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An almost sure central limit theorem for independent random variables

by

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ABSTRACT. – The purpose of this paper is the proof of an almost sure central limit theorem for independent nonidentically distributed random variables.

Key words : Central limit theorem, Wiener measure, Skorokhod representation theorem, almost sure convergence.

RÉSUMÉ. – Le sujet de l'article est la démonstration d'un théorème central limite presque sûr pour des variables aléatoires indépendantes non identiquement distribuées.

1. INTRODUCTION

Let $\{X_n, n \geqq 1\}$ be a sequence of independent random variables, defined on a probability space $(\Omega, \mathscr{A}, P)$, such that $EX_n = 0$ and $EX_n^2 = \sigma_n^2 < \infty$, $n \geqq 1$. Let us put $S_0 = 0$, $S_n = X_1 + \ldots + X_n$, $\mathscr{S}_n^2 = ES_n^2$.

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Let $Y_n(t)$, $t\in[0, 1]$, be the random function defined as follows:

$$(1.1)\quad Y_n(t)=\frac{S_k}{\mathcal{S}_n}+X_{k+1}\left[\frac{t\,\mathcal{S}_n^2-\mathcal{S}_k^2}{\sigma_{k+1}^2\,\mathcal{S}_n^2}\right],\qquad t\in\left[\frac{\mathcal{S}_k^2}{\mathcal{S}_n^2},\frac{\mathcal{S}_{k+1}^2}{\mathcal{S}_n^2}\right],\quad 0\leqq k<n.$$

It is clear that $Y_n(t)=\dfrac{S_k}{\mathcal{S}_n}$ whenever $t=\dfrac{\mathcal{S}_k^2}{\mathcal{S}_n^2}$ and $Y_n(t)$ is the straight line joining $\left(\dfrac{\mathcal{S}_k^2}{\mathcal{S}_n^2},\dfrac{S_k}{\mathcal{S}_n}\right)$ and $\left(\dfrac{\mathcal{S}_{k+1}^2}{\mathcal{S}_n^2},\dfrac{S_{k+1}}{\mathcal{S}_n}\right)$ in the interval $\left[\dfrac{\mathcal{S}_k^2}{\mathcal{S}_n^2},\dfrac{\mathcal{S}_{k+1}^2}{\mathcal{S}_n^2}\right]$. Thus $Y_n(t)$ is continuous with probability one, so that there is a measure P_n in the space $(C[0, 1], \mathcal{C})$, according to which the stochastic process $\{Y_n(t), 0\leqq t\leqq 1\}$ is distributed. Here and in what follows $C[0, 1]$ denotes the space of real-valued, continuous functions on $[0, 1]$ and $\mathcal{C}$ denotes the σ-field of Borel sets generated by the open sets of uniform topology.

It is well known that if $\{X_n, n\geqq 1\}$ satisfies the Lindeberg condition, *i.e.*, for every $\varepsilon>0$

$$(1.2)\qquad \lim_{n\to\infty}\mathcal{S}_n^{-2}\sum_{k=1}^{n}EX_k^2 I(|X_k|\geqq\varepsilon\mathcal{S}_n)=0,$$

then $P_n\xrightarrow{D}W$ as $n\to\infty$, where here and in what follows W denotes the Wiener measure on $C([0, 1], \mathcal{C})$ with the corresponding Wiener process $\{W(t), 0\leqq t\leqq 1\}$, [*cf.* Billingsley (1968), p. 61].

The purpose of this paper is the proof of an almost sure version of this theorem. Namely, for $x\in C[0, 1]$ let δ_x be the probability measure on $C[0, 1]$ which assigns its total mass to x. Let us observe that the distribution P_n of Y_n is just the average of the random measures $\delta_{Y_n(\omega)}$ with respect to P, *i.e.*, for every $A\in\mathcal{C}$,

$$P_n(A)=\int_\Omega \delta_{Y_n(\omega)}(A)\,dP(w).$$

Of course, for every $\omega\in\Omega$, $\{\delta_{Y_n(\omega)}, n\geqq 1\}$ is a sequence of probability measures on the space $(C[0, 1], \mathcal{C})$. We study weak convergence of the sequence

$$\left\{(\log\mathcal{S}_n^2)^{-1}\sum_{k=1}^{n}(\sigma_{i+1}^2/\mathcal{S}_i^2)\,\delta_{Y_i(\omega)},\ n\geqq 2\right\}$$

on the space $(C[0, 1], \mathcal{C})$. The results obtained extend, to nonidentically distributed random variables X_n, $n\geqq 1$, the Theorems given by Brosamler (1988), Schatte (1988), Lacey and Philipp (1990). Other generalizations of the almost sure central limit theorem, based on an argumentation which is purely ergodic and Brownian, are obtained by Atlagh and Weber (1992).

But as in the papers mentioned above only the case of independent and identically distributed random variables is considered.

2. RESULTS

Let $k_0 = 1 < k_1 < k_2 < \ldots$ be an increasing sequence of real numbers such that $k_n \to \infty$ as $n \to \infty$. For $s > 0$ we define $C[0, 1]$-valued random elements $W^{(s)}$ by

$$W^{(s)}(u) = s^{-1/2} W(su), \qquad u \in [0, 1]. \tag{2.1}$$

Let

$$V_s = W^{(k_n)} \quad \text{for} \quad k_n \leqq s < k_{n+1}, \quad n \geqq 0. \tag{2.2}$$

In what follows $h_i = k_{i+1} - k_i$, $i \geqq 0$, and $h_n^* = \max_{1 \leqq i \leqq n} h_i$, $n \geqq 0$.

THEOREM 1. – *If*

$$h_n^* (\log\log k_n)^{1/2} / k_n \to 0 \quad \textit{as } n \to \infty \tag{2.3}$$

and, for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n (h_n^*/k_n)^{1/2} \exp(-\varepsilon k_n / h_n^*) < \infty, \tag{2.4}$$

then

$$(\log k_n)^{-1} \sum_{i=0}^{n-1} (h_i/k_i)\, \delta_{W^{(k_i)}} \xrightarrow{D} W \quad \text{P-}\textit{a.s. as } n \to \infty, \tag{2.5}$$

where $\xrightarrow{D}$ *denotes the weak convergence of measures on* $(C[0, 1], \mathscr{C})$.

THEOREM 2. – *Let* $\{X_n, n \geqq 1\}$ *be a sequence of independent random variables, defined on* $(\Omega, \mathscr{A}, P)$, *with* $EX_n = 0$ *and* $E|X_n|^{2+2\delta} < \infty$, $n \geqq 1$, *for some* $0 < \delta \leqq 1$. *Assume*

$$h_n^* (\log\log \mathscr{S}_n^2)^{1/2} / \mathscr{S}_n^2 \to 0 \quad \textit{as } n \to \infty, \tag{2.6}$$

$$\sum_{n=1}^{\infty} b_n^{-1-\delta} E|X_n|^{2+2\delta} < \infty \tag{2.7}$$

and, for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n (h_n^*/\mathscr{S}_n^2)^{1/2} \exp(-\varepsilon \mathscr{S}_n^2 / h_n^*) < \infty, \tag{2.8}$$

and

$$(2.9)\qquad \sum_{n=1}^{\infty} (b_n/\mathscr{S}_n^2)^{1/2} \exp\{-\varepsilon \mathscr{S}_n^2/b_n\} < \infty,$$

where $\{b_n, n \geqq 1\}$ *is some increasing sequence,* $b_n \to \infty$ *as* $n \to \infty$, *and* $h_n^* = \max_{1 \leqq i \leqq n+1} \sigma_i^2$.

Then

$$(2.10)\qquad (\log \mathscr{S}_n^2)^{-1} \sum_{i=1}^{n} (\sigma_{i+1}^2/\mathscr{S}_i^2)\, \delta_{Y_i(\omega)} \xrightarrow{D} W \quad \text{P-}a.s.\ as\, n \to \infty,$$

where $\xrightarrow{D}$ *denotes the weak convergence of measures on* $(C[0, 1], \mathscr{C})$.

From Theorem 2 we immediately get the following extension of the main result of Brosamler (1988).

THEOREM 3. – *Let* $\{X_n, n \geqq 1\}$ *be a sequence of independent random variables, defined on* $(\Omega, \mathscr{A}, P)$, *with* $EX_n = 0$, $0 < EX_n^2 = \sigma^2$ *and* $E|X_n|^{2+2\delta} = \beta_{2+2\delta} < \infty$ *for some* $0 < \delta < 1$.

Then

$$(\log n)^{-1} \sum_{k=1}^{n} k^{-1} \delta_{Y_k(\omega)} \xrightarrow{D} W \quad \text{P-}a.s.\ as\, n \to \infty,$$

where $\xrightarrow{D}$ *denotes the weak convergence of measures on* $(C[0, 1], \mathscr{C})$.

In the case where $\{X_n, n \geqq 1\}$ is a sequence of independent and identically distributed random variables Theorem 3 gives the result of Brosamler (1988).

3. AUXILLIARY LEMMAS

The proof of Theorem 2 is based on a martingale form of the Skorokhod representation theorem that we recall for the convenience of the reader. This theorem is obtained by Strassen (1967), p. 333.

LEMMA 1. – *Let* $\{Z_n, n \geqq 1\}$ *be a sequence of random variables such that for all* n, $E(Z_n^2 | Z_{n-1}, \ldots, Z_1)$ *is defined and* $E(Z_n | Z_{n-1}, \ldots, Z_1) = 0$ *a.s. Then there exists a probability space supporting a standard Brownian motion* $\{W(t), t \geqq 0\}$ *and a sequence of nonnegative variables* $\{\tau_n, n \geqq 1\}$ *with the following properties. If* $T_n = \sum_{i=1}^{n} \tau_i$, $S'_n = W(T_n)$, $X'_1 = S'_1$,

$X'_n = S'_n - S'_{n-1}$ *for* $n \geqq 2$, *and* G_n *is the* σ*-field generated by* $S'_1, \ldots, S'_n$ *and by* $W(t)$ *for* $0 \leqq t \leqq T_n$, *then*

(i) $\left\{ \sum_{i=1}^{n} Z_i,\ n \geqq 1 \right\}$ *has the same law as* $\{ W(T_n),\ n \geqq 1 \}$,

(ii) T_n *is* G_n*-measurable*,

(iii) *for each real number* $r \geqq 1$

$$E(\tau_n^r \mid G_{n-1}) \leqq C_r E(|X'_n|^{2r} \mid G_{n-1}) = C_r E(|X'_n|^{2r} \mid X'_{n-1}, \ldots, X'_1) \quad a.s.,$$

where $C_r = 2(8/\pi^2)^{r-1} \Gamma(r+1)$, *and*

(iv) $E(\tau_n \mid G_{n-1}) = E(X_n'^2 \mid G_{n-1}) = E(X_n'^2 \mid X'_{n-1}, \ldots, X'_1)\, a.s.$

(v) *If* Z_n, $n \geqq 1$, *are mutually independent, then the random variables* τ_n, $n \geqq 1$, *are mutually independent.*

In the proof of Theorems 1 and 2 we also need the following lemmas.

LEMMA 2. — [Brosamler (1988), Theorem 1.6 (*a*)]. — *If*

$$\mu_t(w) = (\log t)^{-1} \int_1^t s^{-1} \delta_{W(s)(w)}\, ds, \qquad t > 1,$$

then

$$\mu_t \overset{D}{\to} W \quad P\text{-}a.s.,\ as\ t \to \infty,$$

and the convergence $\overset{D}{\to}$ *is weak convergence of measures on* $(C[0, 1], \mathscr{C})$.

LEMMA 3 [Brosamler (1988), Lemma 2.12]. — *If* $X, Z : \mathbf{R}^+ \to C[0, 1]$ *are measurable and are such that*

$$\lim_{s \to \infty} \| X(s) - Z(s) \| = 0,$$

then for all bounded, uniformly continuous functions $f : C[0, 1] \to \mathbf{R}$

$$\lim_{t \to \infty} (\log t)^{-1} \int_1^t s^{-1} \{ f(X(s)) - f(Z(s)) \}\, ds = 0.$$

LEMMA 4. — *Let* $k_0 = 1 < k_1 < \ldots$ *be a sequence of real numbers such that* $k_n \to \infty$ *and* $k_{n+1}/k_n = O(1)$ *as* $n \to \infty$. *If* $\{X_n, n \geqq 1\}$ *and* $\{Y_n, n \geqq 1\}$ *are* $C[0, 1]$*-valued sequences such that*

$$\| X_n - Y_n \| \to 0 \quad as\ n \to \infty,$$

then for every bounded and uniformly continuous function $f : C[0, 1] \to \mathbf{R}$

$$\lim_{n \to \infty} (\log k_n)^{-1} \sum_{i=1}^{n} (h_i/k_i) \{ f(X_i) - f(Y_i) \} = 0,$$

where $h_i = k_{i+1} - k_i$, $i \geqq 0$.

Proof. – Let $f: C[0, 1] \to \mathbf{R}$ be a bounded and uniformly continuous function. Let $\varepsilon > 0$ be given. Then there exists n_0 such that for every $n \geqq n_0$

$$|f(X_n) - f(Y_n)| < \varepsilon$$

since $\|X_n - Y_n\| \to 0$ as $n \to \infty$.

Thus

$$I_n = \sum_{i=1}^{n} (h_i/k_i) |f(X_i) - f(Y_i)|$$

$$\leqq 2 (\sup_{x \in C[0, 1]} |f(x)|) \sum_{i=1}^{n_0} (h_i/k_i)$$

$$+ \varepsilon \sup_n (k_{n+1}/k_n) \int_{k_{n_0}}^{k_n} x^{-1} \, dx,$$

so that

$$I_n/(\log k_n) \to \varepsilon \quad \text{as } n \to \infty;$$

and this achieves the proof.

LEMMA 5. – *Let* $\{b_n, n \geqq 1\}$ *be an increasing sequence of positive numbers such that* $b_n \to \infty$ *as* $n \to \infty$. *If* $\{X_n, n \geqq 1\}$ *is a sequence of independent random variables such that for some constants* $0 < \tau_n \leqq 2$, $\sum_{n=1}^{\infty} E(|X_n|/b_n)^{\tau_n} < \infty$, *where* $EX_n = 0$ *when* $1 \leqq \tau_n \leqq 2$, *then*

$$b_n^{-1} \sum_{k=1}^{n} X_k \to 0 \quad \text{P-}a.s. \ as \, n \to \infty.$$

Lemma 5 is a consequence of Corollary 3 and Kronecker Lemma 2 presented in Chow and Teicher (1978) (p. 114 and p. 111, respectively).

4. PROOFS OF THEOREMS

Proof of the Theorem 1. – At first we prove that

$$\lim_{s \to \infty} \|V_s - W^{(s)}\| = 0 \quad \text{P-a.s.} \tag{4.1}$$

Thus we have to prove that

$$\lim_{n \to \infty} \sup_{0 \leqq h \leqq h_n} \|W^{(k_n)} - W^{(k_n + h)}\| = 0 \quad \text{P-a.s.},$$

where $h_n = k_{n+1} - k_n$, $n \geqq 0$.

By (2.1) and (2.2) we get

$$\| W^{(k_n)} - W^{(k_n+h)} \| = \sup_{0 \leqq u \leqq 1} | k_n^{-1/2} W(uk_n) - (k_n+h)^{-1/2} W(u(k_n+h))$$

$$\leqq k_n^{-1/2} \sup_{0 \leqq u \leqq 1} | W(uk_n) - W(uk_n+uh) |$$

$$+ h \sup_{0 \leqq u \leqq 1} \frac{| W(uk_n+uh) |}{2 k_n^{3/2}} = I_1(n) + I_2(n).$$

On the other hand, by Lemmas 1.11, 1.16 and 1.4 of Freedman (1971), for every $\varepsilon > 0$, we obtain

$$(4.2) \quad P(\sup_{0 \leqq u \leqq 1} \sup_{0 \leqq h \leqq h_n} | W(uk_n+uh) - W(uk_n) | \geqq \varepsilon k_n^{1/2}$$

$$\leqq P(\sup_{0 \leqq u \leqq k_n} \sup_{0 \leqq s \leqq h_n} | W(u+s) - W(u) | \geqq \varepsilon k_n^{1/2})$$

$$\leqq \sum_{i=0}^{n-1} P(\sup_{k_i \leqq u \leqq k_{i+1}} \sup_{0 \leqq s \leqq h_n} | W(u+s) - W(u) | \geqq \varepsilon k_n^{1/2})$$

$$= \sum_{i=0}^{n-1} P(\sup_{0 \leqq u \leqq h_i} \sup_{0 \leqq s \leqq h_n} | W(u+s) - W(u) | \geqq \varepsilon k_n^{1/2})$$

$$\leqq n P(\sup_{0 \leqq u \leqq h_n^*} \sup_{0 \leqq s \leqq h_n^*} | W(u+s) - W(u) | \geqq \varepsilon k_n^{1/2})$$

$$\leqq n P(2 \sup_{0 \leqq u \leqq 2h_n^*} | W(u) | \geqq \varepsilon k_n^{1/2})$$

$$\leqq 4n P(W(1) \geqq \varepsilon (k_n / 8 h_n^*)^{1/2})$$

$$\leqq 4n (2\pi)^{-1/2} (8 h_n^* / k_n \varepsilon^2)^{1/2} \exp(-\varepsilon^2 k_n / 16 h_n^*).$$

Hence, by (2.4), (4.2) and the Borel-Cantelli Lemma

$$(4.3) \qquad I_1(n) \to 0 \quad \text{P-a. s. as } n \to \infty.$$

Moreover

$$\sup_{0 \leqq h \leqq h_n} \sup_{0 \leqq u \leqq 1} | W(uk_n+uh) | h \leqq h_n^* \sup_{0 \leqq u \leqq k_n+h_n^*} | W(u) |,$$

and, by Theorem 1.106 of Freedman (1971), for every n such that $k_n \geqq e^2$,

$$\{ 2(k_n+h_n^*) \log\log(k_n+h_n^*) \}^{-1/2} \sup_{e^2 \leqq u \leqq (k_n+h_n^*)} | W(u) |$$

$$\leqq \sup_{e^2 \leqq u} | W(u) | (2u \log\log u)^{-1/2} < \infty \quad \text{P-a. s.}$$

Thus, by (2.3), we get

$$(4.4) \qquad I_2(n) \to 0 \quad \text{P-a. s. as } n \to \infty.$$

Now (4.1) follows from (4.2), (4.3) and (4.4).

On the other hand, by Lemma 2 for every bounded and continuous function $f: C[0, 1] \to \mathbf{R}$, we have P-a. s.

$$\lim_{n \to \infty} (\log k_n)^{-1} \int_1^{k_n} s^{-1} f(W^{(s)})\, ds = \int_{C[0,1]} f(x)\, dW(x).$$

Thus, by (4.1) and Lemma 3, we get

$$\lim_{n \to \infty} (\log k_n)^{-1} \int_1^{k_n} s^{-1} f(V_s)\, ds = \int_{C[0,1]} f(x)\, dW(x), \quad \text{P-a. s.}$$

for all bounded an uniformly continuous functions $f: C[0, 1] \to \mathbf{R}$, hence for all bounded and continuous functions.

But

$$(4.5) \qquad \int_1^{k_n} s^{-1} f(V_s)\, ds = \sum_{i=0}^{n-1} f(W^{(k_i)}) \log(1 + h_i/k_i).$$

On the other hand

$$\log k_n \to \infty \qquad \text{and} \qquad h_n^*/k_n \to 0 \text{ as } n \to \infty,$$

so that

$$\lim_{n \to \infty} (\log k_n)^{-1} \sum_{i=0}^{n-1} f(W^{(k_i)}) \log(1 + h_i/k_i)$$
$$= \lim_{n \to \infty} (\log k_n)^{-1} \sum_{i=0}^{n-1} (h_i/k_i) f(W^{(k_i)}) \quad \text{P-a. s.,}$$

since $(k_i/h_i) \log(1 + h_i/k_i) \to 1$ as $i \to \infty$. Thus the proof of (2.5) is ended.

Proof of Theorem 2. – By Lemma 1 there exists a probability space $(\Omega, \mathscr{A}, P)$ supporting a standard Brownian motion $\{W(t), t \geqq 0\}$ and a sequence of nonnegative random variables $\{\tau_n, n \geqq 1\}$ such that the sequence $\{W(T_n), n \geqq 1\}$ has the same law as the sequence $\{S_n, n \geqq 1\}$, where $T_n = \tau_1 + \ldots + \tau_n$, $n \geqq 1$. Thus from now on we shall identify S_1, $S_2, \ldots$, and $W(T_1), W(T_2), \ldots$

Define $Z_n: \Omega \to C[0, 1]$ by setting

$$Z_n(\mathscr{S}_k^2/\mathscr{S}_n^2) = W(\mathscr{S}_k^2)/\mathscr{S}_n, \qquad k = 0, 1, \ldots, n,$$

and linear on each interval $[\mathscr{S}_{k-1}^2/\mathscr{S}_n^2, \mathscr{S}_k^2/\mathscr{S}_n^2]$, $k = 1, 2, \ldots, n$.

Let us also define $C[0, 1]$-valued random elements $W^{(s)}$, for $s > 0$, by

$$W^{(s)}(u) = s^{-1/2} W(us), \qquad u \in [0, 1],$$

and put $V_n = W^{(\mathscr{S}_n^2)}$, $n \geqq 1$. Then one gets

$$(4.6) \qquad \| Y_n - V_n \| \leqq \| Y_n - Z_n \| + \| Z_n - V_n \|,$$

where, for $x \in C[0, 1]$, $\|x\| = \sup_{0 \leqq t \leqq 1} |x(t)|$. On the other hand, for every $\varepsilon > 0$,

$$\begin{aligned} &P(\|Z_n - V_n\| \geqq \varepsilon) \\ &\leqq \sum_{k=0}^{n-1} P(\sup_{\mathcal{S}_k^2 \leqq t \mathcal{S}_n^2 \leqq \mathcal{S}_{k+1}^2} |W(t \mathcal{S}_n^2) - W(\mathcal{S}_k^2)| \geqq \varepsilon \mathcal{S}_n/2) \\ &= \sum_{k=0}^{n-1} P(\sup_{\mathcal{S}_k^2 \leqq s \leqq \mathcal{S}_{k+1}^2} |W(s) - W(\mathcal{S}_k^2)| \geqq \varepsilon \mathcal{S}_n/2). \end{aligned}$$

Furthermore, by Lemma 11 (c) of Freedman (1971), for every $0 \leqq k \leqq n-1$ we get

$$P(\sup_{\mathcal{S}_k^2 \leqq s \leqq \mathcal{S}_{k+1}^2} |W(s) - W(\mathcal{S}_k^2)| \geqq \varepsilon \mathcal{S}_n/2) = P(\sup_{0 \leqq t \leqq \sigma_{k+1}^2} |W(s)| \geqq \varepsilon \mathcal{S}_n/2).$$

Moreover, Lemma 16 (c) and Lemma 4 (a) of Freedman (1971) yield

$$\begin{aligned} &P(\sup_{0 \leqq t \leqq \sigma_{k+1}^2} |W(s)| \geqq \varepsilon \mathcal{S}_n/2) \\ &\leqq 2P(|W(\sigma_{k+1}^2)| \geqq \varepsilon \mathcal{S}_n/2) \\ &= 4P(W(1) \geqq \varepsilon \mathcal{S}_n/2\sigma_{k+1}) \\ &\leqq 8\left(\frac{\sigma_{k+1}}{\varepsilon \mathcal{S}_n}\right)(2\pi)^{-1/2} \exp(-\varepsilon^2 \mathcal{S}_n^2/8\sigma_{k+1}^2). \end{aligned}$$

Hence, by the inequalities given above with $h_n^* = \max_{1 \leqq i \leqq n+1} \sigma_i^2$, we get

$$\sum_{n=1}^{\infty} P(\|Z_n - V_n\| \geqq \varepsilon) \leqq (8/\varepsilon)(2\pi)^{-1/2} \sum_{n=1}^{\infty} n(h_n^*/\mathcal{S}_n^2)^{1/2} \exp\{-\varepsilon^2 \mathcal{S}_n^2/8h_n^*),$$

so that (2.8) and the Borel-Cantelli Lemma yield

$$\|Z_n - V_n\| \to 0 \quad \text{P-a. s. as } n \to \infty \tag{4.7}$$

On the other hand

$$\|Y_n - Z_n\| = \sup_{1 \leqq k \leqq n} |W(T_k) - W(\mathcal{S}_k^2)| / \mathcal{S}_n,$$

and

$$\sup_{1 \leqq k \leqq n} |W(T_k) - W(\mathcal{S}_k^2)| / \mathcal{S}_n \to 0 \quad \text{P-a. s. as } n \to \infty.$$

if and only if

$$|W(T_n) - W(\mathcal{S}_n^2)| / \mathcal{S}_n \to 0 \quad \text{P-a. s. as } n \to \infty.$$

By Lemma 1 the random variables τ_n, $n \geqq 1$, are mutually independent, $E\tau_n^{1+\delta} \leqq C_{2+\delta} E|X_n|^{2+2\delta}$ and $E\tau_n = \sigma_n^2$, $n \geqq 1$, so that by (27) and Lemma 5

$$(T_n - \mathscr{S}_n^2)/b_n \to 0 \quad \text{P-a. s. as } n \to \infty.$$

and, in particular,

$$\sup_n |T_n - \mathscr{S}_n^2|/b_n < \infty \quad \text{P-a. s.}$$

Thus, for every $\varepsilon > 0$, there exists $\lambda > 0$ such that

$$P(\sup_n |T_n - \mathscr{S}_n^2|/b_n \geqq \lambda) < \varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrary small, it suffices to prove (4.7) on the set $\Omega_\lambda = \{\sup_n |T_n - \mathscr{S}_n^2|/b_n < \lambda\}$. Let $\delta > 0$ be given. Then, similarly as in Brosamler (1988), p. 569, we get

$$\begin{aligned}
&P\{\{|W(T_n) - W(\mathscr{S}_n^2)| \geqq \delta \mathscr{S}_n\} \cap \Omega_\lambda\} \\
&\leqq P(\sup_{|s - \mathscr{S}_n^2| < \lambda b_n} |W(s) - W(\mathscr{S}_n^2)| \geqq \delta \mathscr{S}_n) \\
&\leqq P(\sup_{\mathscr{S}_n^2 \leqq s \leqq \mathscr{S}_n^2 + \lambda b_n} |W(s) - W(\mathscr{S}_n^2)| \geqq \delta \mathscr{S}_n) \\
&+ P(\sup_{\mathscr{S}_n^2 - \lambda b_n \leqq s \leqq \mathscr{S}_n^2} |W(s) - W(\mathscr{S}_n^2)| \geqq \delta \mathscr{S}_n) \\
&\leqq 2(\sup_{0 < s \leqq \lambda b_n} |W(s)| \geqq \delta \mathscr{S}_n) \\
&\leqq 2P(\sup_{0 < s < 1} |W(s)| \geqq \delta \mathscr{S}_n (\lambda b_n)^{-1/2}) \\
&\leqq 8P(W(1) \geqq \delta \mathscr{S}_n (\lambda b_n)^{-1/2}) \\
&\leqq 8(2\pi)^{-1/2} (\lambda b_n)^{1/2} (\delta^2 \mathscr{S}_n^2)^{-1/2} \exp(-\delta^2 \mathscr{S}_n^2/(2\lambda b_n)).
\end{aligned}$$

Hence by (2.9) and the Borel-Cantelli Lemma

$$\|Y_n - Z_n\| \to 0 \quad \text{P-a. s. as } n \to \infty. \tag{4.8}$$

Now (4.6), (4.7), (4.8) and Lemma 4 with $k_n = \mathscr{S}_n^2$, $n \geqq 1$, yield

$$\lim_{n \to \infty} (\log \mathscr{S}_n^2)^{-1} \sum_{i=1}^{n} (\sigma_{i+1}^2/\mathscr{S}_i^2)\{f(Y_i) - f(V_i)\} = 0 \tag{4.9}$$

for every bounded and uniformly continuous function $f: [0, 1] \to \mathbf{R}$. Thus (4.9) and Theorem 1, also with $k_n = \mathscr{S}_n^2$, $n \geqq 1$, end the proof of Theorem 2 since $V_n = W^{(\mathscr{S}_n^2)}$, $n \geqq 1$.

Proof of Theorem 3. — Under the assumptions of Theorem 3 $\mathscr{S}_n^2 = n\sigma^2$, $h_n^* = \sigma^2$, so that (2.6) and (2.8) hold. On the other hand, putting $b_n = n^{(1+\delta/2)/(1+\delta)}$, $n \geqq 1$, one can easily check that (2.7) and (2.9) hold, too. Thus Theorem 3 follows from Theorem 2.

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