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Covariance inequalities for strongly mixing processes

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ABSTRACT. — Let X and Y be two real-valued random variables. Let α denote the strong mixing coefficient between the two σ-fields generated respectively by X and Y, and $Q_X(u) = \inf\{t : \mathbb{P}(|X| > t) \leq u\}$ be the quantile function of $|X|$. We prove the following new covariance inequality:

$$|\operatorname{Cov}(X, Y)| \leq 2 \int_0^{2\alpha} Q_X(u)\, Q_Y(u)\, du,$$

which we show to be sharp, up to a constant factor. We apply this inequality to improve on the classical bounds for the variance of partial sums of strongly mixing processes.

Key words : Strongly mixing processes, covariance inequalities, quantile transformation, maximal correlation, stationary processes.

RÉSUMÉ. — Soient X et Y deux variables aléatoires réelles. Notons α le coefficient de mélange fort entre les deux tribus respectivement engendrées par X et Y. Soit $Q_X(u) = \inf\{t : \mathbb{P}(|X| > t) \leq u\}$ la fonction de quantile de $|X|$. Nous établissons ici l'inégalité de covariance suivante :

$$|\operatorname{Cov}(X, Y)| \leq 2 \int_0^{2\alpha} Q_X(u)\, Q_Y(u)\, du,$$

et nous montrons son optimalité, à un facteur constant près. Cette inégalité est ensuite appliquée à la majoration de la variance d'une somme de variables aléatoires d'un processus mélangeant.

Classification A.M.S. : 60 F 05, 60 F 17.

1. INTRODUCTION AND RESULTS

Let $(\Omega, \mathscr{T}, \mathbb{P})$ be a probablility space. Given two σ-fields $\mathscr{A}$ and $\mathscr{B}$ in $(\Omega, \mathscr{T}, \mathbb{P})$, the strong mixing coefficient $\alpha(\mathscr{A}, \mathscr{B})$ is defined by

$$\alpha(\mathscr{A}, \mathscr{B}) = \sup_{(A, B) \in \mathscr{A} \times \mathscr{B}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| = \sup_{(A, B) \in \mathscr{A} \times \mathscr{B}} |\operatorname{Cov}(\mathbf{1}_A, \mathbf{1}_B)|$$

[notice that $\alpha(\mathscr{A}, \mathscr{B}) \leqq 1/4$]. This coefficient gives an evaluation of the dependance between $\mathscr{A}$ and $\mathscr{B}$.

The problem of majorizing the covariance between two real-valued r.v.'s X and Y with given marginal distributions and given strong mixing coefficient was first studied by Davydov (1968). He proved that, for any positive reals p, q, and r such that $1/p + 1/q + 1/r = 1$,

$$(1.0) \qquad |\operatorname{Cov}(X, Y)| \leqq C[\alpha(\sigma(X), \sigma(Y))]^{1/p} [\mathbb{E}|X|^q]^{1/q} [\mathbb{E}|Y|^r]^{1/r},$$

where $\sigma(X)$ denotes the σ-field generated by X. Davydov obtained $C = 12$ in (1.0).

Davydov's inequality has the following known application to the control of the variance of partial sums of strongly mixing arrays of real-valued random variables. Let $(X_i)_{i \in \mathbb{Z}}$ be a weakly stationary array of zero-mean real-valued r.v.'s [*i.e.* $\operatorname{Cov}(X_s, X_t) = \operatorname{Cov}(X_0, X_{t-s})$ for any s and any t in $\mathbb{Z}^d$]. For any $n \in \mathbb{Z}^d$, we define a strong mixing coefficient α_n by

$$\alpha_n = \sup_{i \in \mathbb{Z}^d} \alpha(\sigma(X_i), \sigma(X_{i+n})),$$

where $\sigma(X_i)$ denotes the σ-field generated by X_i. We shall say that the array $(X_i)_{i \in \mathbb{Z}}$ is strongly mixing iff $\lim_{|n| \to +\infty} \alpha_n = 0$. Then inequality (1.0) yields the following result.

THEOREM 1.0 (Davydov). – *Let $d \geqq 1$ and let $(X_i)_{i \in \mathbb{Z}}$ be a weakly stationary array of real-valued random variables. Suppose that* $\bigvee_{i \in \mathbb{Z}^d} \mathbb{E}|X_i|^r = M_r < +\infty$ *for some $r > 2$. Let* $S_n = \sum_{i \in]0, n]^d} X_i$*: then*

$$n^{-d} \operatorname{Var} S_n \leqq 2 C M_r \sum_{i \in]-n, n[^d} \alpha_i^{1-2/r}.$$

Under the additional assumption $\sum_{i \in \mathbb{Z}^d} \alpha_i^{1-2/r} < +\infty$, *the series* $\sum_{t \in \mathbb{Z}^d} \operatorname{Cov}(X_0, X_t)$ *is absolutely convergent, has a nonnegative sum* σ^2, *and* $\lim_{n \to +\infty} n^{-d} \operatorname{Var} S_n = \sigma^2$.

Up to now, inequality (1.0) and his corollaries were the main tool for studying mixing processes. We have in view to improve on Davydov's inequality. Let $\mathscr{L}_\alpha(F, G)$ denote the class of bivariate r.v.'s (X, Y) with given marginal distributions functions F and G satisfying the mixing constraint $\alpha(\sigma(X), \sigma(Y)) \leqq \alpha$. Let $F^{-1}(u) = \inf\{t : F(t) \geqq u\}$ denote the usual inverse function of F. In order to maximize Cov(X, Y) over the class $\mathscr{L}_\alpha(F, G)$, it is instructive to look at the extremal case $\alpha = 1/4$ (that is, to relax the mixing constraint). In that case, M. Fréchet (1951, 1957) proved that the maximum of Cov(X, Y) is obtained when $(X, Y) = (F^{-1}(U), G^{-1}(U))$, where U is uniformly distributed over [0, 1] (actually, Fréchet gives a complete proof of this result only when F and G are continuous). In other words, we have:

$$(1.1)\qquad \sup_{(X, Y) \in \mathscr{L}_{1/4}(F, G)} \operatorname{Cov}(X, Y) = \int_0^1 F^{-1}(u) G^{-1}(u)\, du - \int_0^1 F^{-1}(u)\, du \int_0^1 G^{-1}(u)\, du.$$

In view of (1.1), one may think that the maximum of the covariance function over $\mathscr{L}_\alpha(F, G)$ should depend on α, F^{-1} and G^{-1}, rather than on the moments of X and Y. Unfortunately, the exact maximum has a more complicated form in the general case than in the extremal case $\alpha = 1/4$. However, we can provide an upper bound for $|\operatorname{Cov}(X, Y)|$, which is optimal, up to a constant factor.

THEOREM 1.1. – *Let* X *and* Y *be two integrable real-valued* r.v.'*s. Let* $\alpha = \alpha(\sigma(X), \sigma(Y))$. *Let* $Q_X(u) = \inf\{t : \mathbb{P}(|X| > t) \leqq u\}$ *denote the quantile function of* $|X|$. *Assume furthermore that* $Q_X Q_Y$ *is integrable on* [0, 1]. *Then*

$$(a)\qquad |\operatorname{Cov}(X, Y)| \leqq 2 \int_0^{2\alpha} Q_X(u) Q_Y(u)\, du.$$

Conversely, for any symmetric law with distribution function F, *and any* $\alpha \in]0, 1/4]$, *there exists two random variables* X *and* Y *with common distribution function* F, *satisfying the strong mixing condition* $\alpha(\sigma(X), \sigma(Y)) \leqq \alpha$ *and such that*

$$(b)\qquad \operatorname{Cov}(X, Y) \geqq \frac{1}{2} \int_0^{2\alpha} (Q_X(u))^2\, du.$$

Remarks. – Using the same tools as in the proof of inequality (*a*), one can prove the following inequality:

$$(1.2)\quad |\operatorname{Cov}(X, Y)| \leqq \int_0^\alpha (F^{-1}(1-u) - F^{-1}(u))(G^{-1}(1-u) - G^{-1}(u))\, du.$$

Inequality (1.2) is more intrinsic than inequality (*a*), for the upper bound in (1.2) depends only on the "dispersion function" $(s, t) \to F^{-1}(t)$

$-F^{-1}(s)$ of X and on the dispersion function of Y. However, inequality (*a*) is more tractable for the applications.

Theorem 1.1 implies (1.0) with $C=2^{1+1/p}$, which improves on Davydov's constant (note that, when U is uniformly distributed over [0, 1], $Q_X(U)$ has the distribution of $|X|$, and apply Hölder inequality).

The assumptions of moment on the r.v.'s X and Y in Davydov's covariance inequality can be weakened as follows. Assume that $\mathbb{P}(|X|>u)\leqq[C_X(q)/u]^q$ and $\mathbb{P}(|Y|>u)\leqq[C_Y(r)/u]^r$. Then, it follows from Theorem 1.1 that

$$(1.3)\qquad |\operatorname{Cov}(X, Y)| \leqq 2p.(2\alpha)^{1/p} C_X(q)\, C_Y(r).$$

Of course $\|X\|_q \geqq C_X(q)$ by Markov's inequality. Hence, we obtain a similar inequality under weaker assumptions on the distribution functions of X and Y than Davydov's one. We now derive from Theorem 1.1 the following result, which improves on Theorem 1.0.

THEOREM 1.2. — *Let* $(X_i)_{i \in \mathbb{Z}}$ *be an array of real-valued random variables. Define* $\alpha^{-1}(t) = \sum_{i \in \mathbb{Z}^d} \mathbf{1}_{(\alpha_i > t)}$. *For any positive integer n, let* $\bar{Q}_n$ *denote the nonnegative quantile function defined by:*

$$[\bar{Q}_n]^2 = n^{-d} \sum_{i \in]0,\, n]^d} [Q_{X_i}]^2.$$

Then,

$$(a)\quad n^{-d} \operatorname{Var} S_n \leqq n^{-d} \sum_{s \in]0,\, n]^d} \sum_{t \in]0,\, n]^d} |\operatorname{Cov}(X_s, X_t)| \leqq 4 \int_0^1 (\alpha^{-1}(u) \wedge n^d)\, [\bar{Q}_n(2u)]^2\, du.$$

Moreover, if $(X_i)_{i \in \mathbb{Z}}$ *is weakly stationary and if*

$$(1.4)\qquad \bigvee_{n>0} \left[\int_0^1 (\alpha^{-1}(u) \wedge n^d)\, [\bar{Q}_n(2u)]^2\, du\right] \leqq M < +\infty,$$

then,

$$(b)\qquad \sum_{t \in \mathbb{Z}^d} |\operatorname{Cov}(X_0, X_t)| \leqq 4M,$$

and denoting by σ^2 *the sum of the series* $\sum_{t \in \mathbb{Z}^d} \operatorname{Cov}(X_0, X_t)$, *we have:*

$$(c)\qquad \lim_{n \to +\infty} n^{-d} \operatorname{Var} S_n = \sigma^2 \quad \text{and} \quad \sigma^2 \leqq 4M.$$

In particular, if $(X_i)_{i \in \mathbb{Z}}$ *is a strictly stationary array, then* $\bar{Q}_n = Q_{X_0} = Q$, *and so, if*

$$\int_0^1 \alpha^{-1}(u)\,[Q(2u)]^2\,du < +\infty, \tag{1.5}$$

then, (*b*) *and* (*c*) *hold with* $M = \int_0^1 \alpha^{-1}(u)\,[Q(2u)]^2\,du$.

Remark. — In a joint paper with P. Doukhan and P. Massart (1992), we prove that the functional Donsker-Prohorov invariance principle holds for a strictly stationary sequence if a condition related to (1.5) is fulfilled.

Applications. — Let $r > 2$. If the tail functions of the r.v.'s X_i are uniformly bounded as follows: $\mathbb{P}(|X_i| > u) \leq (C_r/u)^r$ for any positive u and any $i \in \mathbb{Z}^d$. Then,

$$\int_0^1 \alpha^{-1}(u)\,[Q(2u)]^2\,du \leq C \sum_{k \in \mathbb{K}^d} \alpha_k^{1-2/r}$$

for some constant C depending on r and C_r. Hence the conclusions of Theorem 1.0 are ensured by a weaker condition on the d.f.'s of the r.v.'s X_i than Davydov's one $\bigvee_{i \in \mathbb{Z}^d} \mathbb{E}|X_i|^r < +\infty$ [this is not surprising in view of (1.3)].

Set-indexed partial sum processes. — Let $(X_i)_{i \in \mathbb{Z}}$ be a strongly mixing array of identically distributed r.v.'s satisfying condition (1.5). Let $A \subset [0, 1]^d$ be a Borel set and let

$$S_n(A) = \sum_{i \in \mathbb{Z}^d} \lambda([i-\mathbf{1}, i] \cap nA)\,X_i,$$

where $[i-\mathbf{1}, i]$ denotes the unit cube with upperright vertice i and λ denotes the Lebesgue measure. Then, we can derive from (*a*) of Theorem 1.2 the following upper bound:

$$n^{-d}\,\mathrm{Var}\,S_n(A) \leq 4\lambda(A) \int_0^1 \alpha^{-1}(u)\,[Q(2u)]^2\,du.$$

[Apply (*a*) of Theorem 1.2 to the array $(Y_i)_{i \in \mathbb{Z}}$ defined by $Y_i = \lambda([i-\mathbf{1}, i] \cap nA)\,X_i$].

We now study the applications of Theorem 1.2 to arrays of r.v.'s satisfying moment constraints. So, we consider the class of functions

$$\mathcal{F} = \left\{ \phi : \mathbb{R}^+ \to \mathbb{R}^+ : \phi \text{ convex, increasing and differentiable, } \phi(0) = 0,\ \lim_{+\infty} \frac{\phi(x)}{x} = \infty \right\}.$$

and, for any $\phi \in \mathcal{F}$, we define the dual function ϕ^* by $\phi^*(y) = \sup_{x>0} [xy - \phi(x)]$. When the Cesaro means of the ϕ-moments of the random variables X_i^2 are uniformly bounded, Theorem 1.2 yields the following result.

COROLLARY 1.2. – *Let* $(X_i)_{i \in \mathbb{Z}}$ *be a stongly mixing array of real-valued random variables. Let* ϕ *be some element of* $\mathcal{F}$ *such that* $\mathbb{E}(\phi(X_i^2)) < +\infty$ *for any* $i \in \mathbb{Z}^d$, *and assume furthermore that the mixing quantile function satisfies*

$$(1.6) \qquad \int_0^1 \phi^*(\alpha^{-1}(u))\, du < +\infty.$$

Then,

$$(a) \qquad n^{-d} \operatorname{Var} S_n \leqq 4 \left[n^{-d} \sum_{i \in]0, n]^d} \mathbb{E}(\phi(X_i^2)) + \int_0^1 \phi^*(\alpha^{-1}(u))\, du \right].$$

Moreover, if $(X_i)_{i \in \mathbb{Z}}$ *is weakly stationary and if*

$$(1.7) \qquad \bigvee_{n>0} [n^{-d} \sum_{i \in]0, n]^d} \mathbb{E}(\phi(X_i^2))] = M_\phi < \infty,$$

then,

$$(b) \qquad \sum_{t \in \mathbb{Z}^d} |\operatorname{Cov}(X_0, X_t)| \leqq 4 [M_\phi + \int_0^1 \phi^*(\alpha^{-1}(u))\, du],$$

and denoting by σ^2 *the sum of the series* $\sum_{t \in \mathbb{Z}^d} \operatorname{Cov}(X_0, X_t)$, *we have:*

$$(c) \quad \lim_{n \to +\infty} n^{-d} \operatorname{Var} S_n = \sigma^2 \qquad \text{and} \qquad \sigma^2 \leqq 4 \left[M_\phi + \int_0^1 \phi^*(\alpha^{-1}(u))\, du \right].$$

Applications. – Suppose that $(X_i)_{i \in \mathbb{Z}}$ is a weakly stationary array satisfying (1.6). Then, $\lim_{|k| \to +\infty} \alpha_k = 0$. Hence, there exists some one to one mapping π from $\mathbb{N}^*$ onto $\mathbb{Z}^d$ such that, for any integer k, $\alpha_{\pi(k+1)} \leqq \alpha_{\pi(k)}$. Let $\alpha_{(k)} = \alpha_{\pi(k)}$. An elementary calculation shows that (1.6) holds if

$$(1.8) \qquad \sum_{k>0} (\phi')^{-1}(k)\, \alpha_{(k)} < +\infty.$$

where $(\phi')^{-1}$ denotes the inverse function of ϕ'. In their note, Bulinskii and Doukhan (1987) obtained similar upper bounds for the variance of sums of Hilbert-valued r.v.'s under the assumption

$$(1.9) \qquad \sum_{k>0} \phi^{-1}(1/\alpha_{(k)})\, \alpha_{(k)} < +\infty.$$

[apply Theorem 2, p. 828, with $p=2$ and $\phi_i(t)=\phi(t^2)$]. Let us now compare this result with (1.8): (1.9) implies (1.8) if, for any large enough k, $\phi^{-1}(1/\alpha_{(k)}) \geqq (\phi')^{-1}(k)$, which is equivalent to the condition

$$(1.10) \qquad (\phi^{-1})'(1/\alpha_{(k)}) \leqq 1/k.$$

Since ϕ^{-1} is a concave function, (1.10) holds if $\alpha_{(k)}\phi^{-1}(1/\alpha_{(k)}) \leqq 1/k$. Now, by the monotonicity of the sequence $(\alpha_{(k)})_{k>0}$, the convergence of the series in (1.9) implies $\lim_{k \to +\infty} k\alpha_{(k)}\phi^{-1}(1/\alpha_{(k)})=0$, therefore establishing (1.10). Hence, in the special case of real-valued r.v.'s, our result implies the corresponding result of Bulinskii and Doukhan. In particular, when $\phi(x)=x^{r/2}$ for some $r>2$, (1.8) holds iff the serie $\sum_{k>0} k^{2/(r-2)}\alpha_{(k)}$ is convergent while Theorem 1.0 of Davydov or condition (1.9) of Bulinskii and Doukhan need $\sum_{k>0} \alpha_{(k)}^{1-2/r} < \infty$. For example, when $d=1$ and $\alpha_n = O(n^{-r/(r-2)}(\log n)^{-\theta})$ for some $\theta>0$ (notice that $r/(r-2)$ is the critical exponent) this condition holds for any $\theta>1$ while Theorem 1.0 or (1.9) need $\theta>r/(r-2)$, which shows that Corollary 1.2 improves on the corresponding results of Davydov or Bulinskii and Doukhan.

Geometrical rates of mixing. – Let $(X_i)_{i \in \mathbb{Z}}$ be a weakly stationary sequence satisfying the mixing condition $\alpha_k = O(a^k)$ for some a in $]0, 1[$. Then there exists some $s>0$ such that (1.6) holds with $\phi^*(x)=\exp(sx)-sx-1$. Since $\phi=(\phi^*)^*$, condition (1.7) holds if

$$\bigvee_{n>0} \left[n^{-d} \sum_{i \in]0, n]^d} \mathbb{E}(X_i^2 \log^+ |X_i|) \right] < +\infty.$$

The organization of the paper is as follows: in section 2, we prove the main covariance inequality. Next, in section 3, we prove Theorem 1.2 and Corollary 1.2.

2. COVARIANCE INEQUALITIES FOR STRONGLY MIXING r.v.'s

Proof of (a) of Theorem 1.1. – Let $X^+ = \sup(0, X)$ and $X^- = \sup(0, -X)$. Clearly,

$$(2.1) \quad \operatorname{Cov}(X, Y) = \operatorname{Cov}(X^+, Y^+) + \operatorname{Cov}(X^-, Y^-) - \operatorname{Cov}(X^-, Y^+) - \operatorname{Cov}(X^+, Y^-).$$

A classical calculation shows that

$$\operatorname{Cov}(X^+, Y^+) = \iint_{\mathbb{R}_+^2} [\mathbb{P}(X>u, Y>v) - \mathbb{P}(X>u)\,\mathbb{P}(Y>v)]\, du\, dv.$$

Now, the strong mixing condition implies:

$$|\mathbb{P}(X>u, Y>v) - \mathbb{P}(X>u)\mathbb{P}(Y>v)| \leqq \inf(\alpha, \mathbb{P}(X>u), \mathbb{P}(Y>v)).$$

Let $\Phi_X(u) = \mathbb{P}(X>u)$. It follows that

$$(2.2) \qquad |\operatorname{Cov}(X^+, Y^+)| \leqq \iint_{\mathbb{R}_+^2} \inf(\alpha, \Phi_X(u), \Phi_Y(v))\, du dv.$$

Apply then (2.1), (2.2) and the elementary inequality

$$[\alpha \wedge a \wedge c] + [\alpha \wedge a \wedge d] + [\alpha \wedge b \wedge c] + [\alpha \wedge b \wedge d] \leqq 2[(2\alpha) \wedge (a+b) \wedge (c+d)]$$

to a $= \Phi_X(u)$, $b = \Phi_{-X}(u)$, $c = \Phi_Y(v)$, $d = \Phi_{-Y}(v)$, to prove that:

$$(2.3) \qquad |\operatorname{Cov}(X, Y)| \leqq 2 \iint_{\mathbb{R}_+^2} \inf(2\alpha, \Phi_{|X|}(u), \Phi_{|Y|}(v)\, du dv.$$

It only remains to prove that, for any r.v.'s X and Y,

$$(2.4) \qquad \iint_{\mathbb{R}_+^2} \inf(2\alpha, \Phi_{|X|}(u), \Phi_{|Y|}(v))\, du dv = \int_0^{2\alpha} Q_X(u) Q_Y(u)\, du.$$

Let U be a r.v. with uniform distribution over [0, 1] and let (Z, T) be the bivariate r.v. defined by $(Z, T) = (0, 0)$ iff $U \geqq 2\alpha$ and $(Z, T) = (Q_X(U), Q_Y(U))$ iff $U < 2\alpha$. So, on one hand

$$\mathbb{E}(ZT) = \int_0^{2\alpha} Q_X(u) Q_Y(u)\, du.$$

On the other hand,

$$(Z>u, T>v) = (U<2\alpha, U<\Phi_{|X|}(u), U<\Phi_{|Y|}(v)).$$

Hence

$$\mathbb{E}(ZT) = \iint_{\mathbb{R}_+^2} \mathbb{P}(Z>u, T>v)\, du dv$$

$$= \iint_{\mathbb{R}^{2+}} \inf(2\alpha, \mathbb{P}(|X|>u), \mathbb{P}(|Y|>v))\, du dv,$$

and (2.4) follows, therefore establishing (a) of Theorem 1.1. ■

Proof of (*b*) *of Theorem* 1.1. — Let F be the distribution function of a symmetric random variable. We construct a bivariate r.v. (U, V) with marginal distributions the uniform distribution over [0, 1] satisfying $\alpha(\sigma(U), \sigma(V)) \leqq \alpha$ in such a way that $(X, Y) = (F^{-1}(U), F^{-1}(V))$ satisfies (b) of Theorem 1.1.

Let a be any real in [0, 1/2]. Let Z and T be two independent r.v.'s with uniform distribution over [0, 1]. Define

$$(2.5) \qquad (U, V) = \mathbf{1}_{(Z \leqq 1-a)}(Z, (1-a)T) + \mathbf{1}_{(Z>1-a)}(Z, Z).$$

Clearly, U and V are uniformly distributed over [0, 1]. We now prove that

$$(2.6)\qquad \alpha(\sigma(\mathrm{U}), \sigma(\mathrm{V})) \leqq \alpha = a - (a^2/2).$$

Proof. – Let $\mathrm{I}=[0, 1]$. Let $\mathrm{P}_{\mathrm{U,V}}$ be the law of (U, V) and P_{U}, P_{V} be the respective marginal distributions of U and V. Clearly, $|\mathrm{P}_{\mathrm{U,V}} - (\mathrm{P}_{\mathrm{U}} \otimes \mathrm{P}_{\mathrm{V}})|(\mathrm{I}^2) = 4a - 2a^2$. Hence (2.6) follows from the known inequality $|\mathrm{P}_{\mathrm{U,V}} - (\mathrm{P}_{\mathrm{U}} \otimes \mathrm{P}_{\mathrm{V}})|(\mathrm{I}^2) \geqq 4\alpha(\sigma(\mathrm{U}), \sigma(\mathrm{V}))$. ■

Now, let $(\mathrm{X}, \mathrm{Y}) = (\mathrm{F}^{-1}(\mathrm{U}), \mathrm{F}^{-1}(\mathrm{V}))$. Clearly,

$$\mathbb{E}(\mathrm{XY}) = \int_{1-a}^{1} (\mathrm{F}^{-1}(u))^2\, du + \frac{1}{1-a}\left(\int_0^{1-a} \mathrm{F}^{-1}(u)\, du\right)^2.$$

Since X has a symmetric law, $\mathrm{F}^{-1}(1-u) = \mathrm{Q}_{\mathrm{X}}(2u)$ for almost every u in $[0, 1/2[$. Hence

$$(2.7)\qquad \mathrm{Cov}(\mathrm{X}, \mathrm{Y}) \geqq \int_0^{a} [\mathrm{Q}_{\mathrm{X}}(2u)]^2\, du \geqq \frac{1}{2}\int_0^{2\alpha} [\mathrm{Q}_{\mathrm{X}}(u)]^2\, du,$$

therefore establishing (*b*) of Theorem 1.1. ■

3. ASYMPTOTIC RESULTS FOR THE VARIANCE OF PARTIAL SUMS

Proof of Theorem 1.2. – First, we prove (*a*). Clearly,

$$(3.1)\qquad \mathrm{Var}\, \mathrm{S}_n \leqq \sum_{s \in]0, n]^d} \sum_{t \in]0, n]^d} |\mathrm{Cov}(\mathrm{X}_s, \mathrm{X}_t)|.$$

Now, by (*a*) of Theorem 1.1 and Cauchy-Schwarz inequality,

$$|\mathrm{Cov}(\mathrm{X}_s, \mathrm{X}_t)| \leqq 2\int_0^{\alpha_{t-s}} ([\mathrm{Q}_{\mathrm{X}_s}(2u)]^2 + [\mathrm{Q}_{\mathrm{X}_t}(2u)]^2)\, du.$$

Hence

$$(3.2)\qquad n^{-d} \sum_{s \in]0, n]^d} \sum_{t \in]0, n]^d} |\mathrm{Cov}(\mathrm{X}_s, \mathrm{X}_t)| \leqq 4\int_0^1 (\alpha^{-1}(u) \wedge n^d)\, [\bar{\mathrm{Q}}_n(2u)]^2\, du.$$

Both (3.1) and (3.2) then imply (*a*) of Theorem 1.2.

Second, we prove (*b*) and (*c*). When $(X_i)_{i \in \mathbb{Z}}$ is a weakly stationary sequence, an elementary calculation shows that

$$(3.3)\quad n^{-d} \sum_{s \in]0,\, n]^d} \sum_{t \in]0,\, n]^d} |\operatorname{Cov}(X_s, X_t)| = \sum_{t \in [-n,\, n]^d} (1-|t_1|/n) \ldots (1-|t_d|/n) |\operatorname{Cov}(X_0, X_t)|.$$

Therefore, under the assumption (1.4),

$$(3.4)\quad \sum_{t \in [-n,\, n]^d} (1-|t_1|/n) \ldots (1-|t_d|/n) |\operatorname{Cov}(X_0, X_t)| \leqq 4M.$$

both (3.4) and Beppo-Levi lemma imply (*b*) of Theorem 1.2. Concluding the proof then needs the following equality:

$$(3.5)\quad n^{-d} \operatorname{Var} S_n = \sum_{t \in [-n,\, n]^d} (1-|t_1|/n) \ldots (1-|t_d|/n) \operatorname{Cov}(X_0, X_t).$$

Since the series $\sum_{t \in \mathbb{Z}^d} \operatorname{Cov}(X_0, X_t)$ is absolutely convergent, (3.5) followed by an application of Lebesgue dominated convergence theorem implies (*c*) of Theorem 1.2. ■

Proof of Corollary 1.2. – By Young's inequality, for any nonnegative numbers x and y, $xy \leqq \phi^*(y) + \phi(x)$, which implies that

$$(3.6)\quad \int_0^1 \alpha^{-1}(u) [\bar{Q}_n(2u)]^2 \, du \leqq \int_0^1 \phi^*(\alpha^{-1}(u)) \, du + \int_0^1 \phi([\bar{Q}_n(u)]^2) \, du.$$

Now, by Jensen inequality,

$$(3.7)\quad \int_0^1 \phi([\bar{Q}_n(u)]^2) \, du \leqq n^{-d} \sum_{i \in]0,\, n]^d} \int_0^1 \phi([Q_{X_i}(u)]^2) \, du = n^{-d} \sum_{i \in]0,\, n]^d} \mathbb{E}(\phi(X_i^2)).$$

Hence

$$(3.8)\quad \int_0^1 \alpha^{-1}(u) [\bar{Q}_n(2u)]^2 \, du \leqq \int_0^1 \phi^*(\alpha^{-1}(u)) \, du + n^{-d} \sum_{i \in]0,\, n]^d} \mathbb{E}(\phi(X_i^2)).$$

(3.8) then implies Corollary 1.2, via Theorem 1.2. ■

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