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Hilbert-valued anticipating stochastic differential equations

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ABSTRACT. – In this paper we show the continuity of the solution of a Hilbert-valued stochastic differential equation with respect to the initial condition. Using an entropy criterium presented in [2], [16] the continuity property is obtained on some compact subsets. We apply this result to prove a theorem on existence of solution for an anticipating Hilbert-valued stochastic differential equation of Stratonovich type. This is done using an infinite dimensional version of a substitution formula for Stratonovich integrals depending on a parameter.

Key words : Stochastic differential equation, Stratonovich integral.

RÉSUMÉ. – Dans cet article nous montrons la continuité, par rapport à la condition initiale, de la solution d'une équation différentielle stochastique hilbertienne. Cette continuité est obtenue dans certains compacts en utilisant un critère d'entropie prouvé dans [2], [16]. Nous appliquons ce résultat pour montrer l'existence de la solution d'une équation différentielle

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stochastique hilbertienne anticipante, de type Stratonovich; la preuve est fondée sur une version en dimension infinie de la formule de substitution pour les intégrales de Stratonovich dépendant d'un paramètre.

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0. INTRODUCTION

Consider the stochastic differential equation on a separable Hilbert space $\mathbb{H}$

$$X_t(x)=x+\int_0^t \sigma(X_s(x))\,dW_s+\int_0^t b(X_s(x))\,ds, \tag{0.1}$$

where $W=\{W_t,\ t\in[0,1]\}$ is a Brownian motion taking values in some Hilbert space $\mathbb{K}$, with covariance operator Q. The coefficient $b(x)$ takes values on $\mathbb{H}$, and $\sigma(x)$ is a linear operator (possibly unbounded) from $\mathbb{K}$ on $\mathbb{H}$ such that $\sigma Q^{1/2}$ is Hilbert-Schmidt. By choosing bases on the Hilbert spaces $\mathbb{H}$ and $\mathbb{K}$, we can interpret this equation as an infinite system of stochastic differential equations driven by an infinite family of independent Brownian motions. This kind of equations are related with certain continuous state Ising-type models in statistical mechanics and also with models of genetic populations. Different problems concerning these diffusions have been studied in [1], [4], [7], [9], [14], [15].

Assume now that we put as initial condition, instead of a deterministic value $x\in\mathbb{H}$ some *random* vector X_0, which depends on the whole path of W. In that case the solution of (0.1), whenever it exists, is no more an adapted process with respect to the filtration associated with W. Then (0.1) becomes an *anticipating* stochastic differential equation, and we should specify the meaning of the stochastic integral.

There has been some recent progress on the stochastic calculus with anticipating integrands (*see* [3], [12]) which has allowed to study some classes of finite dimensional anticipating stochastic differential equations (*see*, for instance, [13]). Roughly speaking, it turns out that equations formulated in terms of the generalized Stratonovich integral are easier to handle than those written using the Skorohod integral, which is an extension of the Itô integral. For this reason, in this paper we will consider a Hilbert valued anticipating stochastic differential equation of the form

$$Y_t=Y_0+\int_0^t \sigma(Y_s)\circ dW_s+\int_0^t b(Y_s)\,ds, \tag{0.2}$$

where the symbol "$\circ$" means the stochastic integral in the Stratonovich sense, and Y_0 is an $\mathbb{H}$-valued random variable which is not necessarily independent of W. The finite dimensional analogue of (0.2) has been studied in [13].

Our main goal has been to prove a theorem on the existence of solution for (0.2). Due to the properties of the extended Stratonovich integral a candidate for the solution of the equation (0.2) will be the composition $X_t(Y_0))$, where $X_t(x)$ is the unique solution of (0.1). In order to show that $X_t(Y_0))$ solves (0.2) we need to establish the substitution result

$$\int_0^t \sigma(X_s(x)) \circ dW_s \big|_{x=Y_0} = \int_0^t \sigma(X_s(Y_0)) \circ dW_s.$$

One of the difficulties in proving such results is to show that the solution $X_t(x)$ of equation (0.1) has a version which is continuous in the variable x. We have been able to establish this continuity property on some particular compact subsets of $\mathbb{H}$, by means of a generalization of the Kolmogorov continuity criterion for processes indexed by a metric space, as presented in Fernique's St. Flour course (*see* [2]). This application of the infinite dimensional continuity criterion has been inspired by a recent result of P. Imkeller [5] on the existence and continuity of local time for some classes of indefinite Skorohod integrals.

In the first section we present, in an abstract setting, some results related with the above mentioned continuity criterion that will be useful in the sequel. Section two is devoted to study the dependence of the solution of (0.1) with respect to the initial condition x. The most important result states the continuity of $X_t(x)$ in the variable x, in the set of elements x whose Fourier coefficients in some fixed basis of $\mathbb{H}$ converge to zero fast enough. This includes the case of an exponential decay with a suitable rate.

Section three deals with the following problem. Assume we are given a Hilbert-valued process $u(., x)$ depending on a Hilbert-valued parameter $x \in \mathbb{H}$, such that the Stratonovich integral $\int_0^1 u(t, x) \circ dW_t$ exists for any x. Consider an $\mathbb{H}$-valued random variable θ. We want to analyze under which conditions $\int_0^1 u(t, \theta) \circ dW_t$ exists and coincides with the value of the random vector $\int_0^1 u(t, x) \circ dW_t$ at $x=\theta$. The corresponding question in the finite dimensional case has been studied in [12], but the methods of their proofs do not have a direct analogue in the infinite dimensional case. Finally, in Section 4, we present an existence theorem on the solution of (0.2), based upon the results proved in Section 3.

1. CONTINUITY CRITERION FOR STOCHASTIC PROCESSES INDEXED BY A METRIC SPACE

In this section we will derive some results on the existence of continuous versions for processes indexed by an arbitrary metric space. Let (T, d) be a metric space and $\mathbb{B}$ a real and separable Banach space. The norm in $\mathbb{B}$ will be denoted by $\|\,.\,\|$. We denote by $\mathrm{N}(\varepsilon)$, or more precisely $\mathrm{N}(\varepsilon, \mathrm{T}, d)$, the smallest number of open balls of radius ε needed to cover T. We recall that a function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a Young function if $\Phi(x) = \int_0^x \phi(y)\, dy$, where ϕ is strictly increasing, continuous and $\phi(0) = 0$. Then we have the following continuity criterion (*see*, for instance, Corollary 3.3 in [2], or Theorem 1.2 of [16]).

THEOREM 1.1. – *Let* $\{\mathrm{X}(t), t \in \mathrm{T}\}$ *be a stochastic process taking its values in* $\mathbb{B}$. *Assume that the stochastic process* $\{\|\mathrm{X}(s) - \mathrm{X}(t)\|, s, t \in \mathrm{T}\}$ *is separable, and there exists a Young function* Φ *such that the following conditions are satisfied*

$$\text{(i)} \qquad \textit{for any } s, t \in \mathrm{T},\ \mathrm{E}\left(\Phi\left\{\frac{\|\mathrm{X}(s) - \mathrm{X}(t)\|}{d(s, t)}\right\}\right) \leqq 1;$$

$$\text{(ii)} \qquad \int_{\{\mathrm{N}(\varepsilon) > 1\}} \Phi^{-1}(\mathrm{N}(\varepsilon))\, d\varepsilon < +\infty.$$

Then, almost surely the paths of X *are continuous and*

$$\mathrm{E}\left\{\sup_{s, t \in \mathrm{T}} \|\mathrm{X}(s) - \mathrm{X}(t)\|\right\} \leqq 8 \int_{\{\mathrm{N}(\varepsilon) > 1\}} \Phi^{-1}(\mathrm{N}(\varepsilon))\, d\varepsilon. \tag{1.1}$$

In the next lemma we will show that condition (i) of Theorem 1.1 is fulfilled by a particular Young function Φ, assuming some L^p-estimates of the involved processes.

LEMMA 1.2. – *Let* $\{\mathrm{X}(t), t \in \mathrm{T}\}$ *be a stochastic process taking its values in* $\mathbb{B}$. *Assume that there exists* $p_0 \geqq 1$ *such that for any* $p \geqq p_0$ *and* $s, t \in \mathrm{T}$

$$\mathrm{E}[\|\mathrm{X}(s) - \mathrm{X}(t)\|^p] \leqq e^{kp^2} [d(s, t)]^p, \tag{1.2}$$

for some positive constant k. *Then, for all* $\beta > 0$ *it holds that*

$$\sup_{s, t \in \mathrm{T}} \mathrm{E}\left(\Phi_\beta\left\{\frac{\|\mathrm{X}(s) - \mathrm{X}(t)\|}{d(s, t)}\right\}\right) \leqq 1,$$

where

$$\Phi_\beta(x) = \beta\, e^{\beta p_0} \int_{p_0}^{\infty} x^p e^{-kp^2 - \beta p}\, dp. \tag{1.3}$$

Proof. – The inequality (1.2) yields

$$\mathrm{E}\int_{p_0}^{\infty}\left(\frac{\|X(s)-X(t)\|}{d(s,t)}\right)^p e^{-kp^2-\beta p}\,dp \leqq \frac{1}{\beta}e^{-\beta p_0},$$

for any $\beta>0$, and this implies the result. ■

Note that $\Phi_\beta(x)$ is a Young function which satisfies the following inequality

$$\Phi_\beta(x) \geqq C_0 \exp[(\log x-\beta)^2/4k], \tag{1.4}$$

where $C_0=\frac{1}{2}\beta e^{\beta p_0}\sqrt{\frac{\pi}{k}}$, for all $x \geqq \exp(2kp_0+\beta)$. Indeed, easy computations show that

$$\int_{p_0}^{\infty} x^p e^{-kp^2-\beta p}\,dp=\sqrt{2\pi}\,\sigma\,e^{m^2/2\sigma^2}\int_{p_0}^{\infty}\frac{1}{\sqrt{2\pi}\,\sigma}e^{-1/2\sigma^2(p-m)^2}\,dp,$$

for $\sigma=\frac{1}{\sqrt{2k}}$ and $m=\frac{1}{2k}(\log x-\beta)$, and we know that

$$\int_{p_0}^{\infty}\frac{1}{\sqrt{2\pi}\,\sigma}e^{-1/2\sigma^2(p-m)^2}\,dp \geqq \frac{1}{2} \qquad \text{if} \quad m \geqq p_0.$$

The inequality (1.4) implies

$$\Phi_\beta^{-1}(y) \leqq \exp\left(\beta+\sqrt{4k\log(y/C_0)}\right), \tag{1.5}$$

provided $y \geqq \Phi_\beta(e^{2kp_0+\beta})$. Hypothesis (ii) of Theorem 1.1 concerns the integrability of the function $\Phi_\beta^{-1}(N(\varepsilon))$ in a neighbourhood of the origin. For this reason we introduce the following definition.

Definition 1.3. – *A metric space* (T, d) *is said to satisfy the property* (E_k), *for some constant* $k>0$, *if the function*

$$\mathbb{R}_+ \to \mathbb{R}_+$$
$$\varepsilon \mapsto \exp[(4k\log N(\varepsilon, T, d))^{1/2}]$$

is integrable at the origin.

As a consequence of Lemma 1.2 and Theorem 1.1 we obtain the following result.

Proposition 1.4. – *Let* $\{X(t), t\in T\}$ *be a* $\mathbb{B}$*-valued stochastic process satisfying the estimate* (1.2) *for some constant* $k>0$. *Assume also that* T *satisfies the property* (E_k). *Then* X *possesses a continuous version.*

By means of the same arguments one can show the next technical result which will be needed later.

PROPOSITION 1.5. — *Let* $\{X_n(t), t\in T\}$ *be a sequence of* $\mathbb{B}$*-valued stochastic processes. Suppose that there exists a constant* $k>0$ *such that the following conditions are satisfied*

(i) *there exists* $p_0\geqq 1$ *and a sequence* $0<\delta_n\leqq 1$ *decreasing to zero such that*

$$E(\|X_n(s)-X_n(t)\|^p)\leqq \delta_n e^{kp^2}[d(s, t)]^p, \tag{1.6}$$

for all $s, t\in T$, *and for all* $p\geqq p_0$;

(ii)

$$\lim_{n\to\infty} E(\|X_n(t_0)\|)=0, \quad \textit{for some } t_0\in T;$$

(iii) *the space* (T, d) *satisfies the property* (E_k).

Then

$$\lim_{n\to\infty} E(\sup_{t\in T}\|X_n(t)\|)=0.$$

Proof. — In view of condition (ii) it suffices to show that

$$\lim_{n\to\infty} E(\sup_{s,t\in T}\|X_n(s)-X_n(t)\|)=0. \tag{1.7}$$

Consider for each $n\geqq 1$ the Young function defined by

$$\Phi_{\beta, n}(x)=\delta_n^{-1}\Phi_\beta(x),$$

where $\Phi_\beta(x)$ is given by (1.3). From condition (i) and Lemma 1.2 we obtain

$$\sup_{s,t\in T} E\left(\Phi_{\beta, n}\left(\frac{\|X_n(s)-X_n(t)\|}{d(s, t)}\right)\right)\leqq 1, \tag{1.8}$$

for all $n\geqq 1$. Furthermore, $\Phi_{\beta, n}^{-1}(y)=\Phi_\beta^{-1}(\delta_n y)$ and, therefore, $\lim_{n\to\infty}\Phi_{\beta, n}^{-1}(y)=0$, for any $y\geqq 0$. Since $\Phi_\beta^{-1}(\delta_n y)\leqq\Phi_\beta^{-1}(y)$, and $\Phi_\beta^{-1}(N(\varepsilon))$ is integrable on a neighbourhood of the origin by condition (iii), we get

$$\lim_{n\to\infty}\int_{\{N(\varepsilon)>1\}}\Phi_{\beta, n}^{-1}(N(\varepsilon))\,d\varepsilon=0. \tag{1.9}$$

Finally the convergence (1.7) follows from Theorem 1.1, (1.8) and (1.9). ■

In Section 2 we will apply Proposition 1.4 to prove the existence of a version of the process $\{X_t(x), (t, x)\in[0, 1]\times\mathbb{H}\}$, solution of (0.1), jointly continuous in (t, x) on some subspace of $[0, 1]\times\mathbb{H}$. Proposition 1.5 will be used in Section 3 to establish the substitution formula for the Stratonovich integral.

2. SOME PROPERTIES ON THE DEPENDENCE OF INFINITE DIMENSIONAL DIFFUSIONS WITH RESPECT TO THE INITIAL CONDITION

Let $\mathbb{H}$ and $\mathbb{K}$ be two real and separable Hilbert spaces. The scalar product and the norm in $\mathbb{H}$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively, and those of $\mathbb{K}$ by $\langle \cdot, \cdot \rangle_{\mathbb{K}}$ and $\|\cdot\|_{\mathbb{K}}$, respectively. Suppose that $W=\{W(t), t\in[0, 1]\}$ is a $\mathbb{K}$-valued Brownian motion defined on some complete probability space $(\Omega, \mathscr{F}, P)$. We will denote by Q the covariance operator of W which is a nuclear operator on $\mathbb{K}$. That is, W is a zero mean Gaussian process such that

$$E[\langle W_t, h_1 \rangle_{\mathbb{K}} \langle W_s, h_2 \rangle_{\mathbb{K}}]=(s\wedge t)Q(h_1, h_2),$$

for all $s, t\in[0, 1]$, and $h_1, h_2\in\mathbb{K}$. We will denote by $\mathscr{L}^2(\mathbb{K}, \mathbb{H})$ the space of Hilbert-Schmidt operators from $\mathbb{K}$ on $\mathbb{H}$, and by $\mathscr{L}^2_Q(\mathbb{K}, \mathbb{H})$ the space of (possibly unbounded) operators $T:\mathbb{K}\to\mathbb{H}$ such that $TQ^{1/2}$ is Hilbert-Schmidt. In $\mathscr{L}^2_Q(\mathbb{K}, \mathbb{H})$ we will consider the norm $\|T\|_Q=\|TQ^{1/2}\|_{HS}$.

We consider measurable functions $b:\mathbb{H}\to\mathbb{H}$, $\sigma:\mathbb{H}\to\mathscr{L}^2_Q(\mathbb{K}, \mathbb{H})$ satisfying the following Lipschitz condition:

(H 1) There exist constants $C_1, C_2>0$ such that

$$\|b(x)-b(y)\|\leqq C_1\|x-y\|,$$
$$\|\sigma(x)-\sigma(y)\|_Q\leqq C_2\|x-y\|,$$

for any $x, y\in\mathbb{H}$, $t\in[0, 1]$.

Under these conditions (*see*, for instance, [9], [14], [17]), we can show that for any fixed initial condition $x\in\mathbb{H}$, there exists a unique continuous $\mathbb{H}$-valued stochastic process $X=\{X_t(x), t\in[0, 1]\}$, solution of the following stochastic differential equation

$$X_t(x)=x+\int_0^t \sigma(X_s(x))\,dW_s+\int_0^t b(X_s(x))\,ds. \tag{2.1}$$

We want to show the joint continuity in (t, x) of $X_t(x)$ in some subset of $[0, 1]\times\mathbb{H}$. To this end we start by proving some general estimates. In the sequel we will denote by C_3 the constant $C_3=[C_2+\|\sigma(0)\|_Q]^2$.

PROPOSITION 2.1. – *Assume that hypothesis* (H 1) *is satisfied. Then, for any constant* $k>C_3/2$, *there exists* $p_0\geqq 2$ *such that*

$$E[\|X_t(x)-X_t(y)\|^p]\leqq e^{kp^2}\|x-y\|^p, \tag{2.2}$$
$$E[(1+\|X_t(x)\|^2)^p]\leqq e^{4kp^2}(1+\|x\|^2)^p, \tag{2.3}$$

for every $p\geqq p_0$, $t\in[0, 1]$, *and* $x, y\in\mathbb{H}$.

Proof. — We will first show the inequality (2.2). Fix $p \geqq 1$, and consider the function $f: \mathbb{H} \to \mathbb{R}$, $f(x) = \| x \|^{2p}$. Itô's formula (cf. [17]) yields

$$\begin{aligned}
\mathrm{E}[\| \mathrm{X}_t(x) - \mathrm{X}_t(y) \|^{2p}] = \| x - y \|^{2p} \\
&+ 2p\mathrm{E} \int_0^t \| \mathrm{X}_s(x) - \mathrm{X}_s(y) \|^{2(p-1)} \\
&\times \langle \mathrm{X}_s(x) - \mathrm{X}_s(y), b(\mathrm{X}_s(x)) - b(\mathrm{X}_s(y)) \rangle \, ds \\
&+ 2p(p-1)\mathrm{E} \int_0^t \| \mathrm{X}_s(x) - \mathrm{X}_s(y) \|^{2(p-2)} \\
&\times \| \mathrm{Q}^{1/2} [\sigma(\mathrm{X}_s(x)) - \sigma(\mathrm{X}_s(y))]^* (\mathrm{X}_s(x) - \mathrm{X}_s(y)) \|^2 \, ds \\
&+ p\mathrm{E} \int_0^t \| \mathrm{X}_s(x) - \mathrm{X}_s(y) \|^{2(p-1)} \| \sigma(\mathrm{X}_s(x)) - \sigma(\mathrm{X}_s(y)) \|_{\mathrm{Q}}^2 \, ds. \quad (2.4)
\end{aligned}$$

By Schwarz's inequality and the Lipschitz hypothesis (H 1),

$$\langle \mathrm{X}_s(x) - \mathrm{X}_s(y), b(\mathrm{X}_s(x)) - b(\mathrm{X}_s(y)) \rangle \leqq \mathrm{C}_1 \| \mathrm{X}_s(x) - \mathrm{X}_s(y) \|^2,$$

and

$$\| \mathrm{Q}^{1/2} [\sigma(\mathrm{X}_s(x)) - \sigma(\mathrm{X}_s(y))]^* (\mathrm{X}_s(x) - \mathrm{X}_s(y)) \|^2 \leqq \mathrm{C}_2^2 \| \mathrm{X}_s(x) - \mathrm{X}_s(y) \|^4.$$

Hence, from (2.4)

$$\begin{aligned}
\mathrm{E}[\| \mathrm{X}_t(x) - \mathrm{X}_t(y) \|^{2p}] \leqq \| x - y \|^{2p} \\
&+ \{ 2p^2 \mathrm{C}_2^2 + p(2\mathrm{C}_1 - \mathrm{C}_2^2) \} \int_0^t \mathrm{E}[\| \mathrm{X}_s(x) - \mathrm{X}_s(y) \|^{2p}] \, ds.
\end{aligned}$$

Note that $\exp(2p^2 \mathrm{C}_2^2 + p(2\mathrm{C}_1 - \mathrm{C}_2^2)) \leqq \exp(4p^2 k)$, for p larger than some value p_0 because $\mathrm{C}_2^2 \leqq \mathrm{C}_3 < 2k$. Then the result follows by Gronwall's lemma.

We can use the same method to show the inequality (2.3). In fact, hypothesis (H 1) ensures

$$\begin{aligned}
\langle \mathrm{X}_s(x), b(\mathrm{X}_s(x)) \rangle &\leqq \mathrm{C}_1 \| \mathrm{X}_s(x) \|^2 + \| b(0) \| \, \| \mathrm{X}_s(x) \| \\
&\leqq (\mathrm{C}_1 + \| b(0) \|)(1 + \| \mathrm{X}_s(x) \|^2)
\end{aligned}$$

and

$$\begin{aligned}
\| \mathrm{Q}^{1/2} [\sigma(\mathrm{X}_s(x))]^* (\mathrm{X}_s(x)) \| &\leqq \mathrm{C}_2 \| \mathrm{X}_s(x) \|^2 + \| \mathrm{Q}^{1/2} [\sigma(0)]^* \| \, \| \mathrm{X}_s(x) \| \\
&\leqq \sqrt{\mathrm{C}_3} (1 + \| \mathrm{X}_s(x) \|^2)^2.
\end{aligned}$$

Then, using the Itô formula we obtain

$$\begin{aligned}
\mathrm{E}[(1 + \| \mathrm{X}_t(x) \|^2)^p] \leqq (1 + \| x \|^2)^p \\
&+ (2p^2 \mathrm{C}_3 + p(2(\mathrm{C}_1 + \| b(0) \|) - \mathrm{C}_3)) \int_0^t \mathrm{E}[(1 + \| \mathrm{X}_s(x) \|^2)^p] \, ds,
\end{aligned}$$

and (2.3) follows again from Gronwall's lemma. ■

PROPOSITION 2.2. — *Under hypothesis* (H 1), *for any constant* $k > C_3/2$ *there exists* $p_0 \geqq 2$ *such that*

$$E[\|X_t(x) - X_s(y)\|^p] \leqq e^{kp^2} \{\|x-y\|^p + |t-s|^{p/2}(1+\|y\|^2)^{p/2}\}, \quad (2.5)$$

for every $p \geqq p_0$, s, $t \in [0, 1]$ *and* x, $y \in \mathbb{H}$.

Proof. — Fix $p \geqq 1$. Since

$$E[\|X_t(x) - X_s(y)\|^{2p}] \leqq 2^{2p-1} \{E(\|X_t(x) - X_t(y)\|^{2p} + \|X_t(y) - X_s(y)\|^{2p})\}$$

and in view of the proof of the estimate (2.2), we need only to show that for any $k > C_3/2$ we have

$$E[\|X_t(y) - X_s(y)\|^{2p}] \leqq e^{4kp^2} |t-s|^p (1+\|y\|^2)^p, \quad (2.6)$$

for any p large enough. This will be checked by using Burkholder's inequality for Hilbert-valued martingales (*see* for instance, [8], p. 212, E.2), together with Hölder's inequality. Indeed, assuming $t \leqq s$, we have

$$\begin{aligned} E[\|X_t(y) - X_s(y)\|^{2p}] &\leqq 2^{2p-1} \left\{ E\left(\left\| \int_t^s \sigma(X_u(y))\, dW_u \right\|^{2p} \right) + E\left(\left\| \int_t^s b(X_u(y))\, du \right\|^{2p} \right) \right\} \\ &\leqq 2^{2p-1} \left\{ (2e)^p p^{2p} E\left(\int_t^s \|\sigma(X_u(y))\|_Q^2\, du \right)^p + E\left(\left\| \int_t^s b(X_u(y))\, du \right\|^{2p} \right) \right\} \\ &\leqq C(p) \{|t-s|^{p-1} + |t-s|^{2p-1}\} \int_t^s E(1+\|X_u(y)\|^2)^p\, du, \end{aligned}$$

where $C(p) = ((C_1 + \|b(0)\|)^{2p} + (C_2 + \|\sigma(0)\|_Q)^{2p}) e^p 2^{3p} p^{2p}$. So, the estimate (2.3) yields (2.6) for p larger than some fixed value, and the inequality (2.5) is proved. ■

On $[0, 1] \times \mathbb{H}$ we consider the metric d' defined by

$$d'((s, x), (t, y)) = |t-s|^{1/2} + d(x, y),$$

for any s, $t \in [0, 1]$, x, $y \in \mathbb{H}$, and where d denotes the metric induced by the norm of the Hilbert space $\mathbb{H}$. Let us fix $M > 0$ and consider the open ball $\{x \in \mathbb{H} : \|x\| < M\}$. Then, for any constant $k > C_3/2$, Proposition 2.2 yields

$$E[\|X_s(x) - X_t(y)\|^p] \leqq e^{kp^2} (d'((s, x), (t, y)))^p, \quad (2.7)$$

for any s, $t \in [0, 1]$, $\|x\|$, $\|y\| \leqq M$, $p \geqq p_1$, where p_1 depends on the constants k, C_1, C_2, $\|b(0)\|$, $\|\sigma(0)\|_Q$ and M. As a consequence of Proposition 1.4 we can now give the main result of this section.

THEOREM 2.3. – *Let* $\{X_t(x),\ t \in [0, 1]\}$ *be the solution of the stochastic differential equation* (2.1), *with initial condition* $x \in \mathbb{H}$. *Let* B *be a bounded subset of* $\mathbb{H}$ *such that* $([0, 1] \times B, d')$ *satisfies the property* (E_k), *that means, the function* $\varepsilon \mapsto \exp[(4k \log N(\varepsilon, [0, 1] \times B, d'))^{1/2}]$ *is integrable at the origin, for some constant* $k > C_3/2$. *Then, there exists a version of* $\{X_t(x),\ (t, x) \in [0, 1] \times B\}$ *with almost surely continuous paths.*

In the sequel we will exhibit some examples of bounded sets B such that $[0, 1] \times B$ verifies property (E_k). Fix a complete orthonormal system $\{e_i,\ i \geqq 1\}$ on $\mathbb{H}$, and consider a sequence $\beta = \{\beta_i,\ i \geqq 1\}$ of positive real numbers such that $\sum_{i=1}^{\infty} (\beta_i)^2 < \infty$. We define the set

$$B_\beta = \{x \in \mathbb{H} : |\langle x, e_i \rangle| \leqq \beta_i,\ i \geqq 1\}. \tag{2.8}$$

Given $\varepsilon > 0$, set

$$j(\varepsilon) = \inf\left\{k \geqq 1 : \sum_{i>k} \beta_i^2 < \frac{\varepsilon^2}{4}\right\}. \tag{2.9}$$

Notice that $j(\varepsilon) < +\infty$, for any $\varepsilon > 0$. The sets B_β are bounded and closed subsets of $\mathbb{H}$. Furthermore, it will be shown that the B_β are totally bounded, and consequently compact. Moreover, for any $\varepsilon > 0$ it will be possible to estimate the number of open balls of radius $\varepsilon > 0$ required to cover the subset B_β.

Before proving these facts we will state an elementary result on totally bounded sets.

LEMMA 2.4. – *Let* (S_i, d_i), $i = 1, 2$ *be two metric spaces. Consider the product space* $S = S_1 \times S_2$ *endowed with a metric d such that*

$$d((x_1, x_2), (y_1, y_2)) \leqq d_1(x_1, y_1) + d_2(x_2, y_2), \tag{2.10}$$

for any $x_i, y_i \in S_i$, $i = 1, 2$. *Then, for any* $\varepsilon > 0$

$$N(\varepsilon, S, d) \leqq N\left(\frac{\varepsilon}{2}, S_1, d_1\right) N\left(\frac{\varepsilon}{2}, S_2, d_2\right). \tag{2.11}$$

In particular, if the spaces (S_i, d_i), $i = 1, 2$, *are totally bounded, the same property holds for the product space* (S, d).

PROPOSITION 2.5. – *Let* B_β *be the subset defined in* (2.8), *and* $D_\beta = [0, 1] \times B_\beta$. *Then* B_β *and* D_β *are totally bounded subsets of* $(\mathbb{H}, d)$ *and* $([0, 1] \times \mathbb{H}, d')$ *respectively. Furthermore, for any* $\varepsilon > 0$,

$$\log N(\varepsilon, B_\beta, d) \leqq j(\varepsilon)\left\{\log \frac{c}{2\varepsilon} + \frac{1}{2} \log j(\varepsilon)\right\}, \tag{2.12}$$

and

$$\log N(\varepsilon, D_\beta, d') \leqq \log \frac{4}{\varepsilon^2} + j\left(\frac{\varepsilon}{2}\right)\left\{ \log \frac{c}{\varepsilon} + \frac{1}{2} \log j\left(\frac{\varepsilon}{2}\right)\right\}, \tag{2.13}$$

where $c = 2(1 + \sup_i 2\beta_i)$, *and with* $j(\varepsilon)$ *defined in* (2.9).

Proof. — We will first estimate the number of open balls of radius ε needed to cover B_β, following the ideas of Imkeller (*see* [5], Proposition 1.1). The corresponding estimation for D_β will follow as an easy consequence. For every $j \geqq 1$ we define

$$T_j = \{ x \in B_\beta : \langle x, e_i \rangle = 0, \quad \text{for} \quad i > j \},$$
$$T'_j = \{ x \in B_\beta : \langle x, e_i \rangle = 0, \quad \text{for} \quad i \leqq j \}.$$

On T_j we consider a metric d_∞ defined by

$$d_\infty(x, y) = \sup_{1 \leqq i \leqq j} |\langle x - y, e_i \rangle|,$$

which verifies

$$d(x, y) \leqq \sqrt{j}\, d_\infty(x, y).$$

As a consequence we obtain

$$\begin{aligned} N(\varepsilon, T_j, d) &\leqq N(\varepsilon/\sqrt{j}, T_j, d_\infty) \\ &\leqq \prod_{i=1}^{j} \left\{ \left[\frac{\beta_i \sqrt{j}}{\varepsilon} \right] + 1 \right\}, \end{aligned} \tag{2.14}$$

where $[.]$ denotes the entire part. Furthermore, T_j is compact because it is a finite dimensional closed rectangle.

Fix $\varepsilon > 0$, and consider the index $j(\varepsilon)$ given by (2.9). For any $x \in T'_{j(\varepsilon)}$, we have

$$\|x\| = \left(\sum_{i > j(\varepsilon)} \langle x, e_i \rangle^2 \right)^{1/2} \leqq \left(\sum_{i > j(\varepsilon)} (\beta_i)^2 \right)^{1/2} < \frac{\varepsilon}{2}.$$

Thus, $T'_{j(\varepsilon)}$ is included in the open ball of $\mathbb{H}$ of radius $\frac{\varepsilon}{2}$ centered at 0, and therefore

$$N\left(\frac{\varepsilon}{2}, T'_{j(\varepsilon)}, d\right) = 1. \tag{2.15}$$

Consider the metric spaces $(T_{j(\varepsilon)}, d)$ and $(T'_{j(\varepsilon)}, d)$. The product $T_{j(\varepsilon)} \times T'_{j(\varepsilon)}$ can be identified with B_β and equipped with the metric d. In that form condition (2.10) is satisfied and we can apply Lemma 2.4 to these metric spaces. The inequality (2.11) together with (2.14) and (2.15) yield

$$N(\varepsilon, B_\beta, d) \leq N\left(\frac{\varepsilon}{2}, T_{j(\varepsilon)}, d\right) < +\infty, \tag{2.16}$$

and consequently (B_β, d) is totally bounded. Furthermore, (2.14) implies

$$N(\varepsilon, B_\beta, d) \leq \prod_{i=1}^{j(\varepsilon)} \left(\left[\frac{2\beta_i \sqrt{j(\varepsilon)}}{\varepsilon}\right] + 1\right).$$

Consequently,

$$\log N(\varepsilon, B_\beta, d) \leq j(\varepsilon) \left\{ \log \frac{c}{2\varepsilon} + \frac{1}{2} \log j(\varepsilon) \right\},$$

with $c = 2(1 + \sup_i 2\beta_i)$. Finally, the inequality (2.13) follows easily from (2.12). ■

As a consequence of the previous Proposition, a set of the form $[0, 1] \times B_\beta$ verifies the property (E_k) if

$$\int_{0^+} \exp\{[8k \log(2/\varepsilon) + 4kj(\varepsilon/2)(\log(c/\varepsilon) + (1/2) \log j(\varepsilon/2))]^{1/2}\} d\varepsilon < \infty,$$

where $c = 2(1 + \sup_i 2\beta_i)$.

We finish this section by giving an example of class of sets B_β and D_β for which the property (E_k), $k > 0$ holds.

Example 2.6. – Consider a square summable sequence of the form $\beta_i = e^{-\delta i}$, where $\delta > 0$. Then the sets B_β and D_β verify the property (E_k) for any k such that $k < \delta/4$. In fact, it holds that

$$j(\varepsilon) = \left[\frac{1}{\delta}\left(\log \frac{2}{\varepsilon\sqrt{1 - e^{-2\delta}}}\right)\right] \tag{2.17}$$

and, from (2.12) and (2.17) we deduce

$$[4k \log N(\varepsilon, B_\beta, d)]^{1/2} \leq \sqrt{\frac{4k}{\delta}} \log \frac{C}{\varepsilon},$$

for some constant $C > 0$, and a similar inequality holds for D_β. Therefore, the function $\varepsilon \mapsto \exp[4k \log N(\varepsilon, B_\beta, d)]^{1/2}$ is integrable at the origin provided $4k/\delta < 1$. As a consequence, the solution $X_t(x)$ of the equation (2.1) has a continuous version on any set of the form $\{x \in \mathbb{H}, |\langle x, e_i \rangle| \leq e^{-i\delta}\}$, provided $\delta > 2C_3$.

The results of this section are still true if the coefficients b and σ depend on the time variable, and in addition to the Lipschitz condition (H 1) they verify a linear growth condition of the form

$$\|b(t, x)\| \leqq K_1 (1+\|x\|^2)^{1/2},$$
$$\|\sigma(t, x)\|_Q \leqq K_2 (1+\|x\|^2)^{1/2}.$$

In that case the constant C_3 appearing in Propositions 2.2, 2.1 and in Theorem 2.3 would be the maximum of C_2^2 and K_2^2.

3. SUBSTITUTION FORMULA FOR THE STRATONOVICH INTEGRAL

This section is devoted to extend the substitution result for the Stratonovich integral, given in Proposition 7.7 of [12], to an infinite dimensional setting. We first recall and introduce some notations and facts on anticipating calculus that will be needed in the sequel.
As in the previous section $W = \{W_t, t \in [0, 1]\}$ will denote a $\mathbb{K}$-valued Brownian motion with covariance operator Q. We will assume that the σ-algebra $\mathscr{F}$ is generated by W. We will denote by D the derivative operator. That is, if F is a $\mathbb{H}$-valued elementary random variable of the form

$$F = f(\langle W(t_1), h_1 \rangle, \ldots, \langle W(t_m), h_m \rangle) v, \tag{3.1}$$

where $f \in \mathscr{C}_b^\infty(\mathbb{R}^m)$, $v \in \mathbb{H}$, and $h_1, \ldots, h_m$ are elements of $\mathbb{H}$, then the derivative of F is the element of $L^2([0, 1] \times \Omega; \mathscr{L}_Q^2(\mathbb{K}, \mathbb{H}))$ defined by

$$D_t F = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(\langle W(t_1), h_1 \rangle, \ldots, \langle W(t_m), h_m \rangle) \mathbf{1}_{[0, t_i]}(t) (v \otimes h_i).$$

Let u be a stochastic process in $L^2([0, 1] \times \Omega; \mathscr{L}_Q^2(\mathbb{K}, \mathbb{H}))$, and suppose that there exists a constant $C > 0$ such that for any elementary random variable F of the form (3.1) we have

$$\left| E\left(\int_0^1 \langle u_t, D_t F \rangle_Q \, dt \right) \right| \leqq C \|F\|_{L^2(\Omega, \mathbb{H})}.$$

Then we can define the Skorohod stochastic integral of u, denoted by $\int_0^1 u_t \, dW_t$, as the adjoint of the operator D. That is, $\int_0^1 u_t \, dW_t$ is the element of $L^2(\Omega, \mathbb{H})$ determined by

$$E\left(\int_0^1 \langle u_t, D_t F \rangle_Q \, dt \right) = E\left(\left\langle F, \int_0^1 u_t \, dW_t \right\rangle \right),$$

for any elementary random variable F. If u is adapted with respect to the natural filtration associated with W, $\{\mathscr{F}_s, s\in[0,1]\}$, this Skorohod integral coincides with the usual Itô integral for Hilbert-valued processes that has been used in the preceding section.

Let Π be a partition of [0, 1] of the form $\Pi=\{0=t_0<t_1<\ldots<t_n=1\}$. We will denote by $|\Pi|$ the norm of the partition. We will also use the notation $\Delta_j=[t_j, t_{j+1}]$, $j=0, 1, \ldots, n-1$, and $|\Delta_j|$ will represent the Lebesgue measure of the interval Δ_j.

Consider a process $v\in L^2([0,1]\times\Omega; \mathscr{L}^2_Q(\mathbb{K}, \mathbb{H}))$, such that for almost all ω we have $v(\omega)\in L^1([0,1]; \mathscr{L}(\mathbb{K}, \mathbb{H}))$. We recall that $\mathscr{L}(\mathbb{K}, \mathbb{H})$ denotes the space of bounded linear operators from $\mathbb{K}$ to $\mathbb{H}$, which is included in $\mathscr{L}^2_Q(\mathbb{K}, \mathbb{H})$. Then to each partition Π of [0, 1] we can associate the $\mathscr{L}^2(\mathbb{K}, \mathbb{H})$-valued step process defined by

$$v^{\Pi}(t)=\sum_{j=0}^{n-1}\frac{1}{|\Delta_j|}\left(\int_{\Delta_j} v(s)\,ds\right)1_{\Delta_j}(t), \tag{3.2}$$

and the corresponding Riemann sums

$$S^{\Pi}=\sum_{j=0}^{n-1}\frac{1}{|\Delta_j|}\left(\int_{\Delta_j} v(s)\,ds\right)W(\Delta_j). \tag{3.3}$$

which are $\mathbb{H}$-valued random variables.

The process v is said to be *Stratonovich integrable* if the family $\{S^{\Pi}\}$ converges in probability as $|\Pi|$ tends to zero. The limit is called the Stratonovich integral of the process v, and is denoted by $\int_0^1 v_t\circ dW_t$ (*see* [3]).

Consider the particular case where, in addition to the above conditions, v is an adapted process continuous in $\mathscr{L}^2(\Omega; \mathscr{L}^2(\mathbb{K}, \mathbb{H}))$. Moreover, assume that the following condition holds:

(C 1) There exists an $\mathbb{H}$-valued process a such that

$$\int_0^1 \|a(t)\|\,dt<\infty \text{ a.s.},$$

and

$$\sum_{j=0}^{n-1}\frac{1}{|\Delta_j|}\int_{\Delta_j}(v(s)-v(t_j))(W_s-W_{t_j})\,ds \underset{|\Pi|\to 0}{\longrightarrow} \frac{1}{2}\int_0^1 a(t)\,dt \tag{3.4}$$

in probability.

Then v is Stratonovich integrable and

$$\int_0^1 v(t)\circ dW_t=\int_0^1 v(t)\,dW_t+\frac{1}{2}\int_0^1 a(t)\,dt. \tag{3.5}$$

The proof of this statement is as follows. Consider the decomposition

$$S^{\Pi}=a^{\Pi}+b^{\Pi},$$

with

$$a^{\Pi}=\sum_{j=0}^{n-1} v(t_j)\,\mathrm{W}(\Delta_j),$$

$$b^{\Pi}=\sum_{j=0}^{n-1} \frac{1}{|\Delta_j|}\int_{\Delta_j} (v(s)-v(t_j))\,\mathrm{W}(\Delta_j)\,ds.$$

The continuity in $\mathrm{L}^2(\Omega;\ \mathscr{L}^2(\mathbb{K},\ \mathbb{H}))$ of the process v yields the convergence of $\{a^{\Pi}\}$ in $\mathrm{L}^2(\Omega;\ \mathbb{H})$ to the Itô integral $\int_0^1 v(t)\,d\mathrm{W}_t$. We can write $b^{\Pi}=b_1^{\Pi}+b_2^{\Pi}$ with

$$b_1^{\Pi}=\sum_{j=0}^{n-1} \frac{1}{|\Delta_j|}\int_{\Delta_j} (v(s)-v(t_j))\,(\mathrm{W}_s-\mathrm{W}_{t_j})\,ds,$$

and

$$b_2^{\Pi}=\sum_{j=0}^{n-1} \frac{1}{|\Delta_j|}\int_{\Delta_j} (v(s)-v(t_j))\,(\mathrm{W}_{t_{j+1}}-\mathrm{W}_s)\,ds.$$

Condition (C 1) yields $\lim_{|\Pi|\downarrow 0} b_1^{\Pi}=\frac{1}{2}\int_0^1 a(t)\,dt$, in probability. Using the fact that v belongs to the space $\mathrm{L}^2([0,\ 1]\times\Omega;\ \mathscr{L}^2_{\mathrm{Q}}(\mathbb{K},\ \mathbb{H}))$, and applying the isometry property of the stochastic integral we can prove that $\lim_{|\Pi|\downarrow 0} b_2^{\Pi}=0$, in $\mathrm{L}^2(\Omega;\ \mathbb{H})$. Hence (3.5) is established.

Consider a family $u(x)=\{u(t,\ x),\ t\in[0,\ 1]\}$ of $\mathscr{L}(\mathbb{K},\ \mathbb{H})$-valued processes indexed by $x\in\mathrm{G}$, where G is some subset of $\mathbb{H}$, and satisfying the following hypotheses

(h 1) the mapping $(t,\ x,\ \omega)\mapsto u(t,\ x,\ \omega)$ is measurable;

(h 2) $u(t,\ x)$ is $\mathscr{F}_t$-measurable, for any $x\in\mathrm{G}$;

(h 3) for any $x\in\mathrm{G}$, $u(.,\ x)\in\mathrm{L}^1([0,\ 1];\ \mathscr{L}(\mathbb{K},\ \mathbb{H}))$;

(h 4) $t\mapsto u(t,\ x)$ is continuous in $\mathrm{L}^2(\Omega;\ \mathscr{L}^2_{\mathrm{Q}}(\mathbb{K},\ \mathbb{H}))$, for any $x\in\mathrm{G}$;

(h 5) there exists a constant $k>0$ and $p_0\geqq 2$ such that for any $s,\ t\in[0,\ 1]$, $x,\ y\in\mathrm{G}$, $p\geqq p_0$, the process

$$\mathrm{V}_{s,t}(x,\ y)=(u(s,\ x)-u(s,\ y))-(u(t,\ x)-u(t,\ y))$$

verifies

$$\mathrm{E}(\|\mathrm{V}_{s,t}(x,\ y)\|_{\mathrm{Q}}^{p})\leqq e^{kp^2}\|x-y\|^p\,|t-s|. \tag{3.6}$$

LEMMA 3.1. — *Suppose that the family of processes* $\{u(x), x \in \mathrm{G}\}$ *satisfies hypotheses* $(h\,1)$ *to* $(h\,5)$. *For any partition* Π *set*

$$\mathrm{A}^{\Pi}(x) = \sum_{j=0}^{n-1} u(t_j, x)\, \mathrm{W}(\Delta_j).$$

Let B *be a bounded subset of* G *which verifies the property* $(\mathrm{E}_{k'})$ *for some constant* $k' > k$ *where* k *is the constant appearing in hypothesis* $(h5)$. *Then,*

$$\lim_{|\Pi| \downarrow 0} \mathrm{E}\left\{\sup_{x \in \mathrm{B}} \left\| \int_0^1 u(t, x)\, d\mathrm{W}_t - \mathrm{A}^{\Pi}(x) \right\| \right\} = 0. \tag{3.7}$$

Proof. — For any point $x_0 \in \mathbb{H}$, it is clear that

$$\lim_{|\Pi| \downarrow 0} \mathrm{E}\left\{\left\| \int_0^1 u(t, x_0)\, d\mathrm{W}_t - \mathrm{A}^{\Pi}(x_0) \right\| \right\} = 0, \tag{3.8}$$

by the well known properties on approximation of the Itô integral. We can write

$$\mathrm{A}^{\Pi}(x) = \int_0^1 u^{\Pi}(t, x)\, d\mathrm{W}_t,$$

where

$$u^{\Pi}(t, x) = \sum_{j=0}^{n-1} u(t_j, x)\, 1_{\Delta_j}(t).$$

Fix $p \geqq 2$, $x, y \in \mathrm{G}$, and set

$$\mathrm{Z}^{\Pi}(x) = \int_0^1 [u(t, x) - u^{\Pi}(t, x)]\, d\mathrm{W}_t.$$

In order to prove the convergence (3.7) we are going to apply Proposition 1.5 to any sequence $\{\mathrm{Z}^{\Pi_n}(x), x \in \mathbb{H}\}$, $n \geqq 1$ such that $|\Pi_n| \downarrow 0$. First notice that condition (ii) of Theorem 1.5 follows from (3.8). In view of the assumptions of the lemma it is sufficient to establish the estimate

$$\mathrm{E}(\|\mathrm{Z}^{\Pi}(x) - \mathrm{Z}^{\Pi}(y)\|^p) \leqq \exp(k' p^2)\, \|x - y\|^p\, |\Pi|, \tag{3.9}$$

for any p larger than some real number. Define

$$u^{\Pi}(s, x, y) = (u(s, x) - u^{\Pi}(s, x)) - (u(s, y) - u^{\Pi}(s, y)).$$

Then, by applying Burkholder's inequality, we obtain

$$\begin{aligned}
\mathrm{E}(\|\mathrm{Z}^{\Pi}(x)-\mathrm{Z}^{\Pi}(y)\|^{p}) &= \mathrm{E}\left(\left\|\int_0^1 u^{\Pi}(s, x, y)\,d\mathrm{W}_s\right\|^p\right) \\
&\leqq \mathrm{C}_p \mathrm{E}\left[\left(\int_0^1 \left\|u^{\Pi}(s, x, y)\right\|_{\mathrm{Q}}^2 ds\right)^{p/2}\right] \\
&\leqq \mathrm{C}_p \sup_{0 \leqq j \leqq n-1} \sup_{s \in \Delta_j} \mathrm{E}(\|u^{\Pi}(s, x, y)\|_{\mathrm{Q}}^p) \\
&\leqq \mathrm{C}_p \sup_{|s-t| \leqq |\Pi|} \mathrm{E}(\|\mathrm{V}_{s,t}(x, y)\|_{\mathrm{Q}}^p),
\end{aligned}$$

where $\mathrm{C}_p=(e/2)^{p/2} p^p$. Since by hypothesis (h5) the right hand side of this inequality is bounded by

$$\exp(kp^2)\,\|x-y\|^p\,|\Pi|,$$

we obtain the estimation (3.9) for p large enough, and this completes the proof of the lemma. ■

LEMMA 3.2. – *Assume that the family of processes* $u(x)$, $x \in \mathrm{G}$ *satisfies hypothesis* (*h*1) *to* (*h*5). *For any partition* Π *of* [0, 1] *set*

$$\mathrm{F}^{\Pi}(x)=\sum_{j=0}^{n-1} \frac{1}{|\Delta_j|}\int_{\Delta_j} [u(s, x)-u(t_j, x)](\mathrm{W}_{t_{j+1}}-\mathrm{W}_s)\,ds.$$

Let B *be a bounded subset of* G *which verifies the property* ($\mathrm{E}_{k'}$) *for some* $k'>k$ *where* k *is the constant appearing in hypothesis* (*h*5). *Then*

$$\lim_{n \to \infty} \mathrm{E}\{\sup_{x \in \mathrm{B}} \|\mathrm{F}^{\Pi}(x)\|\}=0.$$

Proof. – We first establish an estimate of the form

$$\mathrm{E}\{\|\mathrm{F}^{\Pi}(x)-\mathrm{F}^{\Pi}(y)\|^p\} \leqq e^{k'p^2}\|x-y\|^p\,|\Pi|, \qquad (3.10)$$

for any $x, y \in \mathrm{G}$ and p large enough. To prove (3.10) we remark that, using the notation that we have introduced before, we have

$$\mathrm{F}^{\Pi}(x)-\mathrm{F}^{\Pi}(y)=\sum_{j=0}^{n-1} \frac{1}{|\Delta_j|}\int_{\Delta_j} u^{\Pi}(s, x, y)(\mathrm{W}_{t_{j+1}}-\mathrm{W}_s)\,ds.$$

The discrete time $\mathbb{H}$-valued process

$$\left\{\frac{1}{|\Delta_j|}\int_{\Delta_j} u^{\Pi}(s, x, y)(\mathrm{W}_{t_{j+1}}-\mathrm{W}_s)\,ds, j=0, \ldots, n-1\right\}$$

is a martingale difference with respect to $\{\mathscr{F}_{t_j}, j=0, \ldots, n-1\}$. Hence, by Burkholder's and Hölder's inequality, for any $p \geqq 2$ we obtain

$$
\begin{aligned}
&\mathrm{E}[\|\mathrm{F}^{\Pi}(x)-\mathrm{F}^{\Pi}(y)\|^p] \\
&\quad \leqq \mathrm{C}_p \mathrm{E}\left(\sum_{j=0}^{n-1} \frac{1}{|\Delta_j|^2}\left\|\int_{\Delta_j} u^{\Pi}(s, x, y)(\mathrm{W}_{t_{j+1}}-\mathrm{W}_s)\,ds\right\|^2\right)^{p/2} \\
&\quad \leqq \mathrm{C}_p \mathrm{E}\left(\sum_{j=0}^{n-1} \frac{1}{|\Delta_j|}\int_{\Delta_j} \|u^{\Pi}(s, x, y)(\mathrm{W}_{t_{j+1}}-\mathrm{W}_s)\|^2\,ds\right)^{p/2} \\
&\quad \leqq \mathrm{C}_p \sum_{j=0}^{n-1} \frac{1}{|\Delta_j|^{p/2}}\int_{\Delta_j} \mathrm{E}(\|u^{\Pi}(s, x, y)(\mathrm{W}_{t_{j+1}}-\mathrm{W}_s)\|^p)\,ds, \qquad (3.11)
\end{aligned}
$$

where the constant C_p is of the order of $\mathrm{C}^p p^p$, for some constant $\mathrm{C}>0$. Suppose that $\{h_i, i \geqq 1\}$ is a complete orthonormal system in $\mathbb{K}$ such that $\{\langle \mathrm{W}_t, h_i\rangle_{\mathbb{K}}, t \in [0, 1]\}$ are independent real valued Brownian motions with variances γ_i, and $\sum_{i=1}^{\infty} \gamma_i < \infty$. Then using again Burkholder's inequality for Hilbert-valued discrete martingales we obtain

$$
\begin{aligned}
&\mathrm{E}(\|u^{\Pi}(s, x, y)(\mathrm{W}_{t_{j+1}}-\mathrm{W}_s)\|^p) \\
&\quad = \mathrm{E}\left(\left\|\sum_{i=1}^{\infty} u^{\Pi}(s, x, y)(h_i)\langle \mathrm{W}_{t_{j+1}}-\mathrm{W}_s, h_i\rangle_{\mathbb{K}}\right\|^p\right) \\
&\quad \leqq \mathrm{C}_p \mathrm{E}\left(\left|\sum_{i=1}^{\infty} \|u^{\Pi}(s, x, y)(h_i)\|^2 \langle \mathrm{W}_{t_{j+1}}-\mathrm{W}_s, h_i\rangle_{\mathbb{K}}^2\right|^{p/2}\right) \\
&\quad \leqq \mathrm{C}_p \mathrm{E}\Bigg(\|u^{\Pi}(s, x, y)\|_{\mathrm{Q}}^{p-2} \sum_{i=1}^{\infty} \gamma_i^{1-(p/2)} \\
&\qquad\qquad \times \|u^{\Pi}(s, x, y)(h_i)\|^2 |\langle \mathrm{W}_{t_{j+1}}-\mathrm{W}_s, h_i\rangle_{\mathbb{K}}|^p\Bigg).
\end{aligned}
$$

Due to the independence of $u^{\Pi}(s, x, y)$ and $\mathrm{W}_{t_{j+1}}-\mathrm{W}_s$, this last expression is bounded by

$$
\mathrm{C}_p \lambda_p \mathrm{E}(\|u^{\Pi}(s, x, y)\|_{\mathrm{Q}}^p)\,|t_{j+1}-s|^{p/2}, \qquad (3.12)
$$

where λ_p is the p-th moment of the absolute value of a standard normal variable. By substituting (3.12) into the right hand side of (3.11) we obtain that the left hand side of (3.11) is bounded by

$$
\mathrm{C}_p^2 \lambda_p \sup_{|s-t| \leqq |\Pi|} \mathrm{E}(\|\mathrm{V}_{s,t}(x, y)\|_{\mathrm{Q}}^p),
$$

and using hypothesis (h5) we deduce (3.10). Finally by the same arguments used in the proof of (3.10) we get

$$\lim_{|\Pi|\downarrow 0} \mathrm{E}\left(\left\| \sum_{j=0}^{n-1} \frac{1}{|\Delta_j|} \int_{\Delta_j} [u(s, x_0) - u(t_j, x_0)] (\mathrm{W}_{t_{j+1}} - \mathrm{W}_s)\, ds \right\|\right) = 0. \quad (3.13)$$

The estimate (3.10), the convergence (3.13) and the assumptions of the lemma allow us to apply Proposition 1.5 to any sequence $\{ \mathrm{Z}^{\Pi_n}(x), x \in \mathbb{H}, n \geqq 1 \}$ such that $|\Pi_n| \downarrow 0$, and this completes the proof of the lemma. ■

We can now state the main result of this section.

THEOREM 3.3. – *Let* $\{ u(x), x \in \mathrm{G} \}$ *be a family of processes satisfying hypothesis* (h1) *to* (h5). *Let* B *be a bounded subset of* G *which verifies the property* ($\mathrm{E}_{k'}$) *for some* $k' > k$ *where* k *is the constant appearing in hypothesis* (h5). *Let* θ *be a* B*-valued random variable. Suppose that:*

(h6) *There exists a measurable function* $d : [0, 1] \times \mathrm{G} \times \Omega \to \mathbb{H}$ *such that*

$$\mathrm{P}-\lim_{|\Pi|\downarrow 0} \sum_{j=0}^{n-1} \frac{1}{|\Delta_j|} \int_{\Delta_j} (u(t, \theta) - u(t_j, \theta)) (\mathrm{W}_t - \mathrm{W}_{t_j})\, dt = \frac{1}{2} \int_0^1 d(t, \theta)\, dt.$$

Then $\{ u(t, \theta), t \in [0, 1] \}$ *is Stratonovich integrable and*

$$\int_0^1 u(t, \theta) \circ d\mathrm{W}_t = \int_0^1 u(t, x)\, d\mathrm{W}_t \big|_{x=\theta} + \frac{1}{2} \int_0^1 d(t, \theta)\, dt. \quad (3.14)$$

If, in addition

(h7) *for any* $x \in \mathrm{B}$,

$$\mathrm{P}-\lim_{|\Pi|\downarrow 0} \sum_{j=0}^{n-1} \frac{1}{|\Delta_j|} \int_{\Delta_j} (u(t, x) - u(t_j, x)) (\mathrm{W}_t - \mathrm{W}_{t_j})\, dt = \frac{1}{2} \int_0^1 d(t, x)\, dt,$$

then $\{ u(t, x), t \in [0, 1] \}$ *is Stratonovich integrable for any* $x \in \mathrm{B}$ *and*

$$\int_0^1 u(t, x) \circ d\mathrm{W}_t \big|_{x=\theta} = \int_0^1 u(t, \theta) \circ d\mathrm{W}_t. \quad (3.15)$$

Remark. – Hypothesis (h6) [respectively (h7)] ensures the existence of the joint quadratic variation of the process $\{ u(t, \theta), t \in [0, 1] \}$ (respectively $\{ u(t, x), t \in [0, 1] \}$) and the Brownian motion W. These variations exist in the case where the processes $\{ u(t, \theta), t \in [0, 1] \}$ and $\{ u(t, x), t \in [0, 1] \}$ are $\mathscr{L}(\mathbb{K}, \mathbb{H})$-valued adapted continuous semimartingales. In particular if $u(t, x)$ has the integral representation

$$u(t, x) = u(0, x) + \int_0^t \mathrm{A}(s)\, d\mathrm{W}_s + \int_0^t \mathrm{B}(s)\, ds,$$

then hypothesis (h6) holds and the process $d(t, \theta)$ is given by

$$d(t, x) = \mathrm{Tr}\,[\mathrm{A}(t)\,\mathrm{Q}],$$

where we assume that $\mathrm{A}(t)$ is an integrable process taking values in the space $\mathscr{L}(\mathbb{H}, \mathscr{L}^1(\mathbb{K}, \mathbb{K})) \subset \mathscr{L}^2(\mathbb{K}, \mathscr{L}(\mathbb{K}, \mathbb{H}))$.

Proof. — We first decompose the Riemann sums corresponding to $\int_0^1 u(t, \theta) \circ d\mathrm{W}_t$ in the following way

$$\sum_{j=0}^{n-1} \frac{1}{|\Delta_j|} \left(\int_{\Delta_j} u(t, \theta)\, dt \right) \mathrm{W}(\Delta_j) = \mathrm{A}^\Pi(\theta) + \mathrm{F}^\Pi(\theta) + \mathrm{G}^\Pi(\theta)$$

with

$$\mathrm{A}^\Pi(\theta) = \sum_{j=0}^{n-1} u(t_j, \theta)\, \mathrm{W}(\Delta_j),$$

$$\mathrm{F}^\Pi(\theta) = \sum_{j=0}^{n-1} \frac{1}{|\Delta_j|} \int_{\Delta_j} (u(t, \theta) - u(t_j, \theta))\, (\mathrm{W}_{t_{j+1}} - \mathrm{W}_t)\, dt,$$

$$\mathrm{G}^\Pi(\theta) = \sum_{j=0}^{n-1} \frac{1}{|\Delta_j|} \int_{\Delta_j} (u(t, \theta) - u(t_j, \theta))\, (\mathrm{W}_t - \mathrm{W}_{t_j})\, dt.$$

By Lemma 3.1, $\mathrm{A}^\Pi(\theta) \to \int_0^1 u(t, x)\, d\mathrm{W}_t|_{x=\theta}$ in $\mathrm{L}^1(\Omega, \mathbb{H})$ as $|\Pi| \downarrow 0$. By Lemma 3.2, $\lim_{|\Pi| \downarrow 0} \mathrm{F}^\Pi(\theta) = 0$, in $\mathrm{L}^1(\Omega, \mathbb{H})$. Finally, hypothesis (h6) ensures the convergence of $\mathrm{G}^\Pi(\theta)$ to $\frac{1}{2} \int_0^1 d(t, \theta)\, dt$ in probability, as $|\Pi| \downarrow 0$. This proves (3.14).

If (h7) is also satisfied, it is clear that the process $\{u(t, x),\ t \in [0, 1]\}$ is Stratonovich integrable, for any $x \in \mathrm{B}$, and

$$\int_0^1 u(t, x) \circ d\mathrm{W}_t = \int_0^1 u(t, x)\, d\mathrm{W}_t + \frac{1}{2} \int_0^1 d(t, x)\, dt.$$

This shows (3.15) and finishes the proof of the theorem. ■

4. AN EXISTENCE THEOREM

The purpose of this section is to prove an existence theorem for the solution of the infinite dimensional anticipating stochastic differential equation (0.2). We will use some ideas developed in [10] for the finite dimensional case. That means, we will apply the substitution formula proved in Theorem 3.3. In order to check that the hypotheses (h1) to (h7) of this theorem are satisfied, some restrictive hypotheses on the coefficients have to be imposed. We will assume that

$$\sigma:\quad \mathbb{H} \to \mathscr{L}(\mathbb{K}, \mathbb{H}) \qquad \text{and} \qquad b:\quad \mathbb{H} \to \mathbb{H}$$

are functions such that σ is twice continuously differentiable and b is continuously differentiable. In the sequel ∇ will denote the gradient operator. That is, $\nabla \sigma(x)$ is an element of $\mathscr{L}(\mathbb{H}, \mathscr{L}(\mathbb{K}, \mathbb{H}))$, and $\nabla^2 \sigma(x)$ belongs to $\mathscr{L}(\mathbb{H} \otimes \mathbb{H}, \mathscr{L}(\mathbb{K}, \mathbb{H}))$. Along this section we will deal with the following conditions:

Lipschitz properties. – For any $x, y \in \mathbb{H}$,

(H 1) $\|b(x)-b(y)\| \leqq C_1 \|x-y\|, \qquad \|\sigma(x)-\sigma(y)\|_Q \leqq C_2 \|x-y\|;$

(H 2) $\|[(\nabla \sigma)(b)](x)-[(\nabla \sigma)(b)](y)\|_{\mathscr{L}(\mathbb{K}, \mathbb{H})} + \|[(\nabla^2 \sigma)(\sigma Q \sigma^*)](x) - [(\nabla^2 \sigma)(\sigma Q \sigma^*)](y)\|_{\mathscr{L}(\mathbb{K}, \mathbb{H})} + \|[(\nabla \sigma)(\sigma)](x) - [(\nabla \sigma)(\sigma)](y)\|_{\mathscr{L}(\mathbb{K}, \mathscr{L}^2_Q(\mathbb{K}, \mathbb{H}))} \leqq C\|x-y\|.$

Recall that $C_3 = [C_2 + \|\sigma(0)\|_Q]^2$.

We consider the stochastic differential equation on the Hilbert space $\mathbb{H}$

$$X_t(x) = x + \int_0^t \sigma(X_s(x))\, dW_s + \int_0^t b(X_s(x))\, ds, \tag{4.1}$$

$t \in [0, 1]$, $x \in \mathbb{H}$.

Notice that $\nabla \sigma Q^{1/2}$ belongs to $\mathscr{L}(\mathbb{H}, \mathscr{L}^2(\mathbb{K}, \mathbb{H}))$. Indeed, $(\nabla \sigma)(x)$ is a bounded operator from $\mathbb{H}$ in $\mathscr{L}(\mathbb{K}, \mathbb{H})$ and $Q^{1/2} \in \mathscr{L}^2(\mathbb{K}, \mathbb{K})$. Therefore, for any $y \in \mathbb{H}$ the composition $\nabla \sigma(x)(y) Q^{1/2}$ is a Hilbert-Schmidt operator from $\mathbb{K}$ in $\mathbb{H}$, and

$$\|\nabla \sigma(x) Q^{1/2}\|_{\mathscr{L}(\mathbb{H}, \mathscr{L}^2(\mathbb{K}, \mathbb{H}))} \leqq \|\nabla \sigma(x)\|_{\mathscr{L}(\mathbb{H}, \mathscr{L}(\mathbb{K}, \mathbb{H}))} \|Q^{1/2}\|_{\mathscr{L}^2(\mathbb{K}, \mathbb{K})}.$$

Moreover, $\sigma Q^{1/2}$ belongs to $\mathscr{L}^2(\mathbb{K}, \mathbb{H})$. Under these conditions, one can show that their composition $(\nabla \sigma Q^{1/2})^* (\sigma Q^{1/2})$ is an operator in the space $\mathscr{L}(\mathbb{H}, \mathscr{L}^1(\mathbb{K}, \mathbb{K}))$, and we can define its trace, which will be an element of $\mathbb{H}$. Define

$$\tilde{b}(x) = b(x) - \frac{1}{2} \operatorname{Tr}[(\nabla \sigma Q^{1/2})^* (\sigma Q^{1/2})](x). \tag{4.2}$$

Under the conditions stated before, it is obvious that equation (4.1) is a particular case of (2.1). Therefore, all the results obtained in Section 2 apply to the solution $X_t(x)$ of (4.1). In particular, Theorem 2.3 and Example 2.6 yield the existence of suitable bounded subsets B of $\mathbb{H}$ such that the process $X_t(x)$ is jointly continuous in $(t, x) \in [0, 1] \times B$. Set $u(t, x) = \sigma(X_t(x))$. Our first aim is to show that conditions (H 1) and (H 2) ensure the validity of hypotheses (h1) to (h5) of Section 3, with G equal to $\mathbb{H}$. It is clear that (h1), (h2) and (h4) hold. Hypothesis (h3) follows as a consequence of the Lipschitz property (H 1). The next lemma shows that hypothesis (h5) is also satisfied.

LEMMA 4.1. – *Assume that conditions* (H 1) *and* (H 2) *hold. Set* $u(t, x)=\sigma(X_t(x))$, $t\in[0, 1]$, $x\in\mathbb{H}$, *with* $X_t(x)$ *the solution of* (4.1). *Set*

$$V_{s,t}(x, y)=(u(s, x)-u(s, y))-(u(t, x)-u(t, y)). \tag{4.3}$$

Then for any constant $k>C_3/2$ *there exists* $p_0\geqq 2$ *such that*

$$E(\|V_{s,t}(x, y)\|_Q^p)\leqq \exp(kp^2)\|x-y\|^p|t-s|, \tag{4.4}$$

for all s, $t\in[0, 1]$, x, $y\in\mathbb{H}$ *and* $p\geqq p_0$.

Proof. – Fix $p\geqq 1$. We will denote by $\|\cdot\|_{Q\times Q}$ the norm in the space $\mathscr{L}_Q^2(\mathbb{K}, \mathscr{L}_Q^2(\mathbb{K}, \mathbb{H}))$. By means of the Itô formula we obtain

$$\begin{aligned}E(\|V_{s,t}(x, y)\|_Q^{2p})=&\,2pE\int_s^t\|V_{s,u}(x, y)\|_Q^{2p-2}\langle V_{s,u}(x, y), A_u(x, y)\rangle_Q\,du\\&+pE\int_s^t\|V_{s,u}(x, y)\|_Q^{2p-2}\|B_u(x, y)\|_{Q\times Q}^2\,du\\&+2p(p-1)E\int_s^t\|V_{s,u}(x, y)\|_Q^{2p-4}\\&\qquad\times\|\langle V_{s,u}(x, y), Q^{1/2}B_u(x, y)\rangle_Q\|_{\mathbb{K}}\,du,\end{aligned} \tag{4.5}$$

where,

$$\begin{aligned}A_u(x, y)=&\,[(\nabla\sigma)(b)](X_u(x))-[(\nabla\sigma)(b)](X_u(y))\\&+\frac{1}{2}((\nabla^2\sigma)[\sigma Q\sigma^*])(X_u(x))-\frac{1}{2}((\nabla^2\sigma)[\sigma Q\sigma^*])(X_u(y)),\end{aligned}$$

and

$$B_u(x, y)=[(\nabla\sigma)(\sigma)](X_u(x))-[(\nabla\sigma)(\sigma)](X_u(y)).$$

Condition (H 2) yields

$$\|A_u(x, y)\|_Q\leqq C\|X_u(x)-X_u(y)\|, \tag{4.6}$$

and

$$\|B_u(x, y)\|_{Q\times Q}^2\leqq C^2\|X_u(x)-X_u(y)\|^2. \tag{4.7}$$

Therefore, from (4.5), (4.6), (4.7) and Schwarz inequality we obtain:

$$\begin{aligned}E(\|V_{s,t}(x, y)\|_Q^{2p})\leqq&\,2pCE\int_s^t\|V_{s,u}(x, y)\|_Q^{2p-1}\|X_u(x)-X_u(y)\|\,du\\&+(2p^2-p)C^2E\int_s^t\|V_{s,u}(x, y)\|_Q^{2p-2}\|X_u(x)-X_u(y)\|^2\,du.\end{aligned}$$

Using (H 1) we have

$$\|V_{s,u}(x, y)\|_Q^q\leqq C_2^q 2^{q-1}\{\|X_s(x)-X_s(y)\|^q+\|X_u(x)-X_u(y)\|^q\}.$$

So we obtain

$$\begin{aligned}
\mathrm{E}(\|\mathrm{V}_{s,t}(x,y)\|_{\mathrm{Q}}^{2p} &\leqq \mathrm{C}_1(p)\,\mathrm{E}\int_s^t \{\|\mathrm{X}_s(x)-\mathrm{X}_s(y)\|^{2p-1} \\
&\quad\times \|\mathrm{X}_u(x)-\mathrm{X}_u(y)\| + \|\mathrm{X}_u(x)-\mathrm{X}_u(y)\|^{2p}\}\,du \\
&\quad + \mathrm{C}_2(p)\,\mathrm{E}\int_s^t \{\|\mathrm{X}_s(x)-\mathrm{X}_s(y)\|^{2p-2}\|\mathrm{X}_u(x)-\mathrm{X}_u(y)\|^2 \\
&\quad + \|\mathrm{X}_u(x)-\mathrm{X}_u(y)\|^{2p}\}\,du \\
&\leqq 2(\mathrm{C}_1(p)+\mathrm{C}_2(p)) \sup_{s\leqq u\leqq t} \mathrm{E}(\|\mathrm{X}_u(x)-\mathrm{X}_u(y)\|^{2p})\,|t-s|,
\end{aligned}$$

where $\mathrm{C}_1(p)=p\,2^{2p-1}\,\mathrm{C}\mathrm{C}_2^{2p-1}$ and $\mathrm{C}_2(p)=(2p^2-p)\,2^{2p-3}\,\mathrm{C}^2\,\mathrm{C}_2^{p-1}$. It suffices now to apply Proposition 2.1 [estimate (2.2)] to obtain the estimate (4.4). ■

Note that in the proof of Lemma 4.1 we have used the hypothesis (H 3) with the spaces $\mathscr{L}_{\mathrm{Q}}^2(\mathbb{K},\mathbb{H})$ and $\mathscr{L}_{\mathrm{Q}}^2(\mathbb{K},\mathscr{L}_{\mathrm{Q}}^2(\mathbb{K},\mathbb{H}))$ instead of $\mathscr{L}(\mathbb{K},\mathbb{H})$ and $\mathscr{L}(\mathbb{K},\mathscr{L}_{\mathrm{Q}}^2(\mathbb{K},\mathbb{H}))$, respectively. We want to apply Theorem 3.3 to the family of processes $\{\sigma(\mathrm{X}_t(x)),\ t\in[0,1],\ x\in\mathbb{H}\}$. The next lemma will imply the validity of hypothesis (h6).

LEMMA 4.2. – *Assume that conditions* (H1) *and* (H2) *are satisfied. Consider the process* $\{d(t,x),\ (t,x)\in[0,1]\times\mathbb{H}\}$ *defined by*

$$d(t,x)=\mathrm{Tr}\,[(\nabla\sigma\,\mathrm{Q}^{1/2})^*(\sigma\,\mathrm{Q}^{1/2})](\mathrm{X}_t(x)).$$

Let B *be a bounded subset of* $\mathbb{H}$ *which verifies the condition* (E_k) *for some constant* $k>\mathrm{C}_3$. *Then, for any* B-*valued random variable* θ, *the family of random variables*

$$\left(\sum_{j=0}^{n-1}\frac{1}{|\Delta_j|}\int_{\Delta_j}(\sigma(\mathrm{X}_s(\theta))-\sigma(\mathrm{X}_{t_j}(\theta))(\mathrm{W}_s-\mathrm{W}_{t_j})\,ds\right)$$

converges in probability to $\frac{1}{2}\int_0^1 d(t,\theta)\,dt$, *as* $\|\Pi|\downarrow 0$.

Proof. – Using the Itô formula we can write

$$\begin{aligned}
(\sigma(\mathrm{X}_s(x))-\sigma(\mathrm{X}_{t_j}(x)))(\mathrm{W}_s-\mathrm{W}_{t_j}) &= \int_{t_j}^s(\sigma(\mathrm{X}_u(x))-\sigma(\mathrm{X}_{t_j}(x)))\,d\mathrm{W}_u \\
+\int_{t_j}^s[(\nabla\sigma)(\sigma)](\mathrm{X}_u(x))(\mathrm{W}_u-\mathrm{W}_{t_j})\,d\mathrm{W}_u &+ \int_{t_j}^s[(\nabla\sigma)(b)](\mathrm{X}_u(x))(\mathrm{W}_u-\mathrm{W}_{t_j})\,du \\
+\frac{1}{2}\int_{t_j}^s[(\nabla^2\sigma)(\sigma\,\mathrm{Q}\,\sigma^*)]&(\mathrm{X}_u(x))(\mathrm{W}_u-\mathrm{W}_{t_j})\,du \\
&+\int_{t_j}^s\mathrm{Tr}\,[(\nabla\sigma\,\mathrm{Q}^{1/2})^*(\sigma\,\mathrm{Q}^{1/2})](\mathrm{X}_u(x))\,du.
\end{aligned}$$

Then, the proof of the lemma will be done in several steps.

Step 1. – The following convergence holds

$$\mathrm{E}\left\{\sup_{x\in \mathrm{B}}\left\|\sum_{j=0}^{n-1}\frac{1}{|\Delta_j|}\int_{\Delta_j}\left(\int_{t_j}^{s}[\sigma(\mathrm{X}_u(x))-\sigma(\mathrm{X}_{t_j}(x))]\,d\mathrm{W}_u\right)ds\right\|\right\}\to 0, \quad (4.8)$$

as $|\Pi|\downarrow 0$. Indeed, set as in (4.3),

$$\mathrm{V}_{t_j,u}(x,y)=(\sigma(\mathrm{X}_{t_j}(x))-\sigma(\mathrm{X}_{t_j}(y)))-(\sigma(\mathrm{X}_u(x))-\sigma(\mathrm{X}_u(y))).$$

Applying Fubini's theorem for the stochastic integral we can write

$$\int_{\Delta_j}\left(\int_{t_j}^{s}\mathrm{V}_{t_j,u}(x,y)\,d\mathrm{W}_u\right)ds=\int_{\Delta_j}(t_{j+1}-u)\,\mathrm{V}_{t_j,u}(x,y)\,d\mathrm{W}_u.$$

Hence, by Burkholder's and Hölder's inequalities, we get for any $p\geqq 2$,

$$\begin{aligned}&\mathrm{E}\left(\left\|\sum_{j=0}^{n-1}\frac{1}{|\Delta_j|}\int_{\Delta_j}\left(\int_{t_j}^{s}\mathrm{V}_{t_j,u}(x,y)\,d\mathrm{W}_u\right)ds\right\|^p\right)\\&\quad\leqq \mathrm{C}_p\,\mathrm{E}\left[\left(\int_0^1\left\|\sum_{j=0}^{n-1}\frac{1}{|\Delta_j|}(t_{j+1}-u)\,\mathbf{1}_{\Delta_j}(u)\,\mathrm{V}_{t_j,u}(x,y)\right\|_{\mathrm{Q}}^2du\right)^{p/2}\right]\\&\quad\leqq \mathrm{C}_p\,\mathrm{E}\left[\int_0^1\left(\sum_{j=0}^{n-1}\frac{t_{j+1}-u}{t_{j+1}-t_j}\mathbf{1}_{\Delta_j}\|\mathrm{V}_{t_j,u}(x,y)\|_{\mathrm{Q}}\right)^p du\right]\\&\quad\leqq \mathrm{C}_p\sup_{0\leqq j\leqq n-1}\ \sup_{u\in\Delta_j}\mathrm{E}(\|\mathrm{V}_{t_j,u}(x,y)\|_{\mathrm{Q}}^p),\end{aligned}\quad (4.9)$$

where $C_p=(e/2)^{p/2}p^p$. By Lemma 4.1 the right hand side of (4.9) is bounded by $\exp(kp^2)\|x-y\|^p|\Pi|$, for any $p\geqq p_0$, x, $y\in\mathbb{H}$, provided $k>\mathrm{C}_3/2$.

On the other hand, for any fixed $x\in\mathbb{H}$, we have

$$\lim_{n\to\infty}\mathrm{E}\left(\left\|\sum_{j=0}^{n-1}\frac{1}{|\Delta_j|}\int_{\Delta_j}\left(\int_{t_j}^{s}[\sigma(\mathrm{X}_u(x))-\sigma(\mathrm{X}_{t_j}(x))]\,d\mathrm{W}_u\right)ds\right\|^2\right)=0. \quad (4.10)$$

In fact, the arguments used in the proof of (4.9), with $p=2$, show that

$$\begin{aligned}&\mathrm{E}\left(\left\|\sum_{j=0}^{n-1}\frac{1}{|\Delta_j|}\int_{\Delta_j}\left(\int_{t_j}^{s}[\sigma(\mathrm{X}_u(x))-\sigma(\mathrm{X}_{t_j}(x))]\,d\mathrm{W}_u\right)ds\right\|^2\right)\\&\qquad\leqq \mathrm{C}(p)\sup_{0\leqq j\leqq n-1}\ \sup_{u\in\Delta_j}\mathrm{E}(\|\sigma(\mathrm{X}_u(x))-\sigma(\mathrm{X}_{t_j}(x))\|_{\mathrm{Q}}^2),\end{aligned}\quad (4.11)$$

and the right hand side of (4.11) tends to zero as $|\Pi|\downarrow 0$, since the process $\{\sigma(\mathrm{X}_t(x)),\ t\in[0,1]\}$ is continuous in $\mathrm{L}^2([0,1]\times\Omega;\ \mathscr{L}_{\mathrm{Q}}^2(\mathbb{K},\mathbb{H}))$. The results given in (4.9) and (4.10) and the assumption on the set B, allow us to apply Proposition 1.5 to any sequence $\{\mathrm{A}^{\Pi_n}\}$, with $|\Pi_n|\downarrow 0$, where

$$A^{\Pi}=\sum_{j=0}^{n-1}\frac{1}{|\Delta_j|}\int_{\Delta_j}\left(\int_{t_j}^{s}[\sigma(X_u(x))-\sigma(X_{t_j}(x))]\,dW_u\right)ds.$$

This completes the proof of (4.8).

Step 2. – It holds that

$$\lim_{|\Pi|\downarrow 0} E\left\{\sup_{x\in B}\left\|\sum_{j=0}^{n-1}\frac{1}{|\Delta_j|}\int_{\Delta_j} ds \times\int_{t_j}^{s}\nabla\sigma(X_u(x))\,\sigma(X_u(x))(W_u-W_{t_j})\,dW_u\right\|\right\}=0. \quad (4.12)$$

Indeed, set for $u\in[0,1]$ and $x, y\in\mathbb{H}$,

$$Z_u(x, y)=[(\nabla\sigma)(\sigma)](X_u(x))-[(\nabla\sigma)(\sigma)](X_u(y)).$$

Then, analogous arguments as those used in the proof of (4.9) show that, for any $p\geqq 2$,

$$\begin{aligned}E&\left[\left\|\sum_{j=0}^{n-1}\frac{1}{|\Delta_j|}\int_{\Delta_j}ds\int_{t_j}^{s}Z_u(x, y)(W_u-W_{t_j})\,dW_u\right\|^p\right]\\ &\leqq C_p E\left[\left(\int_0^1\left\|\sum_{j=0}^{n-1}\frac{1}{|\Delta_j|}(t_{j+1}-u)\,\mathbf{1}_{\Delta_j}(u)(Z_u(x, y))(W_u-W_{t_j})\right\|_Q^2 du\right)^{p/2}\right]\\ &\leqq C_p\sup_{0\leqq j\leqq n-1}\sup_{u\in\Delta_j}E(\|(Z_u(x, y))(W_u-W_{t_j})\|_Q^p), \quad (4.13)\end{aligned}$$

where $C_p=(e/2)^{p/2}p^p$. Let $\{h_i, i\geqq 1\}$ be a complete orthonormal system in $\mathbb{K}$. Then using Hölder's inequality and condition (H 2) we have

$$\begin{aligned}E&(\|(Z_u(x, y))(W_u-W_{t_j})\|_Q^p)\\ &\leqq E(\|(Z_u(x, y)\|^p_{\mathscr{L}(\mathbb{K}, \mathscr{L}_Q^2(\mathbb{K}, \mathbb{H}))}\|(W_u-W_{t_j})\|_{\mathbb{K}}^p)\\ &\leqq C^{2p}[E\|X_u(x)-X_u(y)\|^{2p}]^{1/2}(u-t_j)^{p/2}\left(E\left(\sum_{i=1}^{\infty}\xi_i^2 Q(h_i, h_i)\right)^p\right)^{1/2}\\ &\leqq C^{2p}[E\|X_u(x)-X_u(y)\|^{2p}]^{1/2}\\ &\quad\times(u-t_j)^{p/2}(\operatorname{Tr} Q)^{p/2}(u-t_j)^{p/2}(\lambda_{2p})^{1/2}, \quad (4.14)\end{aligned}$$

where λ_p is the p-th moment of the absolute value of a standard normal variable, and $\{\xi_i, i\geqq 1\}$ is a sequence of independent $N(0, 1)$ random variables. By Proposition 2.1 [estimate (2.2)] the right hand side of (4.14) is bounded above by

$$e^{2kp^2}\|x-y\|^p|\Pi|^{p/2},$$

provided $k > C_3/2$. Therefore any sequence of $\mathbb{H}$-valued random variables $\{B^{\Pi_n}\}$ where

$$B^{\Pi} = \sum_{j=0}^{n-1} \frac{1}{|\Delta_j|} \int_{\Delta_j} ds \int_{t_j}^{s} \nabla\sigma(X_u(x))\,\sigma(X_u(x))(W_u - W_{t_j})\,dW_u,$$

satisfies the estimate

$$E(\| B^{\Pi_n}(x) - B^{\Pi_n}(y) \|^p) \leq e^{kp^2} \| x-y \|^p |\Pi_n|, \tag{4.15}$$

provided $k > C_3$. Furthermore, for $x \in \mathbb{H}$ fixed, $\lim_{n\to\infty} E(\| B^{\Pi_n}(x) \|^2) = 0$. Therefore we can apply Proposition 1.5 to complete the proof of this step.

Step 3. — For any $x \in \mathbb{H}$, set

$$M_u(x) = [(\nabla\sigma)(b)](X_u(x)) + \frac{1}{2}[(\nabla^2\sigma)(\sigma Q \sigma^*)](X_u(x)).$$

Then for any B-valued random variable θ it holds that

$$\lim_{|\Pi|\downarrow 0} E\left\{ \left\| \sum_{j=0}^{n-1} \frac{1}{|\Delta_j|} \int_{\Delta_j} ds \left(\int_{t_j}^{s} M_u(\theta)(W_u - W_{t_j})\,du \right) \right\|^2 \right\} = 0. \tag{4.16}$$

Indeed, Jensen's inequality yields

$$E\left(\left\| \sum_{j=0}^{n-1} \frac{1}{|\Delta_j|} \int_{\Delta_j} ds \left(\int_{t_j}^{s} M_u(\theta)(W_u - W_{t_j})\,du \right) \right\|^2 \right)$$
$$\leq \int_0^1 \sum_{j=0}^{n-1} \frac{s-t_j}{|\Delta_j|^2} \left(\int_{t_j}^{s} E(\| M_u(\theta)(W_u - W_{t_j}) \|^2)\,du \right) 1_{\Delta_j}(s)\,ds.$$

By Schwarz's inequality, this expression is bounded by

$$\left(\int_0^1 \sum_{j=0}^{n-1} \frac{s-t_j}{|\Delta_j|^2} \int_{t_j}^{s} (E\| W_u - W_{t_j} \|^4))^{1/2}\, 1_{\Delta_j}(s)\,ds \right)$$
$$\times \sup_{0\leq j\leq n-1} \sup_{u\in\Delta_j} (E(\| M_u(\theta) \|^4_{\mathscr{L}(\mathbb{K},\mathbb{H})})^{1/2}$$
$$\leq \frac{\sqrt{3}}{8} (\operatorname{Tr} Q) |\Pi| \sup_{0\leq j\leq n-1} \sup_{u\in\Delta_j} (E(\| M_u(\theta) \|^4_{\mathscr{L}(\mathbb{K},\mathbb{H})})^{1/2}. \tag{4.17}$$

The Lipschitz hypotheses (H 2) ensures that

$$E(\| M_u(\theta) \|^4_{\mathscr{L}(\mathbb{K},\mathbb{H})}) \leq 8(C^4 E(\| \theta \|^4) + \| M_u(0) \|^4).$$

Therefore, the convergence (4.16) holds.

Step 4. — For any B-valued random variable θ it holds that

$$\sum_{j=0}^{n-1} \frac{1}{|\Delta_j|} \int_{\Delta_j} ds \left(\int_{t_j}^{s} \operatorname{Tr}[(\nabla\sigma Q^{1/2})^* (\sigma Q^{1/2})](X_u(\theta))\,du \right), \tag{4.18}$$

converges a. s., as $|\Pi|\downarrow 0$ to $\frac{1}{2}\int_0^1 d(t, \theta)\,dt$, where d has been defined in Lemma 4.2.

Indeed, the expression (4.18) can be written, using Fubini's theorem as

$$\int_0^1 \sum_{j=0}^{n-1} \frac{1}{|\Delta_j|}(t_{j+1}-u)\,d(u, \theta)\,\mathbf{1}_{\Delta_j}(u)\,du.$$

The functions $\sum_{j=0}^{n-1} \frac{1}{|\Delta_j|}(t_{j+1}-u)\,1_{\Delta_j}(u)$ converge to $\frac{1}{2}$ in the weak topology $\sigma(L^2([0, 1]), L^2([0, 1]))$ as the norm of the partition tends to zero. Notice that, hypothesis (H 2) yields

$$E\left(\int_0^1 \|d(t, \theta)\|^2\,dt\right) \leqq K\,(1+E(\sup_{(x, t)\in[0, 1]\times B} \|X_t(x)\|^2)),$$

for some positive constant K. This last quantity is finite, due to the properties of the set B, Proposition 2.1, Lemma 1.2 and Theorem 1.1. Consequently, $d(., \theta)$ belongs to $L^2([0, 1]; \mathbb{H})$, a. s., and the result follows by weak convergence. The lemma is now completely proved. ■

Remark 4.3. – Let d be the process defined in Lemma 4.2. Assume that conditions (H 1) and (H 2) are satisfied. Then, for any $x\in\mathbb{H}$,

$$P-\lim_{|\Pi|\downarrow 0} \sum_{j=0}^{n-1} \frac{1}{|\Delta_j|}\int_{\Delta_j} (\sigma(X_s(x))-\sigma(X_{t_j}(x)))$$
$$\times (W_s-W_{t_j})\,ds = \frac{1}{2}\int_0^1 d(t, x)\,dt.$$

Hence (h8) holds. Indeed, this convergence follows by the arguments developed in the proof of Lemma 4.2.

We can now state an existence theorem for the anticipating stochastic differential equation (0.2).

THEOREM 4.4. – *Assume that* $\sigma:\mathbb{H}\to\mathscr{L}(\mathbb{K}, \mathbb{H})$ *and* $b:\mathbb{H}\to\mathbb{H}$ *are functions satisfying conditions* (*H* 1) *and* (*H* 2). *Let* B *be a bounded subset of* $\mathbb{H}$ *which verifies condition* (E_k) *for some constant* $k>C_3$. *Then for any* B-*valued random variable* Y_0, *the process* $\{Y_t=X_t(Y_0), t\in[0, 1]\}$, *with* $\{X_t(x), t\in[0, 1], x\in\mathbb{H}\}$ *given by* (4.1), *is a solution of the Stratonovich anticipating stochastic differential equation*

$$Y_t=Y_0+\int_0^t \sigma(Y_s)\circ dW_s+\int_0^t \tilde{b}(Y_s)\,ds, \qquad (4.19)$$

where $\tilde{b}$ *is given by* (4.2).

Proof. — The results proved so far show that, under conditions (H 1) and (H 2) the stochastic differential equation (4.1) can also be written in the Stratonovich form, *i.e.*

$$X_t(x) = x + \int_0^t \sigma(X_s(x)) \circ dW_s + \int_0^t \tilde{b}(X_s(x))\, ds.$$

We know, by Theorem 2.3, that $\{X_t(x), (t, x) \in [0, 1] \times K\}$ is jointly continuous. The process $u(t, x) = \sigma(X_t(x))$, $t \in [0, 1]$, $x \in B$ satisfies hypotheses (h1) to (h5) of Section 2. Moreover, Lemma 4.2 implies that for any random variable θ taking values on B, hypothesis (h6) of Theorem 3.3 is satisfied with $d(t, x) = \mathrm{Tr}\,[(\nabla \sigma Q^{1/2})(\sigma Q^{1/2})](x)$. On the other hand, hypotheses (h6) and (h7) of Theorem 3.3 are also satisfied by the set B and the process $u(t, x) = \sigma(X_t(x))$, due to Lemma 4.2 and Remark 4.3. Consequently, Theorem 3.3 yields

$$\int_0^t \sigma(X_s(Y_0)) \circ dW_s = \int_0^t \sigma(X_s(x)) \circ dW_s \big|_{x = Y_0},$$

for any $t \in [0, 1]$. This completes the proof of the Theorem. ■

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