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## Martin boundaries of some branching processes

by

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**ABSTRACT.** — We investigate the Martin boundary of some branching processes, namely Feller's continuous-state branching process, the critical continuous-time Galton-Watson process and the Yule process. The Martin boundary provides an integral representation of the non-negative space-time harmonic functions of these processes.

*Key words :* Space-time Harmonic Function, Martin Boundary, Branching Process.

**RÉSUMÉ.** — Nous décrivons ici la frontière de Martin des processus de branchement suivants : le processus de branchement de Feller, le processus de Galton-Watson critique à temps continu et le processus de Yule. La frontière de Martin donne une représentation intégrale des fonctions harmoniques dans le temps non-négatives de ces processus.

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### 1. INTRODUCTION

In this paper we study the integral representation in the convex set of all non-negative space-time harmonic functions for some branching processes. We specify the extreme points in this set. The set of extreme points

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will be called the Martin boundary. We will apply the general theory of sufficient statistics and extreme points formulated in Dynkin [Dy1]. The extremal space-time harmonic functions are obtained as limits of the *Martin kernel*, that is, as a limit of a quotient of transition densities. Every space-time harmonic function determines a new process, the so-called  $h$ -transform, which describes the original process conditioned on a specific limit behaviour.

Here we consider the critical binary continuous-time Galton-Watson process, the Feller process and the Yule process. For the Feller process and its approximating "particle process", the critical binary continuous-time Galton-Watson process, the Martin boundaries are  $[0, \infty)$  and an extra point  $\emptyset$ . The extremal space-time harmonic function  $h^\emptyset$  corresponding to  $\emptyset$  is the constant function 1 and so the  $h^\emptyset$ -transform is the original process. For  $0 < c < \infty$  the  $h^c$ -transform is the process conditioned on the limit behaviour

$$\lim_{t \rightarrow \infty} \frac{X_t}{t^2} = c.$$

For  $c=0$  we have  $h^0(s, x) = x$  and the limit behaviour of the  $h^0$ -transform is described by

$$\lim_{t \rightarrow \infty} \frac{X_t}{t^2} = 0, \quad X_t > 0, \quad \forall t.$$

The Martin boundary of the Yule process with intensity  $\lambda$  is  $[0, \infty)$ . For  $0 \leq c < \infty$  the  $h^c$ -transform is the Yule process conditioned on the event

$$\lim_{t \rightarrow \infty} e^{-\lambda t} X_t = c.$$

All basic properties of branching processes we use can be found in [AN]. For results on Martin boundaries for supercritical branching processes in discrete time we refer to [AN, ch. II.9], [Lo], [Co] and [Du1], [Du2]. The Martin boundary of some related processes, namely the  $d$ -dimensional Bessel process, which is related to the Feller process, and the Poisson process composed with a symmetric binomial distribution, which is related to the Yule process, are considered in [Sa].

The question of determining the Martin boundary of these branching processes came up in the investigation of superprocesses conditioned on their limit behaviour. The results of this paper on critical branching processes are used in [Ov2] in order to clarify the structure of certain  $H$ -transforms of superprocesses, in the sense of [Dy2], and of their approximating branching diffusions. If  $h$  is a space-time harmonic function of the Feller process, then the function  $H(t, \mu) := h(t, \mu(1))$ , where  $\mu$  is a finite measure on  $\mathbb{R}^d$ , is a space-time harmonic for the superprocess which only depends on the total mass  $\mu(1)$ . In the same way, the results on the Yule

process and the critical Galton-Watson process are used in [Ov1] to investigate the conditional behaviour of the corresponding branching diffusions.

**2. GENERALITIES**

We consider always the canonical model  $(\Omega, \mathcal{F}, P)$  with

$$\Omega = D([0, \infty), S), \quad X_t(\omega) = \omega(t), \quad \mathcal{F} = \sigma \left\{ \bigcup_{t \geq 0} \mathcal{F}_t \right\},$$

where  $\mathcal{F}_t = \sigma \{ X_s, s \leq t \}$ . Here the state space is of the form  $S = [0, \infty), \{0, 1, 2, \dots\}$  or  $\{1, 2, \dots\}$ .  $P$  is the probability measure on  $\Omega$  corresponding to a Markov transition function  $p$  on  $S$  and to the initial state 1 at time 0.

Let  $p(s, x; t, dy)$  be the Markov transition function and  $P_t^s$  the corresponding semigroup acting on functions. A family  $h = (h(t, \cdot))_{t \geq 0}$  of non-negative functions is called space-time harmonic, if

$$P_t^s h(t, \cdot) = h(s, \cdot)$$

for any  $s \leq t$ . We now recall some of the results in Dynkin [Dy1]. Let  $S^p$  be the class of all space-time harmonic functions  $h$  normed by the condition  $h(0, 1) = 1$ . In our cases we have  $p(s, x; t, \cdot) \leq p(0, 1; t, \cdot)$ , and so the assumption 10.2.A of [Dy1] is satisfied. Then the class  $S^p$  admits the following integral representation. Every  $h \in S^p$  is represented by a unique measure  $\mu^h$  on the extreme points  $S_e^p$  of  $S^p$ , i. e.,

$$h = \int_{S_e^p} h' \mu^h(dh'). \tag{1}$$

We call the set  $S_e^p$  the Martin boundary of  $p$ . The extreme points  $h^e$  are characterized by the condition

$$h^e = \int_{S^p} h' \mu(dh') \Rightarrow \mu = \delta_{h^e},$$

where  $\delta$  indicates a Dirac measure.

For every  $h \in S^p$  the  $h$ -transform  $P^h$  is defined as the probability measure on  $(\Omega, \mathcal{F})$  having density  $h(t, X_t)$  with respect to  $P$  on  $\mathcal{F}_t$ :

$$P^h(\Lambda) := E_p[h(t, X_t); \Lambda], \quad \forall \Lambda \in \mathcal{F}_t.$$

By Fubini's theorem, the integral representation (1) implies

$$P^h[\Lambda] = \int_{S_e^p} P^{h'}[\Lambda] \mu^h(dh'). \tag{2}$$

According to the last formula in section 10 of [Dy1] all extremal space-time harmonic functions satisfy

$$h(s, x) = \lim_{u \rightarrow \infty} K(s, x; u, X_u) \text{ a. s. } \mathbf{P}^h, \tag{3}$$

where the *Martin kernel*  $K$  is defined by

$$K(s, x; u, y) := \frac{dp(s, x, u, \cdot)}{dp(0, 1, u, \cdot)}$$

for  $s \leq u$ . In particular, every extremal space-time harmonic function appears as a limit of  $K(s, x; u_n, y_{u_n})$  for  $u_n \rightarrow \infty$  and for some sequence  $(y_{u_n}) \subset S$ .

Suppose we know that  $K(s, x; u, y_u)$  converges to an extremal  $h^c(s, x)$  iff  $\alpha(u, y_u) \rightarrow c$  for some “rescaling” function  $\alpha$ . Then we say that  $(x_u, u)$  converges in the Martin topology to  $c$  iff  $\alpha(u, x_u) \rightarrow c$ , and we write  $(x_u, u) \xrightarrow{\mathcal{M}} c$ . We use the set of these limit points as a parametrization of the Martin boundary  $S_e^p$ .

In the sequel we will calculate all possible limits in our special situations and decide which are space-time harmonic and extremal.

### 3. YULE PROCESS

The Yule process, also known as the linear pure birth process, describes the evolution of a population of particles. Each particle lives for an  $\exp(\lambda)$ -distributed time and produces 2 new particles when it dies, independent of all other particles. Considered as a point process with state space  $S = \{1, 2, \dots\}$ , the Yule process has the intensity  $\lambda X_t$ .

**THEOREM 1.** — *The Martin boundary of a Yule process with parameter  $\lambda$  is  $[0, \infty)$ . A sequence  $(x_t, t)$  converges in the Martin topology as  $t \rightarrow \infty$  to a point  $a$  in the Martin boundary iff  $e^{-\lambda t} x_t$  converges to  $a$  as  $t \rightarrow \infty$ . The extremal space-time harmonic functions are of the form*

$$h^a(s, l) := e^{\lambda s + a} \frac{(ae^{\lambda s})^{l-1}}{(l-1)!} \exp(-ae^{\lambda s}), \tag{4}$$

*in particular we have  $h^0(s, l) = e^{\lambda s} \delta_{1l}$ . According to [AN, p. 111] there exists a non-negative random variable  $W$  such that  $e^{-\lambda t} X_t \rightarrow W$  a. s. as  $t \rightarrow \infty$ . The measure  $\mathbf{P}_\lambda^{h^a}$  is the conditional distribution  $\mathbf{P}_\lambda[\cdot | W = a]$ . Under  $\mathbf{P}_\lambda^{h^a}$ , the process*

$$(X_t - a(e^{\lambda t} - 1))_{t \geq 0} \text{ is a martingale.} \tag{5}$$

Thus the process  $X$  has a state independent but time dependent intensity  $a\lambda e^{\lambda s}$  under  $P_\lambda^{h^a}$ , in contrast to its behaviour under  $P_\lambda$  where the intensity is state dependent but time independent.

*Proof.* – Using the explicit formula for the probability generating function in [AN, p. 109] and the branching property it is easy to prove that

$$P_\lambda[X_t = k | X_0 = l] = e^{\lambda t} (1 - e^{-\lambda t})^{k-l} \binom{k-1}{l-1}.$$

This gives the Martin kernel

$$K(s, l, t_n, k_n) = e^{\lambda s} \left( \frac{1 - e^{-\lambda(t_n - s)}}{1 - e^{-\lambda t_n}} \right)^{k_n - 1} e^{-\lambda(t-1)(t_n - s)} \times (1 - e^{-\lambda(t_n - s)})^{-(l-1)} \binom{k_n - 1}{l-1}.$$

It is immediate that  $K(s, l, k_n, t_n)$  converges as  $t_n \rightarrow \infty$ , iff  $\exp(-\lambda t_n) k_n$  converges to some  $a \in [0, \infty)$ . For a sequence  $(k_n, t_n)_n$  with  $e^{-\lambda t_n} k_n \rightarrow a$  the Martin kernel converges to the function  $h^a(s, l)$  defined by (4).

These functions solve the equation  $Af(s, l) = 0$ , where  $A$  is the generator of the space-time Yule process, *i. e.*

$$Af(s, l) := \frac{\partial}{\partial s} f(s, l) + \lambda l(f(s, l+1) - f(s, l)).$$

Using now the formulation of a *martingale problem* associated with a Markov process as in [EK, ch. 4, sec. 7] one can prove that  $(h^a(s, X_s))_{s \geq 0}$  is a martingale (details can be found in [Ov1]). Therefore  $h^a$  is space-time harmonic for every  $a$ .

Using the set  $\Lambda_a := \{X_t e^{-\lambda t} \rightarrow a\}$  in (2), we get that

$$1 = P_\lambda^{h^a}[\Lambda_a] = \int_0^\infty P_\lambda^{h^c}[\Lambda_a] \mu^{h^a}(dc) = \mu^{h^a}(a).$$

Hence  $\mu^{h^a} = \delta_a$  which is the definition of extremality.

Because

$$P_\lambda^h[W \in A] = \int_0^\infty P_\lambda^{h^c}[W \in A] \mu^h(dc) = \mu^h(A),$$

the measure  $\mu^h$  is the distribution of  $W$  under  $P_\lambda^h$ . This yields for  $h = 1$ :

$$P_\lambda[B \cap \{W \in A\}] = \int_A P_\lambda^{h^c}[B] P[W \in dc]$$

for every  $A \in \mathcal{B}([0, \infty))$  and  $B \in \mathcal{F}$ . So  $P_\lambda^{h^c}$  is the distribution of the Yule process conditioned to have  $W = c$ . Therefore, and because by [AN, p. 127]

$(X(\lambda^{-1} \log(1+tW^{-1}))_{t \geq 0}$  is a Poisson process with intensity 1 under  $P_\lambda$  conditioned on  $W$ , the process  $(X(\lambda^{-1} \log(1+tc^{-1}))_{t \geq 0}$  is a Poisson process under  $P^{hc}$ . This implies, that the process

$$(X_t - c(e^{\lambda t} - 1))_{t \geq 0} \text{ is a martingale. } \square$$

*Remarks 1.* — In [Ov1] we used the fact that the Yule process is a random time change of the Poisson process (as above, cf. [AN], p. 127) in order to investigate the Martin boundary of the Yule process by time-reversing. In a first step, we investigated the Martin boundary of the Poisson process. It turns out that the Martin boundary of the Poisson process with intensity  $\lambda$  is  $[0, \infty)$ , that  $(k_t, t) \rightarrow a$  in the Martin topology iff  $\frac{k_t}{t} \rightarrow a$  and that the extremal space-time harmonic functions are

$$h^a(s, l) = \left(\frac{a}{\lambda}\right)^{l-1} \exp(-(a-\lambda)s);$$

this fact is also mentioned in [Sa]. It follows by the Girsanov transformation, that the  $h^a$ -transform is the distribution of a Poisson process with intensity  $a$ . In a second step, we used again the time change to arrive at the results of Theorem 1.

2. Formula (5) implies that

$$A_s^a f(k) = (a\lambda e^{\lambda s} (f(k+1) - f(k)))$$

is the generator of the  $h^a$ -transform. It can be written as:

$$A_s^a f(k) = \left(a\lambda e^{\lambda s} \frac{1}{k}\right) k \cdot (f(k+1) - f(k)) \quad (6)$$

or as

$$A_s^a f(k) = a\lambda e^{\lambda s} k \left(\frac{1}{k} f(k+1) + \frac{k-1}{k} f(k) - f(k)\right). \quad (7)$$

The formulas (6) and (7) can be interpreted as follows. The generator (6) may be viewed as the generator of a Yule process with state and time dependent intensity  $a\lambda e^{\lambda s} \frac{1}{k}$ . The formula (7) refers to a branching process with time dependent intensity  $a\lambda e^{\lambda s}$  and state dependent splitting behaviour  $p_1 = \frac{k-1}{k}$ ,  $p_2 = \frac{1}{k}$ . This means that each particle on dying gives rise to 2 new particles with a high probability if only a few particles are alive. But if many particles are alive this probability is low.

3. There is a strong connection between the Martin boundary of the Yule process and the Martin boundary of the embedded discrete-time Galton-Watson process which is investigated in [Lo]: Define  $X_n^{(6)} := X_{n8}$

such that  $(X_n^{(\delta)})_{n \geq 0}$  is a supercritical Galton-Watson process with generating function  $f^{(\delta)}(\sigma) = E[\sigma^{X_\delta}]$  and mean  $m^{(\delta)} := \frac{d}{d\sigma} f^{(\delta)}(1) = e^{\lambda\delta}$ . Then a result of [Lo] says that the extremal space-time harmonic functions are

$$h_\delta^a(n, i) = (m^{(\delta)})^n g(i, a(m^{(\delta)})^n), \tag{8}$$

with  $a \in [0, \infty)$  and  $g(i, \cdot)$  the  $i$ -th convolution of the density of  $W$ . In our case  $W$  is exponentially distributed [AN], p. 128. Hence (8) reduces to

$$h_\delta^a(n, i) = e^{\lambda\delta n} \frac{1}{(i-1)!} (ae^{\delta n})^{i-1} \exp(-ae^{\delta n}).$$

With  $h^a(t, i) := e^{\lambda t} h_{(t/n)}^a(n, i)$  one can establish a 1-1-correspondence between the space-time harmonic functions of the Yule process and those of the imbedded Galton-Watson process.

#### 4. CRITICAL BRANCHING PROCESSES

Whereas the Yule process describes a *supercritical* branching process with exponential growth, a critical branching process is a process where the average number of offsprings is one. Because of the random fluctuation every path dies out despite the fact that  $EX_t = 1$  for all  $t$ . The simplest model is binary splitting: Every dying particle gives rise to 2 new particles with probability  $\frac{1}{2}$  and to 0 particles with the same probability. For computational simplification we assume that the intensity of the exponentially distributed lifetime of every particle is 2. The generating function  $F(\sigma, t) := E[\sigma^{X_t}]$  is given by (cf. [Se], chapter I. 8)

$$F(\sigma, t) = 1 - \frac{1 - \sigma}{t(1 - \sigma) + 1}.$$

Using the branching property we are led to the transition function for  $s < t$ :

$$p(s, k, t, n) = \begin{cases} \sum_{l=1}^k \binom{k}{l} \binom{n-1}{l-1} \frac{(t-s)^{n+k-2l}}{(t-s+1)^{n+k}} & \text{if } n > 0, \\ \frac{(t-s)^k}{(t-s+1)^k} & \text{if } n = 0, \end{cases}$$

(with the convention  $\sum_{i=1}^0 = 0$ ). We are now prepared to prove the following theorem:

**THEOREM 2.** — *The Martin boundary of the critical branching process is  $[0, \infty) \cup \{\emptyset\}$ . The Martin topology  $\mathcal{M}$  is given by*

$$(x_t, t) \xrightarrow{\mathcal{M}} \emptyset \quad \text{iff } x_t = 0 \text{ for some } t \in [0, \infty)$$

and

$$(x_t, t) \xrightarrow{\mathcal{M}} c \in [0, \infty) \quad \text{iff } x_t > 0 \text{ for all } t \text{ and } \lim_{t \rightarrow \infty} \frac{x_t}{t^2} = c.$$

The extremal space-time harmonic functions are

$$h^\emptyset(s, k) = 1 \tag{9}$$

$$h^0(s, k) = k \tag{10}$$

$$h^c(s, k) = \exp(-cs) \sum_{l=1}^k \binom{k}{l} \frac{1}{(l-1)!} c^{l-1}, \quad c \in [0, \infty). \tag{11}$$

The  $h^c$ -transform for  $c \in [0, \infty)$  has the generator

$$\begin{aligned} A^c f(k) = & k \cdot \frac{h^c(t, k-1) + h^c(t, k+1)}{h^c(t, k)} \\ & \times \left( (f(k+1) - f(k)) \frac{h^c(t, k+1)}{h^c(t, k-1) + h^c(t, k+1)} \right. \\ & \left. + (f(k-1) - f(k)) \frac{h^c(t, k-1)}{h^c(t, k-1) + h^c(t, k+1)} \right). \end{aligned}$$

*Proof.* — We have to find sequences  $(t_n, x_n)_{n>0}$  such that  $t_n \rightarrow \infty$  and the limit of the Martin kernel  $K(s, k, t_n, x_n)$  exists and is finite.

There are three different cases:

1.  $k=0$ :

$$\frac{p(s, 0, t_n, x_n)}{p(0, 1, t_n, x_n)} = \delta_0(x_n) \cdot \left( \frac{t_n}{t_n + 1} \right)^{-1}$$

This gives  $h(s, 0) = 1$  or  $= 0$  according as  $x_n \rightarrow 0$  or not.

2.  $k \geq 1, x_n = 0$ :

$$\frac{p(s, k, t_n, 0)}{p(0, 1, t_n, 0)} = \left( \frac{t_n - s}{t_n - s + 1} \right)^k \left( \frac{t_n}{t_n + 1} \right)^{-1}$$

The limit is the space-time harmonic function  $h^\emptyset(s, k) = 1$ .

3.  $k \geq 1, x_n \geq 1$ :

$$\frac{p(s, k, t_n, x_n)}{p(0, 1, t_n, x_n)} = \frac{p(s, 1, t_n, x_n)}{p(0, 1, t_n, x_n)} \cdot \left(\frac{t_n - s}{t_n - s + 1}\right)^{k-1} \times \sum_{l=1}^k \binom{k}{l} \binom{x_n - 1}{l-1} (t_n - s)^{-2(l-1)}. \quad (12)$$

Let us first consider the convergence of

$$\begin{aligned} K(s, 1, t_n, x_n) &= \frac{p(s, 1, t_n, x_n)}{p(0, 1, t_n, x_n)} \\ &= \left(\frac{t_n + 1}{t_n - s + 1}\right)^2 \cdot \left(\frac{t_n^2 - t_n(s-1) - s}{t_n^2 - t_n(s-1)}\right)^{x_n - 1}. \end{aligned}$$

The first factor converges to 1 if  $t_n \rightarrow \infty$ . The second factor converges iff  $x_n/t_n^2 \rightarrow c \in [0, \infty]$ . If  $x_n/t_n^2 \rightarrow c \in [0, \infty]$  the second factor converges to  $\exp(-cs)$ . For the sum in (12) there are three cases:

- (a) For  $c=0$ ,  
the summands with  $l > 1$  converge to 0 and the summand with  $l=1$  converges to  $k$ . Hence the function  $h^0(s, k)$  in (10) arises as a limit of the Martin kernel.
- (b) For  $c = \infty$ ,  
the Martin kernel converges to 0 because the exp-function grows faster than any polynomial.
- (c) For  $0 < c < \infty$ ,  
the Martin kernel converges to

$$\exp(-cs) \sum_{l=1}^k \binom{k}{l} \frac{1}{(l-1)!} c^{l-1}.$$

So the possible extremal space-time harmonic functions are the functions  $h^c$  with  $c \in [0, \infty) \cup \{\emptyset\}$  defined in (9)-(11) and the function  $h^\infty(s, k) = \delta_{(1, 0)}(k, s)$ .

The space-time infinitesimal generator of this Markov process is

$$A f(k, t) = \frac{\partial}{\partial t} f(k, t) + k(f(k-1, t) + f(k+1, t) - 2f(k, t)).$$

It is a tedious but not difficult calculation to prove that  $A h(t, k) = 0$  for all these functions except  $h^\infty$ . Hence the same reasoning as in section 3 shows that they are space-time harmonic (see [Ov1]).

Let us now prove extremality of  $h^c$  for  $c \in [0, \infty) \cup \{\emptyset\}$ . We first observe that for  $\Lambda := \{\exists t \text{ with } X_t = 0\} = \bigcup_{n \in \mathbb{N}} \{X_n = 0\}$  we have

$$\begin{aligned} P^{h^\emptyset}[\Lambda] = P[\Lambda] &= 1. \text{ But, because } h^c(t, 0) = 0 \text{ for } c \in [0, \infty), \text{ it follows} \\ P^{h^c}[X_n = 0] &= E[1_{\{X_n = 0\}} h^c(n, X_n)] = 0. \end{aligned} \quad (13)$$

Equations (13) together with (2) implies

$$1 = P^{h^\emptyset}[\Lambda] = \int_{[0, \infty) \cup \{\emptyset\}} P^{h^c}[\Lambda] \mu^{h^\emptyset}(dc) = \mu(\{\emptyset\}).$$

and so  $h^\emptyset$  is extremal. The same reasoning with the set  $\Omega \setminus \Lambda$  shows that the support of the representing measure  $\mu^{h^c}$  of  $h^c$  for  $c \neq \emptyset$  is contained in  $[0, \infty)$ .

Setting  $k = 1$  in the formula (11) and (10) for  $h^c$  we conclude

$$\exp(-cs) = \int_0^\infty h^x(s, 1) \mu^{h^c}(dx) = \int_0^\infty \exp(-xs) \mu^{h^c}(dx).$$

By the unicity of Laplace transforms the equation  $\mu^{h^c} = \delta_c$  follows. This is equivalent to the extremality of  $h^c$ . The formula for  $A^{h^c}$  is proved by using the Girsanov transformation [Ja, p. 225] and the harmonicity of  $h^c$  in order to formulate the martingale problem solved by  $P^{h^c}$ ; cf. [Ov1].

*Remarks.* – 1. We can interpret  $P^{h^c}$  as the distribution of an *interacting* branching process in the following sense. If there are  $k$  particles alive each has an exponentially distributed lifetime with parameter  $\frac{h^c(t, k-1) + h^c(t, k+1)}{h^c(t, k)}$ . When one particle dies it produces 2 (resp. 0) new particles with probability

$$\frac{h^c(t, k+1)}{h^c(t, k-1) + h^c(t, k+1)} \left( \text{resp. } \frac{h^c(t, k-1)}{h^c(t, k-1) + h^c(t, k+1)} \right).$$

For  $c=0$  the generator can be written as

$$A^{h^0} f(k) = 2k(f(k+1) - f(k)) \frac{k+1}{2k} + (f(k-1) - f(k)) \frac{k-1}{2k} \quad (14)$$

or as

$$A^{h^0} f(k) = 2k \left( \frac{1}{k} (f(k+1) - f(k)) + \left(1 - \frac{1}{k}\right) \left( \frac{1}{2} f(k+1) - f(k) \right) + \frac{1}{2} (f(k-1) - f(k)) \right). \quad (15)$$

The generator (14) exposes interaction described above while (15) describes a branching process with state dependent immigration. The waiting time for the next event is  $\exp(2k)$ -distributed. But when the next change happens it is an immigration of one particle with probability  $\frac{1}{k}$ , or a binary branching with the remaining probability  $1 - \frac{1}{k}$ .

2. Because  $(M_t)_{t \geq 0}$  is a  $P^c$ -martingale iff  $(h^c(t, X_t) \cdot M_t)_{t \geq 0}$  is a  $P$ -martingale, the above calculations also solve the problem of finding the Martin boundary of  $P^c := P^{h^c}$  for every  $c \in [0, \infty)$ . In these cases the Martin boundary is given by  $[0, \infty)$ . The extremal space-time harmonic functions of  $P^0$ , e. g., are given by

$$\exp(-cs) \sum_{l=1}^k \frac{1}{k} \binom{k}{l} \frac{1}{(l-1)!} c^{l-1}, \quad c \geq 0.$$

3. This remark explains the special role of  $P^0$ .

We will first prove that  $P^0$  can be viewed as the process conditioned on non-extinction, then comment on its limit behaviour and finally explain the relation with *Palm measures* of critical branching Brownian motions.

Because the set of non-extinction has probability 0 we first condition on the event  $\{X_{t+T} > 0\}$ . Let  $B \in \mathcal{F}_t$ , then

$$\begin{aligned} P[B] X_{t+T} > 0 &= E[\mathbf{1}_B (1 - E[\mathbf{1}_{\{X_{t+T}=0\}} | \mathcal{F}_t]) \cdot (1 - P[X_{t+T}=0])^{-1}] \\ &= E\left[\mathbf{1}_B \left(1 - \left(\frac{T}{T+1}\right)^{X_t}\right) (T+t+1)\right] \end{aligned}$$

The last expression converges to  $E[\mathbf{1}_B \cdot X_t]$  as  $T \rightarrow \infty$ . Hence we have for every  $t > 0$  the equation

$$P^0[B] = \lim_{T \rightarrow \infty} P[B] X_{t+T} > 0 \quad \text{for all } B \in \mathcal{F}_t.$$

In order to exhibit the weak limit behaviour under  $P^0$ , we calculate the generating function of  $X_t$  under  $P^0$  as

$$E_{P^0}[\sigma^{X_t}] = E[X_t \sigma^{X_t}] = \sigma \frac{\partial}{\partial \sigma} F(\sigma, t) = \frac{\sigma}{(t(1-\sigma) + 1)^2}.$$

Setting  $\sigma = e^{-\lambda/t}$  yields the convergence of the Laplace transform of  $\frac{X_t}{t}$  under  $P^0$  to  $\frac{1}{(1+\lambda)^2}$ . This is the Laplace transform of a random variable with density  $x e^{-x} \mathbf{1}_{[0, \infty)}(x)$ , the density of the  $\Gamma$ -distribution.

So  $\frac{X_t}{t}$  converges under  $P^0$  in distribution. It is well known that  $\frac{X_t}{t}$  converges also under the conditioned measure  $P[\cdot | X_t > 0]$ , namely towards the exponential distribution. But the Martin topology is connected with the convergence of  $\frac{X_t}{t^2}$ . That is similar to the case of Brownian motion

where  $B_t/\sqrt{t} \overset{\mathcal{D}}{\sim} \mathcal{N}(0, 1)$  but where the behaviour at the Martin boundary is described by the convergence of  $B_t/t$ .

These results are proved for the discrete time model in [AN, p. 58], where  $P^0$  corresponds to the Q-process, and for the continuous-state model in [EP] and [RR].

Branching Brownian motion and their *Palm measures*  $(P_{t,a})_{t \in (0, \infty), a \in \mathbb{R}^d}$  are considered in [CRW]. The *Palm measure*  $P_{t,a}$  can be viewed as the branching Brownian motion conditioned on the event that one particle populates  $a$  at time  $t$ . It turns out that the particle which populates  $a$  has the branching law of  $P^0$ . This fact is not surprising since the ancestors of this particle build a branching process conditioned on survival.

## 5. FELLER'S PROCESS

Feller's continuous-state branching process is a diffusion process with state space  $[0, \infty)$ , with generator

$$Af(x) = x \cdot f''(x)$$

(cf. [F], [KW]) and it satisfies the stochastic differential equation

$$dX_t = \sqrt{2X_t} dW_t,$$

where  $(W_t)_{t \geq 0}$  is a Brownian motion. Its branching property is expressed in term of Laplace transforms by

$$E_{x+y}[e^{-\lambda X_t}] = E_x[e^{-\lambda X_t}] E_y[e^{-\lambda X_t}], \quad \forall t \in \mathbb{R}.$$

Feller's process is obtained as the limit of a sequence of rescaled critical branching processes studied in section 4. In the  $n$ -th approximation the mass of each particle is  $\frac{1}{n}$  and its lifetime is exponentially distributed with intensity  $2n$ .

In order to identify the Martin kernel we investigate the transition function given by the Laplace transform (cf. e. g. [KW])

$$E_x[e^{-\lambda X_t}] = \exp(-x\lambda(1+\lambda t)^{-1}).$$

At the same time, this is the Laplace transform of  $\sum_{i=1}^{N_t} Y_t^{(i)}$ , where  $N_t$  is Poisson distributed with parameter  $x/t$  and the  $Y_t^{(i)}$  are i. i. d. exponentially with parameter  $1/t$  (with the convention that  $\sum_1^0 = 0$ ). Hence the transition density  $p(s, x, t, y)$  of  $X_t$  with respect to the reference measure

$m = \text{Lebesgue measure} + \delta_0$  is given by

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(x/(t-s)^2)^k y^{k-1}}{k!(k-1)!} \exp(-(x+y)(t-s)^{-1}) \\ \times \mathbf{1}_{(0, \infty)}(y) + \mathbf{1}_{\{0\}}(y) \exp\left(-\frac{x}{t-s}\right) \\ = \frac{1}{t-s} \sqrt{\frac{x}{y}} \exp\left(-\frac{x+y}{t-s}\right) I_{-1}\left(\frac{2\sqrt{xy}}{t-s}\right) \\ \times \mathbf{1}_{(0, \infty)}(y) + \mathbf{1}_{\{0\}}(y) \exp\left(-\frac{x}{t-s}\right), \end{aligned} \tag{16}$$

where  $I_{-1}$  is the modified Bessel function of order  $-1$ .

**THEOREM 3.** — *The Martin boundary of Feller’s continuous state branching process is  $[0, \infty) \cup \{\emptyset\}$ . The Martin topology is the same as in theorem 2. The extremal space-time harmonic functions are*

$$h^\emptyset(s, x) = 1, \tag{17}$$

$$h^0(s, x) = x, \tag{18}$$

$$h^c(s, x) = \sum_{k=1}^{\infty} \frac{(cx)^k}{k!(k-1)!} \left( \sum_{k=1}^{\infty} \frac{c^k}{k!(k-1)!} \right)^{-1} e^{-sc}, \quad 0 < c < \infty. \tag{19}$$

Under  $\mathbf{P}^{h^c}$  the process  $X_t$  satisfies the stochastic differential equation

$$dX_t = \sqrt{2X_t} dW_t + 2 \frac{\partial}{\partial x} \log h^c(t, X_t) \cdot X_t dt.$$

For  $c = 0$  this equation reduces to

$$dX_t = \sqrt{2X_t} dW_t + 2 dt.$$

*Proof.* — First, the Martin kernel for  $y_n > 0, x > 0$  is

$$K(s, x, t_n, y_n) = B\left(\frac{xy_n}{(t_n-s)^2}\right) B^{-1}\left(\frac{y_n}{t_n^2}\right) \exp\left(-\frac{x-1}{t_n-s}\right) \exp\left(-\frac{s(y_n+1)}{(t_n-s)t_n}\right),$$

where  $B(z) := \sum_{k=1}^{\infty} s^k (k!(k-1)!)^{-1}$ .

For  $x = 0$  the Martin kernel is  $\mathbf{1}_{\{0\}}(y_n) \cdot \exp(-1/t_n)$ , which gives  $h(s, 0) = 0$  or  $1$  according as  $y_n = 0$  for all but finitely many  $n$  or not.

For  $x > 0, y_n = 0$  the Martin kernel is  $\exp\left(-\frac{s+t_n(x-1)}{(t_n-s)t_n}\right)$ . This yields the space-time harmonic function  $h^\emptyset(s, x) = 1$ .

For  $y_n > 0, x > 0$  there are again three different possibilities

$$\begin{aligned} K(s, x, t_n, y_n) &\rightarrow x = h^0(s, x) && \text{if } \frac{y_n}{t_n^2} \rightarrow 0 \\ &\rightarrow \frac{B(cx)}{B(c)} \exp(-sc) = h^c(s, x) && \text{if } \frac{y_n}{t_n^2} \rightarrow c > 0 \\ &\rightarrow \delta_{(0,1)}(s, x) =: h^\infty && \text{if } \frac{y_n}{t_n^2} \rightarrow \infty, \end{aligned}$$

where  $h^0$  and  $h^c, 0 < c < \infty$  are defined in (18) and (19). Further, it is easy to show that for every  $c \in [0, \infty)$  the function  $h^c$  satisfies

$$\frac{\partial}{\partial s} h(x, s) + x \frac{\partial^2}{\partial x^2} h(x, s) = 0.$$

It follows that  $h^c$  is space-time harmonic for every  $c \in [0, \infty) \cup \{\emptyset\}$ . The same reasoning as in the section 4 proves the extremality of these functions.

The stochastic differential equation for  $X$  under  $P^{h^c}$  follows as in section 4 by using the Girsanov transformation [Ja], p. 225.  $\square$

*Remarks 1.* – In [Sa] the Martin boundary of Bessel processes of order  $d$  is computed for  $d \in \mathbb{N} \setminus \{0\}$ . Up to a scaling, the Feller process can also be described as the squared Bessel process of order 0 ([RY], p. 409), and so our results are related to the results in [Sa]. The squared Bessel process of order  $r \geq 0$  is the solution of the stochastic differential equation

$$dX_t = 2\sqrt{X_t} dW_t + r dt.$$

In [RY], p. 411, the transition densities with respect to Lebesgue measure of all squared Bessel processes of a strict positive order  $r$  are computed. However, the formula in [RY], p. 411, does not extrapolate to the case  $r=0$  because the transition probabilities of the Feller process give positive mass to 0 [see equation (16)].

2. As in remark 2 of section 4, the Martin boundary of any process  $P^{h^c}$  with  $c \in [0, \infty)$  is also given by  $[0, \infty)$ .

3. Here again the measure  $P^{h^0}$  can be viewed as the process conditioned on survival (cf. [EP], [RR] and [KW]). According to Remark 1., the coordinate process under  $P^{h^0}$  is the squared Bessel process of order 4. This result is already mentioned in the context of Bessel processes in [PY], where also the transition function (16) can be found.

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