ANNALES DE L'I. H. P., SECTION B

PHILIP S. GRIFFIN
TERRY R. MCCONNELL
GREGORY VERCHOTA

Conditioned brownian motion in simply connected planar domains

Annales de l'I. H. P., section B, tome 29, nº 2 (1993), p. 229-249
<http://www.numdam.org/item?id=AIHPB_1993__29_2_229_0>

Ann. Inst. Henri Poincaré,

Vol. 29, n° 2, 1993, p. 229-249. *Probabilités et Statistiques*

Conditioned Brownian motion in simply connected planar domains

by

Philip S. GRIFFIN (*), Terry R. McCONNELL (**) and Gregory VERCHOTA (***)

Department of Mathematics, Syracuse University,
Syracuse, NY 13244-1150, U.S.A

ABSTRACT. — The purpose of this paper is to study Doob's conditioned Brownian motions in simply connected domains in $\mathbb{R}^2$. We obtain the precise value of the best constant in the lifetime inequality of Cranston and McConnell and prove a related maximum principle. We also exhibit a connection between these processes and the classical isoperimetric inequalities.

Key words : Conditioned Brownian motion, expected lifetime, positive harmonic function, isoperimetric inequality.

1. INTRODUCTION

Let D be a domain in $\mathbb{R}^n$ and $p(t, \alpha, \beta)$ the transition densities of Brownian motion killed on exiting D. Let

$$H^+ D = \{ h : h \text{ is a positive harmonic function in } D \}.$$

Classification A.M.S. : (1980) *Subject classification* (1985 *Revision*)*:* 60 J 45, 60 J 65.

(*) Research supported in part by N.S.F. Grant DMS-8700928.

(**) Research supported in part by N.S.F. Grant DMS-8900503.

(***) Research supported in part by N.S.F. Grant DMS-8915413.

For $h \in \mathrm{H}^+ \mathrm{D}$ set

$$p^h(t, \alpha, \beta) = \frac{1}{h(\alpha)} p(t, \alpha, \beta) h(\beta).$$

Let P_α^h denote the measure on path space induced by these transition densities and E_α^h expectation with respect to P_α^h. The canonical process, which we denote by Z_t [or sometimes $\mathrm{Z}(t)$], is then a Doob conditioned Brownian motion or h-process. Its lifetime is given by

$$\tau_\mathrm{D} = \inf\{t : \mathrm{Z}_t \notin \mathrm{D}\}.$$

If no confusion can arise we will often drop the subscript D and simply write τ for the lifetime.

In 1983, Cranston and McConnell [6] proved that for any domain $\mathrm{D} \subseteqq \mathbb{R}^2$, there exists a constant c_D such that

$$\sup_{\substack{\alpha \in \mathrm{D} \\ h \in \mathrm{H}^+ \mathrm{D}}} \mathrm{E}_\alpha^h \tau_\mathrm{D} \leqq c_\mathrm{D} |\mathrm{D}| \tag{1.1}$$

and furthermore

$$\sup_\mathrm{D} c_\mathrm{D} < \infty.$$

Note that (1.1) is only of interest when D has finite area. *See* [11] for references to extensions, applications and related results.

To translate this result into analytic terms, we only need observe that

$$\mathrm{E}_\alpha^h \tau = \int_\mathrm{D} \frac{1}{h(\alpha)} \mathrm{G}(\alpha, z) h(z) \, dm(z)$$

where G is the (probabilist's) Green function for D, *i.e.*, minus two times the analyst's Green function and m is Lebesgue measure.

One of the main results of this paper is an evaluation of the best possible constant among all simply connected domains, *i.e.*

$$c^* = \sup\{c_\mathrm{D} : \mathrm{D} \text{ is simply connected}\}.$$

We will prove that $c^* = \frac{1}{\pi}$ and furthermore that the supremum can not be attained.

Examples of domains for which c_D approaches $\frac{1}{\pi}$ are given in section 3 (*see* Theorem 3.4) and include long thin rectangles. The opposite extreme to these domains is the disc and here we explicitly compute c_D to be $(4 \log 2 - 2)/\pi \cong .7726/\pi$.

If h is a Martin kernel (*i.e.* minimal) function with pole at $\beta \in \partial_\mathrm{M} \mathrm{D}$, where $\partial_\mathrm{M} \mathrm{D}$ = minimal Martin boundary of D, then we write P_α^β and E_α^β for P_α^h and E_α^h respectively. In this case we prove that, as a function of $\alpha \in \mathrm{D}$, $\mathrm{E}_\alpha^\beta \tau_\mathrm{D}$ satisfies a maximum principle; see section 2. It is not clear how important it is that h be minimal in this result, but we should point out that $E_\alpha^h \tau_D$ does not satisfy the maximum principle for all $h \in H^+ D$, for example when h is constant.

In the final section we extend the best constant result to the case of superharmonic h and discuss various topics including the connection with the isoperimetric inequality; *see* Remark 5.4.

The authors would like to thank Tadeusz Iwaniec for informing us of the work of Carleman, Gerry Cargo for a simplification in the proof of Theorem 2.1 and Eugene Poletsky for several enlightening discussions. Finally we would like to thank the referee for pointing out an oversight on our part in the statement of the strong maximum principle in section 2, and for the observation that $c_D \leqq 4/\pi$ for any planar domain. This bound can be bound in Bañuelos [2].

2. A MAXIMUM PRINCIPLE

In this section we will prove a strong maximum principle for

$$\frac{\displaystyle\int_D G^\alpha(z)\, K^\beta(z)\, dm(z)}{K^\beta(\alpha)} = E_\alpha^\beta \tau_D \tag{2.1}$$

as a function of $\alpha \in D$ whenever $D \subset \mathbb{R}^2$ is simply connected and $|D| < \infty$. Here $G^\alpha(z) = G(\alpha, z)$ and K^β is any kernel function for D with pole at $\beta \in \partial_M D$. By replacing K^β with the constant function we see that such a principle cannot hold in general for positive harmonic functions. We will also prove a maximum principle for simply connected domains of infinite area in $\mathbb{R}^2$.

Before giving the strong maximum principle we make a few simple observations. Let B denote the unit disc and ∂B its euclidean boundary. Fix $a \in B$, $b \in \partial B$ and let $\Phi : B \to D$ be the $1-1$, conformal map of B onto D such that $\Phi(a) = \alpha$ and $\Phi(b) = \beta$. Here and elsewhere below we assume that Φ has been extended to a homeomorphism of the Martin closures, and we identify $\partial_M B$ with ∂B. Then by conformal invariance of Green functions and kernel functions

$$E_\alpha^\beta \tau_D = \int_B \frac{G^a(z)\, K^b(z)}{K^b(a)} |\Phi'(z)|^2\, dm(z) \tag{2.2}$$

where,

$$G^a(z) = \frac{1}{2\pi} \log \left| \frac{1 - \bar{a} z}{z - a} \right|^2$$

is the Green function for B with pole at a and

$$K^b(z) = \frac{1-|z|^2}{|b-z|^2}$$

is a kernel function for B with pole at $b \in \partial B$. We can always find such a conformal map for which $a=0$ and $b=1$. In this case Φ maps the interval $[-1, 1]$ onto the hyperbolic geodesic through α and β. Thus the strong maximum principle is a consequence of the following stronger result:

THEOREM 2.1. — *Let* F *not identically zero be holomorphic in the disc* B *and* $F \in L^2(B)$. *For* $-1<a<1$, *let*

$$L(a) = \frac{\int_D G^a(z) K^1(z) |F(z)|^2 \, dm(z)}{K^1(a)}.$$

Then $L'(0)<0$.

Remark. — This result is stronger than the strong maximum principle in two ways. First it allows us to replace Φ' in (2.2) with an arbitrary holomorphic function $F \in L^2(B)$, and secondly it shows that if α_0 is on the hyperbolic geodesic connecting a point $\alpha \in \bar{D} = D \cup \partial_M D$ to β, $E^\beta_{\alpha_0} \tau_D$ increases as α_0 moves along the geodesic away from β.

Proof. — One easily sees that the derivative of the numerator may be taken inside the integral. Evaluating the derivative one gets

$$\frac{1}{\pi}\int_B \left[\frac{\operatorname{Re} z(1-|z|^2)}{|z|^2} - \log\frac{1}{|z|^2}\right] \frac{1-|z|^2}{|1-z|^2} |F(z)|^2 \, dm(z).$$

Let $F(z)(1-z)^{-1} = \sum_{k=0}^{\infty} a_k z^k$. In polar coordinates $dm(z) = r\,dr\,d\theta$ so that by orthogonality

$$\int_{|z|=r} \left|\frac{F(z)}{1-z}\right|^2 d\theta = 2\pi \sum_{k=0}^{\infty} |a_k|^2 r^{2k},$$

while

$$\operatorname{Re}\int_{|z|=r} z\left|\frac{F(z)}{1-z}\right|^2 d\theta = 2\pi \operatorname{Re} \sum_{k=1}^{\infty} a_{k-1} \bar{a}_k r^{2k}.$$

Thus integrating in θ first and then in r, the theorem follows if the inequality

$$\operatorname{Re} \sum_{k=1}^{\infty} a_{k-1} \bar{a}_k \frac{2}{k(k+1)(k+2)} < \sum_{k=0}^{\infty} |a_k|^2 \left[\frac{1}{(k+1)^2(k+2)} + \frac{1}{(k+1)(k+2)^2}\right] \quad (2.3)$$

holds.

By taking absolute values, it suffices to prove (2.3) for nonnegative a_k. Let a_M be the first nonzero coefficient. Then by the geometric-arithmetic mean inequality

$$\sum_{k=M+1}^{\infty} a_{k-1}\, a_k \frac{2}{k(k+1)(k+2)} \leqq \sum_{k=M+1}^{\infty} \left(\frac{a_{k-1}^2}{k^2(k+1)} + \frac{a_k^2}{(k+1)(k+2)^2} \right)$$

$$= \frac{a_M^2}{(M+1)^2 (M+2)} + \sum_{k=M+1}^{\infty} a_k^2 \left[\frac{1}{(k+1)^2(k+2)} + \frac{1}{(k+1)(k+2)^2} \right]. \quad (2.4)$$

This shows the inequality in (2.3) is strict provided all quantities in (2.4) are finite. To see this observe that if $F(z) = \sum_{j=0}^{\infty} b_j z^j$ then $a_k = \sum_{j=0}^{k} b_j$. Thus by the Cauchy-Schwarz inequality,

$$\sum_{k=0}^{\infty} |a_k|^2 \left[\frac{1}{(k+1)^2(k+2)} + \frac{1}{(k+1)(k+2)^2} \right]$$

$$\leqq \sum_{k=0}^{\infty} \left[\frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)^2} \right] \sum_{j=0}^{k} |b_j|^2$$

$$\leqq 2 \sum_{j=0}^{\infty} |b_j|^2 \sum_{k=j}^{\infty} \frac{1}{(k+1)(k+2)} = 2 \sum_{j=0}^{\infty} \frac{|b_j|^2}{j+1},$$

and the last quantity is finite since $F \in L^2(B)$. ■

COROLLARY 2.2. — *Let* D *be a nonempty simply connected plane domain of finite area. Let* α, β *belong to* $\partial_M D$ *and* α_0 *lie on the hyperbolic geodesic* Γ *joining* α *and* β. *Then*

$$E_{\alpha_0}^{\beta} \tau < \frac{2}{\pi} \int_D K^{\alpha}(z) K^{\beta}(z)\, dm(z), \quad (2.5)$$

where the kernel functions are normalized to be 1 *at some point, call it* ξ, *on* Γ.

Proof. — We begin by showing that if α_n converges to α in the Martin topology then

$$\frac{2}{\pi} K^{\alpha}(z) = \lim_{n \to \infty} \frac{G(\alpha_n, z)}{K^{\beta}(\alpha_n)}. \quad (2.6)$$

To see this, map the unit disk 1 − 1 and conformally onto D so that −1, 0, and 1 are mapped to α, ξ, and β respectively. Then (2.6) follows from its counterpart in the unit disk, which reads

$$\frac{2}{\pi} \frac{1-|z|^2}{|1+z|^2} = \lim_{z_n \to -1} \frac{1}{\pi} \frac{\log|(1-\bar{z}_n z)/(z_n - z)|}{1-|z_n|^2} |1-z_n|^2.$$

This is a straightforward computation.

Now let Φ denote the conformal map described above. Choose a strictly decreasing sequence w_n of negative real numbers such that $\lim\limits_{n\to\infty} w_n = -1$. Put $\alpha_n = \Phi(w_n)$. Then by (2.1) and (2.2)

$$E^\beta_{\alpha_n}\tau = \int_D \frac{G(\alpha_n, z)K^\beta(z)}{K^\beta(\alpha_n)}\,dm(z) = \int_B \frac{G(w_n, z)K^1(z)}{K^1(w_n)}|\Phi'(z)|^2\,dm(z).$$

It follows from Theorem 2.1 that $E^\beta_{\alpha_n}\tau$ is strictly increasing. Thus

$$E^\beta_{\alpha_0}\tau < \int_D \frac{G(\alpha_n, z)}{K^\beta(\alpha_n)}K^\beta(z)\,dm(z).$$

By [11], Corollary 1.2, the integrands on the right-hand side of the last inequality are uniformly integrable. Thus we obtain (2.5) from (2.6) by passage to the limit as $n\to\infty$. ■

Remark 2.3. – Essentially the same argument shows that if α_n is *any* sequence converging to α in the Martin topology, then

$$\lim_{n\to\infty} E^\beta_{\alpha_n}\tau = \frac{2}{\pi}\int_D K^\alpha(z)K^\beta(z)\,dm(z),$$

where the kernel functions are normalized at some point ξ lying on the hyperbolic geodesic joining α and β. The right-hand side is the expected lifetime of Brownian motion in D started at the entrance boundary point α and conditioned to die at the boundary point β. (*See* [13].) Accordingly, we denote it by $E^\beta_\alpha\tau$.

COROLLARY 2.4. – *Let* D *be a nonempty simply connected plane domain with a Green function. Let* α, β *belong to* $\partial_M D$ *and* α_0 *lie on the hyperbolic geodesic* Γ *joining* α *and* β. *Then*

$$E^\beta_{\alpha_0}\tau \leqq E^\beta_\alpha\tau. \tag{2.7}$$

Proof. – By conformal invariance it is enough to prove

$$\frac{1}{2\pi}\int_B \frac{1-|z|^2}{|1-z|^2}\log|z|^{-2}|\Phi'(z)|^2\,dm(z) \leqq \frac{2}{\pi}\int_B \frac{(1-|z|^2)^2}{|1-z^2|^2}|\Phi'(z)|^2\,dm(z) \tag{2.8}$$

for any conformal map Φ defined in B.

By monotone convergence the right hand side of (2.8) may be written as

$$\lim_{r\uparrow 1}\frac{2}{\pi}\int_{|z|<r} \frac{(r^2-|z|^2)^2}{|r^2-z^2|^2}|\Phi'(z)|^2\,dm(z). \tag{2.9}$$

The left hand side may be written

$$\lim_{r\uparrow 1}\frac{1}{2\pi}\left[\int_{\substack{|z|<r\\ \operatorname{Re} z<0}}+\int_{\substack{|z|<1/2\\ \operatorname{Re} z>0}}+\int_{\substack{1/2<|z|<r\\ \operatorname{Re} z>0}}\right]\frac{r^2-|z|^2}{|r-z|^2}\log\frac{r^2}{|z|^2}|\Phi'(z)|^2\,dm(z).$$

The first integral converges by monotone convergence and the second by dominated convergence. For the third the functions

$$f_r(z)=\frac{|r+z|^2}{r^2-|z|^2}\log\frac{r^2}{|z|^2}\qquad\text{for }\ \{z:1/2<|z|<r\ \text{and}\ \operatorname{Re} z>0\}$$

may be extended to $\{z:1/2<|z|<1 \text{ and } \operatorname{Re} z>0\}$ so that the $f_r(z)$ converge uniformly there to $f(z)=\dfrac{|1+z|^2}{1-|z|^2}\log\dfrac{1}{|z|^2}$. This function is uniformly bounded away from zero. As has already been noted the functions $g_r(z)=\dfrac{(r^2-|z|^2)^2}{|r^2-z^2|^2}$ for $\{z:|z|<r\}$, extended to be zero outside $\{z:|z|<r\}$, increase monotonically on $\{z:|z|<1\}$ to $g(z)=\dfrac{(1-|z|^2)^2}{|1-z^2|^2}$. Thus given any $\varepsilon>0$, by Fatou's lemma, uniform convergence of f_r to f and monotone convergence, we have

$$\begin{aligned}\int fg|\Phi'|^2\,dm &\leqq \varliminf_{r\uparrow 1}\int f_r g_r|\Phi'|^2\,dm\leqq \varlimsup_{r\uparrow 1}\int f_r g_r|\Phi'|^2\,dm\\ &\leqq \varlimsup_{r\uparrow 1}\varepsilon\int g_r|\Phi'|^2\,dm+\varlimsup_{r\uparrow 1}\int fg_r|\Phi'|^2\,dm\\ &=\varepsilon\int g|\Phi'|^2\,dm+\int fg|\Phi'|^2\,dm.\end{aligned}$$

If $\int g|\Phi'|^2\,dm=\infty$ the same is true of $\int fg|\Phi'|^2\,dm$ since f is bounded away from zero. Thus the left side of (2.8) is

$$\lim_{r\uparrow 1}\frac{1}{2\pi}\int_{|z|<r}\frac{r^2-|z|^2}{|r-z|^2}\log\frac{r^2}{|z|^2}|\Phi'(z)|^2\,dm(z).\tag{2.10}$$

Now by Corollary 2.2 for each $r>0$ the integrals in (2.9) are strictly greater than those in (2.10) and (2.8) follows. ■

Remark 2.5. – If $|D|=\infty$ then it is possible that $E^{\beta}_{\alpha_0}\tau=\infty$ for $\alpha_0\in D$ so that the above proof can not give us a strict inequality in (2.7). However it would be interesting to know whether or not the strict maximum principle holds in the case where the maximum expected lifetime in D is known to be finite but D has infinite area.

3. THE BEST CONSTANT

In this section we show that $\frac{1}{\pi}$ is the best constant in the lifetime inequality when the terminal point lies on the boundary, or, more generally, when the conditioning function is positive harmonic. The inequality continues to hold with the same constant even when the terminal point lies in the interior. The more difficult proof of that fact is deferred to section 5. The precise result to be proved here is the following

THEOREM 3.1. — *Let* D *be a nonempty simply connected plane domain of finite area,* $\alpha \in \bar{\mathrm{D}}$, *and* $h>0$ *harmonic on* D. *Then*

$$\mathrm{E}_\alpha^h \tau < \frac{1}{\pi} |\mathrm{D}|, \tag{3.1}$$

and the constant $\frac{1}{\pi}$ *is best possible.*

It suffices to prove (3.1) when h is minimal, say $h=\mathrm{K}^\beta$, $\beta \in \partial_{\mathrm{M}} \mathrm{D}$. Moreover, by the results of Section 2 we may assume $\alpha \in \partial_{\mathrm{M}} \mathrm{D}$. Recall (*see* Corollary 2.2 and Remark 2.3) that if ξ lies on the hyperbolic geodesic joining α and β, and if K^α and K^β are normalized so that $\mathrm{K}^\alpha(\xi)=\mathrm{K}^\beta(\xi)=1$, then

$$\mathrm{E}_\alpha^\beta \tau = \frac{2}{\pi} \int_{\mathrm{D}} \mathrm{K}^\alpha(z)\, \mathrm{K}^\beta(z)\, dm(z).$$

If Φ is the unique $1-1$ conformal map of the unit disc B onto D such that $\Phi(-1)=\alpha$, $\Phi(0)=\xi$, and $\Phi(1)=\beta$, then

$$\mathrm{E}_\alpha^\beta \tau = \frac{2}{\pi} \int_{\mathrm{B}} \frac{(1-|z|^2)^2}{|1-z^2|^2} |\Phi'(z)|^2\, dm(z). \tag{3.2}$$

The unit disc here may be replaced by any other model simply connected domain, and it is more convenient for our purposes in this section to replace it by the infinite strip

$$\mathrm{S}=\left\{x+iy : -\infty<x<\infty,\ -\frac{\pi}{2}<y<\frac{\pi}{2}\right\}.$$

An easy computation using e.g., the fact that $\log\left(\frac{1+z}{1-z}\right)$ maps B to S, shows that

$$\mathrm{E}_\alpha^\beta \tau = \frac{2}{\pi} \int_{\mathrm{S}} \cos^2(y)\, |\Psi'(x+iy)|^2\, dx\, dy,$$

where Ψ maps S 1−1 and conformally onto D with $\Psi(-\infty)=\alpha$, $\Psi(0)=\xi$, and $\Psi(+\infty)=\beta$. (This formula has been noted independently by R. Bañuelos and T. Carrol [3].)

By the half-angle formula,

$$E_{\alpha}^{\beta}\tau=\frac{1}{\pi}|D|+\frac{2}{\pi}\int_{0}^{\pi/2}\cos(2y)H(y)\,dy, \tag{3.3}$$

where

$$H(y)=\frac{1}{2}\int_{-\infty}^{\infty}(|\Psi'(x+iy)|^{2}+|\Psi'(x-iy)|^{2})\,dx. \tag{3.4}$$

The proof of (3.1) thus reduces to showing that the second term on the right-hand side of (3.3) is strictly negative, which, in turn, is an immediate consequence of the following two results.

LEMMA 3.2. — *Let f be a strictly increasing real-valued function and g a strictly decreasing function on a finite interval* $[a, b]$. *Then*

$$\frac{1}{(b-a)}\int_{a}^{b}f(y)g(y)\,dy<\left[\frac{1}{b-a}\int_{a}^{b}f(y)\,dy\right]\left[\frac{1}{b-a}\int_{a}^{b}g(y)\,dy\right].$$

LEMMA 3.3. — *The function* H *defined in* (3.4) *above is strictly increasing on* $\left[0,\frac{\pi}{2}\right]$.

To complete the proof of (3.1), apply Lemma 3.2 with $[a, b]=\left[0,\frac{\pi}{2}\right]$, $f=H$, and $g(y)=\cos(2y)$.

Proof of Lemma 3.2. — Let $I=\frac{1}{b-a}\int_{a}^{b}f(y)\,dy$ and choose $a<\gamma<b$ such that $f\leqq I$ on $[a,\gamma]$ and $f>I$ on $(\gamma, b]$. Let $\tilde{f}=f-I$. Then

$$\begin{aligned}\int_{a}^{b}\tilde{f}(y)g(y)\,dy&=\int_{a}^{\gamma}\tilde{f}(y)g(y)\,dy+\int_{\gamma}^{b}\tilde{f}(y)g(y)\,dy\\&<g(\gamma)\int_{a}^{\gamma}\tilde{f}(y)\,dy+g(\gamma)\int_{\gamma}^{b}\tilde{f}(y)\,dy=0. \quad \blacksquare\end{aligned}$$

We should remark that this result is a special case of an inequality of Chebyshev. *See*, e.g., Theorem 43 (2.17.1) in [8].

Proof of Lemma 3.3. — We begin by showing that H is bounded on $[0, A]$ for each $0<A<\frac{\pi}{2}$. Fix B such that $A<B<\frac{\pi}{2}$. Since

$$\int_{-\pi/2}^{\pi/2}\int_{-\infty}^{\infty}|\Psi'(x+iy)|^{2}\,dx\,dy=|D|<\infty,$$

it follows from Fubini's Theorem that we may further assume B is chosen so that $H(B)<\infty$. By similar reasoning applied to integrals over vertical line segments we may choose $x_n \to \infty$ and a constant C independent of n such that

$$\int_{-B}^{B} |\Psi'(x_n+iy)|^2\,dy+\int_{-B}^{B} |\Psi'(-x_n+iy)|^2\,dy \leqq C. \tag{3.5}$$

Let $z_0=x_0+iy_0$ satisfy $|y_0|\leqq A$ and let Γ_n denote the positively oriented boundary of the rectangle $[-x_n, x_n]\times[-B, B]$. By the Cauchy integral formula we have for $x_n>|x_0|$

$$\Psi'(z_0)=\frac{1}{2\pi i}\int_{\Gamma_n}\frac{\Psi'(\xi)}{\xi-z_0}\,d\xi.$$

It follows from the Schwarz inequality and (3.5) that the contribution to this integral from the vertical ends of Γ_n vanishes in the limit as $n\to\infty$. Thus we obtain the representation

$$\Psi'(z_0)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\Psi'(x-iB)\,dx}{x-x_0-i(B+y_0)}-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\Psi'(x+iB)\,dx}{x-x_0+i(B-y_0)},$$

valid for all $z_0=x_0+iy_0$ satisfying $|y_0|\leqq A$.

Since $H(B)<\infty$, one may apply results from the theory of H^p-spaces with $p=2$ (e. g., the corollary on p. 172 of [10]) to each of the two integrals above to conclude that

$$\int_{-\infty}^{\infty} |\Psi'(x_0+iy_0)|^2\,dx_0 \text{ is bounded on } |y_0|\leqq A.$$

Next, by a well-known Paley-Wiener Theorem (*see*, e.g. [9], p. 174) $\Psi'|_{\mathbb{R}}$ is the Fourier transform of a nonzero function φ such that $e^{y|\lambda|}\varphi(\lambda)\in L^2(\mathbb{R})$ for each $|y|<\frac{\pi}{2}$. Thus, by Plancherel's Theorem the function $H(y)$ is a positive multiple of $\int_{-\infty}^{\infty}\cosh(2y\lambda)|\varphi(\lambda)|^2\,d\lambda$, which is clearly strictly increasing in y. ■

To show that $\frac{1}{\pi}$ is best-possible, let D_ρ denote the image of B under the mapping $\Phi_\rho(z)=\log\left(\frac{1+\rho z}{1-\rho z}\right)$, $0<\rho<1$. Then a direct, but lengthy, computation shows that

$$\lim_{\rho\uparrow 1}\frac{2}{\pi}\int_B\frac{(1-|z|^2)^2}{|1-z^2|^2}|\Phi'_\rho(z)|^2\,dm(z)\,|D_\rho|^{-1}=\frac{1}{\pi}.$$

Alternatively, one may apply Theorem 3.4 below (*see* esp. example 3.6) but perhaps the easiest way to see this is by means of a probabilistic argument, which we sketch at the end of this section.

Since strict inequality always holds in (3.1) there are no extremal domains, but domains such as D_ρ above for ρ close to 1 may be considered "near extremal". The following result shows that there are many near extremal domains and provides some additional insight into how the geometry of the domain influences the expected lifetime.

THEOREM 3.4. — *Let* D *be a bounded convex domain which is symmetric with respect to one of its diameters. Let* $R=R(D)$ *denote the supremum of radii of all open discs contained in* D, $P=P(D)$ *the perimeter, and* $\Delta=\Delta(D)$ *the diameter of* D. *Then there exist points* $\alpha\in\partial D$ *and* $\beta\in\partial D$ *such that*

$$E_\alpha^\beta \tau \geqq \frac{1}{\pi}|D|-\frac{4}{\pi}R(P-2\Delta). \tag{3.6}$$

Remark 3.5. — J. Xu [15] has shown that there is a positive constant γ so that

$$\sup_{(\alpha,\beta)\in\partial D\times\partial D} E_\alpha^\beta \tau \geqq \gamma|D|,$$

for all convex plane domains.

Proof of Theorem 3.4. — We may assume that D is symmetric with respect to the x-axis and that there are points α and β on the x-axis so that $[\alpha,\beta]$ is a diameter of D. For technical reasons it is convenient to replace D with a slightly smaller convex domain having smooth boundary. Thus, let Φ be a $1-1$ conformal map of B onto D such that $\Phi(-1)=\alpha$, $\Phi(1)=\beta$. Let $\Phi_\rho(z)=\Phi(\rho z)$ for $0<\rho<1$, and let D_ρ be the image of B under Φ_ρ. Then D_ρ is also convex by Study's Theorem (*see*, e.g. [12], p. 224) and symmetric with respect to the x-axis.

Let $\Psi_\rho(z)=\Phi_\rho\left(\dfrac{e^z-1}{e^z+1}\right)$. Note that Ψ_ρ maps S conformally onto D_ρ, is real on the x-axis and maps the upper boundary of S to the upper boundary of D_ρ. It is easy to check that D_ρ has a smooth boundary and also that

$$\Psi_\rho \text{ is analytic in an open set containing the euclidean closure of S,} \tag{3.7}$$

$$|\Psi'_\rho| \text{ is bounded on the euclidean closure of S,} \tag{3.8}$$

$$R(D)=\lim_{\rho\to1^-}R(D_\rho),\ P(D)=\lim_{\rho\to1^-}P(D_\rho),\ \Delta(D)=\lim_{\rho\to1^-}\Delta(D_\rho), \tag{3.9}$$

and

$$\frac{2}{\pi}\int_S \cos^2(y)\,|\Psi'(x+iy)|^2\,dx\,dy$$
$$= \lim_{\rho\to 1^-}\frac{2}{\pi}\int_S \cos^2(y)\,|\Psi'_\rho(x+iy)|^2\,dx\,dy. \quad (3.10)$$

Since the left-hand side of (3.10) equals $E_\alpha^\beta \tau$ and the right-hand side represents the limit of the analogous quantities for the D_ρ, it is sufficient to prove (3.6) for the domains D_ρ, with the quantity $\Delta(D_\rho)$ replaced by the length of the intersection of D_ρ with the x-axis. To simplify the notation, we shall assume for the remainder of the proof that $D=D_\rho$ for some $0<\rho<1$, and drop the "ρ" throughout.

Our method now is to find a lower bound for the second term on the right-hand side of (3.3), *i.e.*,

$$\frac{2}{\pi}\int_0^{\pi/2}\cos(2y)\,H(y)\,dy=\frac{2}{\pi}\int_{-\infty}^{\infty}\int_0^{\pi/2}\cos(2y)\,|\Psi'(x+iy)|^2\,dy\,dx. \quad (3.11)$$

Let $\Psi=u+iv$, u, v real, and note that, by (3.7) and the Cauchy-Riemann equations,

$$\frac{\partial}{\partial y}(|\Psi'|^2)=-2u_x v_{xx}+2v_x u_{xx}=2\kappa|\Psi'|^3.$$

Here all expressions are evaluated at $x+iy$, $-\infty<x<\infty$, $0\leqq y\leqq\pi/2$, and κ is the curvature of the curve $t\to\Psi(t, y)$. Thus, since D is convex we have $\frac{\partial}{\partial y}(|\Psi'|^2)\geqq 0$ for $y=\frac{\pi}{2}$; since Ψ maps the x-axis to the x-axis we have $\frac{\partial}{\partial y}(|\Psi'|^2)=0$ for $y=0$. On the other hand, $\frac{u_{xx}v_x-v_{xx}u_x}{|\Psi'|^2}=\mathrm{Re}\left[i\frac{\Psi''}{\Psi'}\right]$, so that the function $\frac{(\partial/\partial y)(|\Psi'|^2)}{|\Psi'|^2}=2\kappa|\Psi'|$ is harmonic on S, and bounded on S by (3.8). By the maximum principle we may conlude that $\frac{\partial}{\partial y}(|\Psi'|^2)\geqq 0$ for $0\leqq y\leqq\frac{\pi}{2}$, $-\infty<x<\infty$.

Recalling that the expression in (3.11) is negative and integrating by parts on y, we have

$$0<-\frac{2}{\pi}\int_0^{\pi/2}\cos(2y)\,H(y)\,dy=\frac{2}{\pi}\int_{-\infty}^{\infty}\int_0^{\pi/2}\sin(y)\cos(y)\frac{\partial}{\partial y}(|\Psi'|^2)\,dy\,dx$$
$$=\frac{4}{\pi}\int_{-\infty}^{\infty}\int_0^{\pi/2}\sin(y)\cos(y)\,|\Psi'|\frac{\partial}{\partial y}(|\Psi'|)\,dy\,dx.$$

By Koebe's distortion theorem

$$|\Psi'(x+iy)| \leqq \frac{2\,\mathrm{dist}(\Psi(x+iy), \partial\mathrm{D})}{\cos(y)} \leqq \frac{2\mathrm{R}}{\cos(y)}.$$

(*See* e.g., [14], p. 147, where the result is stated for the unit disc. The version we use is easily obtained by conformal mapping.) Thus,

$$\begin{aligned} 0 < -\frac{2}{\pi}\int_0^{\pi/2} \cos(2y)\,\mathrm{H}(y)\,dy &\leqq \frac{8\mathrm{R}}{\pi}\int_{-\infty}^{\infty}\int_0^{\pi/2} \sin(y)\frac{\partial}{\partial y}(|\Psi'|)\,dy\,dx \\ &\leqq \frac{8\mathrm{R}}{\pi}\int_0^{\pi/2}\frac{\partial}{\partial y}\int_{-\infty}^{\infty} |\Psi'(x+iy)|\,dx\,dy \\ &= \frac{8\mathrm{R}}{\pi}\left[\int_{-\infty}^{\infty}\left|\Psi'\left(x+\frac{\pi}{2}i\right)\right|dx - \int_{-\infty}^{\infty}|\Psi'(x)|\,dx\right] \\ &= \frac{4\mathrm{R}}{\pi}[\mathrm{P}-2(\Psi(+\infty)-\Psi(-\infty))], \end{aligned}$$

which completes the proof. ∎

Example 3.6. – Take for D the rectangle $\left[-\frac{\mathrm{L}}{2}, \frac{\mathrm{L}}{2}\right]\times\left[-\frac{1}{2}, \frac{1}{2}\right]$. Then by the proof of Theorem 3.4

$$(\mathrm{E}^{\mathrm{L}/2}_{-\mathrm{L}/2}\,\tau)/|\mathrm{D}| \geqq \frac{1}{\pi} - \frac{4}{\pi\mathrm{L}}.$$

We conclude this section by sketching a simple probabilistic argument to show that long thin rectangles are near-extremal. For given $\varepsilon > 0$ let D_n denote the rectangle $[-\varepsilon n, n]\times\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Let $h(z) = e^x \cos y$, so that h is positive harmonic on D_n. Finally, let $\tau_n = \tau_{\mathrm{D}_n}$. We will show that $\mathrm{E}^h_0\,\tau_n \sim n$ as $n\to\infty$, hence $\mathrm{E}^h_0\,\tau_n/|\mathrm{D}_n| \to \frac{1}{\pi(1+\varepsilon)}$ as $n\to\infty$.

The h-process $\mathrm{Z}_t = (\mathrm{X}_t, \mathrm{Y}_t)$ started from 0 satisfies the stochastic differential equation

$$d\mathrm{Z}_t = d\mathrm{W}_t + \frac{\nabla h}{h}(\mathrm{Z}_t)\,dt$$

with $\mathrm{Z}_0 = 0$. Here W_t is a standard $\mathbb{R}^2$-valued Brownian motion started from 0. Let β_t denote the x-component of W_t. Since the drift is given by $\frac{\nabla h}{h}(z) = (1, -\tan y)$, we have that $\mathrm{X}_t = \beta_t + t$, and moreover,

$$\tau_n = \inf\{t : \beta_t + t \notin (-\varepsilon n, n)\}.$$

(Note that Z_t cannot leave D_n through the top or bottom boundaries since h vanishes there.) Since both points $-\varepsilon n$ and n are accessible for the 1-dimensional diffusion $\beta_t + t$, it follows that $\mathrm{E}\,\tau_n < \infty$. Thus, by Wald's identity, $E\beta_{\tau_n}=0$. We conclude that

$$E_0^h \tau_n = E_0^h X_{\tau_n} = -\varepsilon n P_0^h(X_{\tau_n}=-\varepsilon n) + n P_0^h(X_{\tau_n}=n).$$

The desired result follows if we can show $P_0^h(X_{\tau_n}=n)\to 1$ as $n\to\infty$. One way to see this is to note that $\beta_t/t\to 0$, a. s., as $t\to\infty$ by the strong law, hence β_t+t has sample paths which are almost surely bounded below.

4. THE DISC

As we have already indicated, the extremal domains in the best constant problem are long thin rectangles. The opposite extreme to these domains is the disc. We will explicitly compute the best constant in this case. By the maximum principle of section 2, the worst case occurs on the boundary, *i.e.* Brownian motion conditioned to go from one boundary point to another. The first step is to show that the boundary points are diametrically opposite. We will do this probabilistically, by a coupling argument.

We begin by recalling some basic facts about h-processes. Let D_1 and D_2 be two domains and Φ a $1-1$ conformal map of D_1 onto D_2. We will write $w=\Phi(z)$ where $z=x+iy$ and $w=u+iv$. If h_2 is a positive harmonic function in D_2, then $h_1=h_2\circ\Phi$ is a positive harmonic function in D_1. If Z_t is an h_1-process in D_1 then $\Phi(Z_t)$ is a time change of an h_2-process in D_2; to be precise, there exists an h_2-process W_t such that

$$\Phi(Z_t)=W\left(\int_0^t |\Phi'(Z_s)|^2\,ds\right).$$

As an immediate consequence, we have that

$$\tau_{D_2}=\int_0^{\tau_{D_1}} |\Phi'(Z_s)|^2\,ds.$$

If we apply this with $D_1=B$, $D_2=\{w:u>0\}$, $h_2(w)=u$ and $\Phi(z)=(1+z)(1-z)^{-1}$, we see that $h_1(z)=K^1(z)$. Furthermore the h_2-process $W=(U, V)$ is very simple; U is a Bessel process of index 3 and V is an independent 1-dimensional Brownian motion. If we let $\Psi(w)=(w-1)(w+1)^{-1}$ be the inverse of Φ, and apply the above in reverse, we can conclude that

$$\tau_B=\int_0^{\tau_H} |\Psi'(W_s)|^2\,ds$$

where H is the half space $u>0$. Since $\Phi(-1)=0$ and $\tau_H=\infty$ a. s., to show that $E_z^{h_1}\tau_B$ is maximized when Z starts at $z=-1$ is equivalent to showing

$$E_w^{h_2}\int_0^\infty |\Psi'(W_s)|^2\,ds$$

is maximized when W starts at $w=0$. Since we already know the maximum occurs on the boundary, we only need consider starting points on the imaginary axis $u=0$.

PROPOSITION 4.1. – *Let $w_j = iv_j$, $j=1, 2$, be two points on the boundary of* H *such that $|v_1| \leqq |v_2|$. Then on the same probability space, one can define two h_2-processes W^1 and W^2 starting at w_1 and w_2 respectively, such that*

$$\int_0^\infty |\Psi'(W_s^1)|^2 \, ds \geqq \int_0^\infty |\Psi'(W_s^2)|^2 \, ds \quad \text{a. s.}$$

Proof. – Let $W^1 = (U^1, V^1)$ be an h_2-process started at w_1 defined on some probability space. Let

$$\sigma = \inf\left\{ t > 0 : V_t^1 = \frac{1}{2}(v_1 - v_2) \right\}.$$

Define $W^2 = (U^2, V^2)$ by $U^2 = U^1$ and

$$V_t^2 = \begin{cases} V_t^1 + (v_2 - v_1) & \text{if } t < \sigma \\ -V_t^1 & \text{if } t \geqq \sigma. \end{cases}$$

By the reflection principle V_t^2 is a Brownian motion starting at v_2 independent of U^2. Thus W^2 is an h_2-process started at w_2 and by construction $U_t^1 = U_t^2$ and $|V_t^2| \geqq |V_t^1|$ for all t. Since

$$|\Psi'(w)|^2 = 4((1+u)^2 + v^2)^{-2},$$

it follows that for all s,

$$|\Psi'(W_s^1)|^2 \geqq |\Psi'(W_s^2)|^2,$$

from which the result is immediate. ■

As a consequence of the previous result we have that

$$\sup_{\alpha \in B} E_\alpha^1 \tau_B = E_{-1}^1 \tau_B.$$

Thus by the Poisson representation of positive harmonic functions in B the best possible constant c_B for the unit disk is $\frac{1}{\pi} E_{-1}^1 \tau_B$, which we now compute.

PROPOSITION 4.2. – $E_{-1}^{1}\tau_B = 4\log 2 - 2 \cong .7726$.

Proof. – Recall that by (3.2)

$$\begin{aligned}
E_{-1}^{1}\tau_B &= \frac{2}{\pi}\int_B K^{-1}(z)K^{1}(z)\,dm(z) = \frac{2}{\pi}\int_B \frac{(1-|z|^2)^2}{|1-z|^2|1+z|^2}\,dm(z)\\
&= \frac{2}{\pi}\int_B \frac{(1-r^2)^2}{|1-r^2 e^{i2\theta}|^2}\,r\,dr\,d\theta\\
&= \frac{2}{\pi}\int_B (1-r^2)^2 r\left(\sum_{m=0}^{\infty}(r^2 e^{2i\theta})^m\right)\left(\sum_{n=0}^{\infty}(r^2 e^{-2i\theta})^n\right)dr\,d\theta\\
&= 4\int_0^1 (1-r^2)^2 r\sum_{n=0}^{\infty} r^{4n}\,dr = 4\int_0^1 \frac{(1-r^2)^2 r}{1-r^4}\,dr = 4\int_0^1 \frac{(1-r^2)r}{1+r^2}\,dr\\
&= -4\int_0^1 r\,dr + 4\int_0^1 \frac{2r\,dr}{1+r^2} = 4\log 2 - 2. \quad \blacksquare
\end{aligned}$$

Among convex domains the disc is at the opposite extreme to the long thin rectangles, *i.e.*, it minimizes the perimeter for a given area. Thus it seems reasonable to ask

Open Question. – Amongst all convex domains D of area π, is

$$\sup_{\alpha,\beta\in\bar{D}} E_{\alpha}^{\beta}\tau_D$$

minimized when $D = B$?

If we remove the convexity assumption then the result is false. Indeed by an example of Xu [15] there are simply connected domains of infinite area having $\sup_{\alpha,\beta\in D} E_{\alpha}^{\beta}\tau$ as small as desired.

It is interesting to note that in the case of unconditioned Brownian motion (*i.e.*, $h \equiv 1$), the disc is in fact the worst case, *i.e.*,

$$\sup_{\alpha\in D} E_{\alpha}\tau_D \leqq \frac{1}{2\pi}|D|,$$

and in the simply connected case, equality holds iff D is a disc. This is a consequence of classical isoperimetric theory which says that the distribution function of the Green function is pointwise maximized over domains of equal area only in the ball; *see* [1].

5. THE SUPERHARMONIC CASE

In this section we wish to establish (3.1) in simply connected domains with finite area in the plane when h is superharmonic with nonnegative boundary values. *See* [7]. To do this it suffices by conformal mapping and the Riesz Representation Theorem to prove the following result.

THEOREM 5.1. — *Let* F *not identically zero be holomorphic in the unit disc* B *with* $F \in L^2(B)$. *Then for all* $b \in B$

$$\int_B \frac{G^0(z)\, G^b(z)}{G^b(0)} |F(z)|^2\, dm(z) < \frac{1}{\pi} \int_B |F(z)|^2\, dm(z). \tag{5.1}$$

Our proof is based on a lemma that has a very close relationship with Carleman's generalization of the isoperimetric inequality [5]. Denote $r\,d\theta$ by $d\sigma$ and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta})\, d\theta \text{ by } \fint_{|z|=r} f\, d\theta.$$

LEMMA 5.2. — *Let* $F_1, \ldots, F_{n+1}$, $n = 1, 2, \ldots$, *not identically zero be holomorphic in* B *with* $F_1 F_2 \ldots F_{n+1} \in L^2(B)$. *Then*

$$\frac{1}{2\pi} \int_B (1-|z|^2)^n \log \frac{1}{|z|} |F_1(z) \ldots F_{n+1}(z)|^2\, dm(z)$$
$$< \int_0^1 (1-r^{2n}) \left(\frac{1}{2\pi} \int_{|z|=r} |F_1|^2\, d\sigma \right) \left(\fint_{|z|=r} |F_2|^2\, d\theta \right)$$
$$\times \left(\fint_{|z|=r} |F_3|^2\, d\theta \right) \ldots \left(\fint_{||z|=r} |F_{n+1}|^2\, d\theta \right) dr \tag{5.2}$$

Proof. — Let $F_j(z) = \sum_{k=0}^{\infty} a_k^{(j)} z^k$, $1 \leqq j \leqq n+1$. The square of the modulus of the j_0-th coefficient of $F_1(z) \ldots F_{n+1}(z)$ is

$$\left| \sum_{j_1=0}^{j_0} a_{j_0-j_1}^{(1)} \sum_{j_2=0}^{j_1} a_{j_1-j_2}^{(2)} \sum_{j_3=0}^{j_2} \cdots \sum_{j_n=0}^{j_{n-1}} a_{j_{n-1}-j_n}^{(n)} a_{j_n}^{(n+1)} \right|^2$$
$$\leqq \binom{j_0+n}{n} \sum_{j_1=0}^{j_0} \cdots \sum_{j_n=0}^{j_{n-1}} |a_{j_0-j_1}^{(1)} \ldots a_{j_{n-1}-j_n}^{(n)} a_{j_n}^{(n+1)}|^2 \tag{5.3}$$

by the Schwarz inequality.

Let I denote the left side of (5.2). Integrating in θ and using (5.3)

$$I \leqq \sum_{j_0=0}^{\infty} \binom{j_0+n}{n} \sum_{j_1=0}^{j_0} \cdots \sum_{j_n=0}^{j_{n-1}} |a_{j_0-j_1}^{(1)} \ldots a_{j_n}^{(n+1)}|^2 \int_0^1 r^{2j_0+1} (1-r^2)^n \log \frac{1}{r}\, dr$$
$$= \sum_{j_0=0}^{\infty} |a_{j_0}^{(1)}|^2 \sum_{j_1=0}^{\infty} |a_{j_1}^{(2)}|^2 \ldots \sum_{j_n=0}^{\infty} |a_{j_n}^{(n+1)}|^2 \binom{J+n}{n} \int_0^1 r^{2J+1} (1-r^2)^n \log \frac{1}{r}\, dr$$

where $J = j_0 + j_1 + \ldots + j_n$.

By an induction on $n \geqq 1$,

$$\binom{J+n}{n}\int_0^1 r^{2J+1}(1-r^2)^n \log\frac{1}{r}\,dr < \frac{n/2}{(J+n+1)(J+1)}$$

for all $J \geqq 0$. Thus

$$I < \frac{n}{2}\sum_{j_0=0}^{\infty}|a_{j_0}^{(1)}|^2 \dots \sum_{j_n=0}^{\infty}|a_{j_n}^{(n+1)}|^2 \frac{1}{(J+n+1)(J+1)}$$

$$= 2n\int_0^1 \rho^{2n-1}\,d\rho \int_0^{\rho}\left(\sum_{j_0=0}^{\infty}|a_{j_0}^{(1)}|^2 r^{2j_0+1}\right)\left(\sum_{j_1=0}^{\infty}|a_{j_1}^{(2)}|r^{2j_1}\right)$$

$$\times\left(\sum_{j_2=0}^{\infty}|a_{j_2}^{(3)}|^2 r^{2j_2}\right)\dots\left(\sum_{j_n=0}^{\infty}|a_{j_n}^{(n+1)}|^2 r^{2j_n}\right)dr$$

$$= \int_0^1 (1-r^{2n})\left(\frac{1}{2\pi}\int_{|z|=r}|F_1|^2\,d\sigma\right)\left(⨍_{|z|=r}|F_2|^2\,d\theta\right)$$

$$\times\left(⨍_{|z|=r}|F_3|^2\,d\theta\right)\dots\left(⨍_{|z|=r}|F_{n+1}|^2\,d\theta\right)dr \quad \blacksquare$$

Proof of Theorem 5.1. – It suffices to take $0<b<1$. Since

$$1-\left|\frac{z-b}{1-bz}\right|^2 = \frac{(1-|z|^2)(1-b^2)}{|1-bz|^2}$$

it follows that

$$G^b(z) = \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(1-b^2)^n(1-|z|^2)^n}{n|1-bz|^{2n}}.$$

Thus

$$\int_B \frac{G^0(z)\,G^b(z)}{G^b(0)}|F(z)|^2\,dm$$

$$= \frac{1}{\log b}\sum_{n=1}^{\infty}\frac{(1-b^2)^n}{n}\frac{1}{2\pi}\int_B\frac{(1-|z|^2)^n}{|1-bz|^{2n}}\log|z|\,|F(z)|^2\,dm. \quad (5.4)$$

Apply the lemma for each $n \geqq 1$ with $F_1=F$, $F_2=F_3=\dots F_{n+1}=\dfrac{1}{1-bz}$.

Using the harmonicity of $\dfrac{1-|bz|^2}{|1-bz|^2}$ in B

$$⨍_{|z|=r}\frac{1}{|1-bz|^2}\,d\theta = \frac{1}{1-b^2r^2}.$$

Thus (5.4) is less than

$$\frac{1}{\log 1/b}\sum_{n=1}^{\infty}\frac{(1-b^2)^n}{n}\frac{1}{2\pi}\int_0^1\frac{1-r^{2n}}{(1-b^2r^2)^n}\int_{|z|=r}|F|^2\,d\sigma\,dr.$$

Now use the identity

$$\sum_{n=1}^{\infty} \frac{(1-b^2)^n}{n} \frac{1-r^{2n}}{(1-b^2 r^2)^n} = 2 \log \frac{1}{b}$$

valid for all $0 \leqq r < 1$. ■

Remark 5.3. – An alternative approach to Theorem 5.1 would be to obtain a maximum principle similar to the one in section 2 but with both points lying in the interior. Unfortunately, we have been unable to prove this.

Remark 5.4. – The proof of Lemma 5.2 is a generalization of Carleman's proof when $n=1$ and the log and the weights $(1-|z|^2)^n$ are not present. Many different lemmas of this type may be stated. This one is devised for Theorem 5.1. In [4] Beckenbach and Radó generalized Carleman's result to deal with integrands that are logarithmically subharmonic.

Carleman's result can be used to give an alternative proof of Theorem 3.1, which we very briefly sketch below leaving the details to the interested reader. Using Carleman's isoperimetric inequality, the Cauchy-Schwarz inequality, and the coarea formula, we have

$$\begin{aligned}
\frac{2}{\pi}\int_B \mathrm{K}^{-1}(z)\mathrm{K}^1(z)|F(z)|^2\,dm(z) &= \frac{4}{\pi}\int_0^{\infty} t\,dt \int_{\{\mathrm{K}^1(z)>t\}} \left|\frac{1-z}{1+z}\right|^2 |\mathrm{F}(z)|^2\,dm(z)\\
&\leqq \frac{4}{\pi}\int_0^{\infty} t\,dt \frac{1}{4\pi}\left(\int_{\{\mathrm{K}^1=t\}} \left|\frac{1-z}{1+z}\right| |\mathrm{F}(z)|\,d\sigma\right)^2\\
&\leqq \frac{1}{\pi^2}\int_0^{\infty} t\,dt \int_{\{\mathrm{K}^1=t\}} |1-z|^2|\mathrm{F}(z)|^2\,d\sigma \int_{\{\mathrm{K}^1=t\}} \frac{d\sigma}{|1+z|^2}\\
&= -\frac{1}{\pi}\int_0^{+\infty} \frac{d}{dt}\int_{\{\mathrm{K}^1>t\}} |\mathrm{F}|^2\,dm\,dt\\
&= \frac{1}{\pi}\int_B |\mathrm{F}|^2\,dm.
\end{aligned}$$

The second inequality is strict except when F is a constant multiple of $\dfrac{1}{1-z^2}$, *i.e.*, the derivative of the conformal map taking B to an infinite strip. This is consistent with the observation that long thin domains are extremal for the expected lifetime. Furthermore in this case

$$\left|\frac{1-z}{1+z}\right|^2 |\mathrm{F}|^2 = \frac{1}{|1+z|^4}$$

is the Jacobian of a linear fractional transformation. The isoperimetric inequality is then also sharp since the sets $\{\mathrm{K}^1 > t\}$ are discs.

Remark 5.5. — A simple computation shows that K^{-1}/K^1 is logarithmically harmonic. Hence, by conformal mapping, given any two kernel functions K^α, K^β for a simply connected domain $D \subset \mathbb{R}^2$, K^α/K^β is logarithmically harmonic. Thus

$$\frac{|\nabla_z K^\alpha(z)|}{K^\alpha(z)} = \frac{|\nabla_z K^\beta(z)|}{K^\beta(z)}$$

for all $z \in D$. More generally, given any kernel function K and any positive harmonic function h for D, h/K is logarithmically subharmonic. This is because when transferred to the disc, by the Poisson integral representation, h/K has the representation $I(z) = \int_{-\pi}^{\pi} |F_\theta(z)|^2 \, d\mu(\theta)$ where $F_\theta(z)$ is holomorphic in B for each θ and $d\mu$ is a positive Borel measure. Taking the log and then applying the Laplacian yields

$$\frac{\int |F'_\theta|^2 \, d\mu \int |F_\theta|^2 \, d\mu - \left| \int F'_\theta \bar{F}_\theta \, d\mu \right|^2}{[I(z)]^2}$$

which is positive by the Schwarz inequality. As a consequence

$$\left|\frac{\nabla K}{K}\right| \geqq \left|\frac{\nabla h}{h}\right| \qquad \text{for all} \quad z \in D.$$

Given an h-process the quantity $\frac{\nabla h}{h}$ is called the *drift*. We have thus proved the following.

PROPOSITION 5.6. — *In a simply connected domain* $D \subset \mathbb{R}^2$ *the magnitude of the drift at a point* $z \in D$ *is maximized over all positive harmonic functions* h *in* D *precisely by the kernel functions, and this maximum is attained by the drift associated with every kernel function for* D.

Remark 5.7. — The quantity $\int_D \frac{G^\alpha(z) G^\beta(z)}{G^\alpha(\beta)} \, dm(z)$ can be obtained as the limit of quantities

$$\frac{\int_{D \setminus B_\varepsilon} G^\alpha(z) K^\beta(z) \, dm(z)}{K^\beta(\alpha)}$$

where B_ε is a small disc shrinking to a point and $\beta \in \partial B_\varepsilon$. Thus $1/\pi$ is the best constant in the lifetime inequality with terminal point on the boundary in a doubly connected domain when the hole is small enough. Saying anything more than this about the best constant in multiply connected domains is an open problem.

REFERENCES

[1] C. BANDLE, *Isoperimetric inequalities and applications*, Pitman Publishing, Boston, 1980.

[2] R. BAÑUELOS, On the estimate of Cranston and McConnell for elliptic diffusions in uniform domains, *Prob. Th. Rel. Fields*, Vol. **76**, 1987, pp. 311-323.

[3] R. BAÑUELOS and T. CARROL, *Conditional Brownian motion and hyperbolic geodesics in simply connected domains*, (Preprint).

[4] E. F. BECKENBACH and T. RADO, Subharmonic functions and surfaces of negative curvature, *Trans. Amer. Math. Soc.*, Vol. **35**, 1933, pp. 662-674.

[5] T. CARLEMAN, Zur Theorie der Minimalflächen, *Math Z.*, Vol. **9**, 1921, pp. 154-160.

[6] M. CRANSTON and T. R. MCCONNELL, The lifetime of conditioned Brownian motion, *Z. Warsch. Verw. Gebiete*, Vol. **65**, 1983, pp. 1-11.

[7] J. L. DOOB, *Classical Potential Theory and its Probabilistic Counterpart*, Springer, New York, 1984.

[8] G. HARDY J. E. LITTLEWOOD, and G. PÓLYA, *Inequalities*, 2nd Ed., Cambridge University Press, Cambridge, 1952.

[9] Y. KATZNELSON, *An Introduction to Harmonic Analysis*, 2nd Ed., Dover, New York, 1976.

[10] P. KOOSIS, *Introduction to* H^p *spaces*, Cambridge University Press, Cambridge, 1980.

[11] T. R. MCCONNELL, A conformal inequality related to the conditional gauge theorem, *Trans. Amer. Math. Soc.*, Vol. **318**, 1990, pp. 721-733.

[12] S. SAKS and A. ZYGMUND, *Analytic Functions*, 3rd Ed., American Elsevier Publishing Co., New York, 1971.

[13] T. SALISBURY, A Martin boundary in the plane, *Trans. Amer. Math. Soc.*, Vol. **293**, 1986, pp. 623-642.

[14] W. A. VEECH, *A Second Course in Complex Analysis*, W. A. BENJAMIN Ed., Inc., New York, 1967.

[15] J. XU, The lifetime of conditioned Brownian motion in planar domains of finite area, *Prob. Theory Related Fields*, Vol. **87**, 1991, pp. 469-487.

(*Manuscript received November* 22, 1991;
revised September 9, 1992.)