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On an identity in law obtained by A. Földes and P. Révész

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ABSTRACT. – Using jointly Ray-Knight theorem on Brownian local times, time reversal, and the Ciesielski-Taylor identity in law, another identity in law by A. Földes and P. Révész is recovered, and generalized.

Key words : Brownian local times, time reversal.

RÉSUMÉ. – En utilisant conjointement le théorème de Ray-Knight sur les temps locaux browniens, un résultat classique de retournement du temps, et l'identité en loi de Ciesielski-Taylor, on retrouve et on généralise une identité en loi obtenue par A. Földes et P. Révész.

1. THE FÖLDES-RÉVÉSZ IDENTITY

In their paper [2], A. Földes and P. Révész prove that, for $r>q$:

$$\int_0^\infty dy\, 1_{(0<L(y,T_r)<q)} \overset{\text{(law)}}{=} T_{\sqrt{q}}(R_2) \tag{1}$$

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where, on the left-hand side, $L(y, T_r)$ denotes the local time at level y, up to time $T_r \equiv \inf\{t : L(0, t) > r\}$, of Brownian motion starting at 0, and, on the right-hand side, $T_{\sqrt{q}}(R_2)$ denotes the first hitting time of $\sqrt{q}$ by R_2, a two-dimensional Bessel process starting from 0.

In fact, in [2], it is shown that the Laplace transform, in λ, of the left-hand side of (1) is:

$$\frac{1}{I_0(\sqrt{2\lambda q})},$$

where I_0 is the modified Bessel function, with index 0, but it is well-known that this is the Laplace transform of $T_{\sqrt{q}}(R_2)$ (*see* Kent [3], for example).

In the sequel, we shall write $T_a(R_\delta)$ for the first hitting time of a by a Bessel process of dimension δ starting from 0, and $BESQ_r^\delta$ shall denote the square, starting at r, of a Bessel process with dimension δ.

Here is a quick proof of (1).

a) From Ray-Knight's theorem on Brownian local times, we know that:

$$\int_0^\infty dy\, 1_{(0 < L(y, T_r) < q)} \overset{\text{(law)}}{=} \int_0^{T_0} dy\, 1_{(Y_y < q)},$$

where $(Y_y;\ y \geqq 0)$ denotes a $BESQ_r^0$, and $T_0 = \inf\{y : Y_y = 0\}$. By the strong Markov property, we may as well assume that $Y_0 = q$, hence the explanation of the fact that the right-hand side of (1) does not depend on r, for $r \geqq q$.

b) Using time-reversal for Bessel processes (Getoor-Sharpe [7]; *see* also e. g. Revuz-Yor [8], Chapt. XI, Exercise (1.23), or Yor [9], formula (4.*c*), for another application), we now obtain:

$$\int_0^{T_0} dy\, 1_{(Y_y < q)} \overset{\text{(law)}}{=} \int_0^{\hat{L}_q} dy\, 1_{(\hat{Y}_y < q)} = \int_0^\infty dy\, 1_{(\hat{Y}_y < q)}$$

where $(\hat{Y}_y,\ y \geqq 0)$ is a $BESQ_0^4$ and $\hat{L}_q = \sup\{y : \hat{Y}_y = q\}$.

c) The Ciesielski-Taylor identity in law (*see* [1], [6] for example) tells us that:

$$\int_0^\infty dy\, 1_{(\hat{Y}_y < q)} \overset{\text{(law)}}{=} T_{\sqrt{q}}(R_2),$$

which ends the proof of (1).

2. A GENERALISATION

Let $(B_t,\ t \geqq 0)$ denote Brownian motion starting from 0, and for convenience, we denote now by $(l_t,\ t \geqq 0)$ its local time at 0, instead of

$(L(0, t), t \geqq 0)$. The process $(X_t := |B_t| - \mu l_t, t \geqq 0)$ is, in the case $\mu = 1$, a Brownian motion (as seen from Tanaka's formula, for instance), and in any case, it is a process which possesses a number of very interesting properties. We state two of those.

THEOREM 1 ([4]; *see* also Chapter 8 of [5]). — *We have*

$$\int_0^1 ds\, 1_{(X_s \leqq 0)} \overset{(\text{law})}{=} Z_{(1/2),\,(1/2\,\mu)},$$

where $Z_{a,\,b}$ *denotes a beta variable with parameters* a *and* b, i.e.

$$P(Z_{a,\,b} \in dt) = \frac{t^{a-1}(1-t)^{b-1}\, dt}{B(a, b)} \quad (0 < t < 1).$$

THEOREM 2 (*see* Chapter 9 of [5]). — *Let* $(l_t^\mu, t \geqq 0)$ *be the local time at* 0 *of the process* $(X_t, t \geqq 0)$, *and* $\tau_r^\mu := \inf\{t : l_t^\mu > r\}$. *Then, for fixed* $r > 0$, *the processes* $(l_{\tau_r^\mu}^x(X); x \geqq 0)$ *and* $(l_{\tau_r^\mu}^{-x}(X); x \geqq 0)$ *are independent, and their respective distributions are* Q_r^0, *and* $Q_r^{2-(2/\mu)}$, *where* Q_r^δ *denotes the law of the square of a* δ*-dimensional Bessel process, starting from* r *and absorbed at* 0.

We now prove the following

THEOREM 3. — *Let* $r > q$. *Then, we have*

$$\int_{-\infty}^0 dy\, 1_{(0 < l_{\tau_r^\mu}^y(X) < q)} \overset{(\text{law})}{=} T_{\sqrt{q}}(R_{2/\mu}). \tag{2}$$

Proof. — Following the same sequence of arguments as in the first paragraph, we find with the help of Theorem 2, that the left-hand side of (2) is equal in law, to: $\int_0^\infty dy\, 1_{(\hat{Y}_y \leqq q)}$, where $(\hat{Y}_y, y \geqq 0)$ is a $BESQ_0^{2+(2/\mu)}$.

Then, the Ciesielski-Taylor identity in law tells us that:

$$\int_0^\infty dy\, 1_{(\hat{Y}_y \leqq q)} \overset{(\text{law})}{=} T_{\sqrt{q}}(R_{2/\mu}). \quad \square$$

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