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Intersection Local Times and Tanaka Formulas

by

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ABSTRACT. – A new approach to intersection local times of Brownian motion is given, using additive functionals of a single Markov process and stochastic calculus. New results include the Tanaka formula for the k-multiple points of self-intersection local time and the joint Hölder continuity in all variables of renormalized self-intersection local time for k-multiple points, $k \geqq 4$.

Key words : Intersection local times, Tanaka formula, renormalization, Brownian motion, diffusions, multiple points.

RÉSUMÉ. – Nous donnons une nouvelle approche à l'étude des temps locaux d'intersection du mouvement brownien. Elle se sert de la théorie des fonctionnelles additives d'un seul processus de Markov et du calcul stochastique. Parmi les résultats nouveaux figurent la formule de Tanaka des temps locaux d'intersection pour les points de multiplicité k et la continuité dans toutes les variables du temps local d'intersection renormalisé pour les points de multiplicité k, $k \geqq 4$.

Classification A.M.S. : 60 J 55, 60 J 65.

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1. INTRODUCTION

It has been known for quite some time that 3-dimensional Brownian motion has double points and that 2-dimensional Brownian motion has k-multiple points for every positive integer k. It has been known for not quite as long a time that one can construct a local time for these multiple points, that is, a functional that increases only at the times when Brownian motion has a multiple point and that measures in some sense how many of these times there are. These intersection local times (ILTs) have been constructed by means of Fourier analysis, by means of stochastic calculus, and by the study of additive functionals of several Markov processes. Through the work of Dynkin, LeGall, Rosen, Yor, and others, a great deal is now known about ILTs; *see* [Dy], [L], [R2], [RY], and the references therein.

One of the main purposes of this paper is to introduce a new method of approaching the study of ILTs, via a combination of the theory of additive functionals for a single Markov process and stochastic calculus. This new method allows us to obtain, if not easily, at least systematically, many of the known results about ILTs. We concentrate on Brownian motion in this paper, but the method should also work for other nice diffusions and, to some extent, stable processes.

In addition to discussing our method, we obtain some new results as well. For example, we obtain Tanaka formulas for self-intersections of 2-dimensional Brownian motion of order k for any k (Sections 7, 8). These are similar to some formulas of [R1]. (While we were writing up this paper, we learned of the preprint of Shieh [Sh] who had also obtained the same Tanaka formulas as ours for any k by using white noise analysis.)

Using these Tanaka formulas we prove that one can renormalize ILT for k-multiple points in terms of lower order ILTs in such a way that the renormalized ILT is jointly Hölder continuous in every variable almost surely. This had been previously known only when $k=2$, 3. For other k various sorts of renormalizations were known, but the almost sure joint continuity of any of these renormalizations had been an open problem.

We also can obtain both weak and strong invariance principles that are uniform over all levels x for the convergence of ILTs of lattice valued random walks satisfying suitable moment conditions; these can be found in [BK1] and [BK2].

The basic idea is simple. Let us first consider the intersection of two independent Brownian motions X_t, Y_t. Fix u and define the (random) measure

$$\mu(A)=\int_0^u 1_A(X_s)\,ds.$$

Note μ is supported on the path of X_s. Elementary estimates show that a.s. the measure μ is sufficiently regular so that there is an additive functional of Brownian motion associated to it. That additive functional (for Y_t) is ILT for intersections of X_t and Y_t. Slightly more complicated measures give rise to ILTs for 3 or more Brownian motions. To get self-ILT for a single Brownian motion, we partition $[0, u]$ by points $s_0, s_1, \ldots, s_n$, we look at the intersections of X_t, $s_i \leqq t < s_{i+1}$, with X_r, $0 \leqq r < s_i$, we sum over i, and we then prove we get convergence as we let the partition become finer.

In Section 2, we construct ILTs for the intersection of 2 independent Brownian motions, while in Section 3 we do the same for the intersection of k independent Brownian motions. In Section 4 we obtain the Tanaka formula for these ILTs. Section 5 has some estimates on certain potentials, and Section 6 contains some preliminaries on the Hölder continuity of processes. Section 7 has the construction of ILTs for double points of a single Brownian motion and also the derivation of the Tanaka formula; Section 8 considers multiple points of a single Brownian motion. Finally, the proof of the a.s. joint continuity of renormalized ILTs is in Section 9.

The letter c, with or without subscripts, will denote constants whose exact value is unimportant and may change from line to line. The open ball of radius s about the point y is denoted $B(y, s)$.

2. INTERSECTIONS OF 2 BROWNIAN MOTIONS

Let X_t, Y_t be two independent Brownian motions in $\mathbb{R}^d$, $d=2$ or 3. If $d=3$, let $g(x, y)$ denote the Green function of Brownian motion. If $d=2$, let $g_R(x, y)$ denote the Green function of Brownian motion killed on exiting the ball $B(0, R)$.

Let

$$T_R = T_R(X) = \inf\{t : |X_t| \geqq R\}.$$

For each $x \in \mathbb{R}^d$ and $u \leqq 1$, define the random measure $\mu_{x,u}$ by

$$(2.1) \qquad \mu_{x,u}(A) = \int_0^u 1_A(X_r + x)\, dr.$$

LEMMA 2.1. – *For each* $\varepsilon \in (0, 1]$, *for almost all* ω *there exists* $K_\varepsilon(\omega)$ *such that*

$$(2.2) \qquad \mu_{x,u}(B(y, s))(\omega) \leqq K_\varepsilon(\omega)(s^{2-\varepsilon} \wedge 1)$$

for all $y \in \mathbb{R}^d$.

Proof. – Since $\mu_{x,u}(\mathbb{R}^d)\leqq u$, we may assume $s\leqq 1/2$. Let $R\geqq 2+2|x|$ and let

$$A_t=\int_0^{t\wedge T_R} 1_{B(y,s)}(X_r+x)\,dr.$$

If $d=2$, $g_R(w,z)\leqq c(1\vee\log(1/|w-z|))$, and so if $w\in B(0,R)$,

$$E^w A_{T_R}\leqq c\int_{B(y-x,s)}(1\vee\log(1/|w-z|))\,dz$$
$$\leqq c\int_{B(0,s)}\log(1/|z|)\,dz\leqq cs^2\log(1/s).$$

A similar calculation for $d=3$ gives $E^w A_{T_R}\leqq cs^2$.

Since A_t is an additive functional, the above implies

$$E^0[A_{T_R}-A_t\,|\,\mathscr{F}_t]\leqq E^{X_t}A_{T_R}\leqq\sup_w E^w A_{T_R}\leqq cs^{2-\varepsilon/2}.$$

By [DM], p. 193, $E^0\exp(\lambda A_{T_R})\leqq 2$ if $\lambda\leqq 1/8\sup_w E^w A_{T_R}$. Using Chebyshev, we get

$$P^0(A_{T_R}>c_1 s^{2-\varepsilon})\leqq 2\exp(-c_2 s^{-\varepsilon/2}). \tag{2.3}$$

Now $B(0,3R)$ can be covered by $N=cs^{-d}$ balls of radius $2s$, say $B_1,\ldots,B_N$, so that every ball $B(y,s)$, $y\in B(0,2R)$, is contained in one of the B_i's. Writing

$$D_R=\{\sup_{t\leqq 1}|X_t|\leqq R\},$$

(2.3) yields

$$P^0(\mu_{x,u}(B(y,s))\geqq c_1 s^{2-\varepsilon}\ \text{ for some } y\in B(0,2R);\,D_R)$$
$$\leqq P^0(\mu_{x,u}(B_i)\geqq c_1 s^{2-\varepsilon}\text{ for some } i=1,\ldots,N;\,D_R)$$
$$\leqq c_2 s^{-d}\exp(-c_3 s^{-\varepsilon/2}).$$

By a straightforward Borel-Cantelli argument with $s=2^{-i}$, $i=0,1,2,\ldots$,

$$P^0(\text{for some } y\in B(0,2R),\ \mu_{x,u}(B(y,2^{-i}))/(2^{-i})^{2-\varepsilon}>c,\ \text{i.o.};\,D_R)=0.$$

Hence, if $\omega\in D_R$, then for some $K_{\varepsilon R}(\omega)$,

$$\mu_{x,u}(B(y,2^{-i}))\leqq K_{\varepsilon R}(\omega)(2^{-i})^{2-\varepsilon}$$

for all $y\in B(0,2R)$, $i=0,1,2,\ldots$ If $s\in(0,1]$, then $s\in(2^{-(i+1)},2^{-i}]$ for some i. So, provided $\omega\in D_R$,

$$\mu_{x,u}(B(y,s))\leqq K_{\varepsilon R}(\omega)(2^{-i})^{2-\varepsilon}\leqq c\,K_{\varepsilon R}(\omega)\,s^{2-\varepsilon}. \tag{2.4}$$

for all $y\in B(0,2R)$, all $s\in(0,1]$. If $\omega\in D_R$, $\mu_{x,u}(B(y,s))=0$ if $y\notin B(0,2R)$.

Finally, each $\omega \in D_R$ for some R sufficiently large (except for a null set). This observation with (2.4) yields (2.2). □

Define

$$\mathscr{L} = \{ \psi : \psi \text{ maps } \mathbb{R}^d \text{ to } [-1, 1], \text{ and } \psi \text{ is Lipschitz with Lipschitz constant } 1 \}.$$

Define the metric d_L on the space of finite measures by

$$d_L(\mu, \nu) = \sup \left\{ \left| \int \psi \, d\mu - \int \psi \, d\nu \right| : \psi \in \mathscr{L} \right\}.$$

LEMMA 2.2.
(*a*) $d_L(\mu_{x,u}, \mu_{x,v}) \leq |u-v|$;
(*b*) $d_L(\mu_{x,u}, \mu_{y,u}) \leq u|x-y|$.

Proof. – (*a*) is obvious. For (*b*), notice

$$\left| \int \psi \, d(\mu_{x,u} - \mu_{y,u}) \right| = \left| \int_0^u [\psi(X_t + x) - \psi(X_t + y)] \, dt \right| \leq u|x-y|$$

since $\psi \in \mathscr{L}$. □

Lemma 2.1 implies that for ω not in the exceptional set, $g_R \mu_{x,u}(z)$ is continuous and bounded (*see* [BK1], Section 2). Let $\alpha_2(x, \cdot, u)$ be the continuous additive functional of Y_t associated with $\mu_{x,u}$, that is, the continuous additive functional such that $E^z \alpha_2(x, T_R(Y), u) = g_R \mu_{x,u}(z)$ for all z and R; the existence of $\alpha_2(x, \cdot, u)$ follows from [BG]. In stochastic calculus terms, $\alpha_2(x, \cdot, u)$ is the decreasing part of the supermartingale $g_R \mu_{x,u}(Y_{t \wedge T_R(Y)})$.

We will show that α_2 is jointly Hölder continuous in each variable. Before doing so, we need the following extension of some results of [BK1]. If $\mathscr{M}$ is a collection of positive measures, the d_L metric entropy $\mathscr{H}_L(\delta)$ is defined to be

$$\log(\inf\{n : \text{there exist } n \ d_L\text{-balls of radius } \delta \text{ that cover } \mathscr{M}\}).$$

PROPOSITION 2.3. – *Suppose $c, \gamma > 0$ and $\mathscr{M}$ is a collection of positive measures satisfying* (i) $\mu(B(y, s)) \leq c(s^{d-2+\gamma} \wedge 1)$ *for all* $s \in (0, \infty)$, $y \in \mathbb{R}^d$, $\mu \in \mathscr{M}$ *and* (ii) $\mathscr{H}_L(\delta) \leq c \log(1/\delta)$. *Let* L_t^μ *be the continuous additive functional associated to* μ. *Then* L_t^μ *is jointly Hölder continuous in* μ *and* t, *a.s.*

Remark. – *See* [BK1], Section 2 for the construction of L_t^μ.

Proof. – We will suppose $d=3$, the $d=2$ case being similar. That each L_t^μ is nondecreasing and continuous follows from its construction. So we only need the Hölder continuity. Let g be the Green function. By [BK1], Proposition 2.7, $g\mu$ is Hölder continuous. Hence,

$$E^x |g\mu(X_t) - g\mu(X_0)| \leq c E^x |X_t - X_0|^\alpha \leq c t^{\alpha/2}$$

for some $\alpha>0$, using the Burkholder-Davis-Gundy inequalities ([ReY], p. 151). Since $g\mu(X_t)-g\mu(X_0)+L_t^\mu$ is a mean 0 martingale, $E^x L_h^\mu \leqq c h^{\alpha/2}$, independent of x. By the argument of the first part of Lemma 2.1,

$$P^x(L_h^\mu \geqq c_1 h^{\alpha/2-\varepsilon}) \leqq c_2 \exp(-c_3 h^{-\varepsilon}).$$

Using the Markov property,

$$(2.5) \qquad P^x(L_{t+h}^\mu - L_t^\mu \geqq c_1 h^{\alpha/2-\varepsilon}) \leqq c_2 \exp(-c_3 h^{-\varepsilon}).$$

If we define a metric d_P on $\mathcal{M}\times[0, 1]$ by

$$d_P((\mu, t), (\nu, u)) = d_L(\mu, \nu) + |t-u|$$

and define the d_P metric entropy $\mathcal{H}_P(\delta)$ analogously to the definition of $\mathcal{H}_L(\delta)$, it is easy to see that $\mathcal{H}_P(\delta) \leqq c \log(1/\delta)$. With this, (2.5), and Propositions 2.1 and 2.8 of [BK1], our result now follows by standard metric entropy (*i.e.*, chaining) (*cf.* [Du]). □

THEOREM 2.4. — *There is a version of* $\alpha_2(x, r, u)$ *that is jointly Hölder continuous in* x, r, u.

Proof. — It is enough to let $R \geqq 1$ be arbitrary and to show Hölder continuity for $|x| \leqq R$. In view of Lemma 2.2, the d_L-metric entropy $\mathcal{H}_L(\delta)$ of $\{\mu_{x,u}: x\in B(0, R),\ u\in(0, 1]\}$ satisfies $\mathcal{H}_L(\delta) \leqq c \log(1/\delta)$. By applying Proposition 2.3, there exists a version of $\alpha_2(x, r, u)$ that is jointly Hölder continuous in x, u, and r. □

The question that remains is whether $\alpha_2(x, r, u)$ is actually what one means by ILT.

THEOREM 2.5. — *There exists a null set* N *such that if* $\omega \notin N$, *then*

$$(2.6) \qquad \int_{\mathbb{R}^d} f(x)\,\alpha_2(x, r, u)(\omega)\,dx = \int_0^u \int_0^r f(Y_s(\omega) - X_t(\omega))\,ds\,dt$$

for all bounded measurable f.

Proof. — Suppose $d=2$ and suppose f, h are continuous with compact support. Let

$$B_u^{x,h} = \int_0^u h(X_t - x)\,dt.$$

By a change of variables

$$\int f(x)\,B_u^{x,h}\,dx = \int h(x)\,B_u^{x,f}\,dx, \qquad \text{a.s.,}$$

or

$$(2.7) \qquad \int f(x)\Big(\int h(y)\,\mu_{-x,u}(dy)\Big)dx = \int h(x)\Big(\int f(y)\,\mu_{-x,u}(dy)\Big)dx.$$

We next use monotone convergence to see that (2.7) is valid for nonnegative h.

Now the right-hand side of (2.6) equals $\int_0^r \left(\int f(-y) \mu_{-Y_s, u}(dy) \right) ds$. So its potential in $B(0, R)$, considered as a continuous additive functional of Y, is

$$\int g_R(z, y) \left(\int f(-w) \mu_{-y, u}(dw) \right) dy.$$

By (2.7), this equals

$$\int f(-x) \int g_R(z, y) \mu_{-x, u}(dy)\, dx = \int f(x)\, g_R \mu_{x, u}(z)\, dx,$$

which is the potential of the left-hand side of (2.6). If two additive functionals of Brownian motion have the same potential, they are equal [BG]. Since R is arbitrary, this proves (2.6) when $d=2$ for this particular f. The case $d=3$ is similar but easier. Let N_f be the null set.

Let $\{f_i\}$ be a countable dense subset of the bounded continuous functions on $\mathbb{R}^d$ and let $N = \bigcup_i N_{f_i}$. If $\omega \notin N$, then by taking limits, (2.6) holds for bounded continuous f. It then holds for all bounded measurable f by a monotone class argument. □

3. INTERSECTION OF k BROWNIAN MOTIONS

In this section, we require $d=2$. We construct ILTs for k Brownian motions by induction. Denote the measures $\mu_{x, u}$ of Section 2 by $\mu^2_{x, u}$. Suppose $k \geqq 3$. Let $X^1_t, \ldots, X^{k-1}_t$ be $k-1$ independent Brownian motions and let Y_t be an additional independent Brownian motion. Suppose we have a collection of measures $\mu^{k-1}_{x_1, \ldots, x_{k-2}, r_1, \ldots, r_{k-2}}$ (denoted μ^{k-1} when no confusion results) and associated continuous additive functionals $\alpha_{k-1}(x_1, \ldots, x_{k-2}, r_1, \ldots, r_{k-2}, r_{k-1})$ satisfying

(3.1) for each ε there exists $K_\varepsilon(\omega)$ (depending on $x_1, \ldots, x_{k-2}, r_1, \ldots, r_{k-2}$) such that

$$\mu^{k-1}(B(y, s)) \leqq K_\varepsilon(\omega)(s^{2-\varepsilon} \wedge 1)$$

for all $y \in \mathbb{R}^2$, $s \in (0, \infty)$, and

(3.2) $\alpha_{k-1}(x_1, \ldots, x_{k-2}, r_1, \ldots, r_{k-1})$

is jointly Hölder continuous in all variables.

Define the random measure $\mu^k = \mu^k_{x_1, \ldots, x_{k-1}, r_1, \ldots, r_{k-1}}$ by

(3.3)
$$\mu^k(A) = \int_0^{r_{k-1}} 1_A(X_t^{k-1} + x_{k-1})\, \alpha_{k-1}(x_1, \ldots, x_{k-2}, r_1, \ldots, r_{k-2}, dt).$$

We need the analog of Lemma 2.1.

LEMMA 3.1. – *Suppose* (3.1) *and* (3.2) *hold. If* $\varepsilon > 0$, *there exists* $K_1(\omega)$ *such that*

$$\mu^k(B(y, s)) \leqq K_1(\omega)(s^{2-\varepsilon} \wedge 1)$$

for all $s \in (0, \infty)$, $y \in \mathbb{R}^2$.

Proof. – Define the additive functional A_t of X_t^{k-1} by

$$(3.4) \quad A_t = \int_0^t 1_{B(y, s)}(X_r^{k-1} + x_{k-1})\, \alpha_{k-1}(x_1, \ldots, x_{k-2}, r_1, \ldots, r_{k-2}, dr).$$

Since the potential of α_{k-1} on $B(0, R)$ (considered as an additive functional of X_t^{k-1}) is $g_R \mu^{k-1}$, then the potential of A_t (conditional on the processes $X^1, \ldots, X^{k-1}$) is

$$\int g_R(w, z)\, 1_{B(y, s)}(z)\, \mu^{k-1}(dz).$$

By Hölder's inequality with $p^{-1} + q^{-1} = 1$, this is less than or equal to

$$(3.5) \quad \left(\int g_R(w, z)^p \mu^{k-1}(dz)\right)^{1/p} \left(\int 1_{B(y, s)}(z)\, \mu^{k-1}(dz)\right)^{1/q}.$$

The second term in the product is bounded by $(K_{\varepsilon/3}(\omega)(s^{2-\varepsilon/3} \wedge 1))^{1/q}$, using (3.1). For the first term in the product, we write

$$\begin{aligned}
&\int g_R(w, z)^p \mu^{k-1}(dz) \\
&\quad \leqq c \int (1 \vee \log|w - z|)^p \mu^{k-1}(dz) \\
&\quad \leqq c \sum_{j=0}^{\infty} \int_{2^{-j} \leqq |w - z| \leqq 2^{-j+1}} (1 \vee \log|w - z|)^p \mu^{k-1}(dz) + c\, \mu^{k-1}(\mathbb{R}^2) \\
&\quad \leqq c \sum_{j=0}^{\infty} (j+1)^p \mu^{k-1}(B(w, 2^{-j})) + c\, \mu^{k-1}(\mathbb{R}^2) \\
&\quad \leqq c(\omega),
\end{aligned}$$

using (3.1). Taking q sufficiently close to 1, we get that the potential of A, conditional on the processes $X^1, \ldots, X^{k-1}$, is bounded by $c(\omega)(s^{2-\varepsilon/2} \wedge 1)$.

Using this estimate, we now proceed in a fashion very similar to Lemma 2.1. □

THEOREM 3.2. – *For each k, a version of* α_k *exists that is jointly Hölder continuous in each variable.*

Proof. – The proof is by induction. Suppose (3.1) and (3.2) hold. Write x for $(x_1, \ldots, x_{k-2})$, r for $(r_1, \ldots, r_{k-2})$, and define x' and r' analogously. Condition on $X^1, \ldots, X^{k-1}$. Let μ^k be defined by (3.3) and let

$$\nu^k(A) = \int_0^{r'_{k-1}} 1_A(X_t^{k-1} + x'_{k-1})\,\alpha_{k-1}(x', r', dt).$$

If ψ is in the class $\mathscr{L}$ (defined in Section 2),

$$(3.6)\quad \left|\int \psi\,d\mu^k - \int \psi\,d\nu^k\right| \leq \left|\int_0^{r_{k-1}} \psi(X_t^{k-1} + x_{k-1})\,\alpha_{k-1}(x, r, dt) - \int_0^{r'_{k-1}} \psi(X_t^{k-1} + x_{k-1})\,\alpha_{k-1}(x, r, dt)\right|$$
$$+ \int_0^{r'_{k-1}} |\psi(X_t^{k-1} + x_{k-1}) - \psi(X_t^{k-1} + x'_{k-1})|\,\alpha_{k-1}(x, r, dt)$$
$$+ \left|\int_0^{r'_{k-1}} \psi(X_t^{k-1} + x'_{k-1})\,(\alpha_{k-1}(x, r, dt) - \alpha(x', r', dt))\right|.$$

The first term on the right hand side of (3.6) is bounded by $|r_{k-1} - r'_{k-1}|\,\alpha_{k-1}(x, r, 1)$. By the definition of $\mathscr{L}$, the second term on the right hand side of (3.6) is bounded by $|x_{k-1} - x'_{k-1}|\,\alpha_{k-1}(x, r, 1)$. If $h(t)$ is differentiable in t with $\|h\|_\infty$ and $\|h'\|_\infty$ both bounded by 1, then by the joint Hölder continuity of α_{k-1} and integration by parts,

$$\left|\int_0^{r'_{k-1}} h(t)\,(\alpha_{k-1}(x, r, dt) - \alpha_{k-1}(x', r', dt))\right|$$
$$\leq |h(r'_{k-1})|\,|\alpha_{k-1}(x, r, r'_{k-1}) - \alpha_{k-1}(x', r', r'_{k-1})|$$
$$+ \int_0^{r'_{k-1}} |\alpha_{k-1}(x, r, t) - \alpha_{k-1}(x', r', t)|\,|h'(t)|\,dt$$
$$\leq c\,|(x, r) - (x', r')|^a$$

for some a. Now ψ is Lipschitz and X^{k-1} is a Brownian motion, hence Hölder continuous in t of order 1/4. Using a minor modification of Lemma 4.3 of [BK1], the third term on the right hand side of (3.6) is bounded by $c\,|(x, r) - (x', r')|^{a/4}$. Summing the three estimates, we conclude that for each ω, μ^k is Hölder continuous as a function of all variables with respect to the metric d_L.

Let $\alpha_k(x_1, \ldots, x_{k-1}, r_1, \ldots, r_k)$ be the continuous additive functional of Y_t corresponding to the measure μ^k. The metric entropy of the set $\{\mu^k : x_1, \ldots, x_{k-1} \in B(0, R), r_1, \ldots, r_{k-1} \in [0, 1]\}$ still is bounded by $c \log(1/\delta)$. So as in the proof of Theorem 2.4, there is a version of α_k that is jointly Hölder continuous in each variable. This establishes (3.2) with $k-1$ replaced by k. Lemma 3.1 establishes (3.1) with $k-1$ replaced by k. So by induction, (3.1) and (3.2) hold for all k. □

THEOREM 3.3. — *Except for a null set independent of f,*

$$\int \ldots \int f(x_1, \ldots, x_{k-1})\, \alpha_k(x_1, \ldots, x_{k-1}, r_1, \ldots, r_k)\, dx_1 \ldots dx_{k-1}$$
$$= \int_0^{r_k} \ldots \int_0^{r_1} f(X_{t_2}^2 - X_{t_1}^1, \ldots, Y_{t_k} - X_{t_{k-1}}^{k-1})\, dt_1 \ldots dt_k,$$

for f bounded and measurable on $(\mathbb{R}^2)^{k-1}$, *a.s.*

The proof of Theorem 3.3 is very similar to that of Theorem 2.5 and is left to the reader.

4. TANAKA FORMULAS

The Tanaka formulas for ILTs of independent Brownian motions are actually quite simple. We do the case $d=2$. Let us suppose $k=2$ first. Define

$$G(x) \equiv \frac{1}{\pi} \log(1/|x|). \tag{4.1}$$

Note $G(-x) = G(x)$.

By a formula of Brosamler [Br]

$$g_R \mu_{x,u}(Y_{t \wedge T_R}) - g_R \mu_{x,u}(Y_0) = \int_0^{t \wedge T_R} \nabla g_R \mu_{x,u}(Y_s) \cdot dY_s - \alpha_2(x, t \wedge T_R, u). \tag{4.2}$$

Since $G(\cdot - y) - g_R(\cdot, y)$ is harmonic in $B(0, R)$ for each y, so is $G\mu_{x,u}(\cdot) - g_R \mu_{x,u}(\cdot)$, and we also have by [Br]

$$(G\mu_{x,u} - g_R \mu_{x,u})(Y_{t \wedge T_R}) - (G\mu_{x,u} - g_R \mu_{x,\mu})(Y_0) = \int_0^{t \wedge T_R} \nabla (G\mu_{x,u} - g_R \mu_{x,u})(Y_s) \cdot dY_s. \tag{4.3}$$

Here

$$(4.4)\qquad G\,\mu_{x,u}(y)=\int G(y-z)\,\mu_{x,u}(dz).$$

Adding (4.2) and (4.3) and letting $R\to\infty$,

$$G\,\mu_{x,u}(Y_t)-G\,\mu_{x,u}(Y_0)=\int_0^t \nabla G\,\mu_{x,u}(Y_s)\cdot dY_s-\alpha_2(x,t,u).$$

Finally, recalling the definition of $\mu_{x,u}$, this and (4.4) yield

$$(4.5)\qquad \int_0^u G(Y_t-X_r-x)\,dr-\int_0^u G(Y_0-X_r-x)\,dr$$
$$=\int_0^t\left[\int_0^u \nabla G(Y_s-X_r-x)\,dr\right]\cdot dY_s-\alpha_2(x,t,u).$$

The argument for ILTs of k Brownian motions is the same, and we get

THEOREM 4.1

$$(4.6)\qquad \int_0^{r_k}[G(Y_t-X_r^{k-1}-x_{k-1})$$
$$-G(Y_0-X_r^{k-1}-x_{k-1})]$$
$$\alpha_{k-1}(x_1,\ldots,x_{k-2},r_1,\ldots,r_{k-2},dr)$$
$$=\int_0^{r_{k-1}}\left[\int_0^{r_k}\nabla G(Y_s-X_r^{k-1}-x_{k-1})\right.$$
$$\left.\alpha_{k-1}(x_1,\ldots,x_{k-2},r_1,\ldots,r_{k-2},dr)\right]\cdot dY_s$$
$$-\alpha_k(x_1,\ldots,x_{k-1},r_1,\ldots,r_k).$$

Remark. — Recall that the way Brosamler's formulas are proved is by using Ito's formula and taking limits (*see* also [B1]). Therefore, provided μ is a sufficiently nice measure, we have

$$G\,\mu(Y_t)-G\,\mu(Y_0)=\int_0^t \nabla G\,\mu(Y_s)\cdot dY_s-L_t^{\mu}$$

whenever $Y_0\in\mathscr{F}_0(Y)$, that is, if Y_0 is independent of $\sigma(Y_s-Y_0:s\geqq 0)$. We will apply this fact in Sections 7 and 8 with μ taken to be μ^k.

5. SOME ESTIMATES

Before proceeding to the construction of ILT of double and multiple points of a single Brownian motion, we need some preliminary estimates.

PROPOSITION 5.1. — *Suppose* $a>0$. *Suppose* $\beta(t)$ *is a nondecreasing continuous process with* $\beta(0)\equiv 0$. *Suppose for each* $p\geqq 1$ *there exists* $K(p)\geqq 1$ *such that*

$$\text{(5.1)} \qquad E[\beta(t)-\beta(s)]^p \leqq K(p)\,|t-s|^{ap}, \qquad s,\ t\leqq 1.$$

Let Y_r *be 2-dimensional Brownian motion. Then there exists* $b_2>b_1>0$ *[not depending on p or* $K(p)$*] and constants* $c(p)$ *such that if* $p\geqq 1$, $x\in\mathbb{R}^2$, *and* $\sigma<1$, *then*

$$\text{(5.2)} \qquad P\left[\int_0^1 1_{B(x,\sigma)}(Y_r)\,\beta(dr)>\lambda\right]\leqq c(p)\,K(p)\,\frac{\sigma^{b_1 p}}{\lambda^{b_2 p}}.$$

Proof. — Let us assume $\lambda>2\sigma$, for otherwise the result is trivial. Fix x and define $R_t=|Y_t-x|$. Let $\varepsilon=1/16$. Let $S_1=\inf\{t:R_t\leqq\sigma\}$, $T_1=\inf\{t>S_1:R_t\geqq\sigma^{1-\varepsilon}\}$, $S_{i+1}=\inf\{t>T_i:R_t\leqq\sigma\}$, and $T_{i+1}=\inf\{t>S_{i+1}:R_t\geqq\sigma^{1-\varepsilon}\}$. Let $D_u=\inf\{i:S_i>u\}$. So D_u is greated than or equal to the number of upcrossings of $[\sigma, \sigma^{1-\varepsilon}]$by R_t up to time u.

Since $\log R_t$ is a martingale, by the upcrossing inequality (*see*, e.g., [Ch], p.332)

$$\sup_z E^z D_1 = E^\sigma D_1 \leqq \frac{E^\sigma|\log R_1|+|\log\sigma|}{|\log\sigma^{1-\varepsilon}-\log\sigma|}\leqq c_1.$$

By Chebyshev,

$$\sup_z P^z(D_1\geqq 2c_1)\leqq 1/2.$$

So by the strong Markov property applied at $\inf\{t:D_t\geqq 2nc_1\}$,

$$\sup_z P^z(D_1\geqq 2c_1(n+1))\leqq\frac{1}{2}\sup_z P^z(D_1\geqq 2c_1 n),$$

which leads to

$$\text{(5.3)} \qquad P(D_1\geqq n)\leqq c_2\exp(-c_3 n), \qquad n\geqq 1.$$

By the strong Markov property applied at S_i and standard estimates on Brownian motion,

$$\text{(5.4)} \qquad P(T_i-S_i>M\,\sigma^{2-3\varepsilon})\leqq P^0(T_1>M\,\sigma^{2-3\varepsilon})\leqq c_4\exp(-c_5 M).$$

Let $h\in[0,1]$. If $\beta((t+h)\wedge 1)-\beta(t)\geqq L h^{a/2}$ for some $t\in[0,1]$, then $\beta((j+2)h\wedge 1)-\beta(jh)\geqq L h^{a/2}$ for some $j\leqq[1/h]+1$. Equation (5.1) implies

$$P(\beta(t)-\beta(s)\geqq L|t-s|^{a/2})\leqq K(p)\,|t-s|^{ap/2}/L^p,$$

and so if $p>p_0=8/a$,

$$\text{(5.5)} \qquad P(\sup_{t\leqq 1}[\beta((t+h)\wedge 1)-\beta(t)]\geqq L h^{a/2})\leqq K(p)\,\frac{2}{h}\,\frac{h^{ap/2}}{L^p}\leqq K(p)\,\frac{2\,h^{ap/4}}{L^p}.$$

Note $R_{T_i} \geqq \sigma^{1-\varepsilon}$ and R_t does not return to the interval $[0, \sigma]$ until time S_{i+1}. So if $Y_r \in B(x, \sigma)$, then $r \in [S_i, T_i]$ for some i. Hence

$$(5.6) \qquad \int_0^1 1_{B(x, \sigma)}(Y_r)\, \beta(dr) \leqq \sum_{i=1}^{\infty} [\beta(T_i \wedge 1) - \beta(S_i \wedge 1)].$$

Let $n = [\lambda^d/\sigma^d]$, $M = n^d$, $h = M\sigma^{2-5\varepsilon}$, $L = \lambda/2 h^{a/2} n$, where d will be chosen in a moment. If the sum on the right-hand side of (5.6) is bigger than λ, then either (a) $D_1 > n$ or (b) $T_i - S_i \geqq M\sigma^{2-3\varepsilon}$ for some $i \leqq n$ or (c) $\beta(T_i \wedge 1) - \beta(S_i \wedge 1) > \lambda/2n$ for some $i \leqq n$ and $\max_{i \leqq n} (T_i - S_i) \leqq M\sigma^{2-3\varepsilon}$. So

$$\begin{aligned} P\left(\int_0^1 1_{B(x, \sigma)}(Y_r)\, \beta(dr) > \lambda\right) &\leqq P(D_1 > n) + n \sup_i P(T_i - S_i \geqq M\sigma^{2-3\varepsilon}) \\ &\quad + P(\sup_{t \leqq 1} [\beta((t+h) \wedge 1) - \beta(t)] > \lambda/2n) \\ &\leqq c_2 e^{-c_3 n} + n c_4 e^{-c_5 M} + 2K(p)\, h^{ap/4}/L^p. \end{aligned}$$

If we substitute for n, M, h, and L, recall that $\lambda > 2\sigma$ and $\sigma < 1$, and take d sufficiently small, we obtain our result for $p \geqq p_0$. The result (with the same b_1 and b_2) for $p \in [1, p_0)$ follows if $\sigma^{b_1} < \lambda^{b_2}$, while it is trivial if $\sigma^{b_1} \geqq \lambda^{b_2}$. □

Define, for $\zeta \in (0, 1)$,

$$(5.7) \qquad G_\zeta(x) = G(x) \wedge \frac{1}{\pi} \log(1/\zeta), \qquad H_\zeta(x) = G(x) - G_\zeta(x).$$

A consequence of Proposition 5.1 is

PROPOSITION 5.2. — *Suppose $a > 0$ and β satisfies the hypotheses of Propostion 5.1. There exists $d > 0$ and $n > 0$ (depending on a) and $\zeta_0 < 1$ such that if $p \geqq 1$, $q \geqq 1$ and $K'(p) = \sup_{1 \leqq r \leqq np} K(r)$, then*

$$E\left[\int_0^u |H_\zeta(X_u - X_r - x)|^q\, \beta(dr)\right]^p \leqq c(p, q)\, K'(p)\, \zeta^{dp}$$

if $u \in [0, 1]$, and $\zeta \leqq \zeta_0$.

Proof. — Write $V = \int_0^u |H_\zeta(X_u - X_r - x)|^q\, \beta(dr)$. Let $Y_r = X_u - X_r$. This is again 2-dimensional Brownian motion. Let b_1 and b_2 be the constants in the conclusion of Proposition 5.1 and let $n = [4/b_2] + 4$.

Note

$$H_\zeta^q(z + x) \leqq c_1 \sum_{\{j : 2^{-j} \leqq \zeta\}}^{\infty} j^q\, 1_{B(x, 2^{-j})}(z).$$

Note also that if ζ is sufficiently small, and $2^{-j} \leqq \zeta$, then $(20 c_1 j^{q+2})^{b_2} \leqq 2^{b_1 j/2}$. So, using Proposition 5.1,

$$
\begin{aligned}
P[V>\lambda] &\leqq \sum_{\{j:2^{-j}\leqq\zeta\}}^{\infty} P\left(c_1 j^q \int_0^u 1_{B(x,2^{-j})}(Y_r)\,\beta(dr) \geqq \lambda/20 j^2\right) \\
&\leqq c(np)\,K'(p) \sum_{\{j:2^{-j}\leqq\zeta\}}^{\infty} \frac{(2^{-j})^{b_1 np}}{(\lambda/20 c_1 j^{q+2})^{b_2 np}} \\
&\leqq c(p,q)\,K'(p) \sum_{\{j:2^{-j}\leqq\zeta\}}^{\infty} \frac{2^{-b_1 jnp/2}}{\lambda^{b_2 np}} \\
&\leqq c(p,q)\,K'(p)\,\zeta^{d_1 p}/\lambda^{p+2}
\end{aligned}
$$

if ζ is sufficiently small.

Multiplying by $p\lambda^{p-1}$ and integrating over λ from $\zeta^{1/4}$ to ∞ gives

$$
E[V^p;\ V \geqq \zeta^{1/4}] \leqq c(p,q)\,K'(p)\,\zeta^{d'p}.
$$

Since $E[V^p;\ V \leqq \zeta^{1/4}] \leqq \zeta^{p/4}$, adding gives our result. □

6. STOCHASTIC CALCULUS

When we get to double points and multiple points of a single Brownian motion, the joint Hölder continuity will take some work. In preparation for this, we derive some stochastic calculus results.

Suppose $U_t = M_t - B_t$, where M_t is mean zero martingale, B_t is a continuous non-decreasing process, $B_0 \equiv 0$, U and M have right continuous paths with left limits and U, M, and B are adapted to a filtration satisfying the usual conditions.

PROPOSITION 6.1. — *Let $a>0$. Suppose for each $p \geqq 1$ there exists $K(p) \geqq 1$ such that*

$$
E\,|U_t|^p \leqq K(p), \qquad t \leqq 1 \tag{6.1}
$$

and

$$
E\,|U_t - U_s|^p \leqq K(p)\,|t-s|^{ap}, \qquad s, t \leqq 1. \tag{6.2}
$$

Then there exists $b>0$ and n_1 (independent of p) and constants $c(p)$ such that if $p \geqq 1$ and $K'(p) = \sup_{1 \leqq r \leqq n_1 p} K(r)$, then

$$
EB_1^p \leqq c(p)\,K'(p) \tag{6.3}
$$

and

$$
E(B_t - B_s)^p \leqq c(p)\,K'(p)\,|t-s|^{pb}, \qquad s, t \leqq 1. \tag{6.4}
$$

Remark. — Applying (6.4) with $p>1/b$ implies that there is a dense subset of [0, 1] on which B_t is Hölder continuous, a.s. Since B_t is continuous, this implies B_t is Hölder continuous on [0, 1], a.s.

Proof. — It suffices to prove the result for $p \geqq p_0 = 2/a$, since we can get the result for $p<p_0$ by using Jensen's inequality.

By a standard chaining argument as in the proof of Kolmogorov's theorem (*see* the remark following the proof of Theorem 9.3), (6.1) and (6.2) imply that we can find a version of U_t such that $P(\sup_{t \leqq 1} |U_t|>\lambda) \leqq c(p) K(p+1)(1 \wedge \lambda^{-(p+1)})$. Multiplying by $p\lambda^{p-1}$ and integrating from 0 to ∞, we get $E \sup_{t \leqq 1} |U_t|^p \leqq c(p) K'(p)$. Since U_t and $-B_t$ differ by a martingale, for all $t \leqq 1$

$$E(B_1 - B_t \mid \mathscr{F}_t) = E(U_t - U_1 \mid \mathscr{F}_t) \leqq 2 E(\sup_s |U_s| \mid \mathscr{F}_t).$$

By a standard inequality (*see* for example, [B2], Lemma 2.3),

$$EB_1^p \leqq c(p) E \sup_t |U_t|^p.$$

This and (6.1) proves (6.3).

Similarly, the same chaining argument shows that for some d,

$$E \sup_{s \leqq r \leqq t} |U_r - U_s|^p \leqq c(p) K'(p) |t-s|^{dp}.$$

To get (6.4), apply the above argument to $B'_r = B_{s+r} - B_s$, $U'_r = U_{s+r} - U_s$, $M'_r = M_{s+r} - M_s$, $r \leqq t-s$. □

Now suppose $U_t^i = M_t^i - B_t^i$, $i=1, 2$, with $B_0^i \equiv 0$, B_t^i nondecreasing and continuous, and M_t^i a martingale. Let $B_t = B_t^1 - B_t^2$, and similarly for M_t, U_t. Suppose that $\mathscr{F}_t$ is the filtration generated by a finite number of Brownian motions.

Proposition 6.2. — *Let a, b, $\delta \in (0, 1)$. Suppose for each p there exists $K(p)$ such that*

$$E|U_t^i|^p \leqq K(p), \qquad t \leqq 1, \ i=1, 2,$$
$$E|U_t^i - U_s^i|^p \leqq K(p) |t-s|^{ap}, \qquad s, t \leqq 1, \ i=1, 2,$$

and

$$E|U_t|^p \leqq K(p) \delta^{bp}, \qquad t \leqq 1. \tag{6.5}$$

Then there exists $d>0$ and n_1 such that if $K'(p) = \sup_{1 \leqq r \leqq n_1 p} K(r)$, *then*

$$E|B_t|^p \leqq c(p) K'(p) \delta^{dp}, \qquad t \leqq 1. \tag{6.6}$$

Proof. — As in the proof of Proposition 6.1, for some n_1, $a'>0$

$$E \sup_{s \leqq t \leqq s+h} |U_t^i - U_s^i|^p \leqq c(p) K'(p) h^{a'p}, \qquad i=1, 2.$$

If $n \geqq 1$,

$$\sup_{t \leqq 1} |U_t| \leqq \sup_{j \leqq n} |U_{j/n}| + \sum_{i=1}^{2} \sup_{j \leqq n} \sup_{j/n \leqq t \leqq (j+1)/n} |U_t^i - U_{j/n}^i|.$$

Hence

$$\begin{aligned} \mathrm{E} \sup_t |U_i|^p &\leqq c(p)\, n \sup_{j \leqq n} \mathrm{E} |U_{j/n}|^p \\ &\quad + 2c(p)\, n \max_{1 \leqq i \leqq 2} \sup_{j \leqq n} \mathrm{E}\Big(\sup_{j/n \leqq t \leqq (j+1)/n} |U_t^i - U_{j/n}^i|^p \Big) \\ &\leqq c(p)\, n\, \mathrm{K}'(p)\, \delta^{bp} + 2nc(p)\, \mathrm{K}'(p)\, (1/n)^{a'p}. \end{aligned}$$

We may as before suppose without loss of generality (changing n_1 if necessary) that $p > 2/a' + 2$. Since $a'p > 2$, take $n = [\delta^{-b/2}] + 1$ to get

$$(6.7) \qquad \mathrm{E} \sup_t |U_t|^p \leqq c(p)\, \mathrm{K}'(p)\, \delta^{a'bp/2}.$$

Let $Z = \sup_t |U_t|$ and $W = 1 + B_1^1 + B_1^2$. By Proposition 6.1, $W \in L^p$ for all p. Observe that if $t \leqq 1$,

$$|\mathrm{E}(B_1 - B_t \mid \mathscr{F}_t)| = |\mathrm{E}(U_t - U_1 \mid \mathscr{F}_t)| \leqq 2\,\mathrm{E}(Z \mid \mathscr{F}_t).$$

So as in the proof of [B 2, Lemma 2.3],

$$\begin{aligned} (6.8) \qquad \mathrm{E}[(B_1 - B_t)^2 \mid \mathscr{F}_t] &= 2\,\mathrm{E}\Big[\int_t^1 (B_1 - B_s)\, dB_s \,\Big|\, \mathscr{F}_t \Big] \\ &= 2\,\mathrm{E}\Big[\int_t^1 \mathrm{E}(B_1 - B_s \mid \mathscr{F}_s)\, dB_s \,\Big|\, \mathscr{F}_t \Big] \\ &\leqq 4\,\mathrm{E}\Big[\int_t^1 \mathrm{E}(Z \mid \mathscr{F}_s)\, d(B_s^1 + B_s^2) \,\Big|\, \mathscr{F}_t \Big] \\ &\leqq 4\,\mathrm{E}[Z(B_1^1 + B_1^2) \mid \mathscr{F}_t] \leqq 4\,\mathrm{E}[ZW \mid \mathscr{F}_t]. \end{aligned}$$

Next, let $V_t = \mathrm{E}(B_1 - B_t \mid \mathscr{F}_t)$, and $N_t = \mathrm{E}(B_1 \mid \mathscr{F}_t)$, so that $V_t = N_t - B_t$ (take the right continuous version of V and N). Since $B_1 \in L^p$ for all p, the same is true of N and V. Since $\mathscr{F}_t$ is a Brownian filtration, N_t is continuous. By Jensen's inequality,

$$V_t^2 = (\mathrm{E}(B_1 - B_t \mid \mathscr{F}_t))^2 \leqq \mathrm{E}[(B_1 - B_t)^2 \mid \mathscr{F}_t] \leqq 4\,\mathrm{E}[ZW \mid \mathscr{F}_t].$$

Also, by Ito's lemma,

$$V_1^2 - V_t^2 = 2 \int_t^1 V_s\, dV_s + \langle N \rangle_1 - \langle N \rangle_t$$

Therefore,

$$\begin{aligned} E[\langle N\rangle_1-\langle N\rangle_t \mid \mathcal{F}_t] &\leq \left| E[V_1^2-V_t^2 \mid \mathcal{F}_t]\right| + 2\left| E\left[\int_t^1 V_s\, dV_s \mid \mathcal{F}_t\right]\right| \\ &\leq 8\, E[ZW \mid \mathcal{F}_t] + 2\left| E\left[\int_t^1 V_s\, dB_s \mid \mathcal{F}_t\right]\right| \\ &\leq 8\, E[ZW \mid \mathcal{F}_t] + 2\, E\left[\int_t^1 2\, E(Z \mid \mathcal{F}_s)\, d(B_s^1+B_s^2) \mid \mathcal{F}_t\right] \\ &\leq 12\, E[ZW \mid \mathcal{F}_t]. \end{aligned}$$

Finally, by [B 2, Lemma 2.3], Proposition 6.1, and (6.7),

$$\begin{aligned} E\langle N\rangle_1^p \leq c(p)\, E(ZW)^p &\leq (EZ^{2p})^{1/2}(EW^{2p})^{1/2} \\ &\leq c(2p)\, K'(2p)\, \delta^{2a'bp/2})^{1/2}(K'(2p))^{1/2} \\ &\leq c(p)\, K'(2p)\, \delta^{a'bp/2}. \end{aligned}$$

By Jensen again,

$$E|V_t|^{2p} \leq E[(4\, E[ZW \mid \mathcal{F}_t]^p)] \leq c(p)\, E[(ZW)^p] \leq c(p)\, K'(2p)\, \delta^{a'bp/2}.$$

Therefore,

$$\begin{aligned} E|B_t|^{2p} &\leq c(p)\, E|N_t|^{2p} + c(p)\, E|V_t|^{2p} \\ &\leq c(p)\, E\langle N\rangle_1^p + c(p)\, E|V_t|^{2p} \\ &\leq c(p)\, K'(2p)\, \delta^{a'bp/2}. \end{aligned}$$

Letting $d=a'b/4$ and taking n_1 larger if necessary completes the proof. □

7. DOUBLE POINTS

We now want to construct self-ILT for double points for a single Brownian motion X_t and derive the associated Tanaka formula. These results were first obtained by Yor [Y]. For concreteness, we restrict ourselves to 2-dimensional Brownian motion. Write $\beta(s)=s$ so that $\beta(ds)=ds$.

Fix t, let $\Delta_n=2^{-n}$, and let $s_i=t\wedge i\Delta_n$, $i=0, \ldots, 2^n$. We want to apply the results of Sections 2 and 4 with $u=s_i$ and $Y_r=(X_{s_i+r}-X_{s_i})+X_{s_i}=X_{s_i+r}$, $0\leq r\leq\Delta$. For $x\in\mathbb{R}^2$, let $\mu_{x,u}(A)=\int_0^{s_i} 1_A(X_r)\, dr$. As in Section 2, there exists continuous additive functionals of Y_r, say $\alpha_2^{ni}(x, \cdot)$, that if $A_{n,i,x}=\alpha_2^{ni}(x, s_{i+1}-s_i)$, then

$$(7.1)\quad \int_0^{s_i}[G(X_{s_{i+1}}-X_r-x)-G(X_{s_i}-X_r-x)]\,\beta(dr) = \int_{s_i}^{s_{i+1}}\left[\int_0^{s_i}\nabla G(X_s-X_r-x)\,\beta(dr)\right]\cdot dX_s - A_{n,i,x},$$

$A_{n,i,x}\geqq 0$, $A_{n,i,x}$ is continuous in x, and

$$(7.2)\qquad \int f(x)\,A_{n,i,x}\,dx=\int_{s_i}^{s_{i+1}}\int_0^{s_i} f(X_r-X_s)\,ds\,dr.$$

Note that $X_{s_i+r}-X_{s_i}$ is independent of $Y_0=X_{s_i}$ and recall the remark following Theorem 4.1.

Let

$$U_t^n=U_t^n(x)=\sum_{i=0}^{2^n-1}\int_0^{s_i}[G(X_{s_{i+1}}-X_r-x)-G(X_{s_i}-X_r-x)]\,\beta(dr),$$

$$M_t^n=M_t^n(x)=\sum_{i=0}^{2^n-1}\int_{s_i}^{s_{i+1}}\left[\int_0^{s_i}\nabla G(X_s-X_r-x)\,\beta(dr)\right]\cdot dX_s,$$

$$\beta_t^n(n)=\sum_{i=0}^{2^n-1}A_{n,i,x},$$

and

$$U_t=U_t(x)=\int_0^t[G(X_t-X_r-x)-G(-x)]\,\beta(dr).$$

Summing (7.1) over i, we get

$$(7.3)\qquad U_t^n=M_t^n-\beta_t^n(x).$$

PROPOSITION 7.1. – *Suppose $x\neq 0$. If $p>1$, then*

(*a*) $$\sup_{t\leqq 1} E\,|U_t^n-U_t|^p\to 0;$$

(*b*) $$\sup_n E\,|U_t^n-U_s^n|^p\leqq c\,(E\,[\beta(t)-\beta(s)]^{2p-2})^{1/2}+c\,E\,|U_t-U_s|^p.$$

Remark. – Of course, $\beta(t)=t$; we write the proposition this way so as to be able to use it in Section 8.

Proof. — We have

$$\begin{aligned}U_t^n &= \sum_{i=0}^{2^n-1}\sum_{j=0}^{i-1}\int_{s_j}^{s_{j+1}}[G(X_{s_{i+1}}-X_r-x)-G(X_{s_i}-X_r-x)]\,\beta(dr)\\ &= \sum_{j=0}^{2^n-1}\sum_{i=j+1}^{2^n-1}\int_{s_j}^{s_{j+1}}[G(X_{s_{i+1}}-X_r-x)-G(X_{s_i}-X_r-x)]\,\beta(dr)\\ &= \sum_{j=0}^{2^n-1}\int_{s_j}^{s_{j+1}}[G(X_t-X_r-x)-G(X_{s_{j+1}}-X_r-x)]\,\beta(dr).\end{aligned}$$

So to prove (a), it suffices to prove

$$\int_0^t h_r^n\,\beta(dr)\to 0 \quad \text{in } L^p, \text{ uniformly over } t\leqq 1.$$

where

$$h_r^n=\sum_{j=0}^{2^n-1}[G(-x)-G(X_{s_{j+1}}-X_r-x)]\,1_{(s_j,\,s_{j+1})}(r).$$

By Hölder's inequality and then Cauchy-Schwarz,

$$E\left|\int_0^t h_r^n\,\beta(dr)\right|^p\leqq\left(E\int_0^t |h_r^n|^{2p}\,\beta(dr)\right)^{1/2}(E\,\beta(t)^{2p-2})^{1/2},$$

and since $E\,\beta(t)^{2p-2}=t^{2p-2}$, to prove (a) it suffices to prove

$$(7.4)\qquad E\sum_{j=0}^{2^n-1}\int_{s_j}^{s_{j+1}}|G(-x)-G(X_{s_{j+1}}-X_r-x)|^{2p}\,\beta(dr)\to 0.$$

Also,

$$E\,|U_t^n-U_s^n|^p\leqq c(p)\,E\,|(U_t^n-U_t)-(U_s^n-U_s)|^p+c(p)\,E\,|U_t-U_s|^p.$$

By Hölder and Cauchy-Schwarz again, the first term on the right is less than or equal to

$$c(p)\,E\left|\int_s^t h_r^n\,\beta(dr)\right|^p\leqq c(p)\left(E\int_0^t |h_r^n|^p\,\beta(dr)\right)^{1/2}(E(\beta(t)-\beta(s))^{2p-2})^{1/2}.$$

So proving (7.4) is also sufficient to prove (b).

Choose ζ small enough so that $G_\zeta(z)=G(z)$ for $z\in B(x,|x|/2)$. Observe that G_ζ is differentiable a.e. and $\|\nabla G_\zeta\|_\infty=c\,\zeta^{-1}$.

Note

$$(7.5)\quad \mathrm{E}\sum_{j=0}^{2^n-1}\int_{s_j}^{s_{j+1}} |G_\zeta(-x)-G_\zeta(X_{s_{j+1}}-X_r-x)|^{2p}\,\beta(dr)$$
$$\leqq \|\nabla G_\zeta\|_\infty^{2p}\,\mathrm{E}\Big[\sum_j \int_{s_i}^{s_{j+1}}\beta(dr) \sup_{\substack{u,\,v\leqq 1\\ |u-v|\leqq\Delta_n}} |X_u-X_v|^2 p\Big]$$
$$\leqq c\,\zeta^{-2p}(\mathrm{E}\,\beta(1)^2)^{1/2}(\mathrm{E}(\sup_{\substack{|u-v|\leqq\Delta_n\\ u,\,v\leqq 1}} |X_u-X_v|^{4p}))^{1/2}\leqq c(p)\,\zeta^{-2p}\Delta_n^p.$$

Let

$$V=\{\sup_{\substack{|u-v|\leqq\Delta_n\\ u,\,v\leqq 1}} |X_u-X_v|>|x|/2\}.$$

By standard estimates on the Brownian path (*see*, e.g., [IM], p. 37),

$$PV\leqq \exp(-|x|^2/c_1\Delta_n).$$

By our choice of ζ, $H_\zeta(-x)=0$ and

$$(7.6)\quad \mathrm{E}\Big[\sum_{j=0}^{2^n-1}\int_{s_j}^{s_{j+1}} |H_\zeta(-x)-H_\zeta(X_{s_{j+1}}-X_r-x)|^{2p}\,\beta(dr);\ V^c\Big]=0.$$

On the other hand, noticing the inequality

$$\mathrm{E}\Big(\sum_{j=0}^{2^n-1} Z_j\Big)^2\leqq 2^n\,\mathrm{E}\sum Z_j^2\leqq 2^{2n}\sup_j \mathrm{E}Z_j^2,$$

we get

$$(7.7)\quad \mathrm{E}\Big[\sum\int_{s_j}^{s_{j+1}} |H_\zeta(X_{s_{j+1}}-X_r-x)|^{2p}\,\beta(dr);\ V\Big]$$
$$\leqq\Big(\mathrm{E}\Big(\sum\int_{s_j}^{s_{j+1}} |H_\zeta(X_{s_{j+1}}-X_r-x)|^{2p}\,\beta(dr)\Big)^2\Big)^{1/2}(PV)^{1/2}$$
$$\leqq c\,2^n\Big(\sup_j \mathrm{E}\Big(\int_0^{s_{j+1}} |H_\zeta(X_{s_{j+1}}-X_r-x)|^{2p}\,\beta(dr)\Big)^2\Big)^{1/2}$$
$$\times\exp(-|x|^2/c_1\Delta_n)$$
$$\leqq c\,2^n\exp(-|x|^2/c_1\Delta_n),$$

using Proposition 5.2. If we add (7.5), (7.6), and (7.7) and let $\zeta=\zeta_n\to 0$ as $n\to\infty$ so that $\Delta_n^{1/2}\leqq\zeta_n^2$, we get our desired result. □

PROPOSITION 7.2. — *$\beta_t^n(x)$ increases as $n \to \infty$. If we call the limit $\beta_2(x, t)$, and if f is continuous with compact support, then a.s.*

$$\int f(x)\,\beta_2(x, t)\,dx = \int_0^t \int_0^s f(X_r - X_s)\,dr\,ds. \tag{7.8}$$

Proof. — If φ_ε is a nonnegative symmetric approximation to the identity with compact support, then by (7.2),

$$\int \varphi_\varepsilon(x - x_0)\,\beta_t^n(x)\,dx = \sum_{i=0}^{2^n - 1} \int_{s_i}^{s_{i+1}} \int_0^{s_i} \varphi_\varepsilon(X_r - X_s - x_0)\,dr\,ds. \tag{7.9}$$

For each n, the left-hand side converges a.s. to $\beta_t^n(x_0)$ as $\varepsilon \to 0$ since each $A_{n, i, x}$ is continuous in x. And for each fixed ε, the right-hand side of (7.9) is increasing in n. We conclude that for each $x_0 \neq 0$, $\beta_t^n(x_0)$ increases as $n \to \infty$. Call the limit $\beta_2(x_0, t)$.

By monotone convergence

$$\begin{aligned} \int f(x)\,\beta_2(x, t)\,dt &= \lim_{n \to \infty} \int f(x)\,\beta_t^n(x)\,dx \\ &= \lim_{n \to \infty} \int_0^t \int_0^s f(X_r - X_s)\,1_{(r \leqq s_i \text{ if } s_i \leqq s < s_{i+1})}\,dr\,ds \\ &= \int_0^t \int_0^s f(X_r - X_s)\,dr\,ds, \end{aligned}$$

and (7.8) is proved. □

We define $\beta_2'(x, t)$ to be the limit of $\beta_t^n(x)$ for each $x \in \mathbb{R}^2 - \{0\}$, t rational. By the argument of Proposition 7.2, it is easy to see that $\beta_2'(x, t) \geqq \beta_2'(x, s)$, a.s., if $t \geqq s$. For $t \in [0, 1]$, let

$$\beta_2(x, t) = \inf_{u \geqq t,\ u \text{ rational}} \beta_2'(x, u).$$

Recall $G(-x) = G(x)$.

LEMMA 7.3. — *For each $p \geqq 1$, there exists $\nu(p)$ such that*

(*a*) $$E\,|U_t(x)|^p \leqq c(p)\,(1 \vee |G(x)|)^{\nu(p)}, \qquad t \leqq 1;$$

(*b*) *There exists $a > 0$ such that*

$$E\,|U_t(x) - U_s(x)|^p \leqq c(p)\,(1 \vee |G(x)|)^{\nu(p)}\,|t - s|^{ap}, \qquad s, t \leqq 1.$$

Proof. — $G(x)\,\beta(t)$ trivially has moments of all orders. Take ζ small but fixed. Note that $\int_0^t H_\zeta(X_t - X_r - x)\,\beta(dr)$ has p^{th} moments by

Proposition 5.2, while

$$\left|\int_0^t G_\zeta(X_t - X_r - x)\,\beta(dr)\right| \leqq c \log(1/\zeta)\,\beta(t).$$

This proves (*a*).

For (*b*),

$$\begin{aligned}|U_t - U_s| \leqq |G(x)|\,[\beta(t)-\beta(s)] &+ \left|\int_0^t H_\zeta(X_t - X_r - x)\,\beta(dr)\right| \\ &+ \left|\int_0^s H_\zeta(X_s - X_r - x)\,\beta(dr)\right| + \left|\int_s^t G_\zeta(X_t - X_r - x)\,\beta(dr)\right| \\ &+ \left|\int_0^s [G_\zeta(X_t - X_r - x) - G_\zeta(X_s - X_r - x)]\,\beta(dr)\right|\end{aligned}$$

So by Proposition 5.2,

$$\begin{aligned}(7.10)\quad E|U_t - U_s|^p \leqq\; & c(p)\,|G(x)|^p\,E|\beta(t)-\beta(s)|^p \\ & + c(p)\,\zeta^{dp} + c(p)\,\zeta^{dp} + c(p)\,|\log(1/\zeta)|^p\,E|\beta(t)-\beta(s)|^p \\ & + \|\nabla G_\zeta\|_\infty^p\,E\left(\int_0^s \beta(dr)\,|X_t - X_s|\right)^p \\ \leqq\; & (1 \vee |G(x)|)^p\,|t-s|^p + c(p)\,\zeta^{dp} + |\log(1/\zeta)|^p\,|t-s|^p \\ & + c\,|t-s|^{p/2}/\zeta^p,\end{aligned}$$

using Cauchy-Schwarz to get the last term on the right of (7.10). Taking $\zeta = |t-s|^b$ for suitable b proves (*b*). □

PROPOSITION 7.4. – *For each $p \geqq 1$ there exists $\nu(p)$ such that*

(*a*) $E\,\beta_2(x, t)^p \leqq c(p)\,(1 \vee |G(x)|)^{\nu(p)}, \qquad t \leqq 1,$

(*b*) *There exists $a > 0$ such that*

$$E|\beta_2(x, t) - \beta_2(x, s)|^p \leqq c(p)\,(1 \vee |G(x)|)^{\nu(p)}\,|t-s|^{ap}, \qquad s, t \leqq 1.$$

Proof. – By Propositions 7.1 and 7.3, $\sup_{t \leqq 1} E|U_t^n|^p < \infty$ and $E|U_t^n - U_s^n|^p \leqq c\,|t-s|^{ap}$ if $s \leqq t \leqq 1$, for some a and c depending on x but not n. Since β_t^n is continuous in t, we may apply Proposition 6.1 to conclude

$$(7.11)\qquad E(\beta_t^n)^p \leqq c(p)$$

and

$$(7.12)\qquad E|\beta_t^n - \beta_s^n|^p \leqq c(p)\,|t-s|^{bp}$$

for some c and b independent of n (but depending on x). Using the monotone convergence of $\beta_t^n(x)$ to $\beta_2'(x, t)$ for t rational and the monotonicity of $\beta_2(x, t)$ in t, we see $E\,\beta_2(x, t)^p < \infty$. Since $\beta_t^n(x)\uparrow$, the convergence is in L^p. So taking a limit in (7.12),

$$E|\beta_2(x, t)-\beta_2(x, s)|^p \leqq c(p)|t-s|^{bp}.$$

This implies that for each x, $\beta_2(x, t)$ is continuous on a dense subset of [0, 1], a.s. Since $\beta_2(x, t)$ is increasing in t, $\beta_2(x, t)$ is therefore continuous in t, a.s.

We have

$$E[\beta_1^n(x)-\beta_t^n(x)\,|\,\mathscr{F}_t]=E[U_t^n(x)-U_1^n(x)\,|\,\mathscr{F}_t].$$

Using the monotone convergence of $\beta_t^n(x)$ to $\beta_2'(x, t)$ for t rational, the monotonicity of $\beta_2(x, t)$ in t, and the L^p convergence of $U_t^n(x)$ to $U_t(x)$, we get $E[\beta_2(x, 1)-\beta_2(x, t)\,|\,\mathscr{F}_t]=E[U_t(x)-U_1(x)\,|\,\mathscr{F}_t]$. So $M_t=U_t(x)+\beta_2(x, t)$ is a martingale. Since we showed above that $\beta_2(x, t)$ is continuous in t, our result now follows from Proposition 6.1. □

Remark. – Since β_2 is increasing, Proposition 7.4 (*b*) implies $\beta_2(x, t)$ is Hölder continuous in t. As a consequence $\beta_t^n(x) \to \beta_2(x, t)$, uniformly for $t \in [0, 1]$, a.s., for each x.

PROPOSITION 7.5. – *The Tanaka formula*

$$(7.13)\quad \int_0^t [G(X_t-X_r-x)-G(-x)]\,\beta(dr) = \int_0^t \left[\int_0^s \nabla G(X_s-X_r-x)\,\beta(dr)\right]\cdot dX_s-\beta_2(x, t)$$

holds.

Proof. – As noted in the proof of Proposition 7.4, $\beta_t^n(x) \to \beta_2(x, t)$ in L^p, $p \geqq 1$. Since $U_t^n(x) \to U_t(x)$ in L^p, we conclude $M_t^n(x)$ converges in L^p, say to N_t. Since $M_t^n(x)=\int_0^t h_s^n \cdot dX_s$, where

$$h_s^n=\int_0^s \nabla G(X_s-X_r-x)\,1_{(r \leqq s_i \text{ if } s_i \leqq s < s_{i+1})}\,\beta(dr),$$

then $\int_0^t |h_s^n-h_s^m|^2\,ds=\langle M^n-M^m\rangle_t \to 0$. Since h_s^n converges for each s to $h_s=\int_0^s \nabla G(X_s-X_r-x)\,\beta(dr)$, then $\int_0^t |h_s^n-h_s|^2\,ds \to 0$. It follows that N_t must equal

$$M_t=M_t(x)=\int_0^t \left[\int_0^s \nabla G(X_s-X_r-x)\,\beta(dr)\right]\cdot dX_s,$$

and moreover $M_t(x)$ is square integrable. We then get (7.13) by taking a limit in (7.3). □

PROPOSITION 7.6. — *There exists $a>0$ such that if $x, x' \neq 0$,*

$$E|\beta_2(x,t)-\beta(x',t)|^p c(p)(|x|\wedge|x'|)^{-p}|x-x'|^{ap}.$$

Proof. — We can connect x to x' by an arc of length less than $c|x-x'|$ which never gets closer to the point 0 than $|x|\wedge|x'|$. Along this arc, $|\nabla G|$ is bounded by $c(|x|\wedge|x'|)^{-1}$. So

$$|G(-x)-G(-x')|\beta(t)\leqq \frac{c}{|x|\wedge|x'|}|x-x'|\beta(t),$$

which has pth moments of the desired form.

$E\left|\int_0^t H_\zeta(X_t-X_r-x)\beta(dr\right|^p \leqq c\zeta^{bp}$ by Proposition 5.2 and similarly with x replaced by x'. And finally,

$$E\left|\int_0^t [G_\zeta(X_t-X_r-x)-G_\zeta(X_t-X_r-x')]\beta(dr)\right|^p$$
$$\leqq c\|\nabla G_\zeta\|_\infty^p |x-x'|^p E\beta(t)^p \leqq c|x-x'|^p/\zeta^p.$$

So if we let $\zeta=|x-x'|^{1/2}$ and sum, we get

$$E|U_t^x-U_t^{x'}|^p\leqq c(p)(|x|\wedge|x'|)^{-p}|x-x'|^{ap}. \tag{7.14}$$

Now apply Proposition 6.2, using Lemma 7.3. □

Remark. — The $G(-x)\beta(t)$ term is what contributes the highly singular $(|x|\wedge|x'|)^{-p}$ term.

We finally can prove

THEOREM 7.7. — *There exists a version of $\beta_2(x,t)$ which is jointly Hölder continuous in $t\in[0,1]$ and $x\in\mathbb{R}^2-\{0\}$ and that satisfies* (7.8) *and* (7.13). *Moreover, outside a single null set,* (7.8) *holds for all bounded and measurable f.*

Proof. — By Propositions 7.4 and 7.6, there is a countable dense subset D of $\mathbb{R}^2$ and a countable dense subset T of [0, 1] so that $\beta_2(x,t)$ is uniformly continuous on $(x,t)\in(D\cap B(0,\delta^{-1})-B(0,\delta))\times T$ a.s. for each $\delta\in(0,1)$. For $x\neq 0$, define

$$\hat\beta_2(x,t)=\lim_{\substack{x_n\in D,\, t_n\in T\\ x_n\to x,\, t_n\to t}}\beta_2(x_n,t_n).$$

By the uniform continuity of $\beta_2(x_n,t_n)$, we see that $\hat\beta_2(x,t)$ is jointly continuous in x and t on $(\mathbb{R}^2-\{0\})\times[0,1]$. By Propositions 7.4 and 7.6, in fact $\hat\beta_2(x,t)=\beta_2(x,t)$, a.s., the null set depending on x and t. Since both β_2 and $\hat\beta_2$ are continuous in t, $\hat\beta_2(x,t)=\beta_2(x,t)$, $t\leqq 1$, a.s., the null set depending on x. Hence (7.13) holds with β_2 replaced by $\hat\beta_2$.

By Fubini, there is a null set N such that if $\omega\notin N$, $\hat\beta_2(x,t)=\beta_2(x,t)$ for a.e. x. If f is smooth with compact support in $\mathbb{R}^2-\{0\}$ and $\omega\notin N$, then

$$(7.15)\quad \int f(x)\,\hat{\beta}_2(x, t)\,dx = \int f(x)\,\beta_2(x, t) = \int_0^t \int_0^s f(X_r - X_s)\,dr\,ds.$$

This shows that (7.8) holds for each f with β_2 replaced by $\hat{\beta}_2$. We now proceed as in the last paragraph of the proof of Theorem 2.5 to obtain the last assertion of our theorem. □

Remark. — Using (7.14), it is not hard to show we can find a version of $U_t(x)$ that is jointly continuous in x and t provided $x \neq 0$. Defining $\hat{M}_t(x) = U_t(x) + \hat{\beta}(x, t)$, we see that we can find a single null set outside of which (7.13) holds for all $x \neq 0$ and all t.

For the purposes of the next section, we need

PROPOSITION 7.8. — *If $x \neq 0$, there exists* $K(\omega)$ *and* $\gamma > 0$ *such that*

$$(7.16)\quad \int_0^1 1_{B(y, s)}(X_r)\,\beta_2(x, dr) \leqq K(\omega)(s \wedge 1)^\gamma, \qquad y \in \mathbb{R}^2,\ s \in (0, \infty).$$

Proof. — By the finiteness of $\beta_2(x, 1)$, we may assume $s \leqq 1/2$. By Proposition 7.4 (*b*) and Proposition 5.1 with $Y_r = X_r$, $\beta = \beta_2$, there exists b_1 and b_2 such that

$$P\left[\int_0^1 1_{B(y, s)}(X_r)\,\beta_2(x, dr) > \lambda\right] \leqq c(p)\,s^{b_1 p}/\lambda^{b_2 p}$$

for each $p \geqq 1$. With this estimate for $p \geqq 8/a$ in place of (2.3), we may proceed very much as in the proof of Lemma 2.1. □

8. MULTIPLE POINTS.

We now want to construct ILT for k-multiple points of a single Brownian motion. Here $d = 2$. The proof is by induction. Recall $G(-x) = G(x)$. We let $G^\vee(x_1, \ldots, x_{k-1})$ denote the quantity $1 \vee |G(x_1)| \vee \ldots \vee |G(x_{k-1})|$ and let

$$N^\vee(x_1, \ldots, x_{k-1}) = 1 \vee |x_1|^{-1} \vee \ldots \vee |x_{k-1}|^{-1}.$$

THEOREM 8.1. — *Suppose $k \geqq 2$. Suppose $x_i \neq 0$, $i = 1, \ldots, k-1$. There exists positive reals a, γ, $\nu(p)$ for $p \geqq 1$ and nondecreasing processes $\beta_k(x_1, \ldots, x_{k-1}, t)$ such that*

$$(8.1)\quad E|\beta_k(x_1, \ldots, x_{k-1}, t)|^p \leqq c(p)(G^\vee(x_1, \ldots, x_{k-1}))^{\nu(p)};$$

$$(8.2)\quad E|\beta_k(x_1, \ldots, x_{k-1}, t) - \beta_k(x_1, \ldots, x_{k-1}, s)|^p \leqq c(p)(G^\vee(x_1, \ldots, x_{k-1}))^{\nu(p)}|t - s|^{ap};$$

(8.3) $$E|\beta_k(x_1, \ldots, x_{k-1}, t) - \beta_k(x'_1, \ldots, x'_{k-1}, t)|^p \leq c(p)(N^\vee(x_1, \ldots, x_{k-1}) + N^\vee(x'_1, \ldots, x'_{k-1}))^{\nu(p)} \times |(x_1, \ldots, x_{k-1}) - (x'_1, \ldots, x'_{k-1})|^{ap};$$

(8.4) *there exists* $K(\omega)$ (*depending on* $x_1, \ldots, x_{k-1}$) such that

$$\int_0^1 1_{B(y,s)}(X_r)\,\beta_k(x_1, \ldots, x_{k-1}, dr) \leq K(\omega)(s \wedge 1)^\gamma$$

for $y \in \mathbb{R}^2$, $s \in (0, \infty)$;

(8.5) β_k *is jointly Hölder continuous on* $(\mathbb{R}^2 - \{0\})^{k-1} \times [0, 1]$;

(8.6) $$\int_0^t [G(X_t - X_r - x_1) - G(-x_1)]\,\beta_{k-1}(x_2, \ldots, x_{k-1}, dr) = \int_0^t \left[\int_0^s \nabla G(X_s - X_r - x_1)\,\beta_{k-1}(x_2, \ldots, x_{k-1}, dr)\right] \cdot dX_s - \beta_k(x_1, \ldots, x_{k-1}, t);$$

(8.7) *except for a null set independent of* f,

$$\int \cdots \int f(x_1, \ldots, x_{k-1})\,\beta_k(x_1, \ldots, x_{k-1}, t)\,dx_1 \ldots dx_{k-1} = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-2}} f(X_{s_1} - X_t, \ldots, X_{s_{k-1}} - X_{s_{k-2}})\,ds_{k-1} \ldots ds_1$$

for all bounded measurable f.

Remark. – (8.6) was independently obtained by Shieh ([Sh]).

Proof. – If we write $\beta_1(t) = t$, (8.1)-(8.7) for the case $k = 2$ follow by Section 7. We use induction: we suppose we have the result for k and prove it for $k+1$. We write x for x_1, y for $(x_2, \ldots, x_{k-1})$.

Let $\Delta_n = 2^{-n}$ and let $s_i = t \wedge i\Delta_n$. Fix $u = s_i$ for the moment and set

$$\mu_{x,u}(A) = \int_0^u 1_A(X_r + x)\,\beta_k(y, dr).$$

If $Y_s = (X_{u+s} - X_u) + X_u = X_{u+s}$ for $s \leq \Delta_n$, then by Sections 2 and 4 and the remark following Theorem 4.1, there is a continuous additive functional, $A_{n,x,i}(s)$ say, associated to $\mu_{x,u}$. By Section 4,

$$\int_0^{s_i} [G(X_{s_{i+1}} - X_r - x) - G(X_{s_i} - X_r - x)]\,\beta_k(y, dr) = \int_{s_i}^{s_{i+1}} \left[\int_0^{s_i} \nabla G(X_{s_{i+1}} - X_r - x)\,\beta_k(y, dr)\right] \cdot dX_s - A_{n,x,i}(s_{i+1} - s_i).$$

If we let $A_{n,x}=\sum_{i=0}^{2^n-1} A_{n,x,i}(s_{i+1}-s_i)$ and we sum over i, we get

$$(8.8)\quad \sum_{i=0}^{2^n-1}\left[\int_o^{s_i}[G(X_{s_{i+1}}-X_r-x)-G(X_{s_i}-X_r-x)\right]\beta_k(y, dr)$$
$$=\sum_{i=0}^{2^n-1}\int_{s_i}^{s_{i+1}}\left[\int_0^{s_i}\nabla G(X_{s_{i+1}}-X_r-x)\right]\cdot dX_s-A_{n,x}.$$

We set $\beta=\beta_k(y, dr)$ and then proceed as in Section 7: using (8.5), $A_{n,x}$ increases as $n\to\infty$. We let $\beta_{k+1}(x, y, t)$ denote the limit. As in Proposition 7.1, the left-hand side of (8.8) converges in L^p, uniformly over t, to

$$(8.9)\qquad U_t(x, y)=\int_0^t [G(X_t-X_r-x)-G(-x)]\,\beta_k(y, dr).$$

Continuing exactly as in Section 7, we obtain (8.1), (8.2), (8.4), (8.6), and (8.7). (8.5) will follow, then, once we obtain (8.3).

We have

$$E\,|\beta_{k+1}(x, y, t)-\beta_{k+1}(x', y, t)|^p$$
$$\leqq c(p)(N^\vee(x, y)+N^\vee(x', y'))^{\nu(p)}|x-x'|^{ap};$$

this follows by an argument that is almost identical to the proof of Proposition 7.6. So it remains to show

$$(8.10)\quad E\,|\beta_{k+1}(x, y, t)-\beta_{k+1}(x, y', t)|^p$$
$$\leqq c(p)(N^\vee(x, y)+N^\vee(x', y'))^{\nu(p)}|y-y'|^{ap},\qquad y, y'\in(\mathbb{R}^2-\{0\})^{k-1}.$$

By Section 6, this will follow if we show

$$(8.11)\quad E\,|U_t(x, y)-U_t(x, y')|^p\leqq c(p)(N^\vee(x, y)$$
$$+N^\vee(x', y'))^{\nu(p)}|y-y'|^{bp}$$

for some b.

Now $G(-x)[\beta_k(y, t)-\beta_k(y', t)]$ has pth moments of the desired form by the induction hypothesis. By Proposition 5.2, there exists a such that

$$(8.12)\quad E\left|\int_0^t H_\zeta(X_t-X_r-x)\,\beta_k(y, dr)\right|^p\leqq c(p)(G^\vee(x, y))^{\nu(p)}\zeta^{ap},$$

and similarly with y replaced by y'.

Let

$$V=\{|X_{s+u}-X_u|\geqq u^{1/4}/\zeta \text{ for some } s\in[0, 1],\ u\in[0, 1]\}.$$

By standard estimates on the Brownian path ([IM], p. 37),

$$(8.13)\quad E\left[\int_0^t |G_\zeta(X_t-X_r-x)|\,\beta_k(y,dr);\,V\right]^p$$
$$\leqq \left(E\left(\int_0^t |G_\zeta(X_t-X_r-x)|\,\beta_k(y,dr)\right)^{2p}\right)^{1/2}(PV)^{1/2}$$
$$\leqq c(p)(G^\vee(x,y))^{\nu(p)}\zeta^{dp}$$

for some $d>0$ independent of p.

On V^c, $f(r)=G_\zeta(X_t-X_r-x)$ is Hölder continuous of order 1/4:

$$|G_\zeta(X_t-X_r-x)-G_\zeta(X_t-X_s-x)|\leqq(c/\zeta)|X_r-X_s|\leqq c|r-s|^{1/4}/\zeta.$$

Hence for each $\omega\in V^c$, we can find $f_h(r)$ such that $|f-f_h|\leqq ch^{1/4}/\zeta$ and f_h is Lipschitz with constant $\|f\|_\infty/h\leqq c\log(1/\zeta)/h\leqq c/\zeta h$, namely by letting

$$f_h(t)=\frac{1}{h}\int_t^{t+h} f(u)\,du.$$

Set $h=\zeta^5$. Then

$$(8.14)\quad E\left[\left(\int_0^t |G_\zeta(X_t-X_r-x)-f_h(r)|\,\beta_k(y,dr)\right)^p;\,V^c\right]$$
$$\leqq c(p)(G^\vee(x,y))^{\nu(p)}\zeta^{p/4},$$

and similarly with y replaced by y'.

Finally, by integration by parts and the induction hypothesis,

$$(8.15)\quad E\left[\left|\int_0^t f_h(r)[\beta_k(y,dr)-\beta_k(y',dr)]\right|^p;\,V^c\right]$$
$$\leqq c(p)E|f_h(t)|^p|\beta_k(y,t)-\beta_k(y',t)|^p$$
$$+E\left|\int_0^t[\beta_k(y,r)-\beta_k(y',r)]\,f_h(dr)\right|^p$$
$$\leqq c(p)(G^\vee(x,y)+G^\vee(x',y'))^{\nu(p)}\zeta^{-6p}|y-y'|^{ap}.$$

(Since f_h is Lipschitz in $r\in[0,1]$, it is of bounded variation.)

Adding (8.12)-(8.15) and letting $\zeta=|y-y'|^{a/12}$ yields (8.11). □

9. RENORMALIZATION

Again, $d=2$. For $x\neq 0$, let

$$\xi_2(x,t)=G(x)\,t,\qquad \gamma_2(x,t)=\beta_2(x,t)-\xi_2(x,t).$$

Hence, since $G(-x)=G(x)$,

$$\int_0^t G(X_t - X_r - x)\,dr = \int_0^t [\nabla G(X_s - X_r - x)\,dr]\cdot dX_s - \gamma_2(x, t).$$

Define $\varphi_{ki}:(\mathbb{R}^2)^k \times \{1, \ldots, k\}^i \to (\mathbb{R}^2)^{k-1}$ by letting $\varphi_{ki}(x_1, \ldots, x_k, j_1, \ldots, j_i)$ be the sequence $x_1, \ldots, x_k$ with the $j_1, j_2, \ldots$, and j_i entries deleted. For example,

$$\varphi_{4,2}(x_1, x_2, x_3, x_4; 2, 4) = (x_1, x_3).$$

Let $\beta_1(t) = t$.

Define

$$(9.1)\quad \xi_{k+1}(x_1, \ldots, x_k, t) = \sum_{i=1}^{k} (-1)^{i+1} \sum_{j_1 < \ldots < j_i} G(x_{j_1}) \ldots G(x_{j_i})\, \beta_{k+1-i}(\varphi_{ki}(x_1, \ldots, x_k, j_1, \ldots, j_i), t)$$

and

$$(9.2)\qquad \gamma_{k+1}(x, y, t) = \beta_{k+1}(x, y, t) - \xi_{k+1}(x, y, t).$$

We call γ_{k+1} renormalized ILT.

Remark. – The definition says, for example,

$$\xi_3(x, y, t) = G(x)\beta_2(y, t) + G(y)\beta_2(x, t) - G(x)G(y)t;$$

$$\xi_4(x, y, z, t) = G(x)\beta_3(y, z, t) + G(y)\beta_3(x, z, t) + G(z)\beta_3(x, y, t) - G(x)G(y)\beta_2(z, t) - G(x)G(z)\beta_2(y, t) - G(y)G(z)\beta_2(x, t) + G(x)G(y)G(z)t,$$

and so on.

PROPOSITION 9.1. – *If* $y=(x_2, \ldots, x_{k-1})$ *with* $x_i \neq 0$, $i=2, \ldots, k-1$,

$$(9.3)\quad \xi_{k+1}(x, y, t) = G(x)\beta_k(y, t) - \int_0^t G(X_t - X_r - x)\,\xi_k(y, dr) + \int_0^t \left[\int_0^s \nabla G(X_s - X_r - x)\,\xi_k(y, dr)\right]\cdot dX_s,$$

and

$$(9.4)\quad \gamma_{k+1}(x, y, t) = \int_0^t \left[\int_0^s G(X_s - X_r - x)\,\gamma_k(y, dr)\right]\cdot dX_s - \int_0^t G(X_t - X_r - x)\,\gamma_k(y, dr).$$

Proof. – The proof is by induction: the $(k+1)$ *st* formula follows from the kth formula, (8.6), (9.1), (9.2), and some routine calculations. □

Set

$$(9.5)\quad \gamma_{k+1}^{+}(x_1, \ldots, x_k, t) = \beta_{k+1}(x_1, \ldots, x_k, t) + \sum_{i \leqq k,\, i \text{ even}} \sum_{j_1 < \ldots < j_i} G(x_{j_1}) \ldots G(x_{j_i}) \times \beta_{k+1-i}(\varphi_{ki}(x_i, \ldots, x_k, j_1, \ldots, j_i), t),$$

$$\gamma_{k+1}^{-}(x_1, \ldots, x_k, t) = -(\gamma_{k+1} - \gamma_{k+1}^{+}).$$

If $G^{\vee}(x_1, \ldots, x_{k-1}) = 1 \vee |G(x_1)| \vee \ldots \vee |G(x_{k-1})|$, for each p

$$(9.6)\quad E|\gamma_k^{+}(x_1, \ldots, x_{k-1}, t)|^p \leqq c(p)(G^{\vee}(x_1, \ldots, x_{k-1}))^{\nu(p)}, \qquad t \leqq 1,$$

and

$$(9.7)\quad E|\gamma_k^{+}(x_1, \ldots, x_{k-1}, t) - \gamma_k^{+}(x_1, \ldots, x_{k-1}, s)|^p \leqq c(p)(G^{\vee}(x_1, \ldots, x_{k-1}))^{\nu(p)} |t-s|^{ap}, \qquad s, t \leqq 1$$

for some a and $\nu(p)$, and similarly with γ_k^{+} replaced by γ_k^{-}; this follows by Theorem 8.1 and the representation of γ_k^{+} and γ_k^{-} as linear combinations of the β_i, $i \leqq k$.

We set

$$\bar{U}_t(x, y) = \int_0^t G(X_t - X_r - x)\, \gamma_{k-1}(y, dt),$$

and

$$\bar{M}_t(x, y) = \bar{U}_t(x, y) + \gamma_k(x, y, t),$$

where $x = x_1$, $y = (x_2, \ldots, x_{k-1})$. By (9.4), $\bar{M}_t(x, y)$ is a martingale.

PROPOSITION 9.2. – *There exists $a > 0$ and $\nu(p)$ such that*

(*a*) $E|\bar{U}_t(x, y) - \bar{U}_t(x', y)|^p \leqq c(p)(G^{\vee}(x, y) + G^{\vee}(x', y))^{\nu(p)} |x - x'|^{ap}$;

(*b*) $E|\bar{U}_t(x, y) - \bar{U}_s(x, y)| \leqq c(p)(G^{\vee}(x, y))^{\nu(p)} |t - s|^{ap}$;

(*c*) $E|\bar{U}_t(x, y) - \bar{U}_t(x, y')| \leqq c(p)(G^{\vee}(x, y) + G^{\vee}(x, y'))^{\nu(p)} |y - y'|^{ap}$;

Proof. – The proof is again by induction. Note

$$(9.8)\quad E\left|\int_0^t H_\zeta(X_t - X_r - x)\, \gamma_{k-1}^{+}(y, dt)\right|^p \leqq c(p)(G^{\vee}(x, y))^{\nu(p)} \zeta^{bp}$$

and similarly with γ_{k-1}^{+} replaced by γ_{k-1}^{-} and with x replaced by x', using Proposition 5.2. If we connect x to x' by a curve Γ of length $\leqq c|x - x'|$ so that Γ never gets closer to the point 0 than $|x| \wedge |x'|$,

$$E\left|\int_0^t [G_\zeta(X_t - X_r - x) - G_\zeta(X_t - X_r - x')]\, \gamma_k(y, dr)\right|^p \leqq c\zeta^{-p} |x - x'|^p E|\gamma_k^{+}(y, t) + \gamma_k^{-}(y, t)|^p \leqq c(p)(G^{\vee}(x, y) + G^{\vee}(x', y))^{\nu(p)} |x - x'|^p / \zeta^p.$$

Adding our estimates and setting $\zeta = |x - x'|^{1/2}$, we get (*a*).

The proof of Lemma 7.3 (*b*) with β replaced by γ_k^+, together with (9.6) and (9.7), shows that for some a independent of x, y, and p,

$$E\left|\int_0^t G_\zeta(X_t-X_r-x)\gamma_k^+(y, dr)-\int_0^s G_\zeta(X_s-X_r-x)\gamma_k^+(y, dr)\right|^p \leq c(G^\vee(x, y))^{\nu(p)}|t-s|^{ap},$$

and similarly with γ_k^+ replaced by γ_k^-. Adding these two estimates proves (*b*).

To prove (*c*), we follow the proof of (8.3) almost word for word. We have the estimates (8.12)-(8.15) with $N^\vee$ replaced by $G^\vee$ and β_k replaced by either γ_{k-1}^+ or γ_{k-1}^-, and so obtain

$$E\left|\int_0^t G(X_t-X_r-x)\gamma_{k-1}^+(y, dr)-\int_0^t G(X_t-X_r-x)\gamma_{k-1}^+(y', dr)\right|^p \leq c(p)(G^\vee(x, y)+G^\vee(x, y'))^{\nu(p)}|y-y'|^{ap},$$

and similarly with γ_{k-1}^+ replaced by γ_{k-1}^-. Adding these two estimates gives (*c*) □

THEOREM 9.3. — *$\gamma_k(x_1, \ldots, x_{k-1}, t)$ is jointly Hölder continuous in each variable on the set $(\mathbb{R}^2)^{k-1}\times[0, 1]$.*

Proof. — Let $x=(x_1, \ldots, x_{k-1})$, $z=(x, t)$, $z'=(x', t')$. From Propositions 9.2 and 6.2 and the triangle inequality, we get the existence of $a>0$ and $\nu(p)$ such that

$$(9.9)\qquad E|\gamma_k(z)-\gamma_k(z')|^p\leq c(p)(G^\vee(x)+G^\vee(x'))^{\nu(p)}|z-z'|^{ap}.$$

Fix p large enough so that $ap\geq 12k+8$.

We now proceed to modify the standard chaining argument. Let $\mathscr{B}_n=\{x\in\mathbb{R}^2 : x\neq 0$ and both coordinates of x are integer multiples of $2^{-n}\}$, $n\geq 1$. Let $R\geq 1$ and let $\mathscr{A}_n=\{z=(x_1, \ldots, x_{k-1}, t) : |x_i|\leq R, x_i\in\mathscr{B}_n, i=1, \ldots, k-1, t$ is an integer multiple of $2^{-n}\}$, $n\geq 1$. Let $\mathscr{A}=\bigcup_n \mathscr{A}_n$.

If $z\in\mathscr{A}$, let z_i be the point in $\mathscr{A}_i$ closest to z (with some convention for breaking ties). We write, for any i_0,

$$(9.10)\qquad \gamma_k(z)=\sum_{i=i_0}^{\infty}[\gamma_k(z_{i+1})-\gamma_k(z_i)]+\gamma_k(z_{i_0}),$$

where the sum is actually finite, since $z\in\mathscr{A}$. We do the same for $\gamma_k(z')$. Note $\#\mathscr{A}_i\leq c\,2^{2ik}$.

Let $\lambda>0$. If $|z-z'|<\delta$, and $|\gamma_k(z)-\gamma_k(z')|>\lambda$, then either (*a*) $|\gamma_k(z_{i_0})-\gamma_k(z'_{i_0})|>\lambda/2$ or (*b*) for some $j\geq i_0$ and some $w\in\mathscr{A}_j$, $w'\in\mathscr{A}_{j+1}$

with $|w-w'| \leqq c\,2^{-j}$, we have $|\gamma_k(w)-\gamma_k(w')| \geqq \lambda/40 j^2$. So

$$
\begin{aligned}
(9.11)\quad & P(|\gamma_k(z)-\gamma_k(z')|>\lambda \text{ for some } z, z' \in \mathscr{A} \text{ with } |z-z'|<\delta) \\
& \leqq (\#\mathscr{A}_{i_0}) \sup_{\substack{z, z' \in \mathscr{A}_{i_0} \\ |z-z'| \leqq c\delta}} P(|\gamma_k(z)-\gamma_k(z')|>\lambda/2) \\
& \quad + \sum_{j=i_0}^{\infty} (\#\mathscr{A}_j)(\#\mathscr{A}_{j+1}) \\
& \quad \times \sup \{ P(|\gamma_k(w)-\gamma_k(w')|>\lambda/40 j^2 : \\
& \qquad\qquad w \in \mathscr{A}_j,\ w' \in \mathscr{A}_{j+1},\ |w-w'| \leqq c\,2^{-j} \}.
\end{aligned}
$$

Using Chebyshev with (9.9), we bound (9.11) by

$$
\begin{aligned}
& c(p)\, 2^{2 i_0 k} (1 \vee \sup_{\mathscr{B}_{i_0}} |G|)^{\nu(p)} \delta^{ap}/\lambda^p \\
& \quad + c(p) \sum_{j=i_0}^{\infty} 2^{4jk} (1 \vee \sup_{\mathscr{B}_j} |G|)^{\nu(p)} 2^{-jap} (40 j^2)^p/\lambda^p \\
& \qquad \leqq c(p)\, 2^{2 i_0 k} (i_0)^{\nu(p)} \delta^{ap}/\lambda^p + c(p) \sum_{j=i_0}^{\infty} 2^{-2j} j^{\nu(p)} j^{2p}/\lambda^p
\end{aligned}
$$

by our choice of p and the fact that $\sup_{\mathscr{B}_j} |G| \leqq c \log (2^{-j}) = cj$. Choosing i_0 so that $2^{-i_0} \leqq \delta \leqq 2^{-i_0+1}$, we see the series on the right is summable with a sum $\leqq c(p)\, \delta^{a'p}/\lambda^p$. A standard Borel-Cantelli argument shows that $\gamma_k(z)$ is uniformly Hölder continuous on $\mathscr{A}$, a.s. By Proposition 9.1 and Theorem 8.1, we know that $\gamma_k(z)$ is Hölder continuous on $(\mathbb{R}^2-\{0\})^{k-1} \times [0, 1]$. So we can extend $\gamma_k(z)$ to be continuous on $B(0, R)^{k-1} \times [0, 1]$. Since R is arbitrary, this completes the proof. □

Remark. – In the above proof, we obtained the estimate

$$
(9.12)\qquad P(\sup_{\substack{|z-z'|<\delta \\ z, z' \in \mathscr{A}}} |\gamma_k(z)-\gamma_k(z')|>\lambda) \leqq c(p)\, \delta^{a'p}/\lambda^p.
$$

Let $p_0 \geqq 1$ be given. First take $p=p_0+1$, multiply by $p_0 \lambda^{p_0-1}$ and integrate (9.12) from 1 to ∞; then take $p=(p_0-1) \wedge 1$, multiply by $p_0 \lambda^{p_0-1}$ and integrate (9.12) from 0 to 1; now add. Using the fact that γ_k is continuous, we get

$$
(9.13)\qquad E(\sup_{|z-z'|<\delta} |\gamma_k(z)-\gamma_k(z')|^{p_0}) \leqq c(p)\, \delta^{a'p_0/2}.
$$

Remark. – Theorem 9.3 was conjectured but not proved in [Sh].

REFERENCES

[B1] R. F. BASS, Joint Continuity and Representations of Additive Functionals of *d*-Dimensional Brownian motion, *Stock. Proc. Applic.*, Vol. **17**, 1984, pp. 211-227.

[B2] R. F. BASS, A Probabilistic Approach to the Boundedness of Singular Integral Operators, in: *Séminaire de Probabilités XXIV*, Eds. J. AZÉMA and M. YOR, 15-40, New York, Springer, 1980.

[BK1] R. F. BASS, and D. KHOSHNEVISAN, Local Times on Curves and Uniform Invariance Principles, *Probab. Th. and rel. Fields*, Vol. **92**, 1992, pp. 465-492.

[BK2] R. F. BASS and D. KHOSHNEVISAN, Strong Approximations to Brownian Local Time, in: *Seminar on Stochastic Processes* 1992, E. CINLAR, K. L. CHUNG and M. J. SHARPE Eds, Boston, Birkhäuser, 1993, pp. 43-65.

[BG] R. M. BLUMENTHAL and R. K. GETOOR, *Markov Processes and Potential Theory.* New York, Academic Press, 1968.

[Br] G. A. BROSAMLER, Quadratic Variation of Potentials and Harmonic Functions, *Trans. Amer. Math. Soc.*, Vol. **149**, 1970, pp. 243-257.

[Ch] K. L. CHUNG, *A Course in Probability Theory*, 2nd Ed., New York, Academic Press, 1974.

[DM] C. DELLACHERIE and P.-A. MEYER, *Probabilités et Potentiel : Théorie des Martingales*, Paris, Hermann, 1980.

[Du] R. M. DUDLEY, Sample Functions of the Gaussian process, *Ann. Probab.*, Vol. **1**, 1973, pp. 66-103.

[Dy] E. B. DYNKIN, Self-intersection Gauge for Random Walks and for Brownian Motion, *Ann. Probab.*, Vol. **16**, 1988, pp. 1-57.

[IM] K. ITO and H. P. MCKEAN, *Diffusion Processes and their Sample Paths*, New York, Springer, 1974.

[L] J.-F. LE GALL, Some Properties of Planar Brownian Motion, *École d'été de Saint-Flour XX*, New York, Springer, 1992.

[ReY] D. REVUZ and M. YOR, *Continuous Martingales and Brownian Motion*, New York, Springer, 1991.

[R1] J. ROSEN Tanaka's Formula for Multiple Intersections of Planar Brownian Motion *Stoch. Proc. and their Applic.* Vol. **23**, 1986, pp. 131-141.

[R2] J. ROSEN Random Walks and Intersection Local Time, *Ann. Probab.*, Vol. **18**, 1990, pp. 959-977.

[RY] J. ROSEN and M. YOR, Tanaka Formulae and Renormalization for Triple Intersections of Brownian Motion in the Plane, *Ann. Probab.*, Vol. **19**, 1991, pp. 142-159.

[Sh] N.-R. SHIEH, White Noise Analysis and Tanaka Formula for Intersections of Planar Brownian Motion, *Nagoya Math. J.*, Vol. **122**, 1991, pp. 1-17.

[Y] M. YOR, Compléments aux formules de Tanaka-Rosen, in: *Séminaire de Probabilités XIX*, J. AZÈMA, and M. YOR, New York, Springer, 1985, pp. 332-349.

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