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Probabilities on contractible locally compact groups: The existence of universal distributions in the sense of W. Doeblin

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ABSTRACT. – We show that universally attractable probability distributions in the sense of W. Doeblin exist on locally compact contractible groups. As a consequence we obtain for this class of groups the famous characterisation of infinite divisibility due to Khinchine and Doeblin: A probability measure is infinitely divisible if its domain of attraction is non empty, and vice versa.

Key words : Contractible groups convolution semigroups, domains of partial attraction, Doeblin distributions.

RÉSUMÉ. – Dans ce travail nous montrons l'existence de probabilités universellement attractables dans l'esprit de W. Doeblin pour des groupes localement compacts contractables. Comme conséquence on obtient aussi pour cette classe de groupes un théorème de type Khinchine-Doeblin: Les probabilités infiniment divisibles possèdent des domaines d'attraction partielle non vides et vice versa.

Classification A.M.S. : 43 A 05, 60 B 15, 60 B 10, 60 E 07 (60 F 17), 43 A 70.

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The investigation of limit theorems on locally compact groups leads in a natural way to contractible groups. Let $(X_i)_{i \geq 0}$ be an i.i.d. sequence of G-valued random variables with distribution ν. Let $a \in \mathrm{Aut}(G)$ be a contracting automorphism. Then for sequences of natural numbers $k_n \uparrow \infty$, $l_n \uparrow \infty$ $(a^{l_n}(X_i))_{i=1, \ldots, k_n; n \in \mathbb{N}}$ is an infinitesimal triangular array the row products of which are distributed according to $(a^{l_n} \nu)^{k_n} = a^{l_n}(\nu^{k_n})$. Since the group G is not supposed to be abelian we do not use centering terms to normalize the random variables. ν is in the domain of (normal) partial attraction of μ (abbreviated DNPA $(\mu; a)$ or DNPA(μ)) if $a^{l_n} \nu^{k_n} \to \mu$. (Convergence in the weak sense.)

In the "classical" situation $G = \mathbb{R}$ or $\mathbb{R}^d$ the possible limits μ are infinitely divisible. W. Doeblin [Doe] proved the existence of universal distributions $\nu \in \bigcap \{ \mathrm{DNPA}(\mu)\colon \mu$ infinitely divisible$\}$. Hence the wellknown characterization of infinite divisibility by Doeblin and Khinchine [Doe] follows: $\mu \in M^1(\mathbb{R})$ is infinitely divisible iff its domain of partial attraction is non empty. These results hold for Hilbert spaces ([Ba1], [Ba2]), more generally for Fréchet spaces [Ph] where a is an homothetical automorphism $x \mapsto a.x$ $(a \in \mathbb{R}_+^*)$, and also for Banach spaces with contracting automorphism a [Th].

Our aim is to show analogous results for locally compact groups G with contracting automorphism $a \in \mathrm{Aut}(G)$.

In Section 1 we study on general locally compact groups the limit behaviour of sequences of discrete and continuous convolution semigroups and the embedability of the limit measures. For a new class of groups [called strongly B-root-compact, a slight generalization of strong root-compactness ([He], [Si], [S])] we show that infinitely divisible measures are continuously embeddable up to a shift.

In Section 2 we improve our knowledge of contractible locally compact groups ([M-R], especially [Si1], [Si2]) and show that they belong to the class of groups studied in Section 1. Moreover, to any contractible group G there belongs a submonogeneous group $S_B \subseteqq \mathbb{R}$, such that to every $x \in G$ there exists a unique homomorphism $f\colon S_B \to G$ with $f(1) = x$. Especially, we obtain that infinitely divisible laws are submonogeneously embeddable.

In Section 3 we prove the existence of universal distributions on contractible groups. The proof is similar to the "classical" proofs ([Doe], [Ba1], [Ph], [Th]) but instead of Fourier transforms we have to use the Lévy-Khinchine formula (*see* e. g. [He]) and the interplay between convergence of sequences of convolution semigroups and of the corresponding generating distributions. ([Ha], [Si1]; [H-S], [Kh]). Therefore the proofs only work for locally compact groups. It is an open question if on non locally compact contactible groups always Doeblin distributions exists.

Notations. – Let in the following G be a locally compact group. Let $M^1(G)$ be the convolution semigroup of probability measures, endowed with the topology of weak convergence. $C^b(G)/C_0(G)/\mathscr{D}(G)/\mathscr{E}(G)$ are the function-spaces of bounded continuous/of continuous functions vanishing at infinity/Bruhat-Schwartz test functions/resp. regular functions (*i.e.* $f \in \mathscr{E}(G)$ if $f \in C^b(G)$ and $f.g \in \mathscr{D}(G)$ for $g \in \mathscr{D}(G)$) (*See* e.g. [Ha], [Si1], [He]).

If $(\mu_t, t \geqq 0, \mu_0 = \varepsilon_e)$ is a continuous convolution semigroup (short: c.c.s.) in $M^1(G)$ then the (infinitesimal) generating functional (or infinitesimal generating distribution) $A \in \mathscr{D}'(G)$ is defined $A := \frac{d^+}{dt} \mu_t |_{t=0}$. We use the notation $\operatorname{Exp} tA := \mu_t$, $t \geqq 0$, then, and denote $\mathfrak{W}(G)$ the set of generating functionals. For $\mu \in M^1(G)$ resp. $A \in \mathfrak{W}(G)$ let R_μ resp. R_A be the convolution operators $f \mapsto \mu * f$ resp. $f \mapsto A * f$.

For a compact subgroup K let ω_K be the normalized Haar measure. If $\lambda \in M^1(G)$, $\alpha > 0$ then $A := \alpha(\lambda - \varepsilon_e) \in \mathfrak{W}(G)$ is called Poisson generator, $\left(\operatorname{Exp} tA = \exp t\alpha(\lambda - \varepsilon_e) := e^{-\alpha t} \sum_0^\infty \frac{t^k \alpha^k}{k!} \lambda^k\right)_{t \geqq 0}$ is called Poisson semigroup. More generally, if $\lambda = \omega_K * \lambda * \omega_K$, then $\exp_K t\alpha(\lambda - \varepsilon_e) = \left(\omega_K + \sum_1^\infty \frac{\alpha^k t^k}{k!} \lambda^k\right) e^{-\alpha t}$ is called Poisson measure with idempotent factor ω_K. (For more details *see* e.g. [He] esp. Chap. IV, [Si1], [Ha] ch. O., ch. I.).

1. DOMAINS OF PARTIAL ATTRACTION AND EMBEDDABILITY

Let G be a locally compact group.

DEFINITION 1.1. – Let $\mu \in M^1(G)$.

a) The domain of partial attraction (in the strict sense) is defined as

$$\operatorname{DPA}(\mu) := \{ \nu \in M^1(G) : \text{There exist } a_n \in \operatorname{Aut}(G),\ k_n \in \mathbb{N},\ k_n \uparrow \infty, \text{ such that } a_n \nu \to \varepsilon_e \text{ and } a_n \nu^{k_n} \to \mu \}.$$

The domain of normal partial attraction [w.r.t. $a \in \operatorname{Aut}(G)$] is defined as

$$\operatorname{DNPA}(\mu; a) = \operatorname{DNPA}(\mu) := \{ \nu : \text{There exists } k_n \uparrow \infty,\ l_n \uparrow \infty, \text{ such that } a^{l_n} \nu \to \varepsilon_e \text{ and } a^{l_n} \nu^{k_n} \to \mu \}.$$

If G is strongly root compact and aperiodic or if a is contracting the infinitesimality-conditions $a_n \nu \to \varepsilon_e$, resp. $a^{l_n} \nu \to \varepsilon_e$ are automatically fulfilled. (*cf.* [No1], No2], [H–S]).

b) Let $\nu_n := a_n \nu$ resp. $:= a^{l_n} \nu$. If we consider $(\nu_n^{[k_n t]})_{t \geqq 0}$ as sequence of "discrete semigroups", *i.e.* if we have a functional limit theorem in mind, and if we suppose the existence of a c. c. s. ($\mu_t = \operatorname{Exp} t\mathrm{A}$) with $\mu_1 = \mu$, then we define

$$\mathrm{FDNPA}(\mathrm{A}; a) = \mathrm{FDNPA}(\mathrm{A}) := \{ \nu : a^{l_n} \nu^{[k_n t]} \to \mu_t = \operatorname{Exp} t\mathrm{A},\ t \geqq 0 \}.$$

c) Analogously, if we have the infinitesimal generating functionals in mind we define

$$\mathrm{IDNPA}(\mathrm{A}; a) := \{ \nu : \mathrm{A}_n := k_n (a^{l_n} \nu - \varepsilon_e) \to \mathrm{A} \text{ on } \mathscr{E}(\mathrm{G}) \}.$$

d) Sometimes we have to use normalizing shifts. Then we define e. g.

$$\mathrm{DNPA}_{\mathrm{S}}(\mu; a) := \{ \nu : a^{l_n} \nu^{k_n} * \varepsilon_{x_n} \to \mu \text{ for suitable sequences } l_n \uparrow \infty,\ k_n \uparrow \infty \text{ and } (x_n) \subseteqq \mathrm{G} \}.$$

Remark 1.2. — In the "classical situation" (*i.e.* if $\mathrm{G} = \mathbb{R}$ or $\mathbb{R}^d$) the domains of attraction and the functional resp. infinitesimal versions coincide, *i.e.* we have

$$\mathrm{DNPA}(\mu_1) = \mathrm{FDNPA}(\mathrm{A}) = \mathrm{IDNPA}(\mathrm{A}).$$

In general we can only prove a weaker result:

PROPOSITION 1.3. — *Let* ($\mu_t = \operatorname{Exp} t\mathrm{A}$) *be a c. c. s. with* $\mu_1 = \mu$. *Then*

a) $$\mathrm{IDNPA}(\mathrm{A}; a) \subseteqq \mathrm{FDNPA}(\mathrm{A}; a) \subseteqq \mathrm{DNPA}(\mu; a).$$

b) *If* G *is Lie projective we can prove*

$$\mathrm{IDNPA}(\mathrm{A}; a) = \mathrm{FDNPA}(\mathrm{A}; a).$$

c) *If* G *is strongly root compact and aperiodic then*

$$\mathrm{FDNPA}(\mathrm{A}; a) = \mathrm{DNPA}(\mu; a).$$

(In this case we know that every μ with $\mathrm{DPA}(\mu) \neq \varnothing$ is embeddable into a c. c. s., *see* [No1], [No2].)

⟦*a*) The left inclusion follows from [Ha] I. Satz 2.3, O. Section 2 Satz 4.2, *see* also [No1] (1.1), [No2] remark 2*b*. The right inclusion is obvious.

b) Convergence of discrete semigroups is equivalent to resolvent-convergence of the generating distributions ([Ha], [No1], [No2]), this is again equivalent to convergence of the c. c. s. $\operatorname{Exp} t\mathrm{A}_n \to \operatorname{Exp} t\mathrm{A}$, $t \geqq 0$. If G is Lie-projective this implies $\mathrm{A}_n \to \mathrm{A}$ on $\mathscr{E}(\mathrm{G})$. (The last assertion is proved in [Si1], p. 143, but not explicitely stated, *see* [Kh]. *See* also [H-S].)

c) Let $a^{l_n} \nu^{k_n} \to \mu$. According to [No1] (1.11) resp. [No2] Theorem 1 there exists a subsequence $(n') \subseteqq \mathbb{N}$, and a c. c. s. $\mu_t = \operatorname{Exp} t\mathrm{A}$, $\mu_1 = \mu$, such that $a^{l_n} \nu^{[k_n t]} \xrightarrow[(n')]{} \mu_t$, $t \geqq 0$.⟧

For the construction of universal distributions in paragraph 3 we need the following observation:

PROPOSITION 1.4

a)	$\nu \in \mathrm{FDNPA}(A; a)$	$\Leftrightarrow$	$\forall m \in \mathbb{Z}$,	$\nu \in \mathrm{FDNPA}(a^m A; a)$
a')	$\nu \in \mathrm{IDNPA}(A; a)$	$\Leftrightarrow$	$\forall m \in \mathbb{Z}$,	$\nu \in \mathrm{IDNPA}(a^m A; a)$
b)	$\nu \in \mathrm{FDNPA}(A; a)$	$\Leftrightarrow$	$\forall \alpha > 0$,	$\nu \in \mathrm{FDNPA}(\alpha A; a)$
b')	$\nu \in \mathrm{IDNPA}(A; a)$	$\Leftrightarrow$	$\forall \alpha > 0$,	$\nu \in \mathrm{IDNPA}(\alpha A; a)$.

⟦*a*), *a'*) are obvious.

b) Obviously we have:

$$a^{l_n} \nu^{[k_n t]} \to \mu_t \quad \text{for any } t \geqq 0 \quad \Leftrightarrow \quad a^{l_n} \nu^{[k_n \alpha t]} \to \mu_{\alpha t} \quad \text{for } t \geqq 0,\ \alpha > 0.$$

The representation $[k_n \alpha t] = [[k_n \alpha] t] + s_n$, $0 \leqq s_n \leqq [t] + 1$ yields

$$a^{l_n} \nu^{[k_n \alpha t]} = a^{l_n} (\nu^{[[k_n \alpha] t]} \star \nu^{s_n}) = a^{l_n} \nu^{[[k_n \alpha] t]} \star a^{l_n} \nu^{s_n} \to \mu_{\alpha t}.$$

On the other hand $a^{l_n} \nu \to \varepsilon_e$ implies $a^{l_n} \nu^{s_n} \to \varepsilon_e$, hence $a^{l_n} \nu^{[[k_n \alpha] t]} \to \mu_{\alpha t}$, $t \geqq 0$.

b') Let $k_n (a^{l_n} \nu - \varepsilon_e) \to A$, let $\alpha > 0$. Then

$$\alpha k_n (a^{l_n} \nu - \varepsilon_e) \to \alpha A,$$

and hence because of

$$0 \leqq \alpha k_n - [\alpha k_n] \leqq 1 \quad \text{and} \quad a^{l_n} \nu - \varepsilon_e \to 0$$

we obtain

$$[\alpha k_n] (a^{l_n} \nu - \varepsilon_e) \to \alpha A.⟧$$

DEFINITION 1.6. — Let $S \subseteqq M^1(G)$ be a subset. The common domain of normal partial attraction with respect to a is defined:

$$\mathrm{NDPA}(S; a) := \bigcap_{\mu \in S} \mathrm{NDPA}(\mu; a).$$

And in an analogous way we define for a subset $\mathfrak{S} \subseteqq \mathfrak{W}(G)$ of generating functionals FDNPA ($\mathfrak{S}$; *a*) resp. IDNPA ($\mathfrak{S}$; *a*).

A probability measure $\nu \in M^1(G)$ is called universal for S w. r. t. *a* if $\nu \in$ DNPA (S; *a*).

Obviously we have

PROPOSITION 1.7. — *Let* $S \subseteqq M^1(G)$ *and let* S^- *be the closure* (w. r. *t. the weak topology*). *Then if* G *is metrizable*

$$\mathrm{DNPA}(S; a) = \mathrm{DNPA}(S^-; a).$$

DEFINITION 1.8. — In the following we consider the following subsets of $M^1(G)$:

$\mathfrak{I} := \{$ infinitely divisible measures $\}$,
$\mathfrak{J} := \{$ idempotent measures $\mu = \omega_K$, K a compact subgroup $\}$,
$\mathscr{E} := \{$ continuously embeddable measures $\}$
$= \{ \mu$: there exists a c. c. s. (μ_t) with $\mu_1 = \mu \}$,
$\mathscr{E}_0 := \{ \mu \in \mathscr{E} : \mu = \mu_1$ where (μ_t) is a c.c.s. with $\mu_0 = \varepsilon_e \}$
$= \{$ continuously embeddable measures with trivial idempotent factor $\}$.

DEFINITION 1.9. — ν is called Doeblin-distribution or universal distribution on G w. r. t. a if $\nu \in \mathrm{DNPA}(\mathfrak{I}; a)$.

ν is universal in the wide sense if $\nu \in \mathrm{DNPA}_S(\mathfrak{I}; a)$.

(Universal distributions in [Doe], [Ba1], [Ph], [Th] are by definition universal in the wide sense.)

We define analogously

ν is called

D-universal if $\nu \in \mathrm{DNPA}(\mathscr{E}; a)$
F-universal if $\nu \in \mathrm{FDNPA}(\mathfrak{W}(G); a) (\Rightarrow \quad \nu \in \mathrm{DNPA}(\mathscr{E}_0; a))$
I-universal if $\nu \in \mathrm{INDPA}(\mathfrak{W}(G); a)$.

It is easily shown that the following relations hold,

PROPOSITION 1.10. — *Let* G *be metrizable.*

a) ν *I-universal* $\Rightarrow$ ν *F-universal* $\Rightarrow$ ν *D-universal.*

b) $\nu \in \mathrm{DNPA}(\mathscr{E}_0; a) \Rightarrow \nu \in \mathrm{DNPA}(\mathscr{E}; a)$ ("$\Leftarrow$" *being obvious*)

c) If $\mathscr{E}_0$ *is dense in* $\mathfrak{I}$ *then*

$$\nu \in \mathrm{DNPA}(\mathscr{E}_0; a) \quad \Rightarrow \quad \nu \in \mathrm{DNPA}(\mathfrak{I}; a).$$

d) If $\mathfrak{I} \subseteqq \{ \mu * \varepsilon_x : \mu \in \mathscr{E}, x \in G \}$, i. e. *if every infinitely divisible measure is embeddable up to a shift, then we have:* $\nu \in \mathrm{DNPA}(\mathscr{E}_0; a) \Rightarrow \nu$ *is universal in the wide sense.*

[[*a*) *See* 1.3 *a*)

b) Follows from 1.7 since $\mathscr{E}_0$ is dense in $\mathscr{E}$.

c) Follows from 1.7.

d) is obvious.]]

The assumptions 1.10 *c*, *d* are fulfilled if G is strongly root compact ([He], [Si2]). We need a slight generalization of these groups (*see* [S] for similar definitions):

DEFINITIONS 1.11. — *a*) Let $B \subseteqq \mathbb{N}$ be a nontrivial multiplicative subsemigroup of $\mathbb{N}$. Let

$$S_B := \left\{ \frac{k}{m} : k \in \mathbb{Z}, m \in B \right\}, \qquad S_B^+ := \{ r \in S_B : r > 0 \}.$$

S_B is submonogeneous, *i.e.* there exists a submonogeneous basis

$$(m_i) \subseteq B, \; r_i \in \mathbb{N} \setminus \{1\}, \quad \text{such that } m_i r_i = m_{i+1}, \; i \geqq 1$$

and

$$S_B = \bigcup_1^\infty \frac{1}{m_i} \mathbb{Z} \quad (\textit{cf.}\ \text{e.g. [He]})$$

b) Let $B \subseteq \mathbb{N}$. A locally compact group is B-root-compact if for any compact set $K \subseteq G$, for any $m \in B$ there exist compact sets $C_m \subseteq G$, such that given $x_1, \ldots, x_m \in G$, $x_m = e$, with

$$K x_i K x_j \cap K x_{i+j} \neq \varnothing \qquad \text{for} \quad 1 \leqq i, j \leqq i+j \leqq m$$

we have $x_i \in C_m$, $1 \leqq i \leqq m$. G is strongly B-root compact if we can choose $C_m = C$ independently from $m \in B$. (*See* e.g. [He], [Si1], [S].)

In the following we shall always suppose in addition that B is a multiplicative subsemigroup of $\mathbb{N}$.

Proposition 1.12. — *a*) *Let* G *be* B-*root compact. Let* $\mu \in M^1(G)$ *be* B-*divisible*, i.e. *for any* $m \in B$ *there exists a root* $\mu_{(m)} \in M^1(G)$ *with* $\mu_{(m)}^m = \mu$. *Then* μ *is* S_B^+-*submonogeneously embeddable*, i.e. *there exists a homomorphism* $S_B^+ \ni r \mapsto \mu_r \in M^1(G)$ *with* $\mu_1 = \mu$.

b) *If moreover* G *is strongly* B-*root compact then* $\{\mu_r : 0 < r \leqq 1\}$ *is uniformly tight. Furthermore, there exist a c.c.s.* $(\nu_t)_{t \geqq 0}$ *and a homomorphism* $f: S_B \to G$, $f(r) = : x_r$, *such that* $f(S_B)^-$ *is compact, such that*

$$\nu_t * \varepsilon_{x_r} = \varepsilon_{x_r} * \nu_t, \qquad r \in S_B, \quad t \geqq 0,$$

and

$$\mu_r = \varepsilon_{x_r} * \nu_r, \qquad r \in S_B^+.$$

Moreover the measures $\varepsilon_{x_r} * \omega_K = \omega_K * \varepsilon_{x_r}$, *where* $\nu_0 = \omega_K$, *are accumulation points of* $\{\mu_r\}_{r \to 0}$, $r \in S_B^+$.

⟦The proofs are similar to [Si2] Section 2, Section 6, [He] 3.1, 3.2-3.5, [S] Section 1. Note that if $B \neq \mathbb{N}$ the group $\{x_r K\}^-/K$ is in general not connected.⟧

Next we prove, that on strongly B-root compact groups, measures with non-void domains of partial attraction are shifts of embeddable measures. Indeed, we prove more generally:

Theorem 1.13. — *Let* $B \subseteq \mathbb{N}$ *be a nontrivial multiplicative semigroup. Let* G *be strongly* B-*root-compact. Let* $\nu_n \in M^1(G)$, $\nu_n \to \varepsilon_e$, *and assume* $\nu_n^{k_n} \to \mu \in M^1(G)$, $k_n \uparrow \infty$.

a) *Then there exists a c.c.s.* $(\lambda_t)_{t \geqq 0}$, *a homomorphism* $f: S_B \to G$, $f(r) : = x_r$, *such that* $\varepsilon_{x_r} * \lambda_t = \lambda_t * \varepsilon_{x_r}$, $r \in S_B$, $t \geqq 0$, *such that* $f(S_B)^-$ *is compact, and such that* $(\mu_r : = \varepsilon_{x_r} * \lambda_r)_{r \in S_B^+}$ *is a submonogeneous convolution semigroup with* $\mu_1 = \mu$.

b) Moreover, for any $r \in S_B^+$ *and any* $t>0$ *there exist sequences* α_n, β_n, γ_n, δ_n *in* $\mathbb{N}$ *(depending on r resp. t) such that*

$$\nu_{\beta_n}^{\alpha_n} \to \mu_r, \qquad \nu_{\delta_n}^{\gamma_n} \to \lambda_t, \qquad n \to \infty.$$

Proof. – We start with the following simple observation:

Let ξ_n, η_n, $\rho \in M^1(G)$, such that $\eta_n \to \varepsilon_e$ and $\xi_n * \eta_n \to \rho$. Then $\xi_n \to \rho$.

Fix $m \in \mathbb{N}$ and put $\xi_n := \nu_n^{[k_n/m]\cdot m}$, $\eta_n := \nu_n^{k_n - [k_n/m]\cdot m}$, $\rho := \mu$.

Since $v_n^{k_n} \to \mu$, $\nu_n \to \varepsilon_e$ and $0 \leqq k_n - [k_n/m].m \leqq m$ (hence $\eta_n \to \varepsilon_e$) the observation above yields $\nu_n^{[k_n/m]\cdot m} \to \mu$.

Let (m_i), $(r_i) \subseteq \mathbb{N}$ be a submonogeneous basis for S_B, *i.e.*

$$m_i r_i = m_{i+1}, \qquad i \in \mathbb{N}, \qquad \bigcup_{i=1}^{\infty} \frac{1}{m_i} \mathbb{Z} = S_B.$$

We obtain $\nu_n^{[k_n/m_1]\cdot m_1} \to \mu$, and applying the considerations above again,

$$\nu_n^{[[k_n/m_1]/r_1]\, m_1\, r_1} = \nu_n^{[[k_n/m_1]/r_1]\, m_2} \to \mu, \text{ etc.}$$

So, for $i \in \mathbb{N}$ we obtain a sequence $k_n^{(i)} \uparrow \infty$, such that

$$\nu_n^{k_n^{(i)}} \to \mu \qquad \text{and} \qquad m_j \mid k_n^{(i)}, \qquad j \leqq i, \quad n \in \mathbb{N}.$$

Put $\mathcal{N} := \bigcup_{i=1}^{\infty} \{\nu_n^{k_n^{(i)}}\} \cup \{\mu\}$, and define the root sets

$$R_i := \{ \nu_n^{(k_n^{(i)}/m_i)\cdot s} : 0 \leqq s \leqq m_i,\ n \in \mathbb{N} \}$$

and

$$R_B := \left(\bigcup_{i=1}^{\infty} R_i \right)^{-} \subseteq \{ \rho^s : \exists\, m_i \in B \text{ such that } 0 \leqq s \leqq m_i \text{ and } \rho^{m_i} \in \mathcal{N} \}.$$

An essential step in the proof of the embedding theorem for strongly root compact groups ([Si2] Section 6, Satz 1, [He] Thm. 3.1.13, ($B = \mathbb{N}$) and [S]) yields the compactness of the root set R_B.

Now we continue as in the case $B = \mathbb{N}$ and construct a submonogeneous convolution semigroup $(\mu_r)_{r \in S_B^+}$ in R_B with $\mu_1 = \mu$:

The compactness of R_i^-, $i \in \mathbb{N}$, yields the existence of m_i*-th* roots $\rho_i \in R_i^-$, $\rho_i^{m_i} = \mu$. Hence μ is B-divisible and we can apply proposition 1.12 *a)* to obtain $(\mu_r)_{r \in S_B^+}$.

Apply now 1.12 *b)* to obtain a c.c.s. (λ_t) and $(x_r)_{r \in S_B}$ such that $\mu_r = \varepsilon_{x_r} * \lambda_r$, $r \in S_B^+$.

Moreover the construction of (μ_r) yields the existence of a subsequence $(n') \subseteq \mathbb{N}$ such that

$$\nu_n^{[k_n/m_i]\cdot l} \xrightarrow[(n')]{} \mu_{l/m_i}, \qquad i \in \mathbb{N}, \quad l \in \mathbb{N}.$$

On the other hand the measures $\varepsilon_{x_s} * \omega_K$, $s \in S_B$, $\omega_K = \lambda_0$, are accumulation points of $\{\mu_r\}_{r \to 0}$, therefore for $r \in S_B^+$ $\lambda_r = \varepsilon_{x_r} * \omega_K * \mu_r$ is an accumulation point of $\{\nu_n^l : l\in\mathbb{N},\ n\in(n')\}$. The continuity of $t\mapsto\lambda_t$ proves the last assertion. □

COROLLARY 1.14. – *Let* G *be strongly* B*-root-compact. Let* $\nu\in M^1(G)$ *be* I*-universal w.r.t. a contracting automorphism* $a\in\mathrm{Aut}(G)$. *Then* ν *is universal in the wide sense.*

⟦Follows immediately from 1.10 *d*) and 1.12.⟧

2. THE STRUCTURE OF CONTRACTIBLE GROUPS

G is called contractible if there exists $a\in\mathrm{Aut}(G)$, such that $a^n x \underset{n\to\infty}{\longrightarrow} e$, $x\in G$. In this case [Si3] we have a representation $G=G_0\otimes D$, where G_0 is an a-invariant contractible Lie group (hence especially nilpotent and simply connected) and D is an a-invariant contractible totally disconnected group. There exist compact neighbourhoods $(U_n)_{n\in\mathbb{Z}}\subseteq G$ of e, such that

$$U_n\downarrow,\qquad aU_n=U_{n+1},\quad n\in\mathbb{Z},\qquad \bigcup_n U_n=G,\qquad \bigcap_n U_n=\{e\}.$$

If $G=D$ we can choose U_n as compact open subgroups, such that $U_{n+1}\lhd U_n$, $n\in\mathbb{Z}$ and $U_n/U_{n+1}\cong F$, a fixed finite group. Let $\Delta:=\mathrm{card}(F)$. (U_n) is called a filtration then.

DEFINITION 2.1. – With the notations above put

$$B:=\{m\in\mathbb{N}:(m,\Delta)=1\}.$$

B is an infinite multiplicative semi-group $\subsetneqq\mathbb{N}$. Hence $S_B=\left\{\frac{k}{m}:k\in\mathbb{Z},\ m\in B\right\}$ and S_B^+ are well defined by the totally disconnected part D of the contractible group G. If $G=G_0$, *i.e.* $D=\{e\}$, we put $B:=\mathbb{N}$, $S_B:=\mathbb{Q}$.

PROPOSITION 2.2. – *Let* $G=G_0\otimes D$ *be contractible. Let* F, Δ, B, S_B *and* S_B^+ *as above. Then* D *and hence* G *are strongly* B*-root-compact.*

⟦G_0 is strongly root compact ([Si2], [He]), hence strongly B-root-compact. It is sufficient to prove the strong B-root-compactness of D.

Let $K\subseteq D$ be a compact subset. Let $(U_n)_{n\in\mathbb{Z}}$ be a filtration, $U_n/U_{n+1}\cong F$. Fix $n_0\in\mathbb{Z}$, such that $K\subseteq U_{n_0}$.

Let $m\in B$, $x_1,\ldots,x_m\in D$, $x_m=e$, $m_0\in\mathbb{N}$, such that $m_0<n_0$, and $\bigcup_1^m U_{n_0}x_i\subseteq U_{m_0}$, $U_{n_0}x_iU_{n_0}x_j\cap U_{n_0}x_{i+j}\neq\varnothing$, $1\leqq i,j\leqq i+j\leqq m$.

From $x_m=e$, and $U_{m_0+1}\lhd U_{m_0}$ we obtain

$$U_{m_0+1}x_iU_{m_0+1}x_j=U_{m_0+1}x_{i+j},$$

therefore the elements $\{\bar{x}_i := U_{m_0+1}x_i : 1 \leqq i \leqq m\}$ form a cyclic subgroup of order $o_m \mid m$ in U_{m_0}/U_{m_0+1}. But $U_{m_0}/U_{m_0+1} = F$ and $(o(F), m) = 1$, hence $\bar{x}_i = \bar{e}$, $1 \leqq i \leqq m$, *i.e.* $x_i \in U_{m_0+1}$. Repeating these arguments we obtain finally $x_i \in C := U_{n_0}$, $1 \leqq i \leqq m$.]]

THEOREM 2.3. — *Let* G *be as above. Then the elements of* G *are uniquely m-divisible for any* $m \in B$ *and for* $x \in D$ *the (unique) m-th root of x is contained in the (monothetic, compact, abelian) group* $\langle x \rangle^-$ *generated by x.*

Hence for any $x \in G$ *there exists a uniquely defined homomorphism* $f: S_B \to G$ *such that* $f(1) = x$, *and* $f(S_B) \subseteqq \langle x \rangle^-$.

[[Let $G = G_0 \otimes D$. For $x \in G_0$ the assertion is obvious since G_0 is nilpotent and simply connected. Consider therefore the case $x \in D$. Let $m \in B$. Assume $y \in D$, $x \in D$, such that $y^m = x$. Hence x and y are contained in the compact abelian group $\langle y \rangle^- := A$. We show next that given any compact abelian group $A \subseteqq D$, $x \in A$, $m \in B$ there exists a unique $y \in A$, such that $y^m = x$, and moreover $y \in \langle x \rangle^-$.

[[We have $A \subseteqq U_{n_0}$ for some n_0. Put $A_n := A \cap U_n$, $n \in \mathbb{Z}$ (hence $A_{n_0} = A$). Then $A_n \downarrow \{e\}$ and card $(A_n/A_{n+1}) \mid \Delta$.

Since $(m, \Delta) = 1$ the group A/A_n $(n \geqq n_0)$ is uniquely divisible by m. Hence for fixed $n \geqq n_0$ for any $x \in A$ there exists a unique coset $y A_n \in A/A_n$, such that $(y A_n)^m = y^m A_n = x A_n$, and $y A_n \in \langle x A_n \rangle$.]]

The compactness of $A = A_{n_0}$ and the filtration property $A_n \downarrow \{e\}$ yield the existence of $y \in A$, such that $y^m = x$. Moreover, y is unique and $y \in \langle x \rangle^-$.

Now choose a submonogeneous basis (m_i) for $S_B = \bigcup_1^\infty \frac{1}{m_i} \mathbb{Z}$ and apply the m_i-divisibility successively.

The assertion is proved.]]

Remarks 2.4. — *a*) For p-adic groups $\mathbb{Q}_p$ the strong B-root compactness is proved in [S]. p-adic groups $\mathbb{Q}_p$ (and e.g. p-adic Heisenberg groups) are examples of contractible groups which are root compact but not strongly root-compact ([Si3], ex. 3.5).

b) There exist contractible totally disconnected groups which are strongly B-root compact (for suitable $B \subseteqq \mathbb{N}$) but not root-compact:

Let F be a fixed finite group with order $o(F) = \Delta \in \mathbb{N} \setminus \{1\}$. Let $\Gamma := \bigotimes_{n \in \mathbb{Z}} F$, $U_n := \bigotimes_{k=n}^{\infty} F$. The groups $(U_n)_{-\infty}^{\infty}$ endowed with the product topology are a (normal) filtration of the contractible group $\Gamma^* := \bigcup_n U_n$ [Si3].

It is easily seen that for any $1 < \delta \mid \Delta^k$ $(k \in \mathbb{N})$ and any $n \in \mathbb{Z}$ the unit element $\bar{e} = e U_n$ has infinitely many δ-roots in Γ^*/U_n. Hence Γ^* is not root compact.

c) The results of 1.12, 1.13, 1.14 hold especially for contractible groups $G=G_0\otimes D$. Moreover, in the representation $\mu_r=\varepsilon_{x_r}*\lambda_r*\omega_K$ obtained in 1.12 the shifts x_r are uniquely determined by x_1 (mod K).

d) For *p*-adic matrix groups infinitely divisible measures are continuously embeddable up to a shift. This is proved in [S].

e) Contractible groups are in general not Lie-projective ([Si3], 3.5 *b*). Indeed, the totally disconnected part D is Lie projective if it has a normal filtration.

Hence, since the key result of E. Siebert on convergence of semigroups (*see* Prop. 1.3) is only proved for Lie projective groups, it was necessary in Section 1 to distinguih between the domains FDNPA and IDNPA.

f) Let G be contractible, let Δ, B as above. Let μ be infinitely divisible and let $r\mapsto\mu_r$, $r\mapsto x_r$ be submonogeneous homomorphisms, such that $\mu_r=\varepsilon_{x_r}*\lambda_r$, $r\in S_B^+$, where $\mu_1=\mu$ and (λ_t) is a c.c.s. Then for $(m_n)\subseteq B$, $m_n\uparrow\infty$, $k_n\in\mathbb{N}$, such that $c_n:=\dfrac{k_n\Delta^n}{m_n}\to 1$ we obtain $\mu_r^{(n)}:=\mu_{r\,.\,c_n}\xrightarrow[n\to\infty]{}\lambda_r$, $r\in S_B^+$.

⟦The construction of the homomorphism $f:r\mapsto x_r$ yields: For any sequence $(s_n)\subseteq S_B$ we have $(x_{s_n})^{\Delta^n}=x_{s_n\Delta^n}\to e$. Hence the continuity of (λ_r) yields

$$\mu_r^{(n)}=\lambda_{r\,.\,c_n}*\varepsilon_{[x_{r\,.\,k_n/m_n}]\,\Delta^n}\to\lambda_r.]\!]$$

g) Let G be contractible, B, S_B as above. Let $(\lambda_t)_{t\geqq 0}$ be a continuous convolution semigroup in $M^1(G)$. Let $x\in D$ such that $\lambda_t*\varepsilon_x=\varepsilon_x*\lambda_t$, $t>0$. Let $f:r\to x_r$ be the submonogeneous homomorphism with $x_1=x$. Then $(\mu_r:=\varepsilon_{x_r}*\lambda_r)_{r\in S_B^+}$ is a submonogeneous (in general non-continuous) convolution semigroup.

$$[\![\varepsilon_{x_r}*\lambda_t=\lambda_t*\varepsilon_{x_r} \text{ since } x_r\in\langle x_1\rangle^-]\!].$$

3. THE EXISTENCE OF UNIVERSAL DISTRIBUTIONS ON CONTRACTIBLE GROUPS

We show that on contractible locally compact groups with contracting automorphism $a\in\mathrm{Aut}(G)$ there exist I-universal distributions ν (*see* Definition 1.9). For connected $G=G_0$, especially for vector-spaces $\mathbb{R}^d$, ν is a Doeblin distribution then (proposition 1.10 *c*). For general contractible groups remark 2.5 *c* and corollary 1.14 yield that ν is universal in the wide sense.

3.1. In order to prove the existence of I-universal distributions on G w.r.t. $a\in\mathrm{Aut}(G)$ we reduce the problem along the following steps:

3.1.*a* The set of Poisson generators $\mathscr{P}:=\{\alpha(\lambda-\varepsilon_e):\lambda\in M^1(G),\ \alpha>0\}$ is dense in $\mathfrak{W}(G)$ with respect to the weak topology $\sigma(\mathscr{E}',\mathscr{E})$. Therefore, if $\nu\in\mathrm{IDNPA}(\mathscr{P};a)$ then ν is I-universal.

[[Let $P_m\in\mathscr{P}$, $P_m\to A\in\mathfrak{W}(G)$ on $\mathscr{E}(G)$. For any $m\in\mathbb{N}$ there exist $k_n^{(m)}$, $l_n^{(m)}\uparrow\infty$ such that $k_n^{(m)}(a^{l_n^{(m)}}\nu-\varepsilon_e)\xrightarrow[n\to\infty]{}P_m$. Since G is metrizable we can find suitable subsequences k_n', l_n' such that $a^{l_n'}(\nu^{k_n'}-\varepsilon_e)\to A$.]]

3.1.*b* Let (U_n) be a basis of neighbourhoods of e, such that $U_n\downarrow\{e\}$, $aU_n=U_{n+1}$, $\bigcup U_n=G$. Then $\Gamma:=\{a^m\lambda:m\in\mathbb{Z},\ \lambda\in M^1(G),\ \mathrm{supp}(\lambda)\subseteqq U_0\}$ is dense in $M^1(G)$. Let $\mathscr{P}_1:=\{\alpha(\lambda-\varepsilon_e):\alpha>0,\ \lambda\in M^1(G),\ \mathrm{supp}(\lambda)\subseteqq U_0\}$. We apply proposition 1.4.*a*, *a'*, and similar to 3.1.*a* we have:

If $\nu\in\mathrm{IDNPA}(\mathscr{P}_1;a)$ then ν is I-universal.

3.1.*c* According to proposition 1.4.*b*, *b'* it is sufficient to prove the existence of $\nu\in\mathrm{IDNPA}(\mathscr{P}_2;a)$, where $\mathscr{P}_2:=\{\lambda-\varepsilon_e:\lambda\in M^1(G),\ \mathrm{supp}(\lambda)\subseteqq U_0\}\subseteqq\mathscr{P}_1$.

3.1.*d* Let $\{\lambda_m:m\in\mathbb{N}\}$ be a fixed dense countable subset of $\{\lambda\in M^1(G):\mathrm{supp}(\lambda)\subseteqq U_0\}$. Hence $\{\lambda_m-\varepsilon_e\}$ is dense in $\mathscr{P}_2$. Let further $\alpha(n)$, $\beta(n)$, $\gamma(n)$ be sequences of positive numbers such that

(i) $\sum_1^\infty\alpha(n)=1$,

(ii) $\alpha(n)^{-1}\in\mathbb{N}$, $\quad\alpha(n)\downarrow 0$

(iii) $\alpha(N)^{-1}.\sum_{N+1}^\infty\alpha(n)\xrightarrow[N\to\infty]{}0$

(iv) $\beta(n)\in\mathbb{N}$, $\quad\beta(n)\uparrow\infty$

(v) $\gamma(n):=\beta(n)-\beta(n-1)\uparrow\infty$

(vi) For any $\rho\in(0,1)$, $\quad\rho^{\gamma(n)}/\alpha(n)\xrightarrow[n\to\infty]{}0$.

[[The sequences used in [Doe], [Ba1], [Ph], [Th] $\alpha(n)\sim c.2^{-n^2}$, $\beta(n)\sim n^3$, $\gamma(n)\sim 3n^2-3n+1$ fulfil (i)−(vi).]]

Let $a\in\mathrm{Aut}(G)$ be a fixed contracting automorphism. Define

$$\nu:=\sum_1^\infty\alpha(n)\,a^{-\beta(n)}(\lambda_n)\in M^1(G).$$

THEOREM 3.2. – *The measure* ν *defined above is* I-*universal* w.r.t. *a*.

Proof 1. – According to the reduction steps 3.1.*a*-3.1.*c* it is sufficient to show that for any $\lambda\in M^1(G)$ with $\mathrm{supp}(\lambda)\subseteqq U_0$ there exist $L(k)\uparrow\infty$,

$M(k)\uparrow\infty$, such that

$$M^{(k)}(a^{L(k)}\nu-\varepsilon_e)\to\lambda-\varepsilon_e \quad \text{on } \mathscr{E}(G).$$

[Remark that the M(G)-norms of the approximating Poisson generators are not bounded!].

2. Let $(n_k)\subseteqq\mathbb{N}$ be a subsequence such that $\lambda_{n_k}\to\lambda$ and $n_k\uparrow\infty$. (This is of course possible even if $\lambda\in\{\lambda_m : m\in\mathbb{N}\}$.) We define then $L(k) := \beta(n_k)$, $M(k) := \alpha(n_k)^{-1}$.

The approximating Poisson generators are represented in the form

$$\begin{aligned} M(k)(a^{L(k)}\nu-\varepsilon_e) &= M(k)\left(\sum_1^\infty \alpha(n)(a^{L(k)-\beta(n)}\lambda_n-\varepsilon_e)\right) \\ &= P_k+(\lambda_{n_k}-\varepsilon_e)+Q_k, \end{aligned}$$

where

$$P_k := M(k)\sum_{n=1}^{n_k-1}\alpha(n)(a^{L(k)-\beta(n)}\lambda_n-\varepsilon_e)$$

and

$$Q_k := M(k)\sum_{n_k+1}^{\infty}\alpha(n)(a^{L(k)-\beta(n)}\lambda_n-\varepsilon_e).$$

We have to show $P_k\to 0$ and $Q_k\to 0$ on $\mathscr{E}(G)$.

3. $\|Q_k\|\to 0$.

$[\![\, \|Q_k\| = 2.Q_k(G\setminus\{e\})\leqq 2Q_k(G)$

$$= \frac{2}{\alpha(n_k)}\sum_{n_k+1}^{\infty}\alpha(n)\to 0 \text{ [by 3.1.}d.\text{ (iii)].]\!]}$$

4. Let V be a neighbourhood of the unit. Then we obtain for the Lévy measures $P_k(\complement V)\to 0$.

$$[\![\text{Let } \eta_k := P_k|_{\complement V}.$$

Since

$$\operatorname{supp}(\lambda_n)\subseteqq U_0,\ \operatorname{supp}(\eta_k)\subseteqq\bigcup_{n=1}^{n_k-1}U_{\beta(n_k)-\beta(n)}\subseteqq U_{\beta(n_k)-\beta(n_k-1)}=U_{\gamma(n_k)}.$$

Since $U_n\downarrow\{e\}$ and $\gamma(n_k)\uparrow\infty$ we have $\operatorname{supp}(\eta_k)\subseteqq V$ for sufficiently large k, hence $\eta_k=0$.]\!]

Remark 5. – If $G=D$ is totally disconnected 3. and 4. already imply $P_k+Q_k\to 0$, hence $M(k)(a^{L(k)}\nu-\varepsilon_e)\to\lambda-\varepsilon_e$.

6. Let $G=G_0$ be a contractible Lie group. Let $(\zeta_i)_{i=1}^d$ be local coordinates and let φ be a Hunt function (*cf.* [He] 4.5).

Let $\mathscr{G}$ be the Lie algebra. For functions f on G let $\mathring{f}:=f\circ\exp$ on $\mathscr{G}$, analogously define $\mathring{a}$, $\mathring{\mu}$, $\mathring{A}$ for automorphisms, measures resp. generating distributions. *W.l.o.g.* we may assume that

$$\mathring{\xi}_i(x)=\frac{x_i}{1+\|x\|^2},\qquad \mathring{\varphi}(x)=\frac{\|x\|^2}{1+\|x\|^2}.$$

$\mathring{a}$ is a contracting automorphism on the vectorspace, hence for some $r\geqq 0$ $\|\mathring{a}^r\|=:\delta<1$.

6.1.

$$|\langle P_k,\xi_i\rangle|=|\langle \mathring{P}_k,\mathring{\xi}_i\rangle|\leqq\sum_{n=1}^{n_k-1}\frac{\alpha(n)}{\alpha(n_k)}\cdot\int_{\mathring{U}_0}|\mathring{\xi}_i(\mathring{a}^{\beta(n_k)-\beta(n)}x)|\,d\mathring{\lambda}_n(x).$$

Here $\mathring{U}_0:=\exp^{-1}(U_0)$. Let $R:=\max_{x\in\mathring{U}_0}\|x\|$, $K:=\max_{0\leqq j\leqq r-1}\|\mathring{a}^j\|$.

Then

$$\begin{aligned}|\langle P_k,\xi_i\rangle|&\leqq\frac{1}{\alpha(n_k)}\sum_{n=1}^{n_k-1}\alpha(n).\sup_{x\in\mathring{U}_0}\|\mathring{a}^{\beta(n_k)-\beta(n)}x\|\\&\leqq\frac{R}{\alpha(n_k)}\sum_{n=1}^{n_k-1}\alpha(n)\|\mathring{a}^{\beta(n_k)-\beta(n)}\|\\&\leqq\frac{R.K}{\alpha(n_k)}\sum_{n=1}^{n_k-1}\alpha(n)\,\delta^{[(\beta(n_k)-\beta(n))/r]}\\&\leqq\frac{R.K}{\alpha(n_k)}\sum_{n=1}^{n_k-1}\alpha(n)\,\delta^{[\gamma(n_k)/r]}.\end{aligned}$$

Using $[\gamma(n_k)/r]=\gamma(n_k)/r-\varepsilon$, $\varepsilon\in[0,1]$, we obtain by 3.1.*d* (vi)

$$|\langle P_k,\xi_i\rangle|\leqq\frac{R.K}{\alpha(n_k)}\sum_{1}^{n_k-1}\alpha(n)(\delta^{1/r})^{\gamma(n_k)}.\delta^{-1}\leqq\frac{(\delta^{1/r})^{\gamma(n_k)}}{\alpha(n)}.(R.K.\delta^{-1})\to 0.$$

6.2. Analogously we obtain

$$\begin{aligned}0\leqq\langle P_k,\varphi\rangle&=\langle\mathring{P}_k,\mathring{\varphi}\rangle\\&=\frac{1}{\alpha(n_k)}\sum_{n=1}^{n_k-1}\alpha(n)\int_{\mathring{U}_0}\mathring{\varphi}(\mathring{a}^{\beta(n_k)-\beta(n)}x)\,d\mathring{\lambda}_n(x)\\&\leqq\frac{1}{\alpha(n_k)}.R.K^2.[(\delta^{1/r})^{\gamma(n_k)}]^2\\&=\text{const.}\ (\delta^{2/r})^{\gamma(n_k)}/\alpha(n_k)\to 0\end{aligned}$$

[again by 2.1 (vi)].

Remark 6.3. – If $G=G_0$ is a Lie group we obtain by steps 4., 6.1. and 6.2. $P_k\to 0$ (*see* e.g. [Si1] 5.4, 8.1 remark 4.7).

7. Let $G=G_0\otimes D$ be a contractible group. Let $\tilde{\mathscr{D}}:=\mathscr{D}(G_0)\otimes\mathscr{D}(D)$. $\tilde{\mathscr{D}}$ is a core for the generators of convolution semigroups (e.g. [Ha] 0. Section 4, Satz of F. Hirsch). Moreover the distributions P_k are concentrated on the fixed compact neighbourhood U_0. Hence the assertion $\langle P_k, f\rangle\to 0$, $f\in\mathscr{E}(G)$, is equivalent to $\langle P_k, f\rangle\to 0$, $f\in\tilde{\mathscr{D}}$. Therefore we have to show $\langle P_k, f\otimes g\rangle\to 0$ for $f\in\mathscr{D}(G)$, $g\in\mathscr{D}(D)$.

For any $g\in\mathscr{D}(D)$ there exists a compact open subgroup $V_g\subseteq D$, such that $g(\kappa x)=g(x)$ for $x\in D$, $\kappa\in V_g$.

Let λ_n^1 be the projection of λ_n onto G_0 and let U_0^1 be the projection of U_0 to G_0. Hence we have

$$\begin{aligned}\langle P_k, f\otimes g\rangle &\\ &=\frac{1}{\alpha(n_k)}\sum_{n=1}^{n_k-1}\alpha(n)\int_{G_0\otimes D}[f(x)g(\kappa)-f(e)g(e)]\,da^{\beta(n_k)-\beta(n)}(\lambda_n)(x,\kappa)\\ &=\frac{1}{\alpha(n_k)}\sum_{n=1}^{n_k-1}\alpha(n)\left\{\int\!\!\int[(f(a^{\beta(n_k)-\beta(n)}x)-f(e)]\right.\\ &\qquad\times g(a^{\beta(n_k)-\beta(n)}\kappa)]\,d\lambda_n(x,\kappa)\\ &\qquad+\left.\int[f(e).(g(a^{\beta(n_k)-\beta(n)}\kappa)-g(e))]\,d\lambda_n(x,\kappa)\right\}.\end{aligned}$$

If k is sufficiently large, such that $a^{\beta(n_k)-\beta(n)}\kappa\in V_g$ for $(x,\kappa)\in U_0$, then the second integral is zero and therefore

$$\begin{aligned}\langle P_k, f\otimes g\rangle &=\frac{1}{\alpha(n_k)}\sum_{n-1}^{n_k-1}\alpha(n)\\ &\quad\times\int_{U_0}(f(a^{\beta(n_k)-\beta(n)}x)-f(e)]\,d\lambda_n(x,\kappa).g(e)\\ &=\frac{g(e)}{\alpha(n_k)}\sum_{n=1}^{n_k-1}\int_{U_0^1}(f(a^{\beta(n_k)-\beta(n)}x)-f(e))\,d\lambda_n^1(x)\underset{k\to\infty}{\longrightarrow}0\end{aligned}$$

(as proved in step 6.2).

Theorem 3.2 is proved. □

Now we are able to prove a characterization of infinite divisibility (for $G=\mathbb{R}$ due to Doeblin and Khinchine, *see* [Doe]):

THEOREM 3.3. – [Doe], [Ba1], [Ba2], [Ph], [Th])

Let G *be locally compact with contracting automorphism* a. *Let* $\mu\in M^1(G)$.

Consider the following assertions:

(i) μ *is infinitely divisible*

(i′) μ *is* B-*divisible* (i.e. *for* $m\in B$ *there exists a root* $\mu_{(m)}\in M^1(G)$ *with* $\mu_{(m)}^m=\mu$).

[Here, if $G=G_0$ is a Lie group, B is any infinite subset of $\mathbb{N}$. If $G=G_0 \otimes D$ is contractible and $D \neq \{e\}$ then B is defined as in Definition 2.1.]

(ii) μ *is continuously embeddable*

(ii′) *There exists a shift* ε_x *such that* $\mu * \varepsilon_x = \varepsilon_x * \mu$ *is embeddable into a c. c. s.* λ_t *with* $\varepsilon_x * \lambda_t = \lambda_t * \varepsilon_x$, $t \geqq 0$.

(iii) DNPA $(\mu; a) \neq \varnothing$

(iv) DPA$(\mu) \neq \varnothing$.

a) *We have*

$$\text{(ii)} \Rightarrow \text{(i)} \Rightarrow \text{(i')} \Leftrightarrow \text{(ii')}$$

and

$$\text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(i')}$$

b) *If* $G=G_0$ *is a Lie group, then all the assertions* (i) *to* (iv) *are equivalent.*

⟦*a*) (ii)⇒(i)⇒(i′), (iii)⇒(iv) obvious.

(iv)⇒(ii′) by theorem 1.13.*a*. (ii′)⇒(i′) since by Theorem 2.3 every shift ε_x is submonogeneously embeddable. (i′)⇒(ii′) by proposition 1.12. (ii)⇒(iii) by theorem 3.2.

b) (i)⇔(ii) *see* [He] 3.5.8, 3.5.9 since $G=G_0$ is nilpotent. (i)⇒(i′) obvious, (i′)⇒(ii) by proposition 1.12*b*: $\mu=\varepsilon_x * \lambda = \lambda * \varepsilon_x$, with continuously embeddable λ. But ε_x is continuously embeddable, hence (ii). (ii)⇒(iii) by Theorem 3.1, (iii)⇒(iv) obvious. (iv)⇒(ii) by [No1] I 1.11, [No2] 4, Thm. 1.⟧

Remark 3.4. Let $G=G_0$ be a contractible Lie group with Lie algebra $\mathcal{G}$. Let μ, $\nu \in M^1(G)$ and let $a \in \mathrm{Aut}(G)$ be contracting. Let $\mathring{\mu}$, $\mathring{\nu}$, $\mathring{a}$ be the corresponding objects on the tangent space $\mathcal{G}$ (*see* e.g. [H-S]). Then we have:

$$a^{l_n} \nu^{[k_n t]} \to \mu_t = \operatorname{Exp} t\mathrm{A}, \quad \mu_1 = \mu, \quad t \geqq 0$$

iff

$$\mathring{a}^{l_n} \mathring{\nu}^{[k_n t]} \to \mathring{\gamma}_t := \operatorname{Exp} t\mathring{\mathrm{A}}, \quad t \geqq 0$$

(*See* [H-S]). Therefore we obtain:

ν is a Doeblin distribution w.r.t. a on the group G iff $\mathring{\nu}$ is a Doeblin distribution w.r.t. $\mathring{a}$ on the vectorspace $\mathcal{G}$.

Remark added in proof:

Recently it could be shown that Siebert's characterization of convergence of convolution semigroups and hence proposition 1.3.*b* is valid for arbitrary locally compact groups. Hence the distinction between different domains of attraction is superfluous for contractible groups. ("A generalization of E. Siebert's theorem on convergence of c.c.s. and accompanying laws." To be published.)

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