limit points of $\nu_m^{(n)}$ as $n \to \infty$ and fixed $m \in N$. *i.e.* $\mathscr{A}_m = \mathrm{L\,m\,P}_{n \to \infty} \nu_m^{(n)}$.

THEOREM 3.6. — *For a normal sequence* $\{\mu_n\}_{n=1}^{\infty}$ *in* $\mathscr{M}_1(G)$ *and* $m \in N$ *the following assertions hold:*

a) $A_m = B_m = C_m^- \supset K_m$

b) $\mathscr{A}_m = A_m \lambda_m \ni \lambda_m$.

Proof. — We may suppose $m = 1$.

1) $C_1^- = A_1$. It is obvious that $C_1 \subset A_1 = A_1^-$ and hence $C_1^- \subset A_1$. For every $x \in A_1$ and an arbitrary neighborhood U and of x one can choose a sequence $\{x_1^{(n)}\}_{n=1}^{\infty}$ such that $x_1^{(n)} \in S(\nu_1^{(n)})$, $n \in N$ and $x_1^{(n)} \in U$ for infinitely many of n. By the compactness the sequence $\{x_1^{(n)}\}_{n=1}^{\infty}$ has a limit point in U^-. Hence $C_1 \cap U^- \neq \varnothing$ for every neighborhood U of x and $x \in C_1^-$. Thus $A_1 \subset C_1^-$.

2) $\mathscr{A}_1 = C_1 \lambda_1$ follows from theorem 3.1, since

$$\mathscr{A}_1 = \mathrm{L}\,m\,\mathrm{P}_{n \to \infty} (x_1^{(n)} \lambda_1) = (\mathrm{L}\,m\,\mathrm{P}_{n \to \infty} x_1^{(n)}) \lambda_1$$

for any sequence $\{x_1^{(n)}\}_{n=1}^{\infty}$ with $x_1^{(n)} \in S(\nu_1^{(n)})$.

3) $\mathscr{A}_1 \ni \lambda_1$. By theorem 2.2 $\mathscr{A}_1$ contains an idempotent λ, which has the form $\lambda = x \lambda_1$ by 2). Then $\lambda = \lambda_1$.

4) $A_1 \supset K_1$. Since $\lambda_1 \in \mathscr{A}_1$ there exists a subnet $\{\nu_1^{(n(\alpha))}\}$ of the sequence $\{\nu_1^{(n)}\}$ which converges to λ_1. For any n_0 there exists α_0 such that $n(\alpha) > n_0$ for all $\alpha > \alpha_0$. Hence

$$K_1 = S(\lambda_1) = S(\lim_{\alpha} \nu_1^{(n(\alpha))}) \subset \Big(\bigcup_{\alpha > \alpha_0} S(\nu_1^{(n(\alpha))})\Big)^- \subset \Big(\bigcup_{n > n_0} S(\nu_1^{(n)})\Big)^-$$

On the other hand for $x \notin A_1$ one can choose a number n_0 and a neighborhood U of x such that $S(\nu_1^{(n)}) \cap U = \varnothing$ for all $n > n_0$ and hence $U \cap K_1 = \varnothing$, *i.e.* $x \notin K_1$. Thus $K_1 \subset A_1$.

5) $A_1 \supset B_1$. The equality $\nu_{m+1}^{(n)} * \nu_1^{(m)} = \nu_1^{(m+n)}$ implies

$$S(\nu_{m+1}^{(n)}) . S(\nu_1^{(m)}) = S(\nu_1^{(m+n)}), \quad m, n \in \mathrm{N}.$$

Hence $C_{m+1} S(\nu_1^{(m)}) \subset C_1$, $m \in \mathrm{N}$. Using 1) to A_{m+1} and A_1, we have also $A_{m+1} S(\nu_1^{(m)}) \subset A_1$, $m \in \mathrm{N}$.

Applying 4) to the set A_{m+1} we obtain $A_{m+1} \supset K_{m+1} \ni e$, where e is the unit element of G. Hence $S(\nu_1^{(m)}) \subset A_1$, $m \in \mathrm{N}$ and $B_1 \subset A_1$. The inverse inclusion is obvious. ■

THEOREM 3.7. — *For a normal sequence* $\{\mu_n\}_{n=1}^{\infty}$ *in* $\mathcal{M}_1(G)$ *the following equalities hold for all* $m, n \in \mathrm{N}$ *and* $x_m^{(n)} \in S(\nu_m^{(n)})$

$$\nu_m^{(n)} * \lambda_m = \lambda_{m+n} * \nu_m^{(n)} = x_m^{(n)} \lambda_m = \lambda_{m+n} x_m^{(n)}$$

Proof. — We again may suppose $m = 1$.

Choosen any $x_1^{(k)} \in S(\nu_1^{(k)})$ and $x_{k+1}^{(n)} \in S(\nu_{k+1}^{(n)})$ we deduce by theorem 3.1 as $n \to \infty$

$$(x_{k+1}^{(n)})^{-1} \nu_{k+1}^{(n)} \to \lambda_{k+1} \qquad \textit{and} \qquad (x_{k+1}^{(n)} x_1^{(k)})^{-1} \nu_1^{(n+k)} \to \lambda_1$$

Then taking into account the equality

$$\nu_{k+1}^{(n)} * \nu_1^{(k)} = \nu_1^{(n+k)}$$

we obtain

$$(x_1^{(k)})^{-1} \lambda_{k+1} * \nu_1^{(k)} = \lambda_1$$

that is

$$\lambda_{k+1} * \nu_1^{(k)} = x_1^{(k)} \lambda_1, \qquad k \in \mathrm{N}, \quad x_1^{(k)} \in S(\nu_1^{(k)})$$

Taking integration over $x_1^{(k)} \in S(\nu_1^{(k)})$ by the measures $\nu_1^{(k)}$ we have also

$$\lambda_{k+1} * \nu_1^{(k)} = \nu_1^{(k)} * \lambda_1$$

To prove the last equality

$$x_1^{(n)} \lambda_1 = \lambda_{n+1} x_1^{(n)}$$

we need the following lemma.

LEMMA 3.8. — *Let* $\mathcal{H} = X_m \oplus Y_m$ *be the decomposition of the Hilbert space* $\mathcal{H}$ *defined by the orthoprojector* $L(\lambda_m)$, $m \in \mathrm{N}$. *Then*

$$L(\nu_1^{(m)}) X_1 = X_{m+1}, m \in \mathrm{N}$$

Proof. — It is obvious $L(\nu_1^{(m)}) X_1 \subset X_{m+1}$. Since G is compact the representation L is decomposed into the direct sum of finite dimensional sub-representations $L = \bigotimes_{s \in \Xi} L^s$ acting in the subspaces $\mathcal{H}^s$ where

$\dim \mathcal{H}^s < \infty$, and $\bigoplus_{s \in \Xi} \mathcal{H}^s = \mathcal{H}$. Herewith every operator $L(\mu)$, $\mu \in \mathcal{M}_1(G)$ admits the decomposition (*see* [5], § 27).

$$L(\mu) = \bigoplus_{s \in \Xi} L^s(\mu)$$

Therefore it is enough to check the equalities

$$L^s(\nu_1^{(m)}) X_1^s = X_{m+1}^s, \qquad \textit{where} \quad X_{m+1}^s = \mathcal{H}^s \cap X_{m+1}$$

By the theorem 3.6 $\lambda_1 \in \mathcal{A}_1$ and hence $L^s(\lambda_1)$ is a limit point of the sequence $L^s(\nu_1^{(m)})$ as $m \to \infty$. Since $L^s(\nu_1^{(m)})$ are contractions and $\dim \mathcal{H}^s < \infty$ we obtain for all s

$$\dim L^s(\nu_1^{(m)}) X_1^s = \dim X_1^s = \dim X_{m+1}^s < \infty$$

that implies the required equality. ■

From the above lemma it is seen that

$$\lambda_{k+1} = \nu_1^{(k)} * \lambda_1 * \tilde{\nu}_1^{(k)}, \qquad k \in \mathbb{N}$$

and using $\nu_1^{(k)} * \lambda_1 = x_1^{(k)} \lambda_1$ we conclude

$$\lambda_{k+1} x_1^{(k)} = x_1^{(k)} \lambda_1, \qquad k \in \mathbb{N}, \quad x_1^{(k)} \in S(\nu_1^{(k)})$$

Thus the theorem 3.7 is proved. ■

COROLLARY 3.9. – *For all $x_m^{(n)} \in S(\nu_m^{(n)})$ and $\tilde{x}_m^{(n)} \in S(\tilde{\nu}_m^{(n)})$ the following relations hold*

a) $K_{m+n} = x_m^{(n)} K_m \tilde{x}_m^{(n)}$

b) $A_{m+n} x_m^{(n)} = A_m$.

4. CONVOLUTIONAL ATTRACTORS OF NORMAL SEQUENCES OF MEASURES

The aim of this section is to describe the convolutional attractors for arbitrary normal sequences in $\mathcal{M}_1(G)$.

In common with the sec. 3 let $\{\mu_n\}_{n=1}^{\infty}$ be a fixed normal sequence in $\mathcal{M}_1(G)$ and $\nu_m^{(n)}$, $m, n \in \mathbb{N}$ be its convolutions defined by (3.1). We preserve all notation of the sec. 3 and introduce also the sets:

$$\left.\begin{aligned} &\mathcal{A}^{(n)} = \mathrm{L}m\mathrm{P}_{m \to \infty} \nu_m^{(n)}, && \mathcal{B}^{(n)} = (\nu_m^{(n)}, m \in \mathbb{N})^- \\ &\mathcal{A}^{(\infty)} = \overline{\lim_{n \to \infty}} \mathcal{A}^{(n)}, && \mathcal{B}^{(\infty)} = \left(\bigcup_{n=1}^{\infty} \mathcal{B}^{(n)}\right)^- \\ &\mathcal{A}_\infty = \overline{\lim_{m \to \infty}} \mathcal{A}_m, && \mathcal{B}_\infty = \left(\bigcup_{m=1}^{\infty} \mathcal{A}_m\right)^- \end{aligned}\right\} \tag{4.1}$$

THEOREM 4.1. – *For any normal sequence* $\{\mu_n\}_{n=1}^{\infty}$
a) $\mathscr{A}^{(n)}=\mathscr{B}^{(n)},\ n\in\mathrm{N}$
b) $\mathscr{A}_{\infty}=\mathscr{B}_{\infty}\subset\mathscr{A}^{(\infty)}\subset\mathscr{B}^{(\infty)}$
c) $\mathscr{A}_{\infty}=\mathrm{A}_m\lambda_m\mathrm{A}_m^{-1},m\in\mathrm{N}$

Proof. – By th. 3.6, 3.7 and cor. 3.9

$$\mathscr{A}_m=\mathrm{A}_m\lambda_m=\mathrm{A}_1\lambda_1\mathrm{S}(\mathrm{v}_1^{(m)})^{-1},\qquad m\in\mathrm{N}$$

$$\mathscr{A}_{\infty}=\overline{\lim_{m\to\infty}}\ \mathscr{A}_m=\mathrm{A}_1\lambda_1(\overline{\lim_{m\to\infty}}\ \mathrm{S}(\mathrm{v}_1^{(m)})^{-1})=\mathrm{A}_1\lambda_1\mathrm{A}_1^{-1}$$

$$\mathscr{B}_{\infty}=\left(\bigcup_{m=1}^{\infty}\mathscr{A}_m\right)^-=\mathrm{A}_1\lambda_1\left(\bigcup_{m=1}^{\infty}\mathrm{S}(\mathrm{v}_1^{(m)})^{-1}\right)^-=\mathrm{A}_1\lambda_1\mathrm{B}_1^{-1}=\mathrm{A}_1\lambda_1\mathrm{A}_1^{-1}$$

For $m>1$ and any $x_1^{(m-1)}\in\mathrm{S}(\mathrm{v}_1^{(m-1)}),\ \tilde{x}_1^{(m-1)}\in\mathrm{S}(\tilde{\mathrm{v}}_1^{(m-1)})$

$$\mathrm{A}_m\lambda_m\mathrm{A}_m^{-}=\mathrm{A}_m x_1^{(m-1)}\lambda_1\tilde{x}_1^{(m-1)}\mathrm{A}_m^{-1}=\mathrm{A}_1\lambda_1\mathrm{A}_1^{-1}.$$

Further, for any fixed n the sequence $\{\mathrm{v}_m^{(n)}\}_{m=1}^{\infty}$ is normal since $\{\mu_m\}_{m=1}^{\infty}$ is a such one. Therefore $\mathscr{A}^{(n)}=\mathscr{B}^{(n)},\ n\in\mathrm{N}$

The set $\mathscr{A}^{(\infty)}=\bigcap_{k=1}^{\infty}(\bigcup_{n\geq k}\mathrm{B}^{(n)})^-$ contains of all limit points of all possible sequences $\{\mathrm{v}^{(n)}\}_{n=1}^{\infty}$, where $\mathrm{v}^{(n)}\in\mathscr{B}^{(n)}$. Hence $\mathscr{A}_m\subset\mathscr{A}^{(\infty)}$ for all m and $\mathscr{A}_{(\infty)}\subset\mathscr{A}^{(\infty)}$.

The inclusion $\mathscr{A}^{(\infty)}\subset\mathscr{B}^{(\infty)}$ is obvious. ■

We shall call the set $\mathscr{A}_{\infty}$ *the convolutional attractor* (CA) of the normal sequence $\{\mu_n\}_{n=1}^{\infty}$. The equality

$$\dot{\mu}_n=\mu_n*\lambda_n,\qquad n\in\mathrm{N}$$

defines the "limiting sequence" $\{\dot{\mu}_n\}_{n=1}^{\infty}$ for $\{\mu_n\}_{n=1}^{\infty}$ such that the sequences $\mathrm{v}_m^{(n)}$ and

$$\dot{\mathrm{v}}_m^{(n)}=\dot{\mu}_{m+n-1}*\ldots*\dot{\mu}_m,\qquad m,n\in\mathrm{N}$$

are asymptotically equivalent as $n\to\infty$, that is

$$\lim_{n\to\infty}(\mathrm{v}_m^{(n)}-\dot{\mathrm{v}}_m^{(n)})=0,\qquad m\in\mathrm{N}$$

It is easy to see that the CA

$$\dot{\mathscr{A}}_{\infty}=\overline{\lim_{m\to\infty}}\ \mathrm{L}m\mathrm{P}_{n\to\infty}\dot{\mathrm{v}}_m^{(n)}$$

of the sequence $\{\dot{\mu}_n\}_{n=1}^{\infty}$ coincides with $\mathscr{A}_{\infty}$ and moreover

$$\mathscr{A}_{\infty}=\dot{\mathscr{A}}_{\infty}=(\dot{\mathrm{v}}_m^{(n)},\ m\in\mathrm{N},\ n\in\mathrm{N})^-\tag{4.2}$$

Let us describe now the set $\mathscr{E}_{\infty}$ of all idempotents of $\mathscr{A}_{\infty}$

COROLLARY 4.2. – *For all* $m \in \mathrm{N}$

$$\mathscr{E}_\infty := \{ \alpha \in \mathscr{A}_\infty : \alpha^2 = \alpha \} = \{ \tilde{\alpha} * \alpha, \alpha \in \mathscr{A}_\infty \} = \{ \alpha * \tilde{\alpha}, \alpha \in \mathscr{A}_\infty \} = \mathrm{L}m\mathrm{P}_{n \to \infty} \nu_m^{(n)} * \tilde{\nu}_m^{(n)} = (\lambda_n, n \in \mathrm{N})^- = \{ x \lambda_m x^{-1}, x \in \mathrm{A}_m \}$$

This is a direct consequence of the equality $\mathscr{A}_\infty = \mathrm{A}_m \lambda_m \mathrm{A}_m^{-1}$, $m \in \mathrm{N}$, (*see* th. 4.1 *c*).

COROLLARY 4.3. – *Let* $\mathrm{K} = [\mathrm{K}_m, m \in \mathrm{N}]^-$ *be the smallest compact subgroup containing the subgroups* K_m, $m \in \mathrm{N}$. *Then*

$$\mathrm{K} = [\bigcup_{\nu \in \mathscr{B}^{(\infty)}} \mathrm{S}(\tilde{\nu} * \nu)]^- = [\bigcup_{\nu \in \mathscr{B}^{(\infty)}} \mathrm{S}(\tilde{\nu} * \nu)]^- = [\mathrm{S}(\lambda), \lambda \in \mathscr{E}_\infty]^- = [x \mathrm{K}_m x^{-1}, x \in \mathrm{A}_m]^-, \qquad m \in \mathrm{N}$$

and K *is a subgroup of the group* $\mathrm{H} = [\bigcup_{\nu \in \mathscr{B}^{(\infty)}} \mathrm{S}(\nu)]^-$.

We are going to elucidate now when the CA forms a group of measures and when the sequence $\nu_1^{(n)} * \tilde{\nu}_1^{(n)}$ converges (*cf.* ex. 3.4).

THEOREM 4.4. – *The following conditions are related by* 8) ⇒ 7) ⇔ 6) ⇒ 5) *and* 1) – 5) *are equivalent among themselves:*

1) $\lambda_m = \lambda_1$, $m \in \mathrm{N}$,
2) $\lambda_m = \lambda_\mathrm{K}$, $m \in \mathrm{N}$,
3) *there exists* $\lim_{n \to \infty} \nu_m^{(n)} * \tilde{\nu}_m^{(n)}$,
4) $\lim_{n \to \infty} \nu_m^{(n)} * \tilde{\nu}_m^{(n)} = \lambda_\mathrm{K}$,
5) $\lambda_\mathrm{K} \in \mathscr{B}^{(\infty)}$,
6) $\mathscr{A}_\infty$ *is a subgroup of the semigroup* $\mathscr{M}_1(\mathrm{G})$,
7) $\mathscr{A}_\infty = \lambda_\mathrm{K} \mathrm{H}$,
8) $\mathscr{A}^{(n)} = \mathscr{A}^{(1)} * \ldots * \mathscr{A}^{(1)}$ (*n-times*), $n \in \mathrm{N}$.

Proof. – 1), 2), 3), 4) are equivalent by cor. 4.2 and 4.3.
2) ⇒ 5). $\lambda_\mathrm{K} = \lambda_1 \in \mathscr{A}_1 \subset \mathscr{B}^{(\infty)}$ by th. 3.6 *b*),
5) ⇒ 2). If $\lambda_\mathrm{K} \in \mathscr{B}^{(\infty)} = \{ \nu_m^{(n)}, m, n \in \mathrm{N} \}^-$, then $\lambda_\mathrm{K} \in \{ \nu_m^{(n)} * \lambda_\mathrm{K}, m, n \in \mathrm{N} \}^-$ and $\lambda_\mathrm{K} \in \{ \lambda_m, m \in \mathrm{N} \}^- = \mathscr{E}_\infty$.
Thus $\mathscr{E}_\infty = \{ \lambda_\mathrm{K} \}$ and $\lambda_\mathrm{K} = \lambda_m$, $m \in \mathrm{N}$.
7) ⇒ 5) is obvious
6) ⇒ 7) If $\mathscr{A}_\infty$ is a group, the set $\mathscr{E}_\infty = \{ \lambda_m, m \in \mathrm{N} \}^-$ of all its idempotents coincides to $\{ \lambda_\mathrm{K} \}$. Then K is a normal subgroup of H, the group $\mathscr{A}_\infty$ has the form

$$\mathscr{A}_\infty = \mathrm{A}_1 \lambda_\mathrm{K} \mathrm{A}_1^{-1} = (\mathrm{A}_1 \mathrm{A}_1^{-1}) \lambda_\mathrm{K} \subset \mathrm{H} \lambda_\mathrm{K}$$

The group $\mathscr{A}_\infty$ contains also the sets $(\mathrm{A}_1 \mathrm{A}_1^{-1})^n \lambda_\mathrm{K}$, $n \in \mathrm{N}$ and hence $\mathrm{H} \lambda_\mathrm{K} \subset \mathscr{A}_\infty$.
7) ⇒ 6) since K is a normal subgroup of H in this case.

8) $\Rightarrow$ 6). If 8) holds the set $\mathscr{B}^{(\infty)}=\left(\bigcup_{n=1}^{\infty}\mathscr{A}^{(n)}\right)^{-}$ is a semigroup and $\mathscr{A}^{(\infty)}=\bigcap_{m=1}^{\infty}\left(\bigcup_{n\geq m}\mathscr{A}^{(n)}\right)^{-}$ is a subsemigroup of $\mathscr{B}^{(\infty)}$. Hence $\lambda_m * \lambda_n \in \mathscr{A}^{(\infty)}$ for all m, $n \in \mathrm{N}$, and $\lambda_{\mathrm{K}} \in \mathscr{A}^{(\infty)} \subset \mathscr{B}^{(\infty)}$, since λ_{K} is contained in the compact semigroup generated by $\mathscr{E}_{\infty}=\{\lambda_n, n\in \mathrm{N}\}^{-}$. Using 5) $\Rightarrow$ 2) we see that $\mathscr{E}_{\infty}=\{\lambda_{\mathrm{K}}\}$.

Then $\nu_m^{(n)} * \lambda_{\mathrm{K}} = \lambda_{\mathrm{K}} * \nu_m^{(n)} \in \mathscr{A}_{\infty}$ for all m, $n \in \mathrm{N}$
and

$$\lambda_{\mathrm{K}} * \mathscr{A}^{(\infty)} = \lambda_{\mathrm{K}} * \mathscr{B}^{(\infty)} = \mathscr{A}_{\infty}$$

is the smallest left and in the same time right ideal of the compact semigroups $\mathscr{A}^{(\infty)}$ and $\mathscr{B}^{(\infty)}$. Thus $\mathscr{A}_{\infty}$ is a group ([5], 9.22). ■

Remark 4.5 *a*) The conditions 1)-5) do not imply 6) in a general case. For example, if $\mu_{2k}=\lambda x$, $\mu_{2k-1}=\lambda x^{-1}$, $k\in \mathrm{N}$, where $\lambda^2=\lambda=x\lambda x^{-1}$ and $\lambda x^2 \neq \lambda$, one has the normal sequence $\{\mu_n\}$ with $\mathscr{E}_{\infty}=\{\lambda\}$ and $\mathscr{A}_{\infty}=\{\lambda, \lambda x, \lambda x^{-1}\}$ which is not a group and even semigroup.

b) Remember that the smallest two-sided ideal of a compact semigroup is called its Sushkevich kernel. ([5], 9.21). We have proved now that provided condition 8) of th. 4.4 holds the CA $\mathscr{A}_{\infty}$ of a normal sequence $\{\mu_n\}$ is the Sushkevich kernel of the semigroups $\mathscr{B}^{(\infty)}$ and $\mathscr{A}^{(\infty)}$ and it is a group.

It should be also noted that both inclusions $\mathscr{A}_{\infty} \subset \mathscr{A}^{(\infty)} \subset \mathscr{B}^{(\infty)}$ may be strict (*see* sec. 6).

As a consequence of the above results we can prove now the convergence theorem.

Denote $\mathrm{D}_m = \underline{\lim}_{n\to\infty} \mathrm{S}(\nu_m^{(n)})$, $m\in \mathrm{N}$.

THEOREM 4.6. — *For any normal sequence* $\{\mu_n\}_{n=1}^{\infty}$ *the following conditions are equivalent*

1) $\lim_{n\to\infty} \nu_m^{(n)}$ *exists*,
2) $\lim_{n\to\infty} \nu_m^{(n)} = \lambda_{\mathrm{H}}$ *for all* $m\in \mathrm{N}$,
3) $\mathrm{K}_m = \mathrm{H}$,
4) $\mathrm{A}_m = \mathrm{D}_m$,
5) $\mathrm{D}_m \neq \varnothing$,
6) $\lambda_{\mathrm{H}} \in \mathscr{B}^{(\infty)}$,

Each of the conditions 1)-5) *holds for all* $m\in \mathrm{N}$ *if it does for some one.*

Proof. — 2) $\Rightarrow$ 1) and 4) $\Rightarrow$ 5) are obvious

1) $\Rightarrow$ 3) If $\mathscr{A}_m = \mathrm{A}_m \lambda_m$ consists of the only point then $\mathrm{A}_m \subset \mathrm{K}_m$ and hence $\mathrm{K}_m = \mathrm{H}$,

3) ⇒ 2) If $K_m = H$ then $\mathscr{A}_m = A_m \lambda_m = A_m \lambda_H = \{\lambda_H\}$,
2) ⇒ 4) $H = S(\lim_{n \to \infty} \dot{\nu}_m^{(n)}) \subset D_m \subset A_m \subset H$,
2) ⇒ 6) $\lambda_H = \lim_{n \to \infty} \nu_m^{(n)} \in \mathscr{A}_\infty \subset \mathscr{B}^{(\infty)}$,
6) ⇒ 2) *If* $\lambda_H \in \mathscr{B}^{(\infty)} = (\nu_m^{(n)}, m, n \in N)^-$,
then $\lambda_H \in (\nu_m^{(n)} * \lambda_m, m, n \in N)^- = \mathscr{A}_\infty = A_m \lambda_m A_m^{-1}$,
and $\mathscr{A}_\infty = \{\lambda_H\}$ *i.e.* 2) holds
5) ⇒ 3) If $x \in D_m$ then for every open $U \ni x$ there exists n_0 such that $U \cap S(\nu_m^{(n)})) \neq \varnothing$ for all $n > n_0$. Hence for $x_m^{(n)} \in S(\nu_m^{(n)}) \cap U$ we have

$$S(\nu_m^{(n)}) \subset x_m^{(n)} K_m \subset UK_m, \qquad n > n_0$$

and

$$A_m = \overline{\lim_{n \to \infty}} \, S(\nu_m^{(n)})) \subset UK_m$$

If U runs the filter of open neighborhoods of x the open set UK_m runs the filter of neighborhoods of $x K_m$. We have now

$$K_m \subset A_m \subset x K_m$$

Hence $A_m \subset K_m$ and $H \subset K_m$ and $H = K_m$.

We have proved now 1) ⇒ 2) ⇒ 3) ⇒ 1) and 2) ⇔ 6) and 2) ⇒ 4) ⇒ (5) ⇒ 3). ■

In the simplest case, when $\nu_m^{(n)} = \mu * \ldots * \mu$ (n-times), the theorem proved above is the well known Ito-Kavada theorem (*see* [1], [2], [4], [5], [7] and [8], ch. 2). For Borel normal sequences the implications 1) ⇔ 2) ⇔ 3) have been proved by Urbanik [3]. The convergence of convolutions $\nu_m^{(n)}$ as $n \to \infty$ for every m to the same limit means the compositional convergence in the Maksimov sense [6].

5. THE PROOF OF THE MAIN THEOREMS 1.1-1.4

In this section we shall deduce the main results stated in the introduction from the theorems of the sec. 3 and 4.

Consider a SSRM $\{\mu_n\}_{n=1}^\infty$, $\mu_n = \mu_n(\omega)$, $\omega \in \Omega$, on G which satisfies the conditions A) and B) and let $\nu_m^{(n)} = \nu_m^{(n)}(\omega)$, $m, n \in N$ be the corresponding sequences of their convolutions defined in the sec. 1. We shall use again the notations of the sec. 1.

Let $\mathscr{A}^{(n)} \subset \mathscr{M}_1(G)$ be the support of the distribution $P \circ (\nu_m^{(n)})^{-1}$ of the random measures $\nu_m^{(n)}(\omega)$. (It does not depend on m). Also $\mathscr{A}^{(n)}(\omega) = \mathrm{L}m\mathrm{P}_{m \to \infty} \nu_m^{(n)}(\omega)$, $\mathscr{B}^{(n)}(\omega) = (\nu_m^{(n)}(\omega), m, n \in N)^-$.

Assertion 5.1. – *a)* $\{\nu_m^{(n)}(\omega)\}_{m=1}^\infty$ *is a normal sequence in* $\mathscr{M}_1(G)$ *for a.a.ω.*

b) $\mathscr{A}^{(n)}=\mathscr{A}^{(n)}(\omega)=\mathscr{B}^{(n)}(\omega)$ *for a.a.*ω.

Proof. – One can transfer the SSRM $\{\nu_m^{(n)}\}_{m=1}^{\infty}$ onto the space $(\bar{\Omega}, \bar{P})$ of its realizations by the mapping

$$\varphi: \quad \Omega \ni \omega \to \{\nu_m^{(n)}(\omega)\}_{m=1}^{\infty} \in \bar{\Omega}$$

where $\bar{\Omega}$ is a compact subset of the countable direct product of the copies of $\mathscr{A}^{(n)}$. The compact set $\bar{\Omega}$ has a countable base of the topology by B). Herewith the shift transformation θ on $\bar{\Omega}$ preserves the measure $\bar{P}=P\circ\varphi^{-1}$ and θ is ergodic by A).

One can now deduce *a)* and *b)* from the Poincare recurrence theorem and ergodicity of θ, considering the coutable system of open sets

$$U_{k_1}\times\ldots\times U_{k_m}, \qquad m\in N,$$

where $\{U_k\}_{k=1}^{\infty}$ is a base of the topology on $\mathscr{A}^{(n)}$ (*see* [1], ch. 1 § 1, § 2). ■

Consider now the sets $\mathscr{A}^{(\infty)}$, $\mathscr{B}^{(\infty)}$ and the subgroups K, H defined in sec. 1 and denote

$$\mathscr{A}^{(\infty)}(\omega)=\overline{\lim_{n\to\infty}}\,\mathscr{A}^{(n)}(\omega), \qquad \mathscr{B}^{(\infty)}(\omega)=\left(\bigcup_{n=1}^{\infty}\mathscr{B}^{(n)}(\omega)\right)^{-},$$

$$H(\omega)=[S(\mu_n(\omega)),\, n\in N]^{-},$$

$$K(\omega)=[S(\tilde{\nu}_m^{(n)}(\omega)*\nu_m^{(n)}(\omega)),\, m,\, n\in N]^{-}$$

Assertion 5.2. – *a)* $\mathscr{A}^{(\infty)}(\omega)=\mathscr{A}^{(\infty)}$, $\mathscr{B}^{(\infty)}(\omega)=\mathscr{B}^{(\infty)}$,

b) $H(\omega)=H$, $K(\omega)=K$

for a.a.$\omega\in\Omega$.

This assertion follows immediatly from the above one.

Consider now the CA

$$\mathscr{A}_{\infty}(\omega)=\overline{\lim_{m\to\infty}}\; L\,m\,P_{n\to\infty}\,\nu_m^{(n)}(\omega)$$

of the sequence $\{\mu_n(\omega)\}_{n=1}^{\infty}$ with a fixed ω.

Assertion 5.3. – The mapping $\omega\to\mathscr{A}_{\infty}(\omega)$ is constant a. s.

Proof. – The sequence $\{\mu_n(\omega)\}_{n=1}^{\infty}$ is normal for a.a.ω. For any such ω the limit

$$\lambda_m(\omega)=\lim_{n\to\infty}\tilde{\nu}_m^{(n)}(\omega)*\nu_m^{(n)}(\omega)$$

exists and one can consider the limiting sequence $\dot{\mu}_m(\omega)=\mu_m(\omega)*\lambda_m(\omega)$.

For the corresponding n-th convolutions $\dot{\nu}_m^{(n)}(\omega)=\nu_m^{(n)}(\omega)*\lambda_m(\omega)$ the equality

$$\mathscr{A}_{\infty}(\omega)=\dot{\mathscr{A}}_{\infty}(\omega)=\dot{\mathscr{A}}^{(\infty)}(\omega)=\dot{\mathscr{B}}^{(\infty)}(\omega)$$

holds on account of (4.2), where

$$\dot{\mathscr{A}}_{\infty}(\omega)=\overline{\lim_{m\to\infty}}\; L\,m\,P_{n\to\infty}\,\dot{\nu}_m^{(n)}(\omega)$$

and

$$\dot{\mathscr{B}}^{(\infty)}(\omega)=(\dot{\nu}_m^{(n)}(\omega),\ m,\ n\in \mathrm{N})^-.$$

On the other hand applying the above assertion for the SSRM $\{\dot{\mu}_n\}_{n=1}^{\infty}$, one can see that the mapping $\omega\rightarrow\dot{\mathscr{B}}^{(\infty)}(\omega)$ is constant a. s. ■

In order to prove the rest statements of the theorems 1.1-1.4 and to complete their proofs one can apply now the results of the sec. 3 and 4 for a fixed normal sequence $\{\mu_n(\omega)\}_{n=1}^{\infty}$.

6. EXAMPLES

We shall give here three simple examples of CA to illustrate the objects under consideration.

1) Let $x_n=x_n(\omega)$, $n\in\mathrm{N}$ be i. i. d. random elements on Ω with the values in a compact group G which has a countable base of its topology. Define the SSRM

$$\mu_n(\omega)=\delta_{x_{n+1}(\omega)\,.\,x_n(\omega)^{-1}},\qquad n\in\mathrm{N},\quad \omega\in\Omega$$

and denote by S the essential image of x_n.

In this case we obtain for a.a.ω

$$\nu_m^{(n)}(\omega)=\delta_{x_{n+m}(\omega)\,.\,x_m(\omega)^{-1}},\qquad m, n\in\mathrm{N}$$

$$\mathrm{S}=(x_n(\omega),\ n\in\mathrm{N})^-=\mathrm{L}\,m\,\mathrm{P}_{n\to\infty}\,x_n(\omega),\qquad \mathrm{H}=[\mathrm{S}]^-$$

$$\mathrm{A}_m(\omega)=\mathrm{S}\,x_n(\omega)^{-1},\qquad \mathscr{A}_m(\omega)=\{\delta_x,\ x\in\mathrm{S}\,.\,x_m(\omega)^{-1}\}$$

$$\mathrm{K}=\mathrm{K}_m(\omega)=\{e\},\qquad \mathscr{E}_\infty=\{\delta_e\}$$

$$\mathscr{A}^{(n)}=\mathscr{A}_\infty=\mathscr{A}^{(\infty)}=\mathscr{B}^{(\infty)}=\{\delta_x,\ x\in\mathrm{SS}^{-1}\}$$

One can see that $\mathrm{A}_m(\omega)$ and $\mathscr{A}_m(\omega)$ essentially depend on m and ω and the CA $\mathscr{A}_\infty$ need not coincide with $\lambda_{\mathrm{K}}\,\mathrm{H}$.

2) Let H be a subgroup of a finite group G and K_0 be a *non-normal* subgroup of H.

Denote

$$\Gamma=\{x\lambda_0 y^{-1},\ x,\ y\in\mathrm{H}\}\subset\mathscr{M}_1(\mathrm{G})$$

where λ_0 denotes the Haar measure of K_0.

Consider the SSRM $\mu_n=\mu_n(\omega)$, $n\in\mathrm{N}$, which is a Markov chain with the finite state space Γ, and transition probability matrix

$$\mathrm{Q}=\{q_{\bar{\alpha}\bar{\beta}}\}_{\alpha,\,\beta\in\Gamma},\ \text{where}\ q_{\alpha,\,\beta}=\mathrm{P}\{\mu_{n+1}=\beta\,|\,\mu_n=\alpha\},$$

and stationary vector of probabilities

$$q_\alpha=\mathrm{P}\{\mu_n=\alpha\},\qquad \alpha\in\Gamma$$

We demand that the transition matrix Q satisfies the condition

$$q_{\alpha\beta}>0 \quad \Leftrightarrow \quad \beta * \alpha \in \Gamma \tag{6.1}$$

(ones sees $\beta * \alpha \in \Gamma \Leftrightarrow \alpha * \tilde{\alpha} = \tilde{\beta} * \beta$).

This Markov chain is mixing and for the corresponding convolutions $\nu_m^{(n)}(\omega)$ we have a.s. for $n, m \in \mathbb{N}$

$$\begin{array}{lll} \mathscr{A}^{(n)} = \Gamma, & A_m(\omega) = H, & \mathscr{E}_\infty = \{ x\lambda_0 x^{-1}, x \in H \} \\ \lambda_m(\omega) = \tilde{\mu}_m(\omega) * \mu_m(\omega), & & K_m(\omega) = S(\lambda_m(\omega)) \\ \mathscr{A}_m(\omega) = H\lambda_m(\omega), & & K = [x K_0 x^{-1}, x \in H] \end{array}$$

and

$$\mathscr{A}_\infty = \mathscr{A}^{(\infty)} = \mathscr{B}^{(\infty)} = \Gamma$$

Since K_0 is not a normal subgroup of H the CA $\mathscr{A}_\infty = \Gamma$ contains a nontrivial set of idempotents $\mathscr{E}_\infty = \{ x\lambda_0 x^{-1}, x \in H \}$.

As in the example 1) the SSRM $\{ \mu_n \}$ coincides with its limiting sequence $\{ \dot{\mu}_n \}$ and $\dot{\mathscr{A}}_\infty = \Gamma$.

3) We can change the previous example extending the state space of the considering Markov chain as follows

$$\Gamma' = \Gamma \cup \{ \delta_x, x \in H \}$$

and taking the matrix $Q' = \{ q'_{\alpha,\beta} \}_{a,\beta \in \Gamma'}$ which satisfies the same requirement (6.1) as Q.

Then the CA $\mathscr{A}_\infty$ of the obtained SSRM $\{ \mu_n \}_{n=1}^\infty$ coincides with Γ which is not equal to Γ' and we have the strict inclusion $\mathscr{A}_\infty \subset \mathscr{A}^{(\infty)}$ in such case.

One can construct a lot of different examples of CA replacing the "⇔" in the condition (6.1) on ". . ⇒" or considering generalization on the continuous state space case.

REFERENCES

[1] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai, *Eegodic Theory*, Springer-Verlag, New York, 1981.

[2] I. Csiszar, On Infinite Products of Random Elements and Infinite Convolutions of probability Distributions on Locally Compact Groups, *Z. Wahr. Geb.*, Vol. **5**, 1966, pp. 279-295.

[3] P. R. Halmos, *A Hilbert Space Problembook*, D. Van Nostrand company, INC, Toronto, London, 1967.

[4] H. Heyer, *Probability Measures on Locally Compact Groups*, Springer-Verlag, Berlin, New York, 1977.

[5] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, *Vol. I,II*, Springer-Verlag, Berlin, 1963.

[6] Y. Kawada and K. Itô, On the Probability Distributions on a Compact Group. 1, *Proc. Phys.-Math. Soc. Jap.*, Vol. **22**, 1940, pp. 977-998.

[7] B. M. KLOSS, *Limit Distributions for Sums of Independent Random Variables Taking Values in a Bicompact Group*, Dokt. Akad. Nauk SSSR, Vol. **109**, 1956, pp. 453-455, In Russian.

[8] B. M. KLOSS, Probability Distributions on Bicompact Topological Groups, *Theor. Prob. Appl.*, Vol. **6**, 1959, pp. 237-270.

[9] K. KURATOVSKI, *Topology. vv. I, II*, Academic Press, New York-London, 1966.

[10] A. MUKHERJEA and N. A. TSERPES, Measures on Topological Semigroups, *Lec. Notes in Math.*, Vol. **547**, Springer-Verlag, Berlin, New York, 1976.

[11] V. M. MAKSIMOV, Compositions Sequences of Measures on Compact Groups, *Theory Probab. Appl.*, Vol. **16**, 1971, pp. 54-73.

[12] D. S. MINDLIN and B. A. RUBSHTEIN, Convolutions of Random Measures on Compact Groups, *Theory Prob. Appl.*, Vol. **33**, 1988, pp. 355-357.

[13] D. S. MINDLIN and B. A. RUBSHTEIN, Random Measures on Compact Groups, Izv Uzb. Acad., VINITI report 3, Vol. **28**, 1986, N° 13/192-D.

[14] I. Z. RUZSA, Infinite Convolutions via Representations, *Lect. Notes in Math.*, Vol. **1064**, Springer-Verlag, Berlin-New York, 1988, pp. 398-408.

[15] V. V. SAZONOV and V. N. TUTUBALIN, Probability Distributions on Topological Groups, *Theory Probab. Appl.*, Vol. **11**, 1966, pp. 1-46.

[16] K. URBANIK, On the Limiting Probability Distributions on a Compact Topological Group, *Fund. Math.*, Vol. **44**, 1957, pp. 253-261.

(*Manuscript received Juny* 4, 1992;
revised April 9, 1993.)