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Gaussian chaos and functional laws of the iterated logarithm for Ito-Wiener integrals

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ABSTRACT. — Random samples of centered Gaussian chaos, when properly normalized, converge and cluster in a non-random set. In this paper we study the rates for this convergence and indicate some applications to self-similar processes given by multiple Ito-Wiener integrals.

RÉSUMÉ. — Les échantillons de chaos aléatoires gaussiens centrés, quand ils sont correctement normalisés, convergent et tendent à se grouper dans un ensemble non aléatoire. Dans cet article nous étudions la vitesse de cette convergence et indiquons quelques applications aux processus auto-semblables définis par des intégrales d'Ito-Wiener.

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1. INTRODUCTION

In [MO86] and [MO87] functional laws of the iterated logarithm are established for self-similar processes represented by multiple Wiener integrals. An earlier work [BO77] also obtained tail estimates for multiple Wiener integrals of the type required for laws of the iterated logarithm, and more recently almost sure approximations for U-statistics and von Mises statistics as in [D] have led to a study of similar results. The purpose of this note is to present an approach to these problems via the Gaussian chaos material in [LT90], Chapter 3.

The approach of [LT90] easily allows a formulation where one can examine rates of convergence for the Gaussian chaos, and the first result we present is of this type. Indeed, what we do can be viewed as an attempt to quantify the results in [LT90]. Results of this type have been previously obtained for Brownian motion and other self-similar Gaussian processes, and the reader should consult [GK91] and [GK92] for such results as well as further references. The definitive results for Brownian motion have been recently obtained in [G92] and [T92]. After establishing our Theorem 1 we then apply it to *multiple Wiener integrals*, and combining this with a rescaling lemma, we obtain a functional LIL related to that in [MO86]. All of our results are formulated for chaos of order 2, but can be extended to higher order chaos as well. An earlier version of this paper dealt with the uncentered Gaussian chaos of [LT90], but some useful discussions with Evarist Gine and Murad Taqqu led us to rethink the problem for centered chaos as defined below. This improved the applicability of our results significantly, and we thank Gine and Taqqu for their interest in our work.

2. CENTERED GAUSSIAN CHAOS OF ORDER 2

Let B denote a real separable Banach space with norm $\|.\|$ and topological dual B*. Assume $\{a_{ij}: i, j \geqq 1\}$ is a sequence of elements of B with $\delta(i, j) = 1$ if $i = j$ and zero otherwise, and let $\{g_i : i \geqq 1\}$ be an i.i.d. sequence of N(0, 1) random variables. We then define the *centered Gaussian chaos of order* 2 determined by $\{a_{ij}\}$ to be the B-valued random quadratic form

$$X = \sum_{i, j \geqq 1} a_{ij}(g_i g_j - \delta(i, j)) \tag{2.1}$$

provided the partial sums

$$\pi_d X = \sum_{i, j = 1}^{d} a_{ij}(g_i g_j - \delta(i, j)) \qquad (d \geqq 1) \tag{2.2}$$

converge in probability. Of course, their limit defines X, and with probability one X then takes values in the closed separable subspace F spanned by $\{a_{ij}\}$. Since F is separable it is well known that there exists a countable set D in the unit ball of B* such that

$$\|x\| = \sup_{f \in D} |f(x)| \qquad (x \in F),$$

and hence with probability one we have

$$\|X\| = \sup_{f \in D} |f(X)|.$$

The results in [LT90] were obtained for the uncentered Gaussian chaos

$$\tilde{X} = \sum_{i,j \geq 1} a_{ij} g_i g_j,$$

but similar results hold for the centered chaos in (2.1). In particular, if $\{g'_i : i \geq 1\}$ is a second i.i.d. sequence of N (0, 1) random variables, independent of $\{g_i : i \geq 1\}$, the *decoupled chaos associated with* X is defined to be

$$Y = \sum_{i,j \geq 1} b_{ij} g_i g''_j$$

where $b_{ij} = a_{ij}$ for $i, j \geq 1$. This is the same decoupled chaos as that defined for $\tilde{X}$ in [LT90]. To see that Y makes sense when X exists, we note that by arguing as in [LT90] if $\{a_{ij}\}$ contains only finitely many non-zero terms, then

$$P(\|Y\| > t) \leq 2P(\|X\| > t/2). \tag{2.3}$$

Hence if $\{\pi_d X : d \geq 1\}$ is Cauchy in probability, then so are the partial sums

$$\pi_d Y = \sum_{i,j=1}^{d} b_{ij} g_i g'_j \qquad (d \leq 1).$$

Hence if X exists as above, then Y exists as the limit in probability of the partial sums $\{\pi_d Y : d \geq 1\}$. Furthermore, passing to the limit yields (2.3) whenever X exists.

If E' and P' denote partial expectation and probability with respect to $\{g'_j : j \geq 1\}$, then we define

$$M(d) = \inf\left\{\lambda > 0 : P(\|X - \pi_d X\| > \lambda) < \frac{1}{16}\right\}, \tag{2.4}$$

$$\sigma(d) = \sup\left\{\|x\| : x = \sum_{i \vee j \geq d+1} a_{ij} k_i k_j, \|k\|_{l^2} \leq 1\right\} \tag{2.5}$$

$$m(d) = \inf\left\{\lambda > 0 : P(\sup_{f \in D} E'(f^2(Y - \pi_d Y)) \leq \lambda^2) \geq \frac{3}{4}\right\} \tag{2.6}$$

for $d \geqq 0$ with $\pi_0(x)=0$ for all $x \in B$. Here $\|k\|_{l^2}=(\sum_{i \geqq 1} k_i^2)^{1/2}$ is the usual l^2-norm. and $a \vee b = \max(a, b)$ in (2.5). Since X is the limit in probability of $\{\pi_d X : d \geqq 1\}$, it is obvious $\lim_{d \to \infty} M(d)=0$, and we also have

$$\lim_{d \to \infty} \sigma(d) = \lim_{d \to \infty} m(d) = 0, \tag{2.7}$$

as the arguments on page 68 of [LT90] easily imply

$$m(d) \leqq 4\sqrt{2}\, M(d) \tag{2.8}$$

and

$$\sigma(d) \leqq m(d). \tag{2.9}$$

Furthermore, with these parameters we can now prove the analogue of Lemma 3.8 in [LT90]. The proof is exactly as in [LT90] so we only state the result as:

(*) Let X be a centered Gaussian chaos of order 2. Then, for each $t>0$

$$P(\|X\| > M + mt + \sigma t^2) \leqq \exp\{-t^2/2\}$$

where $M=M(0)$, $m=m(0)$, $\sigma=\sigma(0)$ in (2.4), (2.5), and (2.6). This inequality also holds for $m>2$, and has been obtained in [AG91].

Our first result is the following. Throughout, $Lx = \max\{1, \log_e x\}$ and $L_2 x = L(Lx)$.

THEOREM 1. – *Let* $X, X_1, X_2, \ldots$ *be identically distributed, centered B-valued Gaussian chaos of order 2 determined by* $\{a_{ij}\}$. *Let* $f(0)=1$ *and*

$$f(d) = d^{-1}[\sigma(d) + m(d)\, d^{-1/2}] \qquad (d \geqq 1) \tag{2.10}$$

where $\sigma(d)$ *and* $m(d)$ *are given by* (2.5) *and* (2.6), *and define*

$$d_n = d(n) = \sup\{r \in \mathbb{Z} : r \geqq 1, f(r-1) \geqq L_2 n / L n\}, \tag{2.11}$$

$$\varepsilon_n = \varepsilon(n) = \gamma\, d_n L_2 n / L n \tag{2.12}$$

where $\gamma > 0$. *Let*

$$\Sigma = \{\sum_{i, j \geqq 1} a_{ij} k_i k_j : \|k\|_{l^2} \leqq 1\}. \tag{2.13}$$

Then:

(*a*) $d_n = o(Ln/L_2 n)$ *and* $\varepsilon_n = o(1)$ *as* $n \to \infty$,

(*b*) Σ *is a compact subset of* B,

(*c*) *for* $\gamma > 0$ *sufficiently large and* $E_n = \{X_1, \ldots, X_n\}$

$$P(E_n/(2Ln) \subseteqq \Sigma^{\varepsilon_n} \textit{ eventually}) = 1,$$

(*d*) *for* γ *sufficiently large and* E_n *as in* (*c*)

$$P(\Sigma \subseteqq (E_n/(2Ln))^{\varepsilon_n} \textit{ eventually}) = 1,$$

(2.14) *provided that* $X_1, X_2, \ldots$ *are* i.i.d.

(e) *If* $X_n = \sum_{i, j \geq 1} a_{ij} (g_{i, n} g_{j, n} - \delta(i, j))$ *for* $n = 1, 2, \ldots$ *and*

$$\lim_{\substack{n \to \infty \\ m - n \to \infty}} E(g_{i, m} g_{j, n}) = 0$$

for all $i, j \geq 1$, *then*

$$P(C(\{X_n/(2 L n)\}) = \Sigma) = 1$$

where $C(\{a_n\})$ *denotes all cluster points of the sequence* $\{a_n\}$ *in* B.

In (2.14 *c*), Σ^{ε_n} denotes the set of all points within distance less than ε_n of Σ. Hence

$$\Sigma^{\varepsilon_n} = \Sigma + \varepsilon_n U \tag{2.15}$$

where $U = \{x \in B : \|x\| < 1\}$ and $A + B = \{a + b : a \in A, b \in B\}$.

Remark. – It is possible, using the results in [LT90] in a slightly different fashion, to show for any $\varepsilon > 0$ and $\varepsilon_n = \varepsilon (L n)^{-1/2}$ that (2.14 *c*) always holds. In some situations this rate may be better than what we have obtained here, but our method also produces the same rate for the clustering result in (2.14 *d*). In addition, it is possible to construct examples where the rates for (2.14 *c*) and (2.14 *d*) are better than the universal rate $\varepsilon_n = \varepsilon (L n)^{-1/2}$.

3. SOME USEFUL LEMMAS

To prove Theorem 1 we first establish some lemmas.

LEMMA 1. – *The set* Σ *given in* (2.13) *is a subset of* B *such that*

$$\Lambda = \sup_{x \in \Sigma} \|x\| < \infty. \tag{3.1}$$

Furthermore, Σ *is a compact subset of* B.

Proof. – Fix $\varepsilon > 0$ and choose $d' > d \geq 1$ integers such that

$$P(\|Z\| > \varepsilon) < \frac{1}{16} \tag{3.2}$$

where $Z = \pi_{d'} X - \pi_d X$. Then Z is a centered Gaussian chaos of order 2 with parameters

$$\left\{ \begin{aligned} M_Z &= \inf\left\{\lambda > 0 : P(\|Z\| > \lambda) < \frac{1}{16}\right\}, \\ \sigma_Z &= \sup\left\{\|x\| : x = \sum_{d < i \vee j \leq d'} a_{ij} k_i k_j, \|k\|_{l^2} \leq 1\right\}, \\ m_Z &= \inf\left\{\lambda > 0 : P(\operatorname{up}_{f \in D} E'(f^2(Z'))^{1/2} \leq \lambda) \geq \frac{3}{4}\right\} \end{aligned} \right. \tag{3.3}$$

where Z' is the decoupled Gaussian chaos associated with Z. In this case we can take

$$Z' = \pi_{d'} Y - \pi_d Y$$

where Y is the decoupled chaos associated with X. Then, by (3.2) $M_Z \leqq \varepsilon$, and analogous to (2.8) and (2.9) we thus have

$$\sigma_Z \leqq m_Z \leqq 4\sqrt{2}\,\varepsilon.$$

Now

$$\sigma_Z = \sup_{\|k\|_{l^2} \leqq 1} \Big\| \sum_{d < i \vee j \leqq d'} a_{ij} k_i k_j \Big\|,$$

and since $\sigma_Z \leqq 4\sqrt{2}\,\varepsilon$ with $\varepsilon > 0$ arbitrary this implies the sums

$$\left\{ \sum_{i,\,j=1}^{d} a_{ij} k_i k_j : d \geqq 1 \right\}$$

are Cauchy in B uniformly in each k with $\|k\|_{l^2} \leqq 1$. Since B is complete this implies $\Sigma \subseteqq B$ and obviously $\Lambda < \infty$ as

$$\Sigma_d = \left\{ \sum_{i,\,j=1}^{d} a_{ij} k_i k_j : \|k\|_{l^2} \leqq 1 \right\} \tag{3.4}$$

is a compact subset of B for each $d \geqq 1$. Furthermore, if

$$T(k) = \sum_{i,\,j \geqq 1} a_{ij} k_i k_j, \tag{3.5}$$

the above shows $T : l^2 \to B$ is continuous from the weak topology on l^2 restricted to bounded subsets of l^2 to the norm topology on B. By definition $T(V) = \Sigma$ where $V = \{ k \in l^2 : \|k\|_{l^2} \leqq 1 \}$, and since V is compact in the weak topology we thus have Σ compact in B. Hence Lemma 1 is proved.

LEMMA 2. — *If* X *is a centered Gaussian chaos of order* 2, *then*

$$A = \sup_{f \in D} \sum_{i,\,j \geqq 1} f^2(a_{ij}) < \infty. \tag{3.6}$$

Proof. — Let Y be a decoupled chaos associated to X. Then

$$Y = \sum_{i,\,j \geqq 1} a_{ij} g_i g'_j$$

and Y is the limit in probability of the partial sums $\{ \pi_d Y : d \geqq 1 \}$ as given after (2.3) above. Now

$$E\left(\left(\sum_{i,\,j=1}^{d} f(a_{ij}) g_i g'_j \right)^2 \right) = \sum_{i,\,j=1}^{d} f^2(a_{ij}),$$

so

$$\begin{aligned}A &= \sup_d \sup_{f \in D} \sum_{i,j=1}^{d} f^2(a_{ij}) \\ &\leqq \sup_d E\left(\left(\sup_{f \in D} \sum_{i,j=1}^{d} f(a_{ij}) g_i g_j'\right)^2\right) \\ &= \sup_d E \| \pi_d Y \|^2 \\ &= E \| Y \|^2 < \infty.\end{aligned}$$

The last equality above holds as $\lim_d E \| Y - \pi_d Y \|^2 = 0$, $E \| \pi_d Y \|^2 \uparrow \infty$ as $d \nearrow$, and the final form of (2.3) indicated above implies

$$P(\| \pi_d Y - Y \| > t) \leqq 2 P(\| X - \pi_d X \| > t/(2\sqrt{2})) \qquad (t \geqq 0)$$

with

$$\lim_d E \| X - \pi_d X \|^2 = 0$$

by (*) applied to $X - \pi_d X$. Hence Lemma 2 is verified.

LEMMA 3. – *Let* $\{\varepsilon_n\}$ *and* $\{d_n\}$ *be defined as in Theorem* 1. *Then* (2.14 *a*) *holds, and for* $\gamma > 1$

(3.7) $$P(\| X_n - \pi_{d_n} X_n \| > \varepsilon_n (2 L n) \text{ eventually}) = 0.$$

Proof. – Since the parameters $\sigma(d)$, $m(d)$, and $M(d)$ of (2.4)-(2.6) converge to zero as $d \to \infty$, the definition of $f(d)$ in (2.10) and (2.11) easily implies $d_n = o(Ln/L_2 n)$ as $n \to \infty$. Since $\varepsilon_n = \gamma d_n L_2 n/Ln$ we thus have $\varepsilon_n = o(1)$, and hence (2.14 *a*) is established.

To verify (3.7) we first observe that if

$$\sigma = \sup_{\| k \|_{l^2} \leqq 1} \sup_{f \in D} \left| \sum_{i,j \geqq 1} f(a_{ij}) k_i k_j \right| = 0$$

then $0 = a_{ij}$ for all $i, j \geqq 1$, *i.e.* first choose k so as to examine the diagonal elements and then the off-diagonal elements. Hence, if $\sigma(d) = 0$ for some $d \geqq 1$, we have $P(X - \pi_d(X) = 0) = 1$ and hence $m(d) = M(d) = f(d) = 0$. Thus we may set

$$d_n = \tilde{d} = \inf\{ d \geqq 1 : \sigma(d) = 0 \}$$

for all sufficiently large n, and (3.7) holds as $X, X_1, X_2, \ldots$ are identically distributed. Furthermore, in this situation $\varepsilon_n = \gamma \tilde{d} L_2 n/Ln$.

Now we turn to the proof of (3.7) when $\sigma(d)>0$ for all $d\geqq 1$. Under these conditions $d_n\uparrow\infty$, and (2.11) and (2.12) imply

$$(3.8)\quad 2\varepsilon_n \mathrm{L}n=2\gamma \mathrm{L}_2 n\, d_n=\frac{(\sigma(d_n)+m(d_n)d_n^{-1/2})}{f(d_n)}(2\gamma \mathrm{L}_2 n) \geqq(\sigma(d_n)+m(d_n)d_n^{-1/2})2\gamma \mathrm{L}n$$

as $f(d_n)<\mathrm{L}_2 n/\mathrm{L}n$. Applying (*) following (2.9) we have for each $t>0$ that

$$(3.9)\quad \mathrm{P}(\|\mathrm{X}-\pi_{d_n}\mathrm{X}\|>\mathrm{M}(d_n)+m(d_n)t+\sigma(d_n)t^2)\leqq\exp\left\{-\frac{t^2}{2}\right\},$$

so setting $t^2=2\mathrm{L}n(1+2\mathrm{L}_2 n/\mathrm{L}n)$ we get as $n\to\infty$ that

$$(3.10)\quad \mathrm{M}(d_n)+m(d_m)t+\sigma(d_n)t^2\sim 2\mathrm{L}n(\sigma(d_n) +m(d_n)(2\mathrm{L}n)^{-1/2}+\mathrm{M}(d_n)(2\mathrm{L}n)^{-1}).$$

Now (3.8) and $d_n=o(\mathrm{L}n/\mathrm{L}_2 n)$ imply

$$(3.11)\quad \varepsilon_n\geqq\gamma(\sigma(d_n)+m(d_n)d_n^{-1/2})\geqq\gamma(\sigma(d_n)+m(d_n)(2\mathrm{L}n)^{-1/2}),$$

and since $\varepsilon_n=\gamma d_n \mathrm{L}_2 n/\mathrm{L}n$

$$(3.12)\quad \frac{\mathrm{M}(d_n)}{(2\mathrm{L}n)}\varepsilon_n^{-1}=\frac{\mathrm{M}(d_n)}{2\gamma d_n \mathrm{L}_2 n}\to 0$$

as $n\to\infty$. Thus we have from (3.10), (3.11), and (3.12) that

$$(3.13)\quad \begin{aligned}\mathrm{M}(d_n)+m(d_n)t+\sigma(d_n)t^2 &\sim 2\mathrm{L}n(\sigma(d_n)+m(d_n)(2\mathrm{L}n)^{-1/2} +\mathrm{M}(d_n)(2\mathrm{L}n)^{-1})\\ &\leqq 2\mathrm{L}n\left(\frac{\varepsilon_n}{\gamma}+\mathrm{M}(d_n)(2\mathrm{L}n)^{-1}\right)\\ &=2\mathrm{L}n\frac{\varepsilon_n}{\gamma}(1+\mathrm{M}(d_n)(2\mathrm{L}n)^{-1}\varepsilon_n^{-1}\gamma)\\ &\sim 2\mathrm{L}n\,\varepsilon_n/\gamma.\end{aligned}$$

Hence for $\gamma>1$ and $t^2=2\mathrm{L}n(1+2\mathrm{L}_2 n/\mathrm{L}n)$ we have

$$(3.14)\quad 2\varepsilon_n \mathrm{L}n>\mathrm{M}(d_n)+m(d_n)t+\sigma(d_n)t^2$$

for all n sufficiently large. Combining (3.9) and (3.14) with $t^2=2\mathrm{L}n(1+2\mathrm{L}_2 n/\mathrm{L}n)$, the Borel-Cantelli Lemma easily implies (3.7) and Lemma 3 is proved for $\gamma>1$.

LEMMA 4. – *Let Σ and $\{\varepsilon_n\}$ be as in Theorem* 1, *and for $x\in\Sigma$ of the form $x=\sum_{i,j\geqq 1}a_{ij}k_i k_j$ define the partial sums*

$$\pi_d(x)=\sum_{i,j=1}^{d}a_{ij}k_i k_j\qquad(d\geqq 1).$$

Then

$$\sigma(d_n)=\sup_{x\in\Sigma}\|x-\pi_{d_n}(x)\|<\varepsilon_n/\gamma. \tag{3.15}$$

Proof. — Since d_n is given by (2.11) and ε_n by (2.12) with $f(d_n)$ as in (2.10), we have for all n sufficiently large that

$$\sigma(d_n)\leqq d_n f(d_n)\leqq d_n L_2 n/L n=\varepsilon_n/\gamma.$$

Now, by definition,

$$\sup_{x\in\Sigma}\|x-\pi_{d_n}(x)\|=\sigma(d_n), \tag{3.16}$$

and hence we have (3.15).

LEMMA 5. — *Let $\{\varepsilon_n\}$ and $\{d_n\}$ be as in Theorem* 1. *If $x\in\Sigma$ is of the form*

$$x=\sum_{i,j\geqq 1}a_{ij}k_i k_j,$$

where $\sum_{j=1}^{\infty}k_j^2\leqq 1$ and $d_n\uparrow\infty$, then for all n sufficiently large

$$P(\|\pi_{d_n}X/(2Ln)-\pi_{d_n}x\|<\varepsilon_n)\geqq\exp\left\{-Ln\left(\sum_{j=1}^{d_n}k_j^2+6\varepsilon_n/\gamma\right)\right\}. \tag{3.17}$$

Proof. — Since $\pi_{d_n}X=\sum_{i,j=1}^{d_n}a_{ij}(g_i g_j-\delta(i,j))$, we have

$$\begin{aligned}\|\pi_{d_n}X/(2Ln)-\pi_{d_n}x\| &=\sup_{f\in D}\left|\sum_{i,j=1}^{d_n}f(a_{ij})\left(\frac{g_i g_j}{2Ln}-k_i k_j\right)\right| \\ &\quad+\sup_{f\in D}\left|\sum_{i=1}^{d_n}f(a_{ii})\right|/(2Ln) \\ &\leqq\sup_{f\in D}\left(\sum_{i,j=1}^{dn}f^2(a_{ij})\right)^{1/2}\left[\left(\sum_{i,j=1}^{dn}\frac{g_i^2}{2Ln}\left(\frac{g_j}{\sqrt{2Ln}}-k_j\right)^2\right)^{1/2}\right. \\ &\quad\left.+\left(\sum_{i,j=1}^{d_n}k_j^2\left(\frac{g_i}{\sqrt{2Ln}}-k_i\right)^2\right)^{1/2}\right]+A^{1/2}d_n/(2Ln) \\ &\leqq A^{1/2}\left[\left(\sum_{i=1}^{d_n}\frac{g_i^2}{2Ln}\right)^{1/2}+\left(\sum_{j=1}^{d_n}k_j^2\right)^{1/2}\right]\rho_n^{1/2}+A^{1/2}d_n/(2Ln)\end{aligned} \tag{3.18}$$

where $A<\infty$ is as in (3.6), and

$$\rho_n=\sum_{j=1}^{d_n}\left(\frac{g_j}{\sqrt{2Ln}}-k_j\right)^2. \tag{3.19}$$

Since $\sum_{j=1}^{\infty} k_j^2 \leqq 1$, the triangle inequality implies

$$\left(\sum_{i=1}^{d_n} \frac{g_i^2}{2\,\mathrm{L}\,n}\right)^{1/2} \leqq \rho_n^{1/2}+1, \tag{3.20}$$

and hence (3.18), (3.19), (3.20) imply

$$\|\pi_{d_n} \mathrm{X}/(2\,\mathrm{L}\,n)-\pi_{d_n} x\| \leqq 3\,\mathrm{A}^{1/2}\,\rho_n^{1/2}+\mathrm{A}^{1/2}\,d_n/(2\,\mathrm{L}\,n) \tag{3.21}$$

if $\rho_n^{1/2} \leqq 1$. Hence for large n, $\varepsilon_n/3\,\mathrm{A}^{1/2}<1$, and (3.21) implies that

$$\mathrm{P}(\|\pi_{d_n} \mathrm{X}/(2\,\mathrm{L}\,n)-\pi_{d_n} x\| \leqq \varepsilon_n) \geqq \mathrm{P}(\rho_n^{1/2} \leqq \varepsilon_n/(4\,\mathrm{A}^{1/2})) \tag{3.22}$$

as $\mathrm{A}^{1/2}\,d_n(2\,\mathrm{L}\,n)=\mathrm{A}^{1/2}\,\varepsilon_n/(2\,\gamma\,\mathrm{L}_2\,n)$. Now

$$\begin{aligned}\mathrm{P}(\rho_n^{1/2} \leqq t) &= \mathrm{P}\left(\sum_{j=1}^{d_n} (g_j/(2\,\mathrm{L}\,n)^{1/2}-k_j)^2 \leqq t^2\right) \\ &= \int_{\mathrm{E}_n} \cdots \int \exp\left\{-\sum_{j=1}^{d_n} x_j^2/2\right\}(2\,\pi)^{-d_n/2}\,dx_1 \ldots dx_{d_n}\end{aligned} \tag{3.23}$$

where

$$\begin{aligned}\mathrm{E}_n &= \left\{(x_1, \ldots, x_{d_n}): \sum_{j=1}^{d_n} (x_j-(2\,\mathrm{L}\,n)^{1/2}\,k_j)^2 \leqq 2\,t^2\,\mathrm{L}\,n\right\} \\ &= \int_{\mathrm{F}_n} \cdots \int \exp\left\{-\sum_{j=1}^{d_n} (y_j+(2\,\mathrm{L}\,n)^{1/2}\,k_j)^2/2\right\}(2\,\pi)^{-dn/2}\,dy_1 \ldots dy_{d_n}\end{aligned}$$

where

$$\begin{aligned}\mathrm{F}_n &= \left\{(y_1, \ldots, y_{d_n}): \sum_{j=1}^{d_n} y_j^2 \leqq 2\,t^2\,\mathrm{L}\,n\right\} \\ &= \tau_n \exp\left\{-\mathrm{L}\,n \sum_{j=1}^{d_n} k_j^2\right\} \int_{\mathrm{F}_n} \exp\left\{-\sum_{j=1}^{d_n} y_j k_j (2\,\mathrm{L}\,n)^{1/2}\right\} \frac{d\mu(y_1, \ldots, y_{d_n})}{\tau_n}\end{aligned}$$

where

$$\tau_n = \mathrm{P}\left(\sum_{j=1}^{d_n} g_j^2 \leqq 2\,t^2\,\mathrm{L}\,n\right) = \mu(\mathrm{F}_n) \tag{3.24}$$

and

$$d\mu(y_1, \ldots, y_{d_n}) = \exp\left\{-\sum_{j=1}^{d_n} y_j^2/2\right\}(2\,\pi)^{-d_n/2}\,dy_1 \ldots dy_{d_n}. \tag{3.25}$$

Applying Jensen's inequality to the last term in (3.23) we obtain

$$\mathrm{P}(\rho_n^{1/2} \leqq t) \geqq \tau_n \exp\left\{-\mathrm{L}n \sum_{j=1}^{d_n} k_j^2\right\}. \tag{3.26}$$

Using a trivial lower bound for the density of $\mathscr{L}(g_1, \ldots, g_{d_n})$ we have

$$\begin{aligned} \tau_n &= \mathrm{P}\left(\sum_{j=1}^{d_n} g_j^2 \leqq 2t^2 \mathrm{L}n\right) \\ &\geqq \mathrm{P}\left(\sum_{j=1}^{d_n} g_j^2 \leqq 2t^2 \mathrm{L}n \wedge 1\right) \\ &\geqq \frac{e^{-1}}{(2\pi)^{d_n/2}} (2t^2 \mathrm{L}n \wedge 1)^{d_n} \operatorname{vol}\left(\left\{(x_1, \ldots, x_{d_n}) : \sum_{j=1}^{d_n} x_j^2 \leqq 1\right\}\right) \\ &= e^{-1}(2\pi)^{-d_n/2}(2t^2 \mathrm{L}n \wedge 1)^{d_n} \frac{2\pi^{d_n/2}}{d_n \Gamma(d_n/2)} \\ &= \exp\left\{-1 - \left(\frac{d_n}{2} - 1\right)\mathrm{L}2 - \mathrm{L}d_n - \mathrm{L}(\Gamma(d_n/2))\right\} \min(1, \exp\{d_n \mathrm{L}(2t^2 \mathrm{L}n)\}) \end{aligned} \tag{3.27}$$

Now $d_n = o(\mathrm{L}n/\mathrm{L}_2 n)$ by Lemma 3, and hence

$\mathrm{L}d_n \leqq \mathrm{L}_2 n$, $\mathrm{L}\Gamma(d_n/2) < \mathrm{L}(d_n!) = O(\mathrm{L}(d_n^{d_n+1} e^{-d_n}/\sqrt{2\pi})) = O((d_n+1)\mathrm{L}d_n)$.

Furthermore, since $\varepsilon_n = \gamma d_n \mathrm{L}_2 n/\mathrm{L}n$, if $t^2 = (\varepsilon_n/4\mathrm{A}^{1/2})^2$ we have

$$\begin{aligned} |\mathrm{L}(2t^2 \mathrm{L}n)| &= \left|\mathrm{L}\left(\frac{2\varepsilon_n^2 \mathrm{L}n}{16\mathrm{A}}\right)\right| \\ &= \left|\mathrm{L}\left(\frac{2\gamma^2 d_n^2 (\mathrm{L}_2 n)^2}{16\mathrm{A}\mathrm{L}n}\right)\right| \\ &= O(\mathrm{L}_2 n). \end{aligned} \tag{3.28}$$

Hence (3.27) implies for large n and $t^2 = (\varepsilon_n/(4\mathrm{A}^{1/2}))^2$ that

$$\begin{aligned} \tau_n &\geqq \exp\{-2(d_n+1)\mathrm{L}d_n - 2d_n \mathrm{L}_2 n\} \\ &\geqq \exp\{-6d_n \mathrm{L}_2 n\} \\ &= \exp\left\{-\frac{6}{\gamma}\varepsilon_n \mathrm{L}n\right\} \end{aligned} \tag{3.29}$$

as $d_n \mathrm{L}_2 n = \frac{\varepsilon_n}{\gamma}\mathrm{L}n$. Combining (3.29) and (3.26) with $t^2 = (\varepsilon_n/4\mathrm{A}^{1/2})^2$, (3.22) then implies for all n sufficiently large that (3.17) holds. Hence Lemma 5 is proved.

4. PROOF OF THEOREM 1

The proofs of (2.14 *a*) and (2.14 *b*) are contained in Lemma 3 and Lemma 1, so it remains to verify (2.14 *c*) and (2.14 *d*).

Proof of (2.14 *c*). – The first step is the following

PROPOSITION 1. – *If* $\{\varepsilon_n\}$ *is as in Theorem* 1, *then for* $\gamma > 6\Lambda$

(4.1) $$P(X_n/(2Ln) \in \Sigma^{\varepsilon_n} \text{ eventually}) = 1.$$

Proof. – First observe that

(4.2) $$\begin{aligned} &P(X_n/(2Ln) \notin \Sigma + \varepsilon_n U) \\ &\leqq P\left(\pi_{d_n} X_n/(2Ln) \notin \Sigma + \frac{2}{3}\varepsilon_n U\right) \\ &\quad + P(\|X_n - \pi_{d_n} X_n\| > \varepsilon_n (2Ln)/3 \\ &\leqq P\left(\pi_{d_n} X_n + \sum_{i=1}^{d_n} a_{ii} \notin (2Ln)\left(1 + \frac{\varepsilon_n}{3\Lambda}\right)\Sigma\right) \\ &\quad + P(\|X_n - \pi_{d_n} X_n\| > 2Ln(\varepsilon_n/3)) \end{aligned}$$

since $\left\|\sum_{i=1}^{d_n} a_{ii}\right\|/(2Ln) \leqq A^{1/2}\varepsilon_n/(2\gamma L_2 n)$ and $\Sigma \subseteqq 2\Lambda U$ (with 2 being used only for simplicity). Letting $\sum_{d_n} = \left\{\sum_{i,j=1}^{d_n} a_{ij} k_i k_j : \|k\|_{l^2} \leqq 1\right\}$, we have $\sum_{d_n} \subseteqq \Sigma$ for each $d_n \geqq 1$, and that

$$\pi_{d_n} \tilde{X}_n/(2Ln) \notin (1 + \varepsilon_n/(3\Lambda)) \sum_{d_n}$$

iff

$$(g_1, \ldots, g_{d_n})/(2Ln)^{1/2} \notin \left\{(k_1, \ldots, k_{d_n}) : \sum_{j=1}^{d_n} k_j^2 \leqq 1 + \varepsilon_n/(3\Lambda)\right\}$$

where $\pi_{d_n} \tilde{X}_n + \sum_{i=1}^{d_n} a_{ii} = \sum_{i,j=1}^{d_n} a_{ij} g_i g_j$.

Hence

$$(4.3)\quad P(\pi_{d_n}\tilde{X}_n \notin 2\,L\,n(1+\varepsilon_n/(3\,\Lambda))\,\Sigma)$$
$$\leq P(\pi_{d_n}\tilde{X}_n \notin (2\,L\,n)(1+\varepsilon_n/(3\,\Lambda))\sum_{d_n})$$
$$= P\left(\sum_{j=1}^{d_n} g_j^2 > (2\,L\,n)(1+\varepsilon_n/(3\,\Lambda))\right)$$
$$\sim (2\,\pi\,d_n)^{-1/2}\exp\left\{-\left(1+\frac{\varepsilon_n}{3\,\Lambda}\right)L\,n + \left(\frac{d_n}{2}-1\right)\right.$$
$$\left.\times\log\left(\left(1+\frac{\varepsilon_n}{3\,\Lambda}\right)(2\,L\,n)\,e/(d_n-2)\right)\right\}$$

provided $d_n = o\left(\left(1+\frac{\varepsilon_n}{3\,\Lambda}\right)L\,n\right)$ as $n\to\infty$. Hence for $d_n \geq 3$ and $0 \leq \varepsilon_n \leq 1$ we have for all $n \geq n_0$ that

$$(4.4)\quad P(\pi_{d_n}\tilde{X}_n \notin 2\,L\,n(1+\varepsilon_n/(3\,\Lambda))\sum_{d_n}) \leq (d_n^{1/2}\,n)^{-1}\exp\left\{-\frac{\varepsilon_n\,L\,n}{3\,\Lambda} + d_n\,L_2\,n\right\}$$

provided $d_n = o(L\,n)$ as $n\to\infty$. Thus

$$(4.5)\quad \sum_{n\geq 1} P(\pi_{d_n}\tilde{X}_n \notin (2\,L\,n)(1+\varepsilon_n/(3\,\Lambda))\sum_{d_n}) < \infty.$$

if $\gamma > 6\,\Lambda$ since $\varepsilon_n = \gamma\,d_n\,L_2\,n/L\,n$ and $d_n = o(L\,n)$.

Now Lemma 3 implies

$$(4.6)\quad \sum_{n\geq 1} P(\|X_n - \pi_{d_n}X_n\| > 2\,L\,n(\varepsilon_n/3)) < \infty$$

provided $\gamma > 3$, and hence (4.2), (4.3), (4.5), and (4.6) combine to give

$$(4.7)\quad \sum_{n\geq 1} P(X_n/(2\,L\,n) \notin \Sigma + \varepsilon_n\,U) < \infty$$

provided $\gamma > (6\,\Lambda + 3)$. Hence the Borel-Cantelli lemma yields (4.1) and the proposition is proved.

To finish the proof of (2.14 *c*) recall $E_n = \{X_1, \ldots, X_n\}$. Now (4.1) implies

$$X_k(\omega)/(2\,L\,k) = f_k + \varepsilon_k\,u_k$$

for all $k \geq n_0(\omega)$ where $f_k \in \Sigma$ and $u_k \in U$, and this occurs for almost all ω. Hence for $n \geq k \geq n_0(\omega)$

$$\frac{X_k(\omega)}{2\,L\,n} = f_k\left(\frac{L\,k}{L\,n}\right) + \varepsilon_k\left(\frac{L\,k}{L\,n}\right)u_k$$
$$= \tilde{f}_k + \varepsilon_n\,\tilde{u}_k$$

$\tilde{f}_k \in \Sigma$ and $\tilde{u}_k \in \mathrm{U}$ since Σ and U are convex, $\dfrac{\mathrm{L}k}{\mathrm{L}n} \leqq 1$, and

$$\varepsilon_k\left(\frac{\mathrm{L}k}{\mathrm{L}n}\right) = \gamma\, d_k \frac{\mathrm{L}_2 k}{\mathrm{L}_k} \frac{\mathrm{L}k}{\mathrm{L}n} = \frac{\gamma\, d_k \mathrm{L}_2 k}{\mathrm{L}n} \leqq \varepsilon_n.$$

Hence with probability one for $n \geqq n_0$

$$\left\{\frac{\mathrm{X}_{n_0}}{2\mathrm{L}n}, \ldots, \frac{\mathrm{X}_n}{2\mathrm{L}n}\right\} \subseteqq \Sigma^{\varepsilon_n},$$

and

$$\left\{\frac{\mathrm{X}_1}{2\mathrm{L}n}, \ldots, \frac{\mathrm{X}_{n_0}}{2\mathrm{L}n}\right\} \subseteqq \varepsilon_n \mathrm{U} \text{ eventually.}$$

Thus (2.14 c) holds.

Proof of (2.14 d). – Let K_{d_n} be the closed unit ball of $\mathbb{R}^{d_n}$ in the Euclidean norm. Let $\mathscr{F}_n$ be a finite subset of $\left(1 - \dfrac{\varepsilon_n}{16\,\mathrm{A}^{1/2}}\right)^{1/2} \mathrm{K}_{d_n}$ such that

(i) open balls centered at points of $\mathscr{F}_n$ with radius $\dfrac{\varepsilon_n}{(16\,\mathrm{A}^{1/2})}$ in the Euclidean norm are disjoint, and

(ii) $\mathscr{F}_n$ is maximal, *i.e.* if we add a point of $\left(1 - \dfrac{\varepsilon_n}{16\,\mathrm{A}^{1/2}}\right)^{1/2} \mathrm{K}_{d_n}$ to $\mathscr{F}_n$ we get overlap among the open balls of radius $\dfrac{\varepsilon_n}{16\,\mathrm{A}^{1/2}}$ centered at the larger set.

If we write $\mathrm{K}_{d_n}^0$ for the open unit ball of $\mathbb{R}_{d_n}$ and

$$\mathscr{F}_n = \{x_1, \ldots, x_{\mathrm{N}_n}\}, \tag{4.8}$$

then the balls $x_j + \dfrac{\varepsilon_n}{16\,\mathrm{A}^{1/2}} \mathrm{K}_{d_n}^0$ ($j = 1, \ldots, \mathrm{N}_n$) are disjoint and their union is a subset of $\left(1 + \dfrac{\varepsilon_n}{16\,\mathrm{A}^{1/2}}\right) \mathrm{K}_{d_n}$. Hence if m_{d_n} is Lebesgue measure on $\mathbb{R}^{d_n}$, then

$$m_{d_n}(\lambda\, \mathrm{K}_{d_n}) = \lambda^{d_n} m_{d_n}(\mathrm{K}_{d_n})$$

for each $\lambda \geqq 0$ (and also for $\mathrm{K}_{d_n}^0$), and hence

$$\left(1 + \frac{\varepsilon_n}{16\,\mathrm{A}^{1/2}}\right)^{d_n} m_{d_n}(\mathrm{K}_{d_n}) \geqq \mathrm{N}_n \left(\frac{\varepsilon_n}{16\,\mathrm{A}^{1/2}}\right)^{d_n} m_{d_n}(\mathrm{K}_{d_n}^0).$$

Since $m_{d_n}(K^0_{d_n}) = m_{d_n}(K_{d_n})$, we thus have

$$\#\mathscr{F}_n = N_n \leqq (1 + 16\,A^{1/2}\,\varepsilon_n^{-1})^{d_n}, \tag{4.9}$$

and since $\mathscr{F}_n$ is maximal that

$$\mathscr{F}_n + \frac{\varepsilon_n}{8\,A^{1/2}} K^0_{d_n} \supseteqq \left(1 - \frac{\varepsilon}{16\,A^{1/2}}\right)^{1/2} K_{d_n}. \tag{4.10}$$

Now (4.10) implies

$$\mathscr{F}_n + \frac{\varepsilon_n}{4\,A^{1/2}} K^0_{d_n} \supseteqq K_{d_n}, \tag{4.11}$$

and letting $T : l^2 \to B$ be denoted by

$$T(k) = \sum_{i,\,j \geqq 1} a_{ij} k_i k_j \tag{4.12}$$

as in Lemma 1, we see that

$$T(\lambda V) = \lambda^2 \Sigma \tag{4.13}$$

where $V = \{ k \in l^2 : \| k \|_{l^2} \leqq 1 \}$. Now let

$$\tilde{\mathscr{F}}_n = T(\mathscr{F}_n) \tag{4.14}$$

and observe from (3.18) that if $x = \{x_j\}$, $y = \{y_j\} \in l^2$ then

$$\| T(x) - T(y) \|_B \leqq A^{1/2} (\| x \|_{l^2} + \| y \|_{l^2}) \| x - y \|_{l^2}.$$

Hence if $\| x \|_{l^2} \leqq 1$, $\| y \|_{l^2} \leqq 1$ we see that $\| x - y \|_{l^2} \leqq \varepsilon$ implies

$$\| T(x) - T(y) \|_B \leqq 2\,A^{1/2}\,\varepsilon. \tag{4.15}$$

Since

$$\mathscr{F}_n \subseteqq \left(1 - \frac{\varepsilon_n}{16\,A^{1/2}}\right)^{1/2} K_{d_n},$$

it follows that $\tilde{\mathscr{F}}_n = T(\mathscr{F}_n) \subseteqq \left(1 - \frac{\varepsilon_n}{16\,A^{1/2}}\right) \sum_{d_n}$ and (4.11) and (4.15) combine to imply $\tilde{\mathscr{F}}_n$ is an $\varepsilon_n/2$-net of $\sum_{d_n}$, *i.e.* recall $T(K_{d_n}) = \sum_{d_n}$. Thus by (3.17) we have

$$\begin{aligned} P(\tilde{\mathscr{F}}_n \nsubseteq (\pi_{d_n}(E_n/(2\,L\,n)))^{\varepsilon_n/2}) &\leqq \sum_{f \in \tilde{\mathscr{F}}_n} P\left(\bigcap_{r=1}^{n} \| \pi_{d_n} X_r/(2\,L\,n) - f \|_B \geqq \varepsilon_n/2 \right) \\ &\leqq \sum_{f \in \tilde{\mathscr{F}}_n} \left[1 - \exp\left\{ -L\,n \left(\sum_{j=1}^{d_n} k_j^2 + 3\,\varepsilon_n/\gamma \right) \right\} \right]^n \end{aligned} \tag{4.16}$$

where $k=(k_j)\in\mathcal{F}_n$, $\mathrm{T}(k)=f$. Since $\mathcal{F}_n \subseteqq \left(1-\frac{\varepsilon_n}{16\,\mathrm{A}^{1/2}}\right)^{1/2} \mathrm{K}_{d_n}$ we get from (4.9) and (4.16) that

$$\begin{aligned}
&\mathrm{P}(\tilde{\mathcal{F}}_n \nsubseteq (\pi_{d_n}(\mathrm{E}_n/(2\,\mathrm{L}\,n)))^{\varepsilon_n/2})\\
&\leqq \#\tilde{\mathcal{F}}_n\left[1-\exp\left\{-\mathrm{L}\,n\left(1-\frac{\varepsilon_n}{16\,\mathrm{A}^{1/2}}\right)-3\,\mathrm{L}\,n\,\varepsilon_n/\gamma\right\}\right]^n\\
&\leqq \exp\left\{16\,\mathrm{A}^{1/2}\,d_n\,\varepsilon_n^{-1}-n\exp\left\{-\mathrm{L}\,n\left(1-\frac{\varepsilon_n}{16\,\mathrm{A}^{1/2}}\right)-3\,\gamma^{-1}\,\varepsilon_n\,\mathrm{L}\,n\right\}\right\}\\
&=\exp\left\{16\,\mathrm{A}^{1/2}\,\gamma^{-1}\,\mathrm{L}\,n\,(\mathrm{L}_2\,n)^{-1}-\exp\left\{\frac{\varepsilon_n\,\mathrm{L}\,n}{16\,\mathrm{A}^{1/2}}-3\,\gamma^{-1}\,\varepsilon_n\,\mathrm{L}\,n\right\}\right\}.
\end{aligned}$$

Hence for $\gamma>96\,\mathrm{A}^{1/2}$ we have for n sufficiently large that

$$\begin{aligned}
(4.17)\qquad &\mathrm{P}(\tilde{\mathcal{F}}_n \nsubseteq (\pi_{d_n}(\mathrm{E}_n/(2\,\mathrm{L}\,n)))^{\varepsilon_n/2})\\
&\leqq \exp\{\mathrm{L}\,n\,(\mathrm{L}_2\,n)^{-1}/6-\exp\{3\,d_n\,\mathrm{L}_2\,n\}\}\\
&\leqq \exp\{-(\mathrm{L}\,n)^2\}.
\end{aligned}$$

Since the right terms of (4.17) form a convergent series, the Borel-Cantelli Lemma implies

$$(4.18)\qquad \mathrm{P}(\tilde{\mathcal{F}}_n \subseteqq (\pi_{d_n}(\mathrm{E}_n/(2\,\mathrm{L}\,n)))^{\varepsilon_n/2}\ \text{eventually})=1.$$

Since $\tilde{\mathcal{F}}_n$ is an $\varepsilon_n/2$-net for $\sum_{d_n}$, (4.18) thus implies for γ sufficiently large

$$(4.19)\qquad \mathrm{P}(\textstyle\sum_{d_n} \subseteqq (\tilde{\mathcal{F}}_n)^{\varepsilon_n/2} \subseteqq (\pi_{d_n}(\mathrm{E}_n/2\,\mathrm{L}\,n))^{\varepsilon_n}\ \text{eventually})=1$$

Now Lemma 4 and (2.14 *c*), which has been established, imply for γ sufficiently large that

$$(4.20)\qquad \mathrm{P}((\mathrm{I}-\pi_{d_n})(\mathrm{E}_n/(2\,\mathrm{L}\,n)) \subseteqq \varepsilon_n\,\mathrm{U}\ \text{eventually})=1.$$

By combining (4.19) and (4.20) we thus have for $\gamma>0$ sufficiently large that

$$(4.21)\qquad \mathrm{P}(\textstyle\sum_{d_n} \subseteqq (\mathrm{E}_n/(2\,\mathrm{L}\,n))^{2\,\varepsilon_n}\ \text{eventually})=1.$$

Since $\sum_{d_n}=\pi_{d_n}\Sigma$, and $\sigma(d_n)<\varepsilon_n/\gamma$ by (3.15), (4.21) thus implies for $\gamma>0$ sufficiently large that

$$(4.22)\qquad \mathrm{P}(\Sigma \subseteqq (\mathrm{E}_n/(2\,\mathrm{L}\,n))^{\varepsilon_n}\ \text{eventually})=1.$$

Hence (2.14 *d*) holds.

To prove (2.14 *e*) consider the following notation:

$$(4.23)\quad \begin{cases} V_d=\left\{(x_1, \ldots, x_d): \sum_{j=1}^{d} x_j^2 \leqq 1\right\} \\ T((x_1, \ldots, x_d))= \sum_{i,j=1}^{d} a_{ij} x_i x_j \\ G_n=(g_{1,n}, \ldots, g_{d,n}) \qquad (n \geqq 1) \end{cases}$$

where $\{g_{j,n}: j \geqq 1\}$ are as in the definition of X_n (2.14 *e*).

Now fix $\varepsilon > 0$. By the previous arguments there exists d such that

$$(4.24)\qquad P((I-\pi_d)(E_n/(2\,L\,n)) \subseteqq \varepsilon U \text{ eventually})=1,$$

and

$$(4.25)\qquad (I-\pi_d)(\Sigma) \subseteqq \varepsilon U.$$

Hence (2.14 *e*) holds if we show

$$(4.26)\qquad P(C(\{\pi_d(X_n/(2\,L\,n))\})=\sum_d)=1$$

Now

$$(4.27)\qquad P(G_n/(2\,L\,n)^{1/2} \in (1+\varepsilon)\, V_d \text{ eventually})=1,$$

and since $T: \mathbb{R}^d \to B$ is uniformly continuous on bounded sets of $\mathbb{R}^d$ with $T(V_d)=\sum_d$ we have (4.26) from (4.27) if

$$(4.28)\qquad P(C(\{G_n/(2\,L\,n)^{1/2}\})=V_d)=1$$

Now (4.28) follows from Theorem 4.1 in [CK] since

$$\lim_{\substack{n \to \infty \\ m-n \to \infty}} E(g_{i,n}\, g_{i,m})=0$$

suffices for this result. Hence (2.14 *e*) holds and Theorem 1 is proved.

5. SOME APPLICATIONS TO MULTIPLE ITO-WIENER INTEGRALS

For each $t \geqq 0$ let $Q_t(u_1, u_2)$ be a kernel on $\mathbb{R}^2$ such that

$$(5.1)\qquad \int_{\mathbb{R}^2} |Q_t(u_1, u_2)|^2\, du_1,\, du_2 < \infty.$$

Furthermore, assume

$$(5.2)\qquad X(t)=\int_{\mathbb{R}^2} Q_t(u_1, u_2)\, dB(u_1)\, dB(u_2) \qquad (t \geqq 0)$$

is a multiple Ito-Wiener integral such that the stochastic process $X=\{X(t):0\leqq t<\infty\}$ has a continuous version. Then the following lemma holds.

LEMMA. – *If* $X=\{X(t):0\leqq t<\infty\}$ *is as in* (5.2) *and is sample continuous, then for every* $T<\infty$, $p>0$, *we have*

$$E\|X\|_{\infty, T}^{p}<\infty \tag{5.3}$$

where $\|f\|_{\infty, T}=\sup_{0\leqq t<T}|f(t)|$.

Proof. – It is well known that if $Q_t(u_1, u_2)$ is replaced by the symmetric function $(Q_t(u_1, u_2)+Q_t(u_2, u_1))/2$ in (5.2), then the Ito-Wiener integral is unchanged. Hence we assume Q_t is symmetric from the start. Now let $\{h_n:n\geqq 1\}$ denote an orthonormal basis for $L^2(\mathbb{R}^1)$. Then $\{h_n h_m:n, m\geqq 1\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$ and we let

$$c_{ij}(t)=\int_{\mathbb{R}^2} Q_t(u_1, u_2)\,h_i(u_1)\,h_j(u_2)\,du_1\,du_2 \tag{5.4}$$

for $t\geqq 0$ and $i, j\geqq 1$. We then have

$$Q_t(u_1, u_2)=\lim_N \sum_{i, j=1}^{N} c_{ij}(t)\,h_i(u_1)\,h_j(u_2) \tag{5.5}$$

where for each $t\geqq 0$ the limit is in $L^2(\mathbb{R}^2)$. Thus the theory of Ito-Wiener integrals implies for each $t\geqq 0$ fixed that

$$X(t)=\lim_N \sum_{i, j=1}^{N} c_{ij}(t)\int_{\mathbb{R}^2} h_i(u_1)\,h_j(u_2)\,dB(u_1)\,dB(u_2) \tag{5.6}$$

where the limit is in $L^2(\Omega, \mathscr{F}, P)$ and $\{B(t):-\infty<t<\infty\}$ is a sample path continuous Brownian motion on $(\Omega, \mathscr{F}, P)$ with $B(0)=0$. Elementary facts regarding Ito-Wiener multiple integrals also imply that

$$\int_{\mathbb{R}^2} h_i(u_i)\,h_j(u_2)\,dB(u_1)\,dB(u_2)=g_i g_j-\delta(i, j) \tag{5.7}$$

for $i, j\geqq 1$ where

$$g_i=\int_{-\infty}^{\infty} h_i(u)\,dB(u) \qquad (i\geqq 1)$$

and

$$\delta(i, j)=\begin{cases}1 & i=j\\ 0 & i\neq j.\end{cases}$$

Hence we can rewrite (5.6) as

$$X(t)=\lim_{N} \sum_{i,j=1}^{N} c_{ij}(t)(g_i g_j - \delta(i,j)) \tag{5.8}$$

with the limit being in $L^2(\Omega, \mathscr{F}, P)$ for each $t \geqq 0$. Furthermore, since $\{h_n : n \geqq 1\}$ is orthonormal, the sequence $\{g_i : i \geqq 1\}$ is i.i.d. $N(0, 1)$, and hence it is well known that for each $t \geqq 0$ the convergence in (5.8) is also with probability one.

Now take $\{t_1, t_2, \ldots\}$ dense in $[0, T]$. Then sample function continuity of $X = \{X(t) : t \geqq 0\}$ implies $\|X\|_{\infty, T} = \sup_{n \geqq 1} |X(t_n)|$ with probability one. Furthermore, since $\{t_i : i \geqq 1\}$ is countable and (5.8) converges with probability one for each $t \geqq 0$ we get

$$\|X\|_{\infty, T} = \sup_{n \geqq 1} \Big\| \sum_{i,j \geqq 1} c_{ij}(t_n)(g_i g_j - \delta(i,j)) \Big\| \tag{5.9}$$

Hence $\{X(t) : t \geqq 0\}$ is a centered Gaussian chaos of order two in the sense that is used in [LT90] for the uncentered Gaussian chaos. Thus using their arguments it follows that the quantity on the right hand side of (5.9) has moments of all orders. Thus (5.3) holds and the Lemma is proved. We now can state the following theorem.

THEOREM 2. – *Let* $X, X_1, X_2, \ldots$ *be identically distributed* $C[0, T]$ *valued random vectors with* X *as in* (5.2) *and such that the condition in* (5.1) *holds. Then there exists functions* $\{c_{ij}(t) : i, j \geqq 1\}$ *in* $C[0, T]$ *such that*

$$X(.) = \sum_{i,j \geqq 1} c_{ij}(.)(g_i g_j - \delta(i,j)) \tag{5.10}$$

where the partial sums of X converge uniformly in $C[0, T]$ *with probability one. Furthermore, if* $\sigma(d)$, $m(d)$, *and* $M(d)$ *are defined in terms of the* $C[0, T]$ *valued Gaussian chaos*

$$\sum_{i,j \geqq 1} c_{ij}(.)(g_i g_j - \delta(i,j)) \tag{5.11}$$

as in (2.4), (2.5), *and* (2.6), *and* $\{d_n\}$ *and* $\{\varepsilon_n\}$ *are given by* (2.11) *and* (2.12), *then*

$$P(E_n/2\,L n \subseteq \Sigma^{\varepsilon_n} \textit{ eventually}) = 1 \tag{5.12}$$

where

$$\Sigma = \Big\{ f(t) = \int_{\mathbb{R}^2} Q_t(u_1, u_2)\, k(u_1)\, k(u_2)\, du_1\, du_2 : 0 \leqq t \leqq T, \int_{-\infty}^{\infty} k^2(u)\, du \leqq 1 \Big\} \tag{5.13}$$

is a compact subset of $C[0, T]$ *and* $E_n = \{X_1, \ldots, X_n\}$. *Furthermore, if* $X, X_1, X_2, '''$ *are also independent, then*

$$P(\Sigma \subseteq (E_n/2 L n)^{\varepsilon_n} \text{ eventually}) = 1. \tag{5.14}$$

Remark. — The analogue of (*e*) in Theorem 1 also holds for Theorem 2.

Proof. — Since X takes values in $C[0, T]$, the previous lemma implies $\|X\|_{\infty, T}$ has moments of all order. Thus for $i \neq j$ we have from (5.8) that

$$E(X(t) g_i g_j) = 2 c_{ij}(t) \qquad (t \geqq 0), \tag{5.15}$$

and now we can show $c_{ij}(t)$ is continuous for $t \in [0, T]$ as the left hand term in (5.15) is continuous for $t \in [0, T]$, *i. e.* for any p, $1 < p < \infty$

$$|c_{ij}(t) - c_{ij}(s)| \leqq (E|X(t) - X(s)|^p)^{1/p} (E|g_i g_j|^q)^{1/q},$$

and since $\{X(t)\}$ is assumed continuous,

$$\lim_{t \to s} E|X(t) - X(s)|^p) = 0 \quad \text{for all } s, t \in [0, T]$$

by the DCT with dominating function $2^p \|X\|_{\infty, T}^p$ being integrable by the previous lemma. Similarly, each $c_{ii}(t)$ is continuous on $[0, T]$.

If $\sigma(g_1, \ldots, g_k)$, $k \geqq 1$, denotes the minimal σ-field making $g_1, \ldots, g_k$ measurable, then $\{X(t) : t \geqq 0\}$ being continuous a.s. and $E\|X\|_{\infty, T}^p < \infty$ for some $p > 1$ implies the conditional expectation

$$f_k(.) = E(X(.) | \sigma(g_1, \ldots, g_k)) \tag{5.16}$$

is continuous on $[0, T]$ with probability one.

Now for each t, with probability one

$$f_d(t) = E(X(t) | \sigma(g_1, \ldots, g_d)) = \sum_{i, j=1}^{d} c_{ij}(t)(g_i g_j - \delta(i, j)), \tag{5.17}$$

and since all terms are in $C[0, T]$ and $E\|X\|_{\infty, T} < \infty$, we have $\{f_d : d \geqq 1\}$ a martingale with values in $C[0, T]$ with (5.17) now holding as an element of $C[0, T]$. By the vector valued martingale convergence theorem of S. D. Chatterji as presented in [PK] we get

$$\lim_d f_d = E(X(.) | \sigma(g_1, g_2, \ldots)) \tag{5.18}$$

in the sup-norm $\|.\|_{\infty, T}$. Furthermore, since $X(.)$ is $\sigma(g_1, g_2, \ldots)$ measurable with values in $C[0, T]$ we get with probability one that

$$\lim_d \sup_{0 \leqq t \leqq T} \left| X(t) - \sum_{i, j=1}^{d} c_{ij}(t)(g_i g_j - \delta(i, j)) \right| = 0. \tag{5.19}$$

Applying Theorem 1 to X we now see that the limit set Σ for Theorem 2 is given by

$$\Sigma = \left\{ \sum_{i, j \geqq 1} c_{ij}(t) k_i k_j : \|k\|_2 \leqq 1 \right\} \tag{5.20}$$

and Σ is a compact subset of C[0, T]. Furthermore, if $k(.)\in L^2(\mathbb{R}^1)$ denotes the function whose Fourier coefficients are $\{k_j : j \geqq 1\}$, then by (5.4)

$$\sum_{i,j\geqq 1} c_{ij}(t) k_i k_j = \sum_{i,j\geqq 1} \int_{\mathbb{R}^2} Q_t(u_1, u_2) h_i(u_1) h_j(u_2) \times \int_{-\infty}^{\infty} k(s) h_i(s)\, ds \int_{-\infty}^{\infty} k(v) h_j(v)\, dv\, du_1\, du_2$$

$$= \int_{\mathbb{R}^2} Q_t(u_1, u_2) k(u_1) k(u_2)\, du_1\, du_2.$$

and hence (5.20) is the limit set claimed in (5.13). Hence Theorem 2 is proved.

6. FUNCTIONAL LIL'S FOR SELF-SIMILAR PROCESSES

In [MO86] functional LIL's are obtained for a variety of self-similar processes expressed in terms of multiple Ito-Wiener integrals of dimension m, and having self-similarity parameter H, $\frac{1}{2} < H < 1$. This restriction on H resulted from the implementation of an intricate approximation procedure showing that it sufficed to prove the result for self-similar processes given by multiple Ito-Wiener integrals to which an integration by parts formula could be applied. Here our approach is different, and H is allowed to satisfy $0 < H < \infty$. Further comments and comparisons with the Mori-Oodaira paper are included after the statement of Theorem 3 below. Some comments related to [Ba86] appear in the remark following the proof of Theorem 3 below.

The processes we consider are represented by the multiple Ito-Wiener integrals

$$X(t) = \int_{\mathbb{R}^2} k_t(u_1, u_2)\, dB(u_1)\, dB(u_2) \qquad (t \geqq 0) \tag{6.1}$$

and the kernels $\{k_t : t \geqq 0\}$ are assumed to be of the form

$$k_t(u_1, u_2) = t^{H-1} f\left(\frac{u_1}{t}, \frac{u_2}{t}\right) \qquad (t > 0,\ 0 < H < \infty) \tag{6.2}$$

where $k_0 = 0$ and f satisfies

$$\int_{\mathbb{R}^2} f^2(u_1, u_2)\, du_1\, du_2 < \infty. \tag{6.3}$$

THEOREM 3. — *Let* $\{X(t): t \geqq 0\}$ be a stochastic process given as in (6.1) where $k_0 = 0$ and $\{k_t : t > 0\}$ satisfies (6.2) and (6.3) with $0 < H < \infty$. Let

$$E_n = \{X(k(.))/k^H : k = 1, \ldots, n\}$$

and set

$$(6.4) \quad \Sigma = \left\{ g(t) = \int_{\mathbb{R}^2} k_t(u_1, u_2) h(u_1) h(u_2) du_1 du_2, \ 0 \leqq t \leqq 1, \int_{-\infty}^{\infty} h^2(u) du \leqq 1 \right\}.$$

In addition, assume

$$(6.5) \qquad \{X(t): t \geqq 0\} \text{ has continuous sample paths,}$$

and let

$$(6.6) \qquad \| f(.) \|_\infty = \sup_{0 \leqq t \leqq 1} | f(t) |.$$

Then, $\{X(t): t \geqq 0\}$ is a self-similar process with index H, $0 < H < \infty$, Σ is a compact subset of C [0, 1], and for each $\varepsilon > 0$

$$(6.7) \qquad P(E_n/(2 L_2 n) \subset \Sigma + \varepsilon U \text{ eventually}) = 1$$

where $U = \{f \in C[0, 1] : \| f \|_\infty < 1\}$. Furthermore, we have clustering throughout Σ in the sup-norm on C [0, 1], so that

$$P(C(\{X(n(.))/(2 n^H L_2 n)\}) = \Sigma) = 1.$$

Remark. — For suitable *f*, the kernels k_t defined in (6.2) allow an application of the integration by parts formula for multiple Ito-Wiener integrals established in [MO86]. Under these circumstances one also has $\{X(n(.))/(2 n^H L_2 n) : n \geqq 1\}$ clustering throughout Σ in the $C_\gamma(\mathbb{R}^+)$ topology defined in [MO86]. This is the result of their Theorem 3.3. Convergence in the $C_\gamma(\mathbb{R}^+)$ topology, applied to processes, essentially amounts to uniform convergence on compact subsets of [0, ∞). Hence our Theorem 3 is considerably more general than Theorem 3.3 in [MO86], but $C_\gamma(\mathbb{R}^+)$ is a subspace of the continuous functions on [0, ∞), and we have not shown our processes all live in $C_\gamma(\mathbb{R}^+)$. This could be done, but we chose not to do so.

Remark. — The condition (6.5) of sample function continuity for $\{X(t): t \geqq 0\}$ can be verified through a variety of conditions on the function *f* in (6.2). This is pursued in [MO86] in their Lemma's 6.2 and 6.3, and the reader should note that continuity is really a separate issue from the delicate approximations in [MO86]. Hence it holds for far more general kernels than those applicable for the main results in [MO86]. Also we point out that it is possible to show that $\varepsilon > 0$ in (6.7) can be replaced by suitable $\varepsilon_n \downarrow 0$ obtained from Theorem 2 applied along a subsequence, and then interpolating to the whole sequence. The details can be seen from the proof below.

Remark. – If we had chosen to work in the L^2-norm rather than the sup-norm in Theorems 2 and 3, then our results hold for any kernel f satisfying (6.3) as long as we take a separable measurable version for $\{X(t): t \geqq 0\}$. This follows since it is easy to check that under (6.2) and (6.3) with $k_0 = 0$, $R(s, r) = E(X(s)X(t))$ is continuous for $s, t \geqq 0$. Hence with probability one the jointly measurable separable version of $\{X(t): t \geqq 0\}$ is such that for each $T > 0$

$$E\left(\int_0^T X^2(s)\,ds\right) = \int_0^T R(s, s)\,ds < \infty.$$

Hence $\{X(t): t \geqq 0\}$ has sample paths with finite L^2 norm on $[0, T]$ for any $T > 0$, and our proof applies directly replacing the sup-norm by the L^2 norm with Σ compact in $L^2[0, T]$ in this situation. The only change required is that the evaluation linear functionals used to prove (5.3) need be replaced by countably many linear functionals on $L^2([0, T]$ whose supremum yields the L^2 norm. Then (5.3) would state that $E\|X\|_{T,2}^p < \infty$ for all $p > 0$ where $\|f\|_{T,2} = \left(\int_0^T f^2(s)\,ds\right)^{1/2}$.

Proof of Theorem 3. – That $\{X(t): 0 \leqq t < \infty\}$ is self-similar of index H, $0 < H < \infty$, follows from (6.2) and that the Brownian motion $\{B(t): 0 \leqq t < \infty\}$ is self-similar of index $\frac{1}{2}$. Also, Σ is a compact subset of $C[0, 1]$ by a simple application of Theorem 2 with $T = 1$. The next step of the proof is the following rescaling lemma.

LEMMA 6. – *Let* $n_r = \exp\{r/(L r)^2\}$ *and* $d(n) = 2 n^H L_2 n$. *If* $I(r) = [n_r, n_{r+1}]$, $\varepsilon > 0$, *and*

$$P(X(n_r(.))/d(n_r) \in \Sigma + \varepsilon U \textit{ eventually}) = 1, \tag{6.8}$$

then

$$P(X(n(.))/d(n) \in \Sigma + 2\varepsilon U \textit{ eventually}) = 1. \tag{6.9}$$

Proof. – If (6.8) holds, take $h_r \in \Sigma$ such that

$$\|X(n_{r+1}(.))/d(n_{r+1}) - h_r\| \leqq \varepsilon. \tag{6.10}$$

For $n \in I(r)$, set

$$g(t) = h_r(nt/n_{r+1}) \qquad (0 \leqq t \leqq 1). \tag{6.11}$$

Then g depends on n and r but we suppress that. Furthermore, $g \in \Sigma$ since

$$h(t) = \int_{\mathbb{R}^2} k_t(u_1, u_2)\, x(u_1)\, x(u_2)\, du_1\, du_2 \qquad (0 \leqq t \leqq 1)$$

and (6.2) implies

$$\begin{aligned} g(t) = h(\lambda t) &= \int_{\mathbb{R}^2} k_{\lambda t}(u_1, u_2)\, x(u_1)\, x(u_2)\, du_1\, du_2 \\ &= \int_{\mathbb{R}^2} (\lambda t)^{\mathrm{H}-1} f\left(\frac{u_1}{\lambda t}, \frac{u_2}{\lambda t}\right) x(u_1)\, x(u_2)\, du_1\, du_2 \\ &= \int_{\mathbb{R}^2} (\lambda t)^{\mathrm{H}-1} f\left(\frac{v_1}{t}, \frac{v_2}{t}\right) x(\lambda v_1)\, x(\lambda v_2)\, \lambda^2\, dv_1\, dv_2 \\ &= \lambda^{\mathrm{H}+1} \int_{\mathbb{R}^2} k_t(v_1, v_2)\, x(\lambda v_1)\, x(\lambda v_2)\, dv_1\, dv_2 \\ &= \int_{\mathbb{R}^2} k_t(v_1, v_2)\, (\lambda^{(\mathrm{H}+1)/2} x(\lambda v_1))\, (\lambda^{(\mathrm{H}+1)/2} x(\lambda v_2))\, dv_1\, dv \end{aligned}$$

Hence, for $0 < \lambda \leqq 1$ and $\int_{-\infty}^{\infty} x^2(u)\, du \leqq 1$ we have

$$g(t) = h(\lambda t) \in \Sigma,$$

as

$$\left(\int_{-\infty}^{\infty} \lambda^{\mathrm{H}+1} x^2(\lambda v_1)\, dv_1\right)^{1/2} = \left(\int_{-\infty}^{\infty} \lambda^{\mathrm{H}} x^2(s)\, ds\right)^{1/2} \leqq 1.$$

Since $g \in \Sigma$ and (6.10) holds, for $n \in \mathrm{I}(r)$

$$(6.12) \quad \| \mathrm{X}(n(.))/d(n) - g \|_\infty = \| g - \mathrm{X}(n(.))/d(n_{r+1})) \|_\infty + \| (\mathrm{X}(n(.))/d(n))(1 - d(n)/d(n_{r+1})) \|_\infty.$$

Now by rescaling, (6.10) and (6.11) imply

$$(6.13) \qquad \| g - \mathrm{X}(n(.))/d(n_{r+1}) \|_\infty \leqq \varepsilon$$

Furthermore, if $n \in \mathrm{I}(r)$, then for large r

$$(6.14) \qquad | 1 - d(n)/d(n_{r+1}) | \leqq 2/(\mathrm{L}\, r)^2$$

and hence, by rescaling again and (6.10),

$$(6.15)\qquad \|(X(n(.))/d(n))(1-d(n)/d(n_{r+1}))\|_\infty \leqq \left\|\frac{X(n(.))}{d(n)}\right\|_\infty .2/(L\,r)^2 \leqq \|X(n_{r+1}(.))/d(n_{r+1})\|_\infty . d(n_{r+1})/d(n) . \frac{2}{(L\,r)^2} \leqq 4(\|h_r\|_\infty+\varepsilon)/(L\,r)^2.$$

As $r\to\infty$ we see from (6.13) and (6.15) that for all $n\in I(r)$ with probability one

$$\|X(n(.))/d(n)-g\|_\infty \leqq 2\,\varepsilon$$

and hence (6.9) holds.

In view of Lemma 6 we now turn to the verification of (6.8), and since $\varepsilon>0$ is arbitrary with $L_2\,n_{r+1}\sim L\,r$, it suffices to prove

$$(6.16)\qquad P(X(n_r(.))/(2\,n_r^H\,L\,r)\in\Sigma+\varepsilon\,U \text{ eventually})=1.$$

Fix $\varepsilon>0$ and define

$$X_r=X(n_r(.))/n_r^H$$

for $r\geqq 1$. Then $X, X_1, X_2, \ldots$ are identically distributed with

$$X(t)=\int_{\mathbb{R}^2} k_t(u_1, u_2)\,dB(u_1)\,dB(u_2). \qquad (t\geqq 0).$$

Hence an application of Theorem 2 immediately yields (6.16), and it remains to verify (6.7) with

$$E_n=\{X(k(.))/k^H : k=1. \ldots, n\}.$$

However, this follows from (6.9) and the argument given at the and of the proof of (2.14 c) since $\varepsilon>0$ was arbitrary.

Now we turn to the proof of the clustering result, namely that

$$P(C(\{X(n(.))/(2\,n^H\,L_2\,n)\})=\Sigma)=1.$$

Since $\{X(t):t\geqq 0\}$ is self-similar with parameter H, the processes $\{X(n(.))/n^H\}$ are identically distributed and satisfy (6.5). Hence by Theorem 2 they are Gaussian chaos. In fact, (6.2) and the proof of

Theorem 2 shows that if $\{h_i : i \geqq 1\}$ is any CONS of $L^2(\mathbb{R}^1)$, then

$$X(nt) = \int_{\mathbb{R}^2} k_{nt}(u, v)\, dB(u)\, dB(v)$$
$$= n^{H-1} \int_{\mathbb{R}^2} k_t(u/n, v/n)\, dB(u)\, dB(v)$$
$$= n^{H-1} \sum_{i, j \geqq 1} c_{ij}(t) \int_{\mathbb{R}^2} h_i(u/n)\, h_j(v/n)\, dB(u)\, dB(v),$$

and the series converges uniformly in t, $0 \leqq t \leqq 1$. Letting

$$g_{i,n} = n^{-1/2} \int_{\mathbb{R}^1} h_i(u/n)\, dB(u)$$

for $i = 1, 2$. and $n \geqq 1$, we thus have

$$X(n(t))/n^H = \sum_{i, j \geqq 1} c_{ij}(t)\, (g_{i,n}\, g_{j,n} - \delta(i, j)).$$

To prove the clustering we now apply Theorem 1 and (2.14e) along the subsequence $n_r = r^r$. Since $L_n n_r \sim L r$ as $r \to \infty$, it suffices to prove

$$P(C(\{X(n_r(.))/(2 n_r^H L r)\}) = \Sigma) = 1.$$

Now $\{g_{i, n_r} : i \geqq 1\}$ are independent $N(0, 1)$, and

$$E(g_{i, n_r}\, g_{j, n_s}) = (n_r n_s)^{-1/2} \int_{\mathbb{R}^1} h_i(u/n_r)\, h_j(u/n_s)\, du,$$

hence (2.14e) yields the result provided $\lim_{\substack{r \to \infty \\ s \to r \to \infty}} E(g_{i, n_r}\, g_{j, n_s}) = 0$. To obtain this last condition we specialize our choice of basis $\{h_i : i \geqq 1\}$ to be the Hermite functions, *i.e.* $h_i(x) = H_i(x)\, e^{-x^2/4}$ where $\{H_i(x) : i \geqq 1\}$ are the Hermite polynomials. Then

$$E(g_{i, n_r}\, g_{j, n_s}) = (n_r/n_s)^{1/2} \int_{\mathbb{R}^1} h_i(v)\, h_j\left(\frac{n_r v}{n_s}\right) dv$$
$$= (n_r/n_s)^{1/2} \int_{\mathbb{R}^1} H_i(v)\, H_j\left(\frac{n_r v}{n_s}\right) \exp\left\{ -\frac{v^2}{4}\left(1 + \left(\frac{n_r}{n_s}\right)^2\right)\right\} dv,$$

and since $H_i(x)$ and $H_j(x)$ are polynomials, the dominated convergence theorem easily implies

$$\lim_{\substack{r \to \infty \\ s - r \to \infty}} E(g_{i, n_r}\, g_{i, n_s}) = 0.$$

Thus (2.14e) applies and the clustering holds. Thus Theorem 3 is proved.

Remark. – The special class of processes considered in [MO86] all satisfy (6.1), (6.2), (6.5), and hence Theorem 3 applies to these processes. It is also easy to see that results similar to those in Theorem 2 and 3 can be obtained for multiple stochastic integrals of the form

$$X(t)=\int_{\mathbb{R}^2} k_t(u_1, u_2)\, dB_1(u_1)\, dB_2(u_2) \tag{6.17}$$

where B_1 and B_2 are independent Brownian notions and the k_t are suitable L^2-kernels. These results can be proved in exactly the same fashion since such $X(t)$ are decoupled Gaussian chaos, and an analogue of Theorem 1 holds for not necessarily symmetric decoupled chaos with compact limit set

$$\Sigma=\{x=\sum_{i, j\geqq 1} a_{ij} h_i k_j : \|h+k\|_{l^2}\leqq 1\}. \tag{6.18}$$

When $T=1$, and $X(t)$ is as in (6.17), the limit set derived from (6.18) as in Theorem 2 is easily seen to be

$$\Sigma=\left\{g(t)=\int_{\mathbb{R}^2} k_t(u, v)\, h_1(u)\, h_2(v)\, du\, dv,\ 0\leqq t\leqq 1, \right. \\ \left. \int_{-\infty}^{\infty} (h_1(u)+h_2(u))^2\, du\leqq 1\right\}. \tag{6.19}$$

For example, if

$$k_t(u, v)=I(0<u<v<t) \qquad (t>0),$$

then $k_t(u, v)=f(u/t, v/t)$ where $f(u, v)=I(0<v<1)$ and

$$X(t)=\int_0^t \int_0^v dB_1(u)\, dB_2(v) \qquad (t\geqq 0)$$

is self-similar with parameter $H=1$. Furthermore, $\{X(t): t\geqq 0\}$ satisfies (6.5) and (6.6), and if one believes the proposed analogue of Theorem 3, then for each $\varepsilon>0$

$$P(X(n(.))/(2nL_2 n)\in\Sigma+\varepsilon U \text{ eventually})=1$$

where Σ is the compact set given by (6.19) and $U=\{f\in C[0, 1]\,\|\,\|f\|_\infty<1\}$. We also then have

$$P(C(\{X(n(.))/(2nL_2 n)\})=\Sigma)=1.$$

Hence with probability one

$$\overline{\lim}_n X(n)/(2nL_2 n)=\sup_{g\in\Sigma} g(1)$$

with

$$g(1)=\int_0^1\int_0^v h_1(u)h_2(v)\,du\,dv$$

and h_1 and h_2 are arbitrary functions such that

$$\int_0^1 (h_1(u)+h_2(u))^2\,du \leqq 1$$

Examples of this type were considered previously in [Ba] where further references can be found.

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