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LESZEK SŁOMÍNSKI

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On existence, uniqueness and stability of solutions of multidimensional SDE's with reflecting boundary conditions (*)

by

Leszek SŁOMIŃSKI

Instytut Matematyki, Uniwersytet Mikołaja Kopernika,
ul. Chopina 12/18, 87-100 Toruń, Poland

ABSTRACT. — We study existence, uniqueness and stability of weak and strong solutions to a d-dimensional stochastic differential equations on a domain D with reflecting boundary

$$X_t = H_t + \int_0^t f(X_{s-})\,dZ_s + K_t, \qquad t \in \mathbb{R}^+.$$

We do not assume that $D = \mathbb{R}^+ \times \mathbb{R}^{d-1}$. Moreover, neither H nor the driving semimartingale Z need have continuous trajectories.

Key words : Stochastic differential equations with reflecting boundary, weak and strong stolutions, stability of solutions.

RÉSUMÉ. — Nous étudions l'existence, l'unicité et la stabilité des solutions faibles et fortes de l'équation différentielle stochastique en dimension d sur un domaine D avec frontière réfléchissante

$$X_t = H_t + \int_0^t f(X_{s-})\,dZ_s + K_t, \qquad t \in \mathbb{R}^+.$$

D n'est pas nécessairement $\mathbb{R}^+ \times \mathbb{R}^{d-1}$. De plus, ni H ni la semi-martingale Z ne sont supposées à trajectoires continues.

Classification A.M.S. : 60 H 20, 60 H 99, 60 F 17.

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1. INTRODUCTION

In this paper we investigate a d-dimensional stochastic differential equation (SDE) on a domain D with reflecting boundary condition

$$X_t = H_t + \int_0^t f(X_{s-})\, dZ_s + K_t, \qquad t \in \mathbb{R}^+, \tag{1}$$

where Z is a semimartingale, X is a reflecting process on $\bar{D} = D \cup \partial D$ and K is a bounded variation process with variation $|K|$ increasing only, when $X_t \in \partial D$ (the precise definition will be given in Section 4). This equation is called a Skorokhod SDE with the analogy of the one-dimensional case first discussed by Skorokhod [22] for $D = \mathbb{R}^+$ and for a standard Wiener process W instead of the driving semimartingale Z in (1). Next, many attempts have been made to generalize Skorokhod's results to larger class of domains or larger class of driving processes. And so, the papers [4], [6], [7], [13], [19], [26] are devoted to the study of SDE's with reflection in the half-space *i.e.* $D = \mathbb{R}^+ \times \mathbb{R}^{d-1}$. On the other hand the case of reflecting processes in a domain more general than a half-space has been discussed firstly in the paper by Tanaka [25], where D is any convex subset of $\mathbb{R}^d$ and $Z = W$. Then Lions and Sznitman [11] have investigated domains satisfying the conditions (A) and (B) given in Section 2, and together with the admissibility condition that means, rougly speaking, that D can be approximated in some sense by smooth domains. In their paper Z is any semimartingale with continuous trajectories. Finally, Saisho in [20] has omitted the admissibility condition but he has restricted himself to the case $Z = W$.

In the present paper we assume, as in [20], that D is a general domain satisfying the conditions (A) and (B) and then we discuss the problem of existence and uniqueness of strong and weak solutions to (1) for any driving semimartingale Z. We also consider the question of convergence in the sense of law and in probability of solutions to equations of the type (1).

Now, we describe more precisely the content of the paper. Before solving the SDE (1) we solve simpler d-dimensional Skorokhod problem

$$X_t = Y_t + K_t, \qquad t \in \mathbb{R}^+, \tag{2}$$

on a domain D (for precise definition *see* Section 2).

In Section 2 we consider a deterministic case of (2). We prove existence and uniqueness for such problems provided Y has jumps bounded by some constant r_0, depending on a region D only, *i.e.* $|\Delta Y| < r_0$ (for example if D is convex then $r_0 = \infty$). Let us note that if D is a convex domain the problem (2) has been recently considered by Anulova and Liptzer [2] in order to characterize diffusion approximation for processes with reflection.

In Section 3 we assume that $Y = H + Z$, where H is a process with trajectories in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and Z is a semimartingale. Then we prove some new estimates for the solution (X, K) of (2). These estimates play the key role in Section 4. We discuss also asymptotical behaviour of the sequence of solutions of (2).

In Section 4 we consider a sequence $\{X^n\}$ of solutions of SDE's the form (1) *i.e.*

$$X_t^n = H_t^n + \int_0^t f(X_{s-}^n)\, dZ_s^n + K_t^n, \qquad t \in \mathbb{R}^+, \quad n \in \mathbb{N}. \tag{3}$$

We assume that the sequence of semimartingales $\{Z^n\}$ satisfies the condition introduced by Stricker in his paper [24]. Under this condition called (UT) in [9] limit theorems for stochastic integrals and for solutions of SDE's without reflection has been recently proved in many papers (*see* e. g. [9], [10], [14], [23]). In our main theorem in this section we give some sufficient conditions under which $\{X^n\}$ converges in the law sense to the solution X of the SDE (1). As a consequence we obtain existence of weak solution of the SDE (1) provided that f is continuous and bounded *i. e.* $\|f\| \leqq L < +\infty$, where $\|.\|$ denotes the usual norm in the space of linear operators from $\mathbb{R}^d$ into $\mathbb{R}^d$ and $|\Delta Y| + L|\Delta Z| < r_0$. If additionally f is Lipschitz continuous, using discrete approximations of the solution of the SDE (1), which are constructed with the natural analogy to Euler's formula, we show existence and uniqueness of strong solution of the SDE (1). Next, we consider the convergence in law and in probability of the strong solutions of the SDE (3). Let us note, that if $D = \mathbb{R}^+ \times \mathbb{R}^{d-1}$ the problem of existence and uniqueness of solutions of SDE's like (1) have been examined in papers by Chaleyat-Maurel, El Karoui and Marchal [4] and Protter [19]. Unfortunately, their approach cannot be extended to domain D satisfying conditions (A) and (B).

Let us introduce now some definitions and notations used further on $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ is the space of all mappings $x, x: \mathbb{R}^+ \to \mathbb{R}^d$ which are right continuous and admits left-hand limits with the Skorokhod topology J_1. For $x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $A \subset \mathbb{R}^+$ we denote

$$\omega_x A = \sup_{s, t \in A} |x_t - x_s|, \qquad \omega'_x(h, q) = \inf_{(s_k)} \max_{1 < k \leqq r} \omega_x [s_{k-1}, s_k[,$$

where

$$0 = s_0 < s_1 < \ldots < s_r = q, \qquad s_k - s_{k-1} > h, k = 1, 2, \ldots, r.$$

Every process X appearing in the sequel is assumed to be realized in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $(\mathcal{F}_t)$ be a filtration on $(\Omega, \mathcal{F}, \mathcal{P})$ satisfying the usual conditions. Let X be an $(\mathcal{F}_t)$ adapted process and τ be an $(\mathcal{F}_t)$ stopping time. We write X^τ and $X^{\tau-}$ to denote the stopped processes $X_{\wedge \tau}$ and $X_{\cdot \wedge \tau -}$, respectively. If

$X=(X^1, \ldots, X^d)$ is a semimartingale then $[X]_t$ stands for $\sum_{i=1}^{d} [X^i]_t$, where for $i=1, \ldots, d$ $[X^i]$ is a quadratic variation process of X^i. Similarly, if $k=(k^1, \ldots, k^d)$ is a process with locally finite variation, then $|k|_t = \sum_{i=1}^{d} |k^i|_t$, where $|k^i|_t$ is a total variation of k^i on $[0, t]$.

In the paper we use results from general theory of stochastic processes. Here the books by Dellacherie and Meyer [5] and Jacod and Shiriayev [8] are good source. For information on the space $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ we defer the reader to Billingsley [3] and once more to Jacod and Shiriayev [8].

2. A DETERMINISTIC CASE

Let D be a domain in $\mathbb{R}^d$. Define the set $\mathcal{N}_x$ of inward normal unit vectors at $x \in \partial D$ by

$$\mathcal{N}_x = \bigcup_{r>0} \mathcal{N}_{x,r}, \qquad \mathcal{N}_{x,r} = \{\mathbf{n} \in \mathbb{R}^d;\ |\mathbf{n}|=1,\ B(x-r\mathbf{n}, r) \cap D = \varnothing\},$$

where

$$B(z, r) = \{y \in \mathbb{R}^d;\ |y-z| < r\},\ z \in \mathbb{R}^d,\ r>0.$$

Following Lions and Sznitman [11] and Saisho [20] we introduce two assumptions.

(A) There exists a constant $r_0 > 0$ such that

$$\mathcal{N}_x = \mathcal{N}_{x, r_0} \neq \varnothing \quad \text{for every } x \in \partial D.$$

(B) There exist constants $\delta > 0$, $\beta \geqq 1$ such that for every $x \in \partial D$ there exists a unit vector $\mathbf{l}_x$ with the following property

$$\langle \mathbf{l}_x, \mathbf{n} \rangle \geqq \frac{1}{\beta} \quad \text{for every } \mathbf{n} \in \bigcup_{y \in B(x, \delta) \cap \partial D} \mathcal{N}_y$$

where $\langle .\,, . \rangle$ denotes the usual inner product in $\mathbb{R}^d$.

Remark 1 ([11], [20]):

(i) $\mathbf{n} \in \mathcal{N}_{x,r}$ if and only if $\langle y-x, \mathbf{n} \rangle + \frac{1}{2r}|y-x|^2 \geqq 0$ for every $y \in \bar{D}$,

(ii) if $\operatorname{dist}(x, \bar{D}) < r_0$, $x \notin \bar{D}$ then there exists a unique $[x]_\partial \in \bar{D}$ such that $|x-[x]_\partial| = \operatorname{dist}(x, \bar{D})$ and moreover $([x]_\partial - x)/|[x]_\partial - x| \in \mathcal{N}_{[x]_\partial}$.

(iii) if D is a convex domain in $\mathbb{R}^d$ then $r_0 = +\infty$.

The Skorokhod deterministic problem is stated in the following manner.

DEFINITION 1. — *Let $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and $y_0 \in \bar{D}$. We will say that a pair $(x, k) \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ is a solution of the Skorokhod problem associated with* *y if*

(i) $x_t = y_t + k_t$, $t \in \mathbb{R}^+$,

(ii) $x_t \in \bar{D}$, $t \in \mathbb{R}^+$,

(iii) *k is a function with bounded variation on each finite interval such that* $k_0 = 0$ *and*

$$k_t = \int_0^t \mathbf{n}_s \, d|k|_s, \qquad |k|_t = \int_0^t \mathbf{1}_{\{x_s \in \partial D\}} \, d|k|_s,$$

where $\mathbf{n}_s \in \mathcal{N}_{x_s}$ *if* $x_s \in \partial D$.

(iv) $|\Delta x_t| \leqq |\Delta y_t|$, $t \in \mathbb{R}^+$.

Remark 2. — As compred with classical definitions of the Skorokhod problem we have added the condition (iv). However it is known, that if either D is convex or the condition (A) is satisfied and y is continuous, then for every pair (x, k) satisfying (i)-(iii) we have also (iv) (*see* e.g. [20], [25]). On the other hand, as in our paper, if D is not necessarily connected then under (i)-(iii) only the solution of the Skorokhod problem associated with fixed discontinuous y is in general not unique. Moreover, the set of solutions must be neither relatively compact in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ nor locally bounded as the following example shows.

Example 1. — Put $D = (-\infty, 0) \cup \bigcup_{n=1}^{\infty} (4n-2, 4n)$ and $y_t = t - \frac{9}{8}$ for $t < 1$ and $y_t = \frac{1}{8}$ for $t \geqq 1$. Then the conditions (A), (B) are satisfied with $r_0 = 1$ and one can check that the function $x_t = y_t$ for $t < 1$ and $x_t = 0$ for $t \geqq 1$ solves the problem associated with y. Moreover, for each $n \in \mathbb{N}$ the function $x_t = y_t$ for $t < 1$ and $x_t = 0$ for $t \geqq 1$ solves the problem associated with y. Moreover, for each $n \in \mathbb{N}$ the function x^n defined as $x_t^n = y_t$ for $t < 1$ and $x_t^n = 4n - 2$ for $t \geqq 1$ is a solution of the problem, too.

LEMMA 1. — *Assume the condition* (A). *If* $y_0 \in \bar{D}$ *and* $|\Delta y| < r_0$ *then the Skorokhod problem has at most one solution.*

Proof. — We will use the inequality proved by Saisho (*see* [20], 2.6). Let $y, \hat{y} \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and let (x, k), $(\hat{x}, \hat{k})$ *be the solutions associated with* y and $\hat{y}$, respectively. Then

$$|x_t - \hat{x}_t|^2 \leqq |y_t - \hat{y}_t|^2 + \frac{1}{r_0} \int_0^t |x_s - \hat{x}_s|^2 \, d(|k|_s + |\hat{k}|_s) + 2 \int_0^t \langle y_t - y_s - \hat{y}_t + \hat{y}_s, \, d(k_s - \hat{k}_s) \rangle. \quad (4)$$

First assume $|\Delta y| \leqq \frac{r_0}{4}$. Suppose (x, k), $(\hat{x}, \hat{k})$ be two solutions of the Skorokhod problem associated with $y=\hat{y}$. By (4)

$$|x_t-\hat{x}_t|^2 \leqq \frac{1}{r_0}\int_0^{t-} |x_s-\hat{x}_s|^2\, d(|k|_s+|\hat{k}|_s)+\frac{1}{r_0}|x_t-\hat{x}_t|^2(|\Delta k_t|+|\Delta \hat{k}_t|).$$

Since $|\Delta x_t|$, $|\Delta \hat{x}_t| \leqq |\Delta y_t|$ and $|\Delta y_t| \leqq \frac{r_0}{4}$ it is clear that $|\Delta k_t|+|\Delta \hat{k}_t| \leqq \frac{r_0}{2}$ and has a consequence

$$|x_t-\hat{x}_t|^2 \leqq \frac{2}{r_0}\int_0^{t-} |x_t-\hat{x}_t|^2\, d(|k|_s+|\hat{k}|_s).$$

Therefore by Gronwall's lemma (*see* e.g. [20, Lemma 2.2]) $x=\hat{x}$.

Now, let $|\Delta y| < r_0$. Define

$$s_1=\inf\left\{t>0;\ |\Delta y_t|>\frac{r_0}{4}\right\},\qquad s_m=\inf\left\{t>s_{m-1};\ |\Delta y_t|>\frac{r_0}{4}\right\},$$
$$m \geqq 2.$$

Obviously on each finite interval there exists only finitely many jumps bigger than $\frac{r_0}{4}$. By the arguments used previously $x_t=\hat{x}_t$ for $t \in [0, s_1)$ and in particular $x_{s_1-}=\hat{x}_{s_1-}$. Therefore by using the notation from Remark 1 (ii), and additionally setting $[x]_\partial = x$ for every $x \in \bar{\mathrm{D}}$ we have

$$x_{s_1}=\hat{x}_{s_1}=[x_{s_1-}+\Delta y_{s_1}]_\partial.$$

Hence $x_t=\hat{x}_t$ for $t \in [0, s_1]$. Analogously we obtain the equality $x=\hat{x}$ on every interval $[s_{m-1}, s_m]$. The proof is finished. □

PROPOSITION 1. — *Let* D *satisfies the conditions* (A) *and* (B). *Assume that* (x, k) *is a solution of the Skorokhod problem associated with* $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $|\Delta y| \leqq c$ *for some constant* $c < r_0$. *If* $\sup_t |y_t| \leqq a$ *then there exists a constant* C *depending on* a *and* c (*and also on* r_0, δ, β) *such that*

$$|k|_{t_m}-|k|_{t_{m-1}} \leqq \mathrm{C}\,\omega_y[t_{m-1}, t_m]$$

provided that $t_m < +\infty$, *where*

$$t_0=\inf\{t;\ x_t \in \partial \mathrm{D}\},$$
$$t_m^\delta=\inf\{t>t_{m-1};\ |x_t-x_{t_{m-1}}| \geqq \delta\},\qquad t_m=\inf\{t \geqq t_m^\delta;\ x_t \in \partial\mathrm{D}\},$$
$$m \in \mathbb{N}.$$

Proof. — If $t\in[t_{m-1}, t_m^\delta[$, $\mathbf{l}=\mathbf{l}_{x_{t_{m-1}}}$ then by (B) we have

$$\langle \mathbf{l}, x_t-x_{t_{m-1}}\rangle=\langle \mathbf{l}, y_t-y_{t_{m-1}}\rangle+\langle \mathbf{l}, k_t-k_{t_{m-1}}\rangle$$
$$\geqq\langle \mathbf{l}, y_t-y_{t_{m-1}}\rangle+\frac{1}{\beta}(|k|_t-|k|_{t_{m-1}}).$$

Therefore,

$$|k|_{t_m^\delta -}-|k|_{t_{m-1}}\leqq\beta\{\sup_{t_{m-1}\leqq t<t_m^\delta}|x_t-x_{t_{m-1}}|+\omega_y[t_{m-1}, t_m^\delta[\}. \tag{5}$$

On the other hand by the inequality 2.7 in [20] for $t\in[t_{m-1}, t_m^\delta[$

$$|x_t-x_{t_{m-1}}|^2$$
$$\leqq|y_t-y_{t_{m-1}}|^2+\frac{1}{r_0}\int_{t_{m-1}}^t |x_s-x_{t_{m-1}}|^2\,d|k|_s+2\int_{t_{m-1}}^t (y_t-y_s)\,dk_s$$
$$\leqq\omega_y^2[t_{m-1}, t_m^\delta[+\frac{1}{r_0}\int_{t_{m-1}}^{t-} |x_s-x_{t_{m-1}}|^2\,d|k|_s$$
$$+\frac{1}{r_0}|x_t-x_{t_{m-1}}|^2\,|\Delta k_t|+2\,\omega_y[t_{m-1}, t_m^\delta[(|k|_{t_m^\delta -}-|k|_{t_{m-1}}).$$

Hence for $t\in[t_{m-1}, t_m^\delta[$

$$|x_t-x_{t_{m-1}}|^2\leqq\frac{r_0}{r_0-c}\left\{\omega_y^2[t_{m-1}, t_m^\delta[+\frac{1}{r_0}\int_{t_{m-1}}^{t-} |x_s-x_{t_{m-1}}|^2\,d|k|_s\right.$$
$$\left.+2\,\omega_y[t_{m-1}, t_m^\delta[(|k|_{t_m^\delta -}-|k|_{t_{m-1}})\right\}$$

and by Gronwall's lemma

$$\sup_{t_{m-1}\leqq t<t_m^\delta}|x_t-x_{t_{m-1}}|^2$$
$$\leqq\frac{r_0}{r_0-c}\{\omega_y^2[t_{m-1}, t_m^\delta[+2\,\omega_y[t_{m-1}, t_m^\delta[(|k|_{t_m^\delta -}-|k|_{t_{m-1}})\}$$
$$\times\exp\left\{\frac{1}{r_0-c}(|k|_{t_m^\delta}-|k|_{t_{m-1}})\right\}.$$

Therefore using (5) and putting $w=\sup_{t_{m-1}\leqq t<t_m^\delta}|x_t-x_{t_{m-1}}|$ and $b=\omega_y[t_{m-1}, t_m^\delta[$ we have

$$w^2\leqq\frac{r_0}{r_0-c}\{b^2+2\,b\,\beta(w+b)\}\exp\left\{\frac{1}{r_0-c}\beta(w+b)\right\}.$$

By easy calculations $w\leqq\delta$ and $b\leqq 2\,a$. Hence it is clear that there exists a constant C_1 such that

$$w^2\leqq C_1(b^2+2\,bw).$$

Since b, $w \geqq 0$, there exists also a constant C_2 for which $w \leqq C_2 b$. As a consequence, using (5) once more we get

$$|k|_{t_m^\delta -} - |k|_{t_{m-1}} \leqq \beta (C_2+1)\, \omega_y [t_{m-1}, t_m^\delta[.$$

Thus

$$\begin{aligned} |k|_{t_m^\delta} - |k|_{t_{m-1}} &\leqq \beta (C_2+1)\, \omega_y [t_{m-1}, t_m^\delta[+ |\Delta k_{t_m^\delta}| \\ &\leqq \beta (C_2+1)\, \omega_y [t_{m-1}, t_m^\delta[+ |\Delta y_{t_m^\delta}| \\ &\leqq \{ \beta (C_2+1)+1 \}\, \omega_y [t_{m-1}, t_m^\delta]. \end{aligned}$$

Therefore if $t_m = t_m^\delta$ the proof is complete. Assume now $t_m^\delta < t_{m-1}$. Since k is constant in the interval $]t_m^\delta, t_{m-1}[$,

$$\begin{aligned} |k|_{t_m} - |k|_{t_{m-1}} &\leqq \{ \beta (C_2+1)+1 \}\, \omega_y [t_{m-1}, t_m^\delta] + |\Delta k_{t_m}| \\ &\leqq \{ \beta (C_2+1)+2 \}\, \omega_y [t_{m-1}, t_m] \end{aligned}$$

and it is enough to put $C = \beta (C_2+1)+2$. □

COROLLARY 1. – *Under assumptions of Proposition* 1 *there exists a constant* C *depending on a and c such that*

$$\delta \leqq |x_{t_m^\delta} - x_{t_{m-1}}| \leqq C\, \omega_y [t_{m-1}, t_m^\delta],$$

provided that $t_m^\delta < +\infty$, $m \in \mathbb{N}$.

Proof. – It follows easily by the definition of t_m^δ and by the estimation $w \leqq C_2$ from the proof of Proposition 1. □

COROLLARY 2. – *Assume the conditions* (A) *and* (B). *Let* $\{y^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $|\Delta y^n| \leqq c < r_0$, *for every* $n \in \mathbb{N}$. *Let* $\{(x^n, k^n)\}$ *be a sequence of solutions of the Skorokhod problem associated with the sequence* $\{y^n\}$. *If* $\{y^n\}$ *is relatively compact in* $\mathbb{D}([0, q], \mathbb{R}^d)$ *then*

(i) *there exists a constant* C *depending on* q, c, $\sup_n \sup_{t \leqq q} |y_t^n|$ *and on the modulus of continuity* ω'_{y^n} *on* $[0, q]$ *such that*

$$|k^n|_t - |k^n|_s \leqq C\, \omega_{y^n} [s, t], \qquad n \in \mathbb{N}, \quad 0 \leqq s < t \leqq q,$$

(ii) $|k^n|_q \leqq 2C \sup_{t \leqq q} |y_t^n|$,

(iii) $\{(x^n, y^n, k^n, |k|^n\}$ *is relatively compact in* $\mathbb{D}([0, q], \mathbb{R}^{3d+1})$.

Proof. – (i) By the version of Arzela-Ascoli theorem in $\mathbb{D}([0, q], \mathbb{R}^d)$, [3, Theorem 14.3] the sequence $\{y^n\}$ is relatively compact in $\mathbb{D}([0, q], \mathbb{R}^d)$ if and only if the following two conditions are satisfied

$$\sup_n \sup_{t \leqq q} |y_t^n| < +\infty \tag{6}$$

$$\lim_{h \downarrow 0} \sup_n \omega'_{y^n}(h, q) = 0 \tag{7}$$

It is obvious that we can rewrite (7) into the form

$$\left.\begin{array}{c}\forall_{\varepsilon>0}\,\exists_{\gamma>0,\,(s_k)}\ \text{such that}\ \omega_{y^n}[s^n_{k-1}, s^n_k[<\varepsilon,\\ k=1,\dots,r_n,\quad n\in\mathbb{N},\end{array}\right\}\tag{8}$$

where $0=s^n_0<\ldots<s^n_{r_n}=q$, $s^n_k-s^n_{k-1}>\gamma$, $k=1,\ldots,r_n$. Similarly to t_m, t^δ_m we define the times t^n_m, $t^{n,\delta}_m$ for x^n, $n, m\in\mathbb{N}$. By Corollary 1 there exists a constant C depending on q, c and $a=\sup_n\sup_{t\leq q}|y^n_t|$ (and also on r_0, δ, β) such that

$$\delta\leqq|x^n_{t^{n,\delta}_m}-x^n_{t^n_{m-1}}|\leqq C\,\omega_{y^n}[t^n_{m-1}, t^{n,\delta}_m],\tag{9}$$

provided that $t^{n,\delta}_m\leqq q$. Hence for such m and $\varepsilon=\dfrac{\delta}{C}$

$$0<\varepsilon\leqq\omega_{y^n}[t^n_{m-1}, t^{n,\delta}_m]\leqq\omega_{y^n}[t^n_{m-1}, t^n_m].$$

As a consequence every interval $[s^n_{k-1}, s^n_k[$ in (8) contains at most one point t^n_m for every $n\in\mathbb{N}$. Therefore by Proposition 1

$$|k^n|_t-|k^n|_s\leqq\left(\frac{q}{\gamma}+1\right)C\,\omega_{y^n}[s, t],\qquad s, t\leqq q,$$

which yields (i).

(ii) It is a trivial consequence of (i) and of an estimate $\omega_{y^n}[o, q]\leqq 2\sup_{t\leq q}|y^n_t|$.

(iii) By using (i) and (ii) we can estimate $\sup_{t\leq q}|x^n_t|$, $\sup_{t\leq q}|k^n_t|_q$ by $\sup_{t\leq q}|y^n_t|$ and the moduli of continuity $\omega_{x^n}[s, t]$, $\omega_{k^n}[s, t]$, $\omega_{|k^n|}[s, t]$ by $\omega_{y^n}[s, t]$, $s, t\leqq q$. Thus, by simple calculations we deduce that the conditions (6), (8) are satisfied for $\{(x^n, y^n, k^n, |k|^n)\}\subset\mathbb{D}([0, q], \mathbb{R}^{3d+1})$ instead of $\{y^n\}\subset\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. □

THEOREM 1. — *Suppose that a domain* D *satisfies the conditions* (A) *and* (B). *Let* $\{y^n\}\subset\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $y^n_0\in\bar{D}$ *and let* $\{(x^n, k^n)\}$ *be a sequence of solutions of the Skorokhod problem associated with* $\{y^n\}$. *If* $y^n\to y$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ *and* $|\Delta y|<r_0$ *then*

$$(x^n, k^n, y^n)\to(x, k, y)\quad\text{in }\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}),$$

where (x, k) *is a solution of the Skorokhod problem corresponding to* y.

Proof. — Let $\{c_k\}$ be a sequence of constants, $c_k\uparrow r_0$ such that $|\Delta y_t|\neq c_k$ for all $t\in\mathbb{R}^+$, $k\in\mathbb{N}$. Define

$$s_k=\inf\{t>0;\ |\Delta y_t|\geqq c_k\}\qquad\text{and}\qquad s^n_k=\inf\{t>0;\ |\Delta y^n_t|\geqq c_k\},\qquad n, k\in\mathbb{N}.$$

By Proposition 2.7 in [8] $s^n_k\to s_k$ and $y^{n, s^n_k-}\to y^{s_k-}$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. Let us fix $q\in\mathbb{R}^+$ such that $\Delta y_q=0$. Then $y^{n, s^n_k-}\to y^{s_k-}$ in $\mathbb{D}([0, q], \mathbb{R}^d)$. Since

$|\Delta y^{n, s_k^n -}| \leqq c_k$ then by Corollary 2 (iii)

$$\{(x^{n, s_k^n -}, k^{n, s_k^n -}, |k|^{n, s_k^n -}, y^{n, s_k^n -})\}$$

is relatively compact in $\mathbb{D}([0, q], \mathbb{R}^{4d})$.

On the other hand $\lim_{k \to +\infty} \lim_{n \to +\infty} s_k^n = +\infty$. Therefore putting $q \uparrow +\infty$ we deduce

$$\{(x^n, k^n, |k|^n, y^n)\} \text{ is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}). \tag{10}$$

Now, assume that there exists a subsequence $\{n'\} \subset \{n\}$ such that

$$(x^{n'}, k^{n'}, |k|^{n'}, y^{n'}) \to (x, k, b, y) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}). \tag{11}$$

Due to Lemma 1 we have to check that (x, k) is a solution of the Skorokhod problem associated with y, only. Firstly, since for each $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$ $|\Delta x_t^{n'}| \leqq |\Delta y_t^{n'}|$, it follows by (11) that $|\Delta x_t| \leqq \Delta y_t|$ for $t \in \mathbb{R}^+$. Therefore all we have to do is to show that

$$|k|_t = \int_0^t \mathbf{1}_{\{x_s \in \partial D\}} d|k|_s, \qquad t \in \mathbb{R}^+, \tag{12}$$

$$k_t = \int_0^t \mathbf{n}_s \, d|k|_s, \quad \text{where } \mathbf{n}_s \in \mathcal{N}_{x_s} \text{ if } x_s \in \partial D, \qquad t \in \mathbb{R}^+. \tag{13}$$

By Remark 1 for every continuous z with values in $\bar{D}$ and every $n' \in \mathbb{N}$,

$$\int_0^t \langle z_s - x_s^{n'}, dk_s^{n'} \rangle + \frac{1}{2 r_0} \int_0^t |z_s - x_s^{n'}|^2 \, d|k^{n'}|_s \geqq 0. \tag{14}$$

Now, we will need the following simple lemma which is a consequence of [9], Proposition 2.9.

LEMMA 2. — *Let* $\{f^n\}$, $\{g^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ *and for every* $q \in \mathbb{R}^+$ $\sup_n |g^n|_q < +\infty$. *If* $(f^n, g^n) \to (f, g)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ *then*

$$\left(f^n, g^n, \int_0^{\cdot} f_s^n \, dg_s^n\right) \to \left(f, g, \int_0^{\cdot} f_s \, dg_s\right) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}). \quad \square$$

Since z is continuous, by (11)

$$(z - x^{n'}, |z - x^{n'}|^2, k^{n'}, |k^{n'}|) \to (z - x, |z - x|^2, k, b)$$

in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{5d})$, and it follows by Corollary 2 (ii) and Lemma 2 that (14) implies

$$\int_0^t \langle z_s - x_s, dk_s \rangle + \frac{1}{2 r_0} \int_0^t |z_s - x_s|^2 \, db_s \geqq 0. \tag{15}$$

The last part of the proof follows the proof of [11, Theorem 1.1] and we give it below for completness, only. By the definition of b,

$k_t - k_s \leqq |k|_t - |k|_s \leqq b_t - b_s$ for $s \leqq t$ and hence there exists a bounded measurable function h_s, $|h_s| \leqq 1$ such that $dk_s = h_s\, db_s$, $|k|_s = |h_s|\, db_s$. Therefore by (15)

$$\langle z_s - x_s, h_s \rangle + \frac{1}{2r_0}|z_s - x_s|^2 \geqq 0 \quad db_s\, a.e. \tag{16}$$

Let $D^0 = \bar{D} \setminus \partial D$ and let $\{\phi_m\}$ be a sequence of continuous functions $\phi_m : \mathbb{R}^d \to \mathbb{R}$, $\phi_m \uparrow \mathbf{1}_{D^0}$, $\phi_m = 0$ on $\mathbb{R}^d \setminus D^0$. Since by Definition 1 $\forall_m$, $\int_0^t \phi_m(X_s^{n'})\, d|k^{n'}|_s = 0$, it follows by Lemma 2 that $\int_0^t \mathbf{1}_{\{x_s \in D^0\}}\, db_s = 0$, which yields $x_s \in \partial D$ $d|k|_s$ a.e. and (12). On the other hand by (16)

$$\langle z_s - x_s, h_s \rangle + \frac{1}{2r_0}|z_s - x_s|^2 \geqq 0 \quad dk_s \text{ a.e.} \tag{17}$$

In order to finish the proof it is sufficient to deduce from (12) that $\mathbf{n}_s = \frac{h_s}{|h_s|} d|k|_s$ a.e. We have

$$k_t = \int_0^t h_s\, db_s = \int_0^t \frac{h_s}{|h_s|} |h_s|\, db_s = \int_0^t \mathbf{n}_s\, d|k|_s,$$

which gives (13). The equality $\mathbf{n}_s = \frac{h_s}{|h_s|}$ is trivial by Remark 1 if $|h_s| = 1$. Assume $0 < |h_s| < 1$ and define $z = x_s - r_0 h_s$, $C = |z - [z]_\partial|$, where $[z]_\partial$ is uniquely determined by Remark 1. Then $|z - x_s| = r_0 |h_s|$ and by (17)

$$\begin{aligned} 0 &\leqq \left\langle [z]_\partial - x_s, \frac{x_s - z}{r_0 |h_s|} \right\rangle + \frac{1}{2r_0|h_s|} |[z]_\partial - x_s|^2 \\ &= -r_0|h_s| + \left\langle [z]_\partial - z, \frac{x_x - z}{r_0 \| h_s|} \right\rangle + \frac{1}{2r_0|h_s|} |[z]_\partial - x_s|^2 \\ &= -r_0|h_s| + \frac{1}{2r_0|h_s|} (|[z]_\partial - z|^2 + |z - x_s|^2) \\ &= -r_0|h_s| + \frac{1}{2r_0|h_s|} (C^2 + r_0^2 |h_s|^2) \end{aligned}$$

Hence $C \geqq r_0|h_s|$ and using again Remark 1, $x_s = [z]_\partial$. Since $\frac{[z]_\partial - z}{|[z]_\partial - z|} = \frac{h_s}{|h_s|}$, the proof is finished. □

It is proved in Saisho [20] that if y is continuous then the solution of the Skorokhod problem corresponding to y can be approximated by the solutions of discrete Skorokhod problems. Now, we discuss such a problem for $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. Let us consider an array $\{\{t_{nk}\}\}$ of nonnegative numbers. In each n-th row let the sequence $\{t_{nk}\}$ form a partition on $\mathbb{R}^+$ such that $0 = t_{n\,0} < t_{n\,1} < \ldots, \lim_{k \to +\infty} t_{nk} = +\infty$. Assume that

$$\max_{t_{nk} \leqq q} (t_{nk} - t_{n,\,k-1}) \to 0, \qquad q \in \mathbb{R}^+.$$

In addition for the array $\{\{t_{nk}\}\}$ we define the sequence of summation rules $\{\rho_n\}$, $\rho_n : \mathbb{R}^+ \to \mathbb{R}^+$ by the equality $\rho_n(t) = \max\{t_{nk} : t_{nk} \leqq t\}$. For every $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ we define the sequence $\{y^{\rho_n}\}$ of discretizations of y. More precisely,

$$y_t^{\rho_n} = y_{\rho_n(t)} = y_{t_{nk}} \quad \text{for } t \in [t_{nk}, t_{n,\,k+1}), \qquad k \in \mathbb{N} \cup \{0\}, \quad n \in \mathbb{N}.$$

It is easy to see that

$$y^{\rho_n} \to y \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d). \tag{18}$$

Now, let us fix $n \in \mathbb{N}$. It is observed in [20] that if $|\Delta y| < r_0$ then for sufficiently large $n \in \mathbb{N}$ the solution (x^n, k^n) of the Skorokhod problem associated with a discretization $y^{\rho_n} \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ has the form

$$x_t^n = \begin{cases} y_0 & \text{if } t \in [0, t_{n1}), \\ [x_{t_{n,\,k-1}}^n + y_{t_{nk}} - y_{t_{n,\,k-1}}]_\partial & \text{if } t \in [t_{nk}, t_{n,\,k+1}), \quad k \in \mathbb{N}, \end{cases} \tag{19}$$

and

$$k_t^n = \begin{cases} 0 \quad \text{if } t \in [0, t_{n\,1}), \\ k_{t_{n,\,k-1}}^n + x_t^n - x_{t_{n,\,k-1}}^n - y_{t_{nk}} + y_{t_{n,\,k-1}} \quad \text{if } t \in [t_{nk}, t_{n,\,k+1}), \\ k \in \mathbb{N}. \end{cases} \tag{20}$$

Corollary 3. — *Assume the conditions* (A) *and* (B). *Let* $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $|\Delta y| < r_0$ *and* $y_0 \in \bar{D}$. *Then there exists a unique solution* (x, k) *of the Skorokhod problem associated with* y. *Moreover if* $\{(x^n, k^n)\}$ *is a sequence of solutions of the Skorokhod problem corresponding to the sequence of discretizations* $\{y^{\rho_n}\}$ *then*

(i) $(x^n, k^n, y^{\rho_n}) \to (x, k, y)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$,

(ii) $\sup_{t \leqq q} |x_t^n - x_t^{\rho_n}| \to 0$ *and* $\sup_{t \leqq q} |k_t^n - k_t^{\rho_n}| \to 0$, $q \in \mathbb{R}^+$.

Proof. — The uniqueness was proved in Lemma 1. The existence part and property (i) follow immediately from Theorem 1 and (18). To see (ii) first observe that by (i) and, for example, by Lemma A in [23]

$$(x^n, x^{\rho_n}) \to (x, x) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

Hence $x^n - x^{\rho_n} \to 0$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ which leads to (ii). □

Corollary 4. — *Assume the conditions* (A) *and* (B). *Let* $\{y^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ *and let* $\{(x^n, k^n)\}$ *be a sequence of solutions of the*

Skorokhod problem corresponding to the sequence $\{y^n\}$. *If* $\sup_{t \leq q} |y_t^n - y_t| \to 0$, $q \in \mathbb{R}^+$, *where* $|\Delta y| < r_0$ *then* $\sup_{t \leq q} |x_t^n - x_t| \to 0$ *and* $\sup_{t \leq q} |k_t^n - k_t| \to 0$, $q \in \mathbb{R}^+$, *where* (x, k) *is a solution of the Skorokhod problem associated with* y.

Proof. — Since uniform convergence on compact subsets of $\mathbb{R}_+$ is stronger than convergence in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, it follows by Theorem 1 that in particular

$$(x^n, y^n) \to (x, y) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

On the other hand, if $|\Delta x_t| > 0$ then by the nature of the Skorokhod problem $|\Delta y_t| > 0$. Using Corollary C in [23] we completes the proof. □

Now, let μ be a probability measure on $\mathbb{R}^+$, which is equivalent to the Lebesgue measure. For $x, y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ write

$$d_\mu(x, y) = \inf\{\varepsilon > 0 : \mu\{t \in \mathbb{R}^+; |x_t - y_t| \geqq \varepsilon\} \leqq \varepsilon\}.$$

So d_μ metrizes the topology of convergence in measure, which is weaker than the usual topology J_1 on $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. For $y^n, y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ we will write $y^n \to_\mu y$ if and only if $d_\mu(y^n, y) \to 0$. This kind of convergence turns out to be very useful in many problems connected with convergence of semimartingales and SDE's (*see* e.g. [1], [17], [18], [24]). There arises the problem if it is possible to obtain some stability results for the sequence of solutions of the Skorokhod problems by using the notion of the convergence in measure. The answer is not positive. It is not true even in the classical case $d = 1$, $D = \mathbb{R}^+$.

Example 2. — Define

$$y_t^n = \begin{cases} -nt & \text{if } t \in [0, n^{-1}), \\ nt - 2 & \text{if } t \in [n^{-1}, 2n^{-1}), \\ 0 & \text{if } t \in [2n^{-1}, +\infty). \end{cases}$$

Then it is well known that

$$k_t^n = \sup_{s \leqq t} \max(0, -y_s^n) = \begin{cases} nt & \text{if } t \in [0, n^{-1}), \\ 1 & \text{otherwise.} \end{cases}$$

In this case $y^n \to_\mu y = 0$ but $k^n \to_\mu 1$ and $x^n \to_\mu 1$. On the other hand it is obvious that the solution of the Skorokhod problem associated with $y = 0$ is of the form $(0, 0)$. □

3. APPLICATIONS FOR STOCHASTIC PROCESSES

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $(\mathcal{F}_t)$ be a filtration on $(\Omega, \mathcal{F}, \mathcal{P})$ satisfying the usual conditions.

DEFINITION 2. – *Let* Y *be an* $(\mathscr{F}_t)$ *adapted process and* $Y_0 \in \bar{D}$. *We will say that a pair* (X, K) *of* $(\mathscr{F}_t)$ *adapted processes solves the Skorokhod problem associated with* Y *if and only if for every* $\omega \in \Omega$, $(X(\omega), K(\omega))$ *is a solution of the Skorokhod problem corresponding to* $Y(\omega)$.

Let us note that as a consequence of Corollary (3) for every process Y such that $Y_0 \in \bar{D}$, $|\Delta Y| < r_0$, there exists a unique solution of the Skorokhod problem associated with Y.

In this section we will give some estimates for the solution (X, K) assuming Y is of the form

$$Y_t = H_t + Z_t = H_t + M_t + V_t, \qquad t \in \mathbb{R}^+, \tag{21}$$

where H is an $(\mathscr{F}_t)$ adapted process, Z is an $(\mathscr{F}_t)$ adapted semimartingale, $Z_0 = 0$, decomposed into the sum of a local martingale M and of a process with bounded variation V, $M_0 = V_0 = 0$.

THEOREM 2. – *Assume the conditions* (A) *and* (B). *Let* $Y_0 \in \bar{D}$, $|\Delta Y| \leqq c < r_0$ *and let* Y *is as in* (21), *where* $\sup_t |Z_t|$, $\sup_t |H_t|$ *are bounded by some constant* a, *and* M, V *are square integrable martingale and a process with square integrable variation, respectively* (i.e. $E[M]_\infty$, $E|V|_\infty^2 < +\infty$). *Then for every* $(\mathscr{F}_t)$ *stopping time* σ *there exists a sequence of* $(\mathscr{F}_t)$ *stopping times* $\{\sigma_j\}$ *and a sequence of constants* $\{C_j\}$ *such that*

$$|K|_{\sigma_j \wedge \sigma} \leqq C_j \sup_{t \leqq \sigma_j \wedge \sigma} |Y_t| \quad \text{on the set } [\sigma < +\infty],$$

where $\mathscr{P}[\sigma_j \leqq \sigma, \sigma < +\infty] \to 0$ *and for every* $j \in \mathbb{N}$ *the constant* C_j *depends only on* σ, a, c, $E[M]_\infty$, $E|V|_\infty^2$ *and on the modulus of continuity* ω'_H.

Proof. – Without loss of generality we assume $\mathscr{P}[\sigma < +\infty] = 1$. Define

$$\gamma_0^i = 0, \qquad \gamma_{k+1}^i = \min(\gamma_k^i + b_k^i, \inf\{t > \gamma_k^i; |\Delta H_t| > b^i\}), \tag{22}$$

where $\{b^i\}$, $\{\{b_k^i\}\}$ are two families of constants such that $b^i \downarrow 0$, $\frac{b^i}{2} \leqq b_k^i \leqq b^i$ and $\mathscr{P}[|\Delta H_t| = b^i, t \in \mathbb{R}^+] = 0$, $\mathscr{P}[\Delta H_{\gamma_k^i + b_k^i} = 0] = 1$. With this notation we introduce a sequence of discretizations to the process H. We set

$$\begin{aligned} H_t^i &= H_{\gamma_k^i} \quad \text{if } t \in [\gamma_k^i, \gamma_{k+1}^i), \qquad k \in \mathbb{N} \cup \{0\}, \\ &= \sum_{\gamma_k^i \leqq t} (H_{\gamma_{k+1}^i} - H_{\gamma_k^i}), \qquad t \in \mathbb{R}^+. \end{aligned} \tag{23}$$

Let q_j be such that $\mathscr{P}[q_j \leqq \sigma] \leqq \frac{1}{j}$ and $\mathscr{P}[\Delta H_{q_j \wedge \sigma} = 0] = 1$. By simple calculations (*see* e.g. [23]) $|\Delta(H - H^i)| \leqq b^i$ and

$$\sup_{t \leqq q_j \wedge \sigma} |(H - H^i)_t| \leqq b^i + 2\,\omega'_H(b^i, q_j \wedge \sigma).$$

Obviously if $i \uparrow +\infty$ then $\sup_{t \leqq q_j \wedge \sigma} |(\mathrm{H}-\mathrm{H}^i)_t| \to 0$ $\mathscr{P}$-*a.e.* Analogously as in Proposition 1 we define $\tau_0 = \inf\{t;\ \mathrm{X}_t \in \partial\mathrm{D}\}$, $\tau_m^\delta = \inf\{t > \tau_{m-1};\ |\mathrm{X}_t - \mathrm{X}_{\tau_{m-1}}| \geqq \delta\}$, $\tau_m = \inf\{t \geqq \tau_m^\delta;\ \mathrm{X}_t \in \partial\mathrm{D})$. By Corollary 1 there exists a constant C depending on a such that

$$\delta \leqq |\mathrm{X}_{\tau_m^\delta} - \mathrm{X}_{\tau_{m-1}}| \leqq \mathrm{C}\,\omega_{\mathrm{Y}}[\tau_{m-1}, \tau_m^\delta], \quad \text{provided that } \tau_m^\delta \leqq \sigma. \tag{24}$$

For $\varepsilon = \dfrac{\delta}{\mathrm{C}}$ set $\xi_i = \inf\left\{t;\ |(\mathrm{H}-\mathrm{H}^i)_t| \geqq \dfrac{\varepsilon}{8} \text{ or } |(\mathrm{H}-\mathrm{H}^i)_{t-}| \geqq \dfrac{\varepsilon}{8}\right\}$, $i \in \mathbb{N}$. Then it is easy to see that

$$\sup_{t \leqq \xi_i \wedge q_j \wedge \sigma} |(\mathrm{H}-\mathrm{H}^i)_t| \leqq \frac{\varepsilon}{8} + b^i. \tag{25}$$

Therefore we can choose $i = i(j)$ as large that $b^{i(j)} \leqq \dfrac{\varepsilon}{8}$ and

$$\mathscr{P}[\xi_{i(j)} \leqq q_j \wedge \sigma] = \mathscr{P}\left[\sup_{t \leqq q_j \wedge \sigma} |(\mathrm{H}-\mathrm{H}^i)_t| \geqq \frac{\varepsilon}{8}\right] \leqq \frac{1}{j}.$$

Also we can choose h_j as small that $\mathscr{P}[\omega'_{\mathrm{H}}(h_j, q_j) \geqq b^{i(j)}] \leqq \dfrac{1}{j}$. Now, let us observe that if $\omega'_{\mathrm{H}}(h_j, q_j) < b^{i(j)}$ then every interval of length less or equal h_j contains at most one jump of H bigger than $b^{i(j)}$. Hence on the set $\omega'_{\mathrm{H}}(h_j, q_j) < b^{i(j)}$, $\gamma_k^{i(j)} > q_j$ for $k \geqq k(j)$, where

$$k(j) = \left[\frac{q_j}{h_j \wedge 2^{-1} b^{i(j)}}\right] + 1,$$

and γ_k^i is defined by (22). On the other hand

$$\mathscr{P}[\gamma_{k(j)}^{i(j)} \leqq \sigma] \leqq \mathscr{P}[q_j \leqq \sigma] + \mathscr{P}[\omega'_{\mathrm{H}}(h_j, q_j) \geqq b^{i(j)}] \leqq \frac{2}{j}$$

and

$$\mathscr{P}[q_j \wedge \xi_{i(j)} \wedge \gamma_{k(j)}^{i(j)} \leqq \sigma] \leqq \frac{3}{j}.$$

Put $\alpha_j = q_j \wedge \xi_{i(j)} \wedge \gamma_{k(j)}^{i(j)}$. By (24) and (25)

$$0 < \frac{\varepsilon}{2} \leqq \omega_{\mathrm{M}}[\tau_{m-1}, \tau_m^\delta] + \omega_{\mathrm{H}^{i(j)}+\mathrm{V}}[\tau_{m-1}, \tau_m^\delta], \qquad \text{if } \tau_m^\delta \leqq \alpha_j \wedge \sigma.$$

In the next considerations we discuss only processes stopped in $\alpha_j \wedge \sigma$. For simplicity we write M, $\mathrm{H}^{i(j)}$, V instead of the stopped processes

$M^{\alpha_j \wedge \sigma}$, $H^{i(j), \alpha_j \wedge \sigma}$, $V^{\alpha_j \wedge \sigma}$. We have for every $m \in \mathbb{N}$

$$\mathbf{1}_{\{\tau_m^\delta \leqq \alpha_j \wedge \sigma\}} \frac{\varepsilon^2}{2} \leqq \omega_M^2 [\tau_{m-1}, \tau_m^\delta] + \omega_{H^{i(j)}+V}^2 [\tau_{m-1}, \tau_m^\delta]$$
$$\leqq 2 \sup_{\tau_{m-1} \leqq s \leqq \tau_m^\delta} |M_s - M_{\tau_{m-1}}|^2 + 2(|H^{i(j)}|_{\tau_m^\delta} - |H^{i(j)}|_{\tau_{m-1}})^2 + 2(|V|_{\tau_m^\delta} - |V|_{\tau_{m-1}})^2.$$

Define $L_u^m = M_{\tau_{m-1}+u} - M_{\tau_{m-1}}$, $m \in \mathbb{N}$. Then L^m is an $(\mathcal{F}_{\tau_{m-1}+u})$ adapted square integrable martingale and $\eta_m = \tau_m^\delta - \tau_{m-1}$ is an $\mathcal{F}_{\tau_{m-1}+u})$ stopping time. Hence, summing to n and integrating the above inequalities we get

$$\mathcal{P}[\tau_n^\delta \leqq \alpha_j \wedge \sigma]\, n \frac{\varepsilon^2}{2}$$
$$\leqq \sum_{m=1}^n \{2 E \sup_{u \leqq \eta_m} |L_u^m|^2 + 2(|H^{i(j)}|_{\tau_m^\delta} - |H^{i(j)}|_{\tau_{m-1}})^2 + 2(|V|_{\tau_m^\delta} - |V|_{\tau_{m-1}})^2\}$$
$$\leqq \sum_{m=1}^n 8 E |L_{\eta_m}^m|^2 + 2 E |H^{i(j)}|_\infty^2 + 2 E |V|_\infty^2$$
$$\leqq \sum_{m=1}^n 8 E([M]_{\tau_m^\delta} - [M]_{\tau_{m-1}}) + 2(2ak(j))^2 + 2 E |V|_\infty^2$$
$$\leqq 8 E [M]_\infty + 8 a^2 k^2(j) + 2 E |V|_\infty^2$$

Since the right hand of the last inequality is finite, for sufficiently large $n = n(j)$,

$$\mathcal{P}[\tau_{n(j)}^\delta \leqq \alpha_j \wedge \sigma] \leqq \frac{1}{j} \quad \text{and} \quad \mathcal{P}[\tau_{n(j)}^\delta \wedge \alpha_j \leqq \sigma] \leqq \frac{4}{j}.$$

Finally, if we define $\sigma_j = \tau_{n(j)}^\delta \wedge \alpha_j$, by Proposition 1 we conclude that for a constant C in (24) we have

$$|K|_{\sigma_j \wedge \sigma} \leqq (n(j)+1)\, 2C \sup_{t \leqq \sigma_j \wedge \sigma} |Y_t| = C_j \sup_{t \leqq \sigma_j \wedge \sigma} |Y_t|,$$

which is our claim. □

Assume now, we have given another $(\mathcal{F}_t)$ adapted process $\hat{Y}$ such that $\hat{Y}_0 \in \bar{D}$, $|\Delta \hat{Y}| < r_0$ and $\hat{Y}$ admits the decomposition

$$\hat{Y}_t = H_t + \hat{Z}_t = H_t + \hat{M}_t + \hat{V}_t, \qquad t \in \mathbb{R}^+, \tag{26}$$

where $\hat{Z}$ is an $(\mathcal{F}_t)$ adapted semimartingale, $\hat{Z}_0 = 0$ and $\hat{M}$ is an $(\mathcal{F}_t)$ adapted local martingale, $\hat{M}_0 = 0$ and $\hat{V}$ is an $(\mathcal{F}_t)$ adapted process with bounded variation, $\hat{V}_0 = 0$. Assume that $(\hat{X}, \hat{K})$ is a solution of the Skorokhod problem corresponding to $\hat{Y}$. In Theorem 3 below we estimate $E \sup_s |X_s - \hat{X}_s|^2$, which is the crucial step in proofs of uniqueness in Section 4.

THEOREM 3. — *Assume the conditions* (A) *and* (B). *Let* Y_0, $\hat{Y}_0 \in \bar{D}$, $|\Delta Y|$, $|\Delta \hat{Y}| \leqq \frac{r_0}{4}$ *and let processes* Y, $\hat{Y}$ *satisfy* (21) *and* (26), *respectively, where* M, $\hat{M}$ *are square integrable martingales and* V, $\hat{V}$ *are processes with square integrable variation. If* $r_0 < +\infty$ *we assume additionally that there exists a constant a such that* $|K|_\infty$, $|\hat{K}|_\infty \leqq a$. *Then there exists a constant* C *depending on a (and also on* r_0, β, δ) *such that for every* $(\mathcal{F}_t)$ *stopping time* σ

$$E \sup_{t \leqq \sigma} |X_s - \hat{X}_s|^2 \leqq C E \{ [M - \hat{M}]_\sigma + |V - \hat{V}|_\sigma^2 \}.$$

Proof. — For simplicity we consider the processes stopped in σ, only. First we discuss the more complicated case $r_0 < +\infty$. By (4) for every $t \in \mathbb{R}^+$

$$|X_t - \hat{X}_t|^2 \leqq 2 \left\{ |Z_t - \hat{Z}_t|^2 + \frac{1}{r_0} \int_0^{t-} |X_s - \hat{X}_s|^2 \, d(|K|_s + |\hat{K}|_s) \right.$$
$$\left. + 2 \int_0^t \langle Z_t - Z_s - \hat{Z}_t + \hat{Z}_s, d(K_s - \hat{K}_s) \rangle \right\}.$$

Since $K_0 = \hat{K}_0 = 0$, by the integration by parts formula

$$2 \int_0^t \langle Z_t - Z_s - \hat{Z}_t + \hat{Z}_s, d(K_s - \hat{K}_s) \rangle$$
$$= 2 \int_0^t \langle (K_{s-} - \hat{K}_{s-}), d(Z_s - \hat{Z}_s) \rangle$$
$$= 2 \int_0^t \langle (X_{s-} - \hat{X}_{s-}), d(Z_s - \hat{Z}_s) \rangle$$
$$- 2 \int_0^t \langle (Z_{s-} - \hat{Z}_{s-}), d(Z_s - \hat{Z}_s) \rangle$$
$$= 2 \int_0^t \langle (X_{s-} - \hat{X}_{s-}), d(Z_s - \hat{Z}_s) \rangle + [Z - \hat{Z}]_t - |Z_t - \hat{Z}_t|^2.$$

Hence for every stopping time τ

$$E \sup_{t \leqq \tau} |X_t - \hat{X}_t|^2 \leqq 2 \left\{ \frac{1}{r_0} E \int_0^{\tau-} |X_s - \hat{X}_s|^2 \, d(|K|_s + |\hat{K}|_s) \right.$$
$$\left. + 2 E \sup_{t \leqq \tau} \int_0^t \langle X_{s-} - \hat{X}_{s-}, d(Z_s - \hat{Z}_s) \rangle + E [Z - \hat{Z}]_\tau \right\}.$$

By simple calculations based on Burkholder-Davies-Gundy and Schwartz's inequalities

$$\mathrm{E}\sup_{t\leqq\tau}\left|\int_0^t \langle (\mathrm{X}_{s-}-\hat{\mathrm{X}}_{s-}),\, d(\mathrm{M}_s-\hat{\mathrm{M}}_s)\rangle\right|$$
$$\leqq \mathrm{C}_0\,\mathrm{E}\left(\int_0^\tau |\mathrm{X}_{s-}-\hat{\mathrm{X}}_{s-}|^2,\, d[\mathrm{M}-\hat{\mathrm{M}}]_s\right)^{1/2}$$
$$\leqq \mathrm{C}_0\,(\mathrm{E}\sup_{s\leqq\tau}|\mathrm{X}_{s-}-\hat{\mathrm{X}}_{s-}|)^{1/2}\,(\mathrm{E}[\mathrm{M}-\hat{\mathrm{M}}]_\tau)^{1/2},$$

and

$$\mathrm{E}\sup_{t\leqq\tau}\left|\int_0^t \langle (\mathrm{X}_{s-}-\hat{\mathrm{X}}_{s-}),\, d(\mathrm{V}_s-\hat{\mathrm{V}}_s)\rangle\right|$$
$$\leqq \mathrm{E}\sup_{s\leqq\tau}|\mathrm{X}_{s-}-\hat{\mathrm{X}}_{s-}|\,.\,|\mathrm{V}-\hat{\mathrm{V}}|_\tau$$
$$\leqq (\mathrm{E}\sup_{s\leqq\tau}|\mathrm{X}_{s-}-\hat{\mathrm{X}}_{s-}|^2)^{1/2}\,(\mathrm{E}|\mathrm{V}-\hat{\mathrm{V}}|_\tau^2)^{1/2}.$$

On the other hand

$$\mathrm{E}[\mathrm{Z}-\hat{\mathrm{Z}}]_\tau \leqq 2\,\mathrm{E}[\mathrm{M}-\hat{\mathrm{M}}]_\tau + 2\,\mathrm{E}[\mathrm{V}-\hat{\mathrm{V}}]_\tau \leqq 2\,\mathrm{E}[\mathrm{M}-\hat{\mathrm{M}}]_\tau + 2\,\mathrm{E}|\mathrm{V}-\hat{\mathrm{V}}|_\tau.$$

If we denote

$$x=(\mathrm{E}\sup_{t\leqq\tau}|\mathrm{X}_t-\hat{\mathrm{X}}|^2)^{1/2},$$
$$b_1=(\mathrm{E}[\mathrm{M}-\hat{\mathrm{M}}]_\tau+\mathrm{E}|\mathrm{V}-\hat{\mathrm{V}}|_\tau^2)^{1/2},$$
$$b_2=\frac{1}{r_0}\mathrm{E}\int_0^{\tau-} |\mathrm{X}_s-\hat{\mathrm{X}}_s|^2\, d(|\mathrm{K}|_s+|\hat{\mathrm{K}}|_s)$$

then by the above calculations we deduce that there exists a constant C_1 such that

$$x^2 \leqq \mathrm{C}_1\,(b_2+2\,b_1\,x+b_1^2) \tag{27}$$

Since $x,\ b_1>0$ it is clear that $x^2\leqq \mathrm{C}_2\,(b_2+b_1^2)$ for some constant C_2. To finish the proof, we will need the following version of Gronwall's lemma.

LEMMA 3. — *Let* Y^1, Y^2 *be two increasing processes,* $\mathrm{Y}_0^1=\mathrm{Y}_0^2=0$ *such that* $\mathrm{EY}_\infty^1<+\infty$, $\mathrm{Y}_\infty^2<a_1$ *for some constant* a_1. *If one of the following two conditions is satisfied*

(i) *for every stopping time* τ

$$\mathrm{EY}_\tau^1 \leqq a_2+\mathrm{E}\int_0^{\tau-} \mathrm{Y}_s^1\, d\mathrm{Y}_s^2,$$

(ii) *for every stopping time* τ

$$EY^1_{\tau-} \leqq a_2 + E\int_0^{\tau-} Y^1_{s-}\, dY^2_s,$$

then $EY^1_\infty \leqq a_2 e^{a_1}$.

Proof. — (i) We give proof of part (i) for the sake of completeness, only. We denote $\tau_t = \inf\{s;\ Y^2_s \geqq t\}$. Then by [5], VI, Theorem 55, we have

$$EY^1_{\tau_t} \leqq E\int_0^{\tau_t-} Y^1_s\, dY^2_s + a_2 = E\int_0^\infty Y^1_s\, \mathbf{1}_{\{s<\tau_t\}}\, dY^2_s + a_2$$
$$\leqq E\int_0^\infty Y^1_{\tau_u}\, \mathbf{1}_{\{\tau_u<\tau_t,\ \tau_u<+\infty\}}\, du + a_2 \leqq \int_0^t EY^1_{\tau_u}\, du + a_2$$

and it is sufficient to use classical Gronwall's lemma for $f(t) = EY^1_{\tau_t}$. Hence $EY^1_{\tau_t} \leqq a_2 e^t$ assuming $t \uparrow a_1$ we obtain our claim.

(ii) *See* [12], Lemma 2. □

If we set in Lemma 3

$$Y^1_t = \sup_{s\leqq t} |X_s - \hat{X}_s|^2, \qquad Y^2_t = C_2(|K|_t + |\hat{K}|_t) < C_2 2a + 1$$

and

$$a_2 = C_2 E\{[M-\hat{M}]_\sigma + |V-\hat{V}|^2_\sigma\}$$

then the proof in the case $r_0 < +\infty$ is complete.

Finally, let us note that in the case $r_0 = +\infty$ instead of (27) we obtain simpler inequality, namely $x^2 \leqq \hat{C}_1(2b_1 x + b_1^2)$. Hence there exists a constant $\hat{C}_2$ such that $x^2 \leqq \hat{C}_2 b_1^2$ and we get the desired result putting $\tau = \sigma$. □

Since $X = Y + K$ it is easy to obtain from Theorem 3 the following result.

COROLLARY 5. — *Under assumptions of Theorem* 3, *there exists a constant* C *such that for every stopping time* σ

$$E\sup_{t\leqq\sigma} |K_t - \hat{K}_t|^2 \leqq CE\{[M-\hat{M}]_\sigma + |V-\hat{V}|^2_\sigma\}. \quad \square$$

The estimates proved in Theorem 3 and Corollary 5 are very similar to those obtained previously by Chaleyat-Maurel, El Karoui and Marchal [4], Proposition 8, in the case $D = \mathbb{R}^+ \times \mathbb{R}^{d-1}$. Moreover, using their method based on [15], Theorem 1, we can deduce from Corollary 5 estimates on the interval $[0, \sigma[$ in place of $[0, \sigma]$.

COROLLARY 6. — *Under assumptions of Theorem* 3, *there exists a constant* C *such that for every stopping time* σ

$$E\sup_{t<\sigma} |K_t - \hat{K}_t|^2 \leqq CE\{[M-\hat{M}]_{\sigma-} + \langle M-\hat{M}\rangle_{\sigma-} + |V-\hat{V}|^2_{\sigma-}\}. \quad \square$$

In turn, we will discuss applications of Theorems 1-3 for sequences of solutions of the Skorokhod problems. Let $\{Y^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted processes and let $\{(X^n, K^n)\}$ be a sequence of solutions of the Skorokhod problems associated with $\{Y^n\}$. We will assume that Y^n is of the form

$$Y^n = H^n + Z^n, \tag{28}$$

where $\{Z^n\}$ is a sequence of $(\mathcal{F}_t^n)$ adapted semimartingales, satisfying the condition (UT) introduced by Stricker in [24], $Z_0^n = 0$. The condition (UT) is given as follows,

(UT) for every $q \in \mathbb{R}^+$ the family of random variables

$$\left\{ \int_0^q U_s^n \, dZ_s^n;\ n \in \mathbb{N},\ U^n \in \mathbf{U}_q^n \right\} \text{ is tight in } \mathbb{R},$$

where $\mathbf{U}_q^n$ is a class of predictable processes of the form $U_s^n = U_9^n + \sum_{i=0}^{k} U_i^n \mathbf{1}_{\{t_i < s \leq t_{i+1}\}}$ such that $0 = t_0 < t_1 < \ldots < t_k = q$ and every U_i^n is $\mathcal{F}_{t_i}^n$ measurable, $|U_i^n| \leq 1$ for every $i \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$.

We recall a simple characterization of (UT), which has been recently proved in [14]. Let for every $n \in \mathbb{N}$, Z^n be decomposed into the sum of three processes

$$Z^n = J^n + M^n + B^n, \tag{29}$$

where $J_t^n = \sum_{0 < s \leq t} \Delta Z_s^n \mathbf{1}_{\{|\Delta Z_s^n| > 1\}}$, M^n is a locally square integrable martingale, $M_0^n = 0$ and B^n is a predictable process with bounded variation, $B_0^n = 0$. Then

Proposition 2. — *The following three conditions are equivalent.*

(i) $\{Z^n\}$ *satisfies the condition* (UT),

(ii) *for every* $q \in \mathbb{R}^+$ *the families of random variables* $\{|J^n|_q\}$, $\{|B^n|_q\}$, $\{[M^n]_q\}$ *are tight in* $\mathbb{R}$.

(iii) *for every* $q \in \mathbb{R}^+$ *and* $\varepsilon > 0$ *there exists* $\alpha > 0$ *such that for every* $n \in \mathbb{N}$ *and every* $(\mathcal{F}_t^n)$ *adapted processes* U^n

$$\mathcal{P}[\sup_{t \leq q} |U_{t-}^n| > \alpha] < \alpha \quad \Rightarrow \quad \mathcal{P}\left[\sup_{t \leq q} \left| \int_0^t U_{s-}^n \, dZ_s^n \right| > \varepsilon \right] < \varepsilon. \quad \square$$

Immediately by (ii) we can obtain the following result proved earlier by Stricker.

Corollary 7. — *If* $\{Z^n\}$ *satisfies* (UT) *then for every* $q \in \mathbb{R}^+$ *the families of random variables* $\{\sup_{t \leq q} |Z_s^n|\}$, $\{[Z^n]_q\}$ *are tight in* $\mathbb{R}$. $\square$

The following proposition play the key role in the proof of existence of solutions to the SDE (1).

PROPOSITION 3. — *Assume the conditions* (A) *and* (B). *Let* $\{Y^n\}$ *be a sequence of* $(\mathcal{F}_t^n)$ *adapted processes*, $|\Delta Y^n| \leqq c < r_0$, *and let* $\{(X^n, K^n)\}$ *be a sequence of solutions of the Skorokhod problem associated with* $|\{Y^n\}$. *We assume that every* Y^n *is of the form* (28), *where* $\{H^n\}$ *is a tight in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ *sequence of* $(\mathcal{F}_t^n)$ *adapted processes and* $\{Z^n\}$ *is a sequence of* $(\mathcal{F}_t^n)$ *adapted semimartingales satisfying* (UT). *Then*

(i) *for every* $q \in \mathbb{R}^+$ *the family of random variables* $\{|K^n|_q\}$ *is tight in* $\mathbb{R}$,

(ii) *if* $H^n = Y_0^n$ *then* $\{X^n\}$ *satisfies* (UT).

Proof. — (i) Define $\tau^{n,a} = \inf\{t;\ |H_t^n| \vee |Z_t^n| \geqq a\}$, $a \in \mathbb{R}^+$, $n \in \mathbb{N}$. Since by Corollary 7 $\{\sup_{t \leqq q} |Z_t^n|\}$, $q \in \mathbb{R}^+$ is tight in $\mathbb{R}$, and by the tightness of $\{H^n\}$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ $\{\sup_{t \leqq q} |H_t^n|\}$, $q \in \mathbb{R}^+$ is tight in $\mathbb{R}$, it is clear that

$$\lim_{a \to \infty} \limsup_{n \to \infty} \mathcal{P}(\tau^{n,a} \leqq q) = 0, \qquad q \in \mathbb{R}^+. \tag{30}$$

Further, by easy calculations

$$\omega'_{H^{n,\tau^{n,a}-}}(h, q) \leqq \omega'_{H^n}(h, q) \qquad \text{and} \qquad \sup_{t \leqq q} |H_t^{n,\tau^{n,a}-}| \leqq \sup_{t \leqq q} |H_t^n|,$$
$$q \in \mathbb{R}^+,$$

and as a consequence the sequence $\{H^{n,\tau^{n,a}-}\}$ is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, too. On the other hand by the definition of (UT) also the sequence $\{Z^{n,\tau^{n,a}}\}$ satisfies (UT). But

$$\begin{aligned} Z_t^{n,\tau^{n,a}-} &= Z_t^{n,\tau^{n,a}} - \Delta Z_{\tau^{n,a}}^n \mathbf{1}_{\{t \geqq \tau^{n,a}\}} \\ &= Z_t^{n,\tau^{n,a}} + V_t^n, \end{aligned}$$

where $|V^n|_q \leqq 2 \sup_{t \leqq q} |Z_t^n|$, $q \in \mathbb{R}^+$. Thus using once more Corollary 7 and the definition of (UT), it is clear that $\{Z^{n\,\tau^{n,a}-}\}$ satisfies (UT), too. Therefore by (30), without loss of generality, we can assume $Z^n = Z^{n,\tau^{n,a}-}$, $H^n = H^{n,\tau^{n,a}-}$ and $\sup_t |Z_t^n| \leqq a$, $\sup_t |H_t^n| \leqq a$ for some constant $a \in \mathbb{R}^+$. Analogously as in (29) we decompose Z^n as a sum of J^n, M^n, and B^n. We define $\gamma^{n,b} = \inf\{t;\ |J^n|_t \vee [M^n]_t \vee |B^n|_t \leqq b\}$. By Proposition 2 (ii)

$$\lim_{b \to \infty} \limsup_{n \to \infty} \mathcal{P}(\gamma^{n,b} \leqq q) = 0, \qquad q \in \mathbb{R}^+,$$

and as before we can assume $H^n = H^{n,\gamma^{n,b}}$, $J^n = J^{n,\gamma^{n,b}}$, $M^n = M^{n,\gamma^{n,b}}$, $B^n = B^{n,\gamma^{n,b}}$ for some constant b. As a consequence we can assume that $Z^n = M^n + V^n$, $V^n = J^n + B^n$, where

$$[M^n]_\infty \leqq b + 4,$$
$$|V^n|_\infty^2 \leqq 2|J^n|_\infty^2 + 2|B^n|_\infty^2 \leqq 2(b^2 + 4a^2) + 2(b^2 + 1).$$

Finally, it is enough to use Theorem 2. Indeed. Setting $\sigma=q$ we have

$$|K^n|_{\sigma_j^n \wedge q} \leqq C_j^n \sup_{t \leqq \sigma_j^n \wedge q} |Y_t^n| \leqq C_j^n a, \qquad j, n \in \mathbb{N}. \tag{31}$$

By the definition of σ_j^n, C_j^n, we have

$$\lim_{j \to \infty} \limsup_{n \to \infty} \mathscr{P}(\sigma_j^n \leqq q) = 0$$

and

$$\limsup_{n \to \infty} C_j^n < +\infty$$

for every $j \in \mathbb{N}$, which together with (31) gives (i).

(ii) Immediately by (i) the sequence of processes $\{K^n\}$ satisfies (UT). Since by our assumption $\{Y_0^n\}$ is bounded in probability and $\{Z^n\}$ satisfies (UT), the sequence $\{X^n = Y_0^n + Z^n + K^n\}$ satisfies (UT), too. □

Assume for the moment that a sequence of processes $\{Y^n\}$ is of the form considered in (ii)) *i.e.* $\{Y^n\}$ satisfies (UT). By using the theory of convergence in Meyer and Zheng's sense ([17], [24]) we conclude by (ii) that there exist processes X, K, Y, a subsequence $\{n'\} \subset \{n\}$ and a set A of full Lebesgue measure such that the finite dimensional distributions of $(X_t^{n'}, K_t^{n'}, Y_t^{n'})_{t \in A}$ converge to those of $(X_t, K_t, Y_t)_{t \in A}$. Since the convergence in Meyer and Zheng's sense is given by the topology of the convergence in measure, Example 2 in Section 2 shows that (X, K) need not be a solution associated with Y. Therefore the notion of convergence in Meyer and Zheng's sense is not appropriate in stability theorems for solutions of the Skorokhod problem. This kind of convergence seems to be to weak.

On the other hand assume additionally that all the semimartingales $Y^n = Y_0^n + Z^n$ are adapted to the same filtration $(\mathscr{F}_t)$. If Y^n tends to Y in the space of semimartingales $\mathscr{H}^2$ (*see* e. g. [5]) then by Theorem 3 $E \sup_t |X_t^n - X_t|^2 \to 0$ and $E \sup_t |K_t^n - K_t|^2 \to 0$, where (X, K) is a solution corresponding to Y *i.e.* we have convergence in the space $\mathscr{S}^2$. To see that this kind of convergence can not be strenghten to convergence in $\mathscr{H}^2$ we consider the following example from [21].

Example 3. – Let $d=1$, $D=\mathbb{R}^+$ and let M be a continuous square integrable martingale. Then from the definition of local time we have

$$M_t^+ = \int_0^t \mathbf{1}_{\{M_s > 0\}} dM_s + \frac{1}{2} L_t^0(M),$$

$$(M_t - a_n)^+ = \int_0^t \mathbf{1}_{\{M_s > a_n\}} dM_s + \frac{1}{2} L_t^{a_n}(M),$$

where $a_n>0$, $a_n \downarrow 0$. Then it is clear that $\left((M_. - a_n)^+, \frac{1}{2} L^{a_n}_.(M)\right)$ is a solution of the Skorokhod problem associated with $\int_0^. \mathbf{1}_{\{M_s > a_n\}} dM_s$, and $\left((M_.^+, \frac{1}{2} L^0_.(M)\right)$ is a solution of the Skorokhod problem associated with $\int_0^. \mathbf{1}_{\{M_s>0\}} dM_s$, and

$$\left\| \int_0^. \mathbf{1}_{\{0<M_s \leqq a_n\}} dM_s \right\|_{\mathscr{H}^2} = \left(E \int_0^\infty \mathbf{1}_{\{0<M_s \leqq a_n\}} d[M]_s \right)^{1/2} \to 0.$$

On the other hand

$$| L^0_.(M) - L^{a_n}_.(M) |_q \to 2 L^0_q(M) \quad \text{P-a.e.}$$

and as a consequence the convergence $\| (M - a_n)^+ - M^+ \|_{\mathscr{H}^2} \to 0$ does not hold. □

What we can do, is to give convergence results in law and in probability in the Skorokhod topology, which is stronger than Meyer and Zheng's topology.

Proposition 4. — *Assume the conditions* (A) *and* (B). *Let* $\{H^n\}$, $\{Z^n\}$, $\{Y^n\}$ *be the sequences of processes,* $Y^n_0 \in \bar{D}$ *and let* $\{(X^n, K^n)\}$ *be a sequence of solutions of the Skorokhod problem associated with* $\{Y^n\}$. *If*

$$(H^n, Z^n, Y^n) \underset{\mathscr{D}}{\to} (H, Z, Y) \quad \textit{in} \ \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$$

[*resp.* $(H^n, Z^n, Y^n) \to_{\mathscr{P}} (H, Z, Y)$] *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$), *provided that* $|\Delta Y| < r_0$ *then*

$$(H^n, Z^n, X^n, K^n, Y^n) \underset{\mathscr{D}}{\to} (H, Z, X, K, Y) \quad \textit{in} \ \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{5d})$$

[*resp.* $(H^n, Z^n, X^n, K^n, Y^n) \to_{\mathscr{P}} (H, Z, X, K, Y)$], *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{5d})$, *where* (X, K) *is a solution of the Skorokhod problem associated with the process* Y.

Proof. — It easy follows from Theorem 1 and the Skorokhod representation theorem. □

By Corollaries 3 and 4 we have the next

Corollary 8. — *Assume the conditions* (A) *and* (B). *Let* (X, K) *be a solution of the Skorokhod problem corresponding to a process* Y, $Y_0 \in \bar{D}$, $|\Delta Y| < r_0$.

(i) *Let* $\{(X^n, K^n)\}$ *be a sequence of solutions of the Skorokhod problem associated with a sequence of processes* $\{Y^n\}$. *If* $\sup_{t \leqq q} |Y^n_t - Y_t| \to_{\mathscr{P}} 0$, $q \in \mathbb{R}^+$, *then*

$$\sup_{t\leqq q}|X_t^n - X_t| \underset{\mathscr{P}}{\to} 0 \qquad \textit{and} \qquad \sup_{t\leqq q}|K_t^n - K_t| \underset{\mathscr{P}}{\to} 0$$

for every $q\in\mathbb{R}^+$.

(ii) *Let for every* $n\in\mathbb{N}$, Y^n *be of the form* Y^{ρ_n}, *where* $\{\rho_n\}$ *is the sequence of summation rules defined in Section* 2. *Then*

$$\sup_{t\leqq q}|X_t^n - X_t^{\rho_n}| \underset{\mathscr{P}}{\to} 0 \qquad \textit{and} \qquad \sup_{t\leqq q}|K_t^n - K_t^{\rho_n}| \underset{\mathscr{P}}{\to} 0$$

for every $q\in\mathbb{R}^+$. □

4. SDE's WITH REFLECTION

Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a probability space and let $(\mathscr{F}_t)$ be a filtration on $(\Omega, \mathscr{F}, \mathscr{P})$ satisfying the usual conditions. Given a function $f:\bar{D}\to\mathbb{R}^d\otimes\mathbb{R}^d$, $f(x)=\{f_{ik}\}_{i,k=1,\dots,d}$ we consider the SDE (1) *i.e.*

$$X_t^i = H_t^i + \sum_{k=1}^d \int_0^t f_{ik}(X_{s-})\,dZ_s^i + K_t^i, \qquad i=1,\dots,d, \quad t\in\mathbb{R}^+, \tag{32}$$

where $H_t=(H_t^1,\dots,H_t^d)$ is an $(\mathscr{F}_t)$ adapted process and $Z_t=(Z_t^1,\dots,Z_t^d)$, $Z_0=0$ is an $(\mathscr{F}_t)$ adapted semimartingale. We will say that the SDE (1) has a strong solution if there exists a pair (X, K) of $(\mathscr{F}_t)$ adapted processes satisfying the conditions

$$X \text{ is } \bar{D}\text{-valued}, \tag{33}$$

K is a process with locally bounded variation such that $K_0=0$ and

$$K_t=\int_0^t \mathbf{n}_s\,d|K|_s, \qquad |K|_t=\int_0^t \mathbf{1}_{\{X_s\in\partial D\}}\,d|K|_s, \qquad t\in\mathbb{R}^+,$$

where $\mathbf{n}_s\in\mathscr{N}_{X_s}$ if $X_s\in\partial D$.

$$|\Delta X_t| \leqq |\Delta H_t + f(X_{t-})\,\Delta Z_t|, \qquad t\in\mathbb{R}^+. \tag{34}$$

Remark 3. — *Similarly to the deterministic case, if either* D *is convex or* D *satisfies* (A) *and the processes* H, Z *have continuous trajectories then* (32)-(34) *imply* (35).

If any two $(\mathscr{F}_t)$ adapted solutions (X, K), (X′, K′) on $(\Omega, \mathscr{F}. \mathscr{P})$ of the SDE (32) satisfy $\mathscr{P}[(X_t, K_t)=(X'_t, K'_t),\ t\in\mathbb{R}^+]=1$ then we will say that strong uniqueness holds for (32).

We will say that the SDE (1) has a weak solution if there exists a probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathscr{P}})$ with filtration $(\hat{\mathscr{F}}_t)$ satisfying the usual conditions and $(\hat{\mathscr{F}}_t)$ adapted processes $\hat{X}$, $\hat{K}$, $\hat{H}$, $\hat{Z}$ such that $\mathscr{L}(\hat{H}, \hat{Z})=\mathscr{L}(H, Z)$ and (32)-(35) hold for processes $\hat{X}$, $\hat{K}$, $\hat{H}$, $\hat{Z}$ instead of X, K. H, Z. If any two weak solutions $(\hat{X}, \hat{K})$ and $(\hat{X}', \hat{K}')$ of SDE (1), possibly defined on the different probability spaces, are such that $\mathscr{L}(\hat{X}, \hat{K}) = \mathscr{L}(\hat{X}', \hat{K}')$ we will say that the weak uniqueness for the SDE (1) holds.

Now, we are ready to formulate our main theorem.

THEOREM 4. – *Assume the conditions* (A) *and* (B). *Let* $\{Z^n\}$ *be a sequence of* $\mathscr{F}^n$ *adapted semimartingale satisfying* (UT) *and let* $\{H^n\}$ *be a sequence of* $\mathscr{F}^n$ *adapted processes. Let* $\{(X^n, K^n)\}$ *be a sequence of strong solutions of the SDE* (1). *If* f *is continuous,* $\|f(x)\| \leqq L < +\infty$, $x \in \bar{D}$ *and* $(H^n, Z^n) \to_{\mathscr{D}} (H, X)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$, *where* $|\Delta H| + L|\Delta Z| < r_0$ *then*

(i) $\{(X^n, K^n, H^n, Z^n)\}$ *is tight in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d})$ *and every limit point of* $\{(X^n, K^n)\}$ *is a weak solution of the SDE* (1),

(ii) *if additionally the SDE* (1) *has a unique weak solution* (X,K) *then*

$$(X^n, K^n) \underset{\mathscr{D}}{\to} (X, K) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

Proof. – (i) First let us note that joint weak convergence of (H^n, Z^n) to (H, Z) and the continuous mapping theorem imply

$$(H^n, Z^n, |H^n_{\cdot}| + L|Z^n_{\cdot}|) \underset{\mathscr{D}}{\to} (H, Z, |H_{\cdot}| + L|Z_{\cdot}|)$$

in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d+1})$. (36)

Let $\{c_k\}$ be a sequence of constants $c_k \uparrow r_0$, such that $\mathscr{P}[|\Delta H_t| + L|\Delta Z_t| = c_k,\ t \in \mathbb{R}^+] = 0$, $k \in \mathbb{N}$. Define

$$\sigma_k = \inf\{t > 0;\ |\Delta H_t| + L|\Delta Z_t| \geqq c_k\},$$
$$\sigma_k^n = \inf\{t > 0;\ |\Delta H_t^n| + L|\Delta Z_t^n| \geqq c_k\},$$
$$k, n \in \mathbb{N}.$$

By (36) and [8], Proposition 3.15, for every $k \in \mathbb{N}$,

$$(H^{n, \sigma_k^n -}, Z^{n, \sigma_k^n -}, \sigma_k^n) \underset{\mathscr{D}}{\to} (H^{\sigma_k -}, Z^{\sigma_k -}, \sigma_k),$$

in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}) \times \mathbb{R}$.

Since $|\Delta H| + L|\Delta Z| < r_0$, $\sigma_k \uparrow +\infty$ $\mathscr{P}$-a.e. and as a consequence

$$\lim_{k \to \infty} \limsup_{n \to \infty} \mathscr{P}[\sigma_k^n \leqq q] = 0.$$

Therefore without loss of generality we may assume that there exists a constant $c < r_0$ such that

$$|\Delta H^n| + L|\Delta Z^n| \leqq c, \quad n \in \mathbb{N}.$$

Due to [14], Lemma 1.6, a sequence of stochastic integrals $\left\{\int_0^{\cdot} f(X_{s-}^n)\, dZ_s^n\right\}$ satisfies (UT), hence by Proposition 3 and Corollary 7 the sequences of random variables $\{|K^n|_q\}$, $\{\sup_{t\leq q}|X_t^n|\}$ are tight in $\mathbb{R}$ for every $q\in\mathbb{R}^+$.

Now, assume that $\{b^i\}$, $\{b_k^i\}$ are exactly the same as in (22). Put

$$\gamma_0^{ni}=0, \qquad \gamma_{k+1}^{ni}=\min(\gamma_k^{ni}+b_k^i, \inf\{t>\gamma_k^{ni}; |\Delta H_t^n|>b^i\}),$$

and

$$\begin{aligned} H_t^{ni}&=H_{\gamma_k^{ni}}^n, \qquad \text{if} \quad t\in[\gamma_k^{ni}, \gamma_{k+1}^{ni}[,\\ &=\sum_{\gamma_k^{ni}\leq t}(H_{\gamma_{k+1}^{ni}}^n-H_{\gamma_k^{ni}}^n), \qquad t\in\mathbb{R}^+. \end{aligned}$$

Then, as it is observed in [23], for every $i\in\mathbb{N}$,

$$(H_{\gamma_0^{ni}}^n, \gamma_0^{ni}, H_{\gamma_1^{ni}}^n, \gamma_1^{ni}, \ldots)\underset{\mathcal{D}}{\rightarrow}(H_{\gamma_0^i}, \gamma_0^i, H_{\gamma_1^i}, \gamma_1^i, \ldots) \quad \text{in } \mathbb{R}^\infty.$$

As a consequence, for every $i\in\mathbb{N}$

$$H^{ni}\underset{\mathcal{D}}{\rightarrow}H^i \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d), \tag{37}$$

$$\left.\begin{aligned} &|H_q^{ni}|\underset{\mathcal{D}}{\rightarrow}|H_q^i| \quad \text{in } \mathbb{R},\\ &\text{provided that } \mathcal{P}[|\Delta H_q^i|=0]=1, \qquad q\in\mathbb{R}^+. \end{aligned}\right\} \tag{38}$$

On the other hand it is well known that for continuous f, $f: \mathbb{R}^d\rightarrow\mathbb{R}^d\otimes\mathbb{R}^d$, we can construct a sequence of functions $\{f_i\}$ such that for every $i\in\mathbb{N}$, $f_i\in\mathcal{C}^2$ and

$$\sup_{x\in K}\|f_i(x)-f(x)\|\rightarrow 0, \tag{39}$$

for all compact subsets K of $\mathbb{R}^d$. Define for n, $i\in\mathbb{N}$

$$X^{ni}=H^{ni}+\int_0^\cdot f(X_{s-}^n)\,dZ_s^n+K_s^n,$$

$$Y^{ni}=H^n+\int_0^\cdot f_i(X_{s-}^{ni})\,dZ_s^n,$$

$$Y^n=H^n+\int_0^\cdot f(X_{s-}^n)\,dZ_s^n.$$

By simple calculations based on (37) (*see* e.g. [23]),

$$\lim_{i\rightarrow\infty}\limsup_{n\rightarrow\infty}\mathcal{P}[\sup_{t\leq q}|H_t^{ni}-H_t^n|\geq\varepsilon]=0, \qquad \varepsilon>0, \quad q\in\mathbb{R}^+.$$

Therefore due to (39) and Proposition 3 (iii) we have

$$\lim_{i\rightarrow\infty}\limsup_{n\rightarrow\infty}\mathcal{P}[\sup_{t\leq q}|Y_t^{ni}-Y_t^n|\geq\varepsilon]=0, \qquad \varepsilon>0, \quad q\in\mathbb{R}^+. \tag{40}$$

On the other hand by (38) $\{|H^{ni}|_q\}$ is tight in $\mathbb{R}$. Hence $\{X^{ni}\}$ is a sum of three sequences of processes satisfying (UT). As a consequence $\{X^{ni}\}$

satisfies (UT), too. Moreover, since $f_i \in \mathscr{C}^2$, by [14], Lemma 1.7, for every $i \in \mathbb{N}$ we have

$$\{ f_i(X^{ni}) \} \quad \text{satisfies the condition (UT).} \tag{41}$$

Now, we will use the following tightness criterion [14], Proposition 3.3.

LEMMA 4. – *Let* $\{H^n\}$, $\{U^n\}$, $\{Z^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ *and let* $\{(H^n, Z^n)\}$ *be tight in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$. *Assume that the sequences of processes* $\{U^n\}$, $\{Z^n\}$ *satisfy the condition* (UT). *Then*

$$\left\{ \left(H^n, Z^n, \int_0^{\cdot} U^n_{s-}\, dZ^n_s \right) \right\} \quad \textit{is tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}). \quad \square$$

Setting $U^n = f_i(X^{ni})$ for every $i \in \mathbb{N}$ we obtain

$$\left\{ \left(H^n, Z^n, \int_0^{\cdot} f_i(X^{ni}_{s-}\, dZ^n_s \right) \right\} \quad \text{is tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$$

Hence by (40)

$$\{(H^n, Z^n, Y^n)\} \quad \text{is tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}).$$

Assume that there exists a subsequence $\{n'\} \subset \{n\}$ such that

$$(H^{n'}, Z^{n'}, Y^{n'}) \underset{\mathscr{D}}{\rightarrow} (\hat{H}, \hat{Z}, \hat{Y}) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}),$$

where $\mathscr{L}(\hat{H}, \hat{Z}) = \mathscr{L}(G, Z)$. Then by Proposition 4

$$(H^{n'}, Z^{n'}, X^{n'}, K^{n'}, Y^{n'}) \underset{\mathscr{D}}{\rightarrow} (\hat{H}, \hat{Z}, \hat{X}, \hat{K}, \hat{Y})$$

in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{5d})$, where $(\hat{X}, \hat{K})$ is the solution of the Skorokhod problem corresponding to $\hat{Y}$. Moreover, by using the limit theorem for stochastic integrals [9], Theorem 2.6

$$\left(H^{n'}, X^{n'}, K^{n'}, \int_0^{\cdot} f(X^{n'}_{s-})\, dZ^{n'}_s \right) \underset{\mathscr{D}}{\rightarrow} \left(\hat{H}, \hat{X}, \hat{K}, \int_0^{\cdot} f(\hat{X}_{s-})\, d\hat{Z}_s \right)$$

in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d})$.

Finally, by the continuous mapping theorem we conclude that $(\hat{X}, \hat{K})$ is a solution of the SDE (1).

(ii) It follows immediately by (i). $\square$

COROLLARY 9. – *Assume the conditions* (A) *and* (B). *Let f be continuous and bounded,* $\|f(x)\| \leqq L < +\infty$, $x \in \bar{D}$. *If* $|\Delta H| + L|\Delta Z| < r_0$ *and* $H_0 \in \bar{D}$ *then there exists a weak solution of the SDE* (1).

Proof. – Let $\{\rho_n\}$ be a sequence of summation rules defined in Section 2. By simple calculations we can prove that for sufficiently large n there exists a unique strong solution of the following discrete stochastic differential equation

$$X_t^n = H_t^{\rho_n} + \int_0^t f(X_{s-}^n)\, dZ_s^{\rho_n} + K_t^n. \tag{42}$$

Moreover, the special form of H^{ρ_n} and Z^{ρ_n} implies that

$$X_t^n = \begin{cases} H_0 \quad \text{if} \quad t \in [0, t_{n1}[, \\ [X_{t_{n,k-1}}^n + H_{t_{nk}} - H_{t_{n,k-1}} + f(X_{t_{n,k-1}}^n)(Z_{t_{nk}} - Z_{t_{n,k-1}})]_\partial \\ \text{if} \\ \quad t \in [t_{nk}, t_{n,k+1}[, \quad k \in \mathbb{N}, \end{cases} \tag{43}$$

and

$$K_t^n = \begin{cases} 0 \quad \text{if} \quad t \in [0, t_{n1}[, \\ K_{t_{n,k-1}}^n + X_t^n - X_{t_{n,k-1}}^n - \{H_{t_{nk}} - H_{t_{n,k-1}} + f(X_{t_{n,k-1}}^n)(Z_{t_{nk}} - Z_{t_{n,k-1}})\} \\ \text{if} \\ \quad t \in [t_{nk}, t_{n,k+1}[, \quad k \in \mathbb{N}. \end{cases} \tag{44}$$

By (18)

$$(H^{\rho_n}, Z^{\rho_n}) \to (H, Z) \ \mathscr{P}\text{-a.e.} \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

Since in view of Dellacherie and Mokobodzki's theorem (*see* e.g. [5], p. 401) the sequence $\{Z^{\rho_n}\}$ of discrete $(\mathscr{F}_t^{\rho_n})$ adapted semimartingales, where $\mathscr{F}_t^{\rho_n} = \mathscr{F}_{\rho_n(t)}$, $t \in \mathbb{R}^+$, $n \in \mathbb{N}$ satisfies the condition (UT), the conclusion easy follows by Theorem 4 (i). □

THEOREM 5. — *Assume the conditions* (A) *and* (B). *Let f be Lipschitz continuous and bounded* i.e. *there exists a constant* $L > 0$ *such that* $\|f(x) - f(y)\| \leqq L|x - y|$, $\|f(x)\| \leqq L$ *for all* $x, y \in \bar{D}$. *If* $|\Delta H| + L|\Delta Z| < r_0$, $H_0 \in \bar{D}$ *then there exsits a unique strong solution of the* SDE (1).

Proof. — By using Corollary 9 there exists a probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathscr{P}})$, a filtration $(\hat{\mathscr{F}}_t)$ and $(\hat{\mathscr{F}}_t)$ adapted processes $\hat{H}$, $\hat{Z}$, $\hat{X}$ defined on $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathscr{P}})$ such that $\mathscr{L}(\hat{H}, \hat{Z}) = \mathscr{L}(H, Z)$ and (32)-(35) hold. Assuming the Lipschitz continuity of f we will show a little more, namely that there exists a measurable map F (depending only on f), $F : \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}) \to \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ such that

$$(\hat{X}, \hat{K}) = F(\hat{H}, \hat{Z}), \tag{45}$$

which shows pathwise uniqueness of the SDE (1) on $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathscr{P}})$. Moreover, if we come back to our basic probability space $(\Omega, \mathscr{F}, \mathscr{P})$ with filtration $(\mathscr{F}_t)$ and with $(\mathscr{F}_t)$ adapted processes H, Z then defining $(X, K) = F(H, Z)$ we obtain a strong solution on the space $(\Omega, \mathscr{F}, \mathscr{P})$. This means also, that the solution of the SDE (1) on $(\Omega, \mathscr{F}, \mathscr{P})$ is unique in the strong sense. Without loss of generality we will assume that the solution (X, K) of the SDE (1) is defined on the basic probability space $(\Omega, \mathscr{F}, \mathscr{P})$ and satisfies (32)-(35). Let $\{(X^n, K^n)\}$ be a sequence of solutions of the discrete SDE (42) defined by (43) and (44). We will show that

$$(X^n, K^n) \underset{\mathscr{P}}{\rightarrow} (X, K) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}), \tag{46}$$

which leads to (45) for (X, K) and (H, Z) instead of $(\hat{X}, \hat{K})$ and $(\hat{H}, \hat{Z})$.

The proof of (46) is long, so it will be devided into two steps. Before giving them define $Y = \int_0^{\cdot} f(X_{s-})\, dZ_s$. Let $\{(\bar{X}^n, \bar{K}^n)\}$ be a sequence of solutions of the Skorokhod problem associated with the sequence $\{Y^{\rho_n}\}$ [we recall that for sufficiently large n $(\bar{X}^n, \bar{K}^n)$ is defined by (19) and (20)]. By using Corollary 7 (ii) for every $q \in \mathbb{R}^+$,

$$\sup_{t \leq q} |\bar{X}_t^n - X_t^{\rho_n}| \rightarrow 0, \quad \mathscr{P}\text{-a. e.}, \tag{47}$$

and

$$\sup_{t \leq q} |\bar{K}_t^n - K_t^{\rho_n}| \rightarrow 0, \quad \mathscr{P}\text{-a. e.} \tag{48}$$

Step 1. – *We will assume additionally that* $\sup_t |H_t|$, $\sup_t |Z_t|$ *are bounded by constant* $\dfrac{r_0}{8(L+1)} \wedge 1$.

By our assumption Z is a special semimartingale. Hence Z is uniquely decomposed into the sum

$$Z = M + V,$$

where M is a locally square integrable martingale, $M_0 = 0$, and V is a process with bounded variation, $V_0 = 0$. Define

$$\tau^a = \inf\{t > 0;\ |M_t| \vee |V|_t \geq a\}.$$

Obviously $\tau^a \uparrow +\infty$ and we can assume that the processes H, M, V are stopped in τ^a for some fixed $a \in \mathbb{R}^+$.

For $n, i \in \mathbb{N}$ let us denote

$$\tau_n^i = \inf\left\{t > 0;\ \left|\int_0^t f(X_{s-}^n)\, dZ_s^{\rho_n}\right| \vee \left|\int_0^{\rho_n(t)} f(X_{s-})\, dZ_s\right| \vee [M^{\rho_n}]_t \vee \langle M^{\rho_n} \rangle_t \vee |V^{\rho_n}|_t \geq i\right\}.$$

Then it is clear that for every $q \in \mathbb{R}^+$,

$$\lim_{i \to \infty} \limsup_{n \to \infty} \mathscr{P}[\tau_n^i \leq q] = 0, \tag{49}$$

and the processes $\int_0^{\cdot} f(X^n_{s-})\,dZ^{\rho_n}_s$, $\int_0^{\rho_n(\cdot)} f(X_{s-})\,dZ_s$, $[M^{\rho_n}]$, $\langle M^{\rho_n}\rangle$, $|V^{\rho_n}|$. stopped in τ^i_n are uniformly in n bounded by some constant K_1. In view, of (49) we can restrict our attention to the processes stopped in τ^i_n. Then by Theorem 2 for fixed $q\in\mathbb{R}^+$ there exists a sequence of $(\mathcal{F}^{\rho_n}_t)$ stopping times $\{\sigma^j_n\}$ and a sequence of constants $\{C^j_n\}$ such that

$$\lim_{j\to\infty}\ \limsup_{n\to\infty} \mathcal{P}[\sigma^j_n \leqq q]=0, \tag{50}$$

$$\lim_{j\to\infty}\ \limsup_{n\to\infty} C^j_n < +\infty, \tag{51}$$

and $|K^n|_{q\wedge\sigma^j_n}$, $|\bar{K}^n|_{q\wedge\sigma^j_n} \leqq C^j_n\left(K_1+\frac{r_0}{8(L+1)}\right)$. In the sequel due to (50) and (51), we can and will assume that every $(\mathcal{F}^{\rho_n}_t)$ adapted process is stopped in $q\wedge\sigma^j_n\wedge\tau^i_n$ and $|K^n|_{q\wedge\sigma^j_n\wedge\tau^i_n}$, $|\bar{K}^n|_{q\wedge\sigma^j_n\wedge\tau^i_n}\leqq K_2$.

Now, suppose that γ_n is an $(\mathcal{F}^{\rho_n}_t)$ stopping time. Then

$$\begin{aligned} E\sup_{t<\gamma_n}|\bar{X}^n_t - X^n_t|^2 &\leqq 3\,E\sup_{t<\gamma_n}\left|\int_0^t f(\bar{X}^n_{s-})-f(X^n_{s-})\,dZ^{\rho_n}_s\right|^2 \\ &\quad + 3\,E\sup_{t<\gamma_n}|\bar{K}^n_t - K^n_t|^2 + 3\,\varepsilon_n = 3\,I^n_1 + 3\,I^n_2 + 3\,\varepsilon_n, \end{aligned}$$

where $\varepsilon_n = E\sup_{t<\gamma_n}\left|\int_0^{\rho_n(t)} f(X_{s-})\,dZ_s - \int_0^t f(\bar{X}^n_{s-})\,dZ^{\rho_n}_s\right|^2$. We recall that by the Doob type inequality proved in [16]

$$\begin{aligned} I^n_1 &\leqq 8\,E\int_0^{\gamma_n-}\|f(\bar{X}^n_{s-})-f(X^n_{s-})\|^2\,d([M^{\rho_n}]_s+\langle M^{\rho_n}\rangle_s) \\ &\quad + 2\,E\,|V^{\rho_n}|_{\gamma_n-}\int_0^{\gamma_n-}\|f(\bar{X}^n_{s-})-f(X^n_{s-})\|^2\,d|V^{\rho_n}|_s \\ &\leqq (8\,L^2+2\,L^2\,K_1) \\ &\quad \times E\int_0^{\gamma_n-}\sup_{u\leqq s}|\bar{X}^n_{u-}-X^n_{u-}|^2\,d([M^{\rho_n}]_s+\langle M^{\rho_n}\rangle_s+\langle V^{\rho_n}|_s) \end{aligned}$$

On the other hand by Corollary 6

$$\begin{aligned} I^n_2 &\leqq 2\,CE\int_0^{\gamma_n-}\|f(\bar{X}^n_{s-})-f(X^n_{s-})\|^2\,d([M^{\rho_n}]_s+\langle M^{\rho_n}\rangle_s) \\ &\quad + 2\,CE\,|V^{\rho_n}|_{\gamma_n-}\int_0^{\gamma_n-}\|f(\bar{X}^n_{s-})-f(X^n_{s-})\|^2\,d|V^{\rho_n}|_s \end{aligned}$$

$$+2\,\mathrm{C}\mathrm{E}\left[\int_0^{\rho_n(\cdot)} f(\mathrm{X}_{s-})\,d\mathrm{M}_s-\int_0^{\cdot} f(\bar{\mathrm{X}}^n_{s-})\,d\mathrm{M}^{\rho_n}_s\right]_{\gamma_n-}$$

$$+2\,\mathrm{C}\mathrm{E}\left|\int_0^{\rho_n(\cdot)} f(\mathrm{X}_{s-})\,d\mathrm{V}_s-\int_0^{\cdot} f(\bar{\mathrm{X}}^n_{s-})\,d\mathrm{V}^{\rho_n}_s\right|^2_{\gamma_n-}$$

$$\leqq (2\,\mathrm{CL}^2+2\,\mathrm{CL}^2\,\mathrm{K}_1)$$

$$\times \mathrm{E}\int_0^{\gamma_n-}\sup_{u\leqq s}|\bar{\mathrm{X}}^n_{u-}-\mathrm{X}^n_{u-}|^2\,d([\mathrm{M}^{\rho_n}]_s+\langle \mathrm{M}^{\rho_n}\rangle_s+|\mathrm{V}^{\rho_n}|_s)+2\,\mathrm{C}\,(\varepsilon^1_n+\varepsilon^2_n).$$

Therefore for every $(\mathscr{F}^{\rho_n}_t)$ stopping time γ_n

$$\mathrm{E}\sup_{t<\gamma_n}|\bar{\mathrm{X}}^n_t-\mathrm{X}^n_t|^2\leqq (8\,\mathrm{L}^2+2\,\mathrm{L}^2\,\mathrm{K}_1)(2\,\mathrm{C}+1)$$

$$\times \mathrm{E}\int_0^{\gamma_n-}\sup_{u\leqq s}|\bar{\mathrm{X}}^n_{u-}-\mathrm{X}^n_{u-}|^2\,d([\mathrm{M}^{\rho_n}]_s+\langle \mathrm{M}^{\rho_n}\rangle_s+|\mathrm{V}^{\rho_n}|_s)$$

$$+3\,\varepsilon_n+2\,\mathrm{C}\,\varepsilon^1_n+2\,\mathrm{C}\,\varepsilon^2_n.$$

If we denote $\varepsilon(n)=3\,\varepsilon_n+2\,\mathrm{C}\,\varepsilon^1_n+2\,\mathrm{C}\,\varepsilon^2_n$ then due to Lemma 3 (ii)

$$\mathrm{E}\sup_{t<q\,\wedge\,\sigma^j_n\,\wedge\,\tau^i_n}|\bar{\mathrm{X}}^n_t-\mathrm{X}^n_t|^2\leqq\varepsilon(n)\exp\{(8\,\mathrm{L}^2+2\,\mathrm{L}^2\,\mathrm{K}_2)(2\,\mathrm{C}+1)\,3\,\mathrm{K}_1\}.$$

Finally, let us observe that by simple calculations based on (47) and by the arguments from the proof of [5], Theorem 15, VIII, $\varepsilon(n)\to 0$. Hence

$$\mathrm{E}\sup_{t<q\,\wedge\,\sigma^j_n\,\wedge\,\tau^i_n}|\bar{\mathrm{X}}^n_t-\mathrm{X}^n_t|\to 0,\qquad j,\,i\in\mathbb{N},\quad q\in\mathbb{R}^+.$$

Since

$$\mathrm{E}\sup_{t<q\,\wedge\,\sigma^j_n\,\wedge\,\tau^i_n}|\bar{\mathrm{K}}^n_t-\mathrm{K}^{\rho_n}_t|^2\leqq (2\,\mathrm{CL}^2+2\,\mathrm{CL}^2\,\mathrm{K}_1)$$

$$\times \mathrm{E}\int_0^{q\,\wedge\,\sigma^j_t\,\wedge\,\tau^i_n}\sup_{u\leqq s}|\bar{\mathrm{X}}^n_{u-}-\mathrm{X}^n_{u-}|^2\,d([\mathrm{M}^{\rho_n}]_s+,\,\mathrm{M}^{\rho_n}\rangle_s+|\mathrm{V}^{\rho_n}|_s)$$

$$+2\,\mathrm{C}\,(\varepsilon^1_n+\varepsilon^2_n),$$

it follows by (47)-(50) that

$$\sup_{t\leqq q}|\mathrm{X}^n_t-\mathrm{X}^{\rho_n}_t|\underset{\mathscr{P}}{\to},\qquad q\in\mathbb{R}^+,$$

and

$$\sup_{t\leqq q}|\mathrm{K}^n_t-\mathrm{K}^{\rho_n}_t|\underset{\mathscr{P}}{\to},\qquad q\in\mathbb{R}^+,$$

which gives (46).

Step 2. — *We will show, how to omit the assumption* $\sup_t |H_t|$, $\sup_t |Z_t| < \frac{r_0}{8(L+1)} \wedge 1$.

Define

$$\sigma_0 = 0,$$
$$\sigma_{k+1} = \inf\left\{ t > \sigma_k;\ |H_t - H_{\sigma_k} + X_{\sigma_k}| \vee |Z_t - Z_{\sigma_k}| \geqq \frac{r_0}{8(L+1)} \wedge 1 \right\},$$
$$k \in \mathbb{N}.$$

By the arguments used in Step 1

$$\sup_{t \leqq q} |X^{n, \rho_n^*(\sigma_1)} - X^{\sigma_1 -}_{\rho_n(t)}| \underset{\mathscr{P}}{\rightarrow} 0, \qquad q \in \mathbb{R}^+, \tag{52}$$

and as a consequence

$$X^{n}_{\rho_n^*(\sigma_1)-} \underset{\mathscr{P}}{\rightarrow} X_{\sigma_1 -}, \tag{53}$$

where $\rho_n^*(t) = \inf\{t_{nk};\ t_{nk} \geqq t\}$ [observe that contrary to $\rho_n(\sigma_1)$, $\rho_n^*(\sigma_1)$ is an $(\mathscr{F}_t^{\rho_n})$ stopping time!]. By simple calculations

$$X_t^{n, \rho_n^*(\sigma_1)} - X^{\sigma_1}_{\rho_n(t)} = X_t^{n, \rho_n^*(\sigma_1 -)} - X^{\sigma_1 -}_{\rho_n(t)} + (\Delta X^{n}_{\rho_n^*(\sigma_1)} - \Delta X_{\sigma_1})\, 1_{\{t \geqq \rho_n^*(\sigma_1)\}}. \tag{54}$$

On the other hand

$$\Delta X^n_{\rho_n^*(\sigma_1)} = [X^n_{\rho_n^*(\sigma_1)-} + \Delta X^{\rho_n}_{\rho_n^*(\sigma_1)} + f(X^n_{\rho_n^*(\sigma_1)-})\, \Delta Z^{\rho_n}_{\rho_n^*(\sigma_1)}]_\partial - X^n_{\rho_n^*(\sigma_1)-}$$

and

$$\Delta X_{\sigma_1} = [X_{\sigma_1 -} + \Delta H_{\sigma_1 -} + f(X_{\sigma_1 -})\, \Delta Z_{\sigma_1}]_\partial - X_{\sigma_1 -}.$$

Therefore by (53) and by the convergence

$$\Delta H^{\rho_n}_{\rho_n^*(\sigma_1)} \to \Delta H_{\sigma_1},\ \Delta Z^{\rho_n}_{\rho_n^*(\sigma_1)} \to \Delta Z_{\sigma_1},\ \mathscr{P}\text{-a. e.}$$

we get $\Delta X^{\rho_n}_{\rho_n^*(\sigma_1)} \to \Delta X_{\sigma_1}$ $\mathscr{P}$-a. e. Due to (54)

$$\sup_{t \leqq q} |X_t^{n, \rho_n^*(\sigma)-} - X^{\sigma_1}_{\rho_n(t)}| \underset{\mathscr{P}}{\rightarrow} 0, \qquad q \in \mathbb{R}^+.$$

By replacing the processes H, Z by $H_{\sigma_1 + \cdot} - H_{\sigma_1} + X_{\sigma_1}$ and $Z_{\sigma_1 + \cdot} - Z_{\sigma_1}$, respectively, we can obtain the convergence X^n to X on the interval $[\sigma_1, \sigma_2[$, etc. Since $\sigma_k \uparrow +\infty$ the proof is complete. □

Immediately from the proof of Theorem 5 we obtain.

COROLLARY 10. — *Assume the conditions* (A) *and* (B). *Let* $|\Delta H| + L|\Delta Z| < r_0$ *and let* (X, K) *be a unique strong solution of the SDE* (1), *where* f *is Lipschitz continuous and bounded* i.e. $\|f(x)\| \leqq L$,

$\| f(x)-f(y) \| \leqq L|x-y|$ *for* $x, y \in \bar{D}$. *If* $\{(X^n, K^n)\}$ *is a sequence of solutions of the discrete* SDE (42) *defined by* (43) *and* (44) *then*

(i) $\sup_{t \leqq q} |X_t^n - X_t^{\rho_n}| \to_{\mathscr{P}} 0$ *and* $\sup_{t \leqq q} |K_t^n - K_t^{\rho_n}| \to_{\mathscr{P}} 0$, $q \in \mathbb{R}^+$,

(ii) $(X^n, K^n, H^{\rho_n}, Z^{\rho_n}) \to_{\mathscr{P}} (X, K, H, Z)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d})$. □

COROLLARY 11. — *Assume the conditions* (A) *and* (B). *Let* $\{Z^n\}$ *be a sequence of* $(\mathscr{F}_t^n)$ *adapted semimartingales, satisfying the condition* (UT) *and let* $\{H^n\}$ *be a sequence of* $(\mathscr{F}_t^n)$ *adapted processes. Let* $\{(X^n, K^n)\}$ *be a sequence of strong solutions of the SDE* (1), *where f is Lipschitz continuous and bounded* i. e. $\|f(x)\| \leqq L$, $\| f(x)-f(y) \| \leqq L|x-y|$ *for* $x, y \in \bar{D}$. *Then*

(i) *If* $(H^n, Z^n) \to_{\mathscr{D}} (H, Z)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ *and* $|\Delta H| + L|\Delta Z| < r_0$ *then*

$$(X^n, K^n, H^n, Z^n) \underset{\mathscr{D}}{\to} (X, K, H, Z) \quad \textit{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}),$$

(ii) *if* $(H^n, Z^n) \to^{\mathscr{P}} (H, Z)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ *and* $|\Delta H| + L|\Delta Z| < r_0$ *then*

$$(X^n, K^n, H^n, Z^n) \underset{\mathscr{P}}{\to} (X, K, H, Z) \quad \textit{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}),$$

(iii) *if* $\sup_{t \leqq q} |H_t^n - H_t| \to_{\mathscr{P}} 0$, $\sup_{t \leqq q} |Z_t^n - Z_t| \to_{\mathscr{P}} 0$, $q \in \mathbb{R}^+$ *and*

$$|\Delta H| + L|\Delta Z| < r_0$$

then

$$\sup_{t \leqq q} |X_t^n - X_t| \underset{\mathscr{P}}{\to} 0 \quad \textit{and} \quad \sup_{t \leqq q} |K_t^n - K_t| \underset{\mathscr{P}}{\to} 0,$$
$$q \in \mathbb{R}^+,$$

where (X, K) *is a unique strong solution of the SDE* (1).

Proof. — (i) Due to Corollary 10 (ii) the SDE (1) has a unique weak solution. Thus by Theorem 4 (i)

$$(X^n, K^n) \underset{\mathscr{D}}{\to} (X, K), \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

Moreover, in this case the joint law of (X, K, H, Z) is uniquely determined. Therefore we can strengthen the conclusion of Theorem 4 (ii) to the convergence

$$(X^n, K^n, H^n, Z^n) \underset{\mathscr{D}}{\to} (X, K, H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}).$$

(ii) We use the method of the proof of [23], Theorem 1. Since we assume the convergence in probability, all processes are defined on the same probability space $(\Omega, \mathscr{F}, \mathscr{P})$. Let $B \in \mathscr{F}$, $\mathscr{P}(B) > 0$. Define $Q_B(A) = \mathscr{P}(A|B)$ for every $A \in \mathscr{F}$. By [23], Lemma 4 $\{Z^n\}$ is a sequence of semimartingales on $(\Omega, \mathscr{F}, Q_B)$, for which the condition (UT) is satisfied.

Then

$$(H^n, Z^n) \underset{\mathscr{D}(Q_B)}{\rightarrow} (H, Z), \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}),$$

and

$$X_t^n = H_t^n + \int_0^t f(X_{s-}^n)\, dZ_s^n + K_t^n, \qquad t \in \mathbb{R}^+, \quad Q_B\text{-a. e.}$$

As a consequence, by part (i),

$$(X^n, K^n, H^n, Z^n) \underset{\mathscr{D}(Q_B)}{\rightarrow} (X, K, H, Z), \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}),$$

Hence for all bounded and continuous mappings Φ, $\Phi: \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}) \rightarrow \mathbb{R}$,

$$\lim_{n \to \infty} \int_\Omega \Phi(X^n, K^n, H^n, Z^n)\, dQ_B = \int_\Omega \Phi(X, K, H, Z)\, dQ_B. \tag{55}$$

Since (55) holds for all $B \in \mathscr{F}$, $\mathscr{P}(B) > 0$ and all bounded continuous mappings Φ: $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}) \rightarrow \mathbb{R}$ we have

$$(X^n, K^n, H^n, Z^n) \underset{\mathscr{P}}{\rightarrow} (X, K, H, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}). \tag{56}$$

(iii) It is clear that by the part (ii) the condition (55) is satisfied. Now, let us note that

$$\Delta X_t = \Delta H_t + f(X_{t-})\, \Delta Z_t + \Delta K_t, \tag{57}$$

and if $\Delta K_t \neq 0$ then $\Delta H_t \neq 0$ or $\Delta Z \neq 0$. Using [23], Corollary C, we obtain that

$$\sup_{t \leq q} |K_t^n - K_t| \underset{\mathscr{P}}{\rightarrow} 0, \qquad q \in \mathbb{R}^+.$$

Finally, let us observe that by (56) we have

$$(X^n, K^n, |H_\cdot^n| + L|Z_\cdot^n|) \underset{\mathscr{P}}{\rightarrow} (X, K, |H_\cdot| + L|Z_\cdot|),$$

in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d+1})$.

Since it is clear that if $\Delta X_t \neq 0$ then $|\Delta H_t| + L|\Delta Z_t| \neq 0$ or $\Delta K_t \neq 0$, by using once more [23], Corollary C, we get

$$\sup_{t \leq q} |X_t^n - X_t| \underset{\mathscr{P}}{\rightarrow} 0, \qquad q \in \mathbb{R}^+,$$

and the proof is complete. □

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