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Some results on the influence of extremes on the bootstrap

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ABSTRACT. – We study the influence of the extremes in the construction of consistent bootstraps in three illustrative situations. These are bootstrapping maxima. bootstrapping intermediate trimmed means and bootstrapping means. In the process, we shed new light upon the problem of bootstrapping the mean and obtain refinements and improvements of previous results. We also expose some interesting asymptotic distributional curiosities in connection with this problem. Our approach throughout is novel, in the sense that we work purely within the quantile function uniform empirical process methodology.

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RÉSUMÉ. — Nous étudions l'influence des valeurs extrêmes sur les propriétés de convergence du bootstrap, dans trois exemples qui sont propres à mettre en évidence les phénomèmes associés. Il s'agit du bootstrap du maximum, ainsi que des bootstraps de sommes tronquées intermédiaires et de la moyenne d'un échantillon. Nous arrivons au passage à éclairer de manière nouvelle les questions liées au comportement asymptotique du bootstrap de la moyenne, et obtenons ainsi des améliorations et raffinements de résultats antérieurs portant sur le sujet. Nous exposons également des propriétés inattendues liées au comportement en loi de ces statistiques. Notre approche du problème est entièrement nouvelle au sens que nous raisonnons intégralement dans le cadre méthodologique des processus empiriques uniformes et de la transformation de quantile.

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1. INTRODUCTION

We study how the extremes influence the construction of consistent bootstratps in three representative situations. Besides that of bootstrapping the extremes themselves, these are bootstrapping intermediate trimmed means and bootstrapping the mean.

In Section 2, we show how to consistently bootstrap the maximum $X_{n,n}$ of an i. i. d. sample of size n. Bickel and Freedman (1981) pointed out that the natural approach to bootstrapping $X_{n,n}$ fails. This means that one cannot choose a boostrap sample size equal to n. Swanepoel (1986) proved that the situation can be salvaged by choosing a bootstrap sample of size $m=m(n)$ such that $m(n)$ converges to infinity and $m(n)/n$ goes to zero at a particular rate. This was in the same spirit as Brétagnolle (1983), who was the first to introduce such bootstrap sampling rates. He found them necessary to obtain consistent bootstrap for certain U-statistics. We refine Swanepoel's result for bootstrapping the maximum by deriving a natural range of rates for $m(n)$, which not suprizingly are in agreement with those obtained by Athreya (1985) and by Arcones and Giné (1989), respectively, for the in probability and almost sure consistency of the bootstrapped mean in the infinite variance case.

Next, in Section 3, we investigate the problem of bootstrapping intermediate trimmed means. Such trimmed means are formed when the $k(n)$ smallest and $m(n)$ largest observations are removed from the average of a sample size of size n, where it is assumed that both $k(n)$ and $m(n)$ converge to infinity at a rate such that both $k(n)/n$ and $m(n)/n$ go to zero. The consistency of the bootstrap for such trimmed means with bootstrap sample size n is obtained under conditions that are in the same format as the necessary and sufficient conditions for their asymptotic normality given by S. Csörgő, Haeusler and Mason (1988 *a*). The quantile inequalities of Shorack (1991 *a*, 1991 *b*) play a crucial role in our proofs.

Finally, in Section 4, we turn to the problem of bootstrapping the mean when the underlying distribution is in the domain of attraction of a stable law. We provide a new derivation and interpretation of the results of Athreya (1985) and Arcones and Giné (1989) on this problem. In the process, we introduce natural centering and norming constants for the bootstrapped mean, which are functions of the data only. Surprizingly, it turns out that these are empirical versions of the normalizing constants used in Section 3 for the asymptotic normality of intermediate trimmed means. When combined with the results in Section 3, this leads to a curious asymptotic distributional comparison between bootstrapped means and intermediate trimmed means.

In each of the three situations we consider, the large sample behavior of the extremes play a fundamental role in determining how the statistic in question can be consistently bootstrapped.

2. BOOTSTRAPPING EXTREMES

Let $X_1, X_2, \ldots,$ be a sequence of independent Uniform $(0, \theta)$, $\theta>0$, random variables. It is trivial to show that if

$$X_{n,n}=\max\{X_1, \ldots, X_n\},$$

then

$$n(\theta-X_{n,n})/\theta \underset{d}{\rightarrow} Y, \quad \text{as } n\rightarrow\infty, \tag{2.1}$$

where here and elsewhere Y denotes an exponential random variable with mean 1.

It was pointed out by Bickel and Freedman (1981) that the distribution of $X_{n,n}$ cannot be bootstrapped in the following sense. Let $X^*_{n,n}$ denote the maximum of $X^*_1, \ldots, X^*_n$ sampled from $\mathbb{F}_n$, the empirical distribution of $X_1, \ldots, X_n$ (we define $\mathbb{F}_n(x)=n^{-1}\#\{i:X_i\leqq x, \ 1\leqq i\leqq n\}$ for $-\infty<x<\infty$). Then, as $n\rightarrow\infty$,

$$P\{n(X_{n,n}-X^*_{n,n})/X_{n,n}>x \mid \mathbb{F}_n\}$$

does not converge to $\exp(-x)$ for all $x \geqq 0$, either in probability or almost surely. The reason is that, with probability one,

$$P\{X_{n,n} - X^*_{n,n} > 0 \mid \mathbb{F}_n\} = \left(1 - \frac{1}{n}\right)^n,$$

which converges to $\exp(-1) \neq 1$.

Swanepoel (1986) showed that a modification of the bootstrap does work. Suppose now that $X_1^*, \ldots, X_n^*$ are sampled from $\mathbb{F}_n$, where $m(n)$ is a positive integer function of n. He proved that if $m(n) \to \infty$ at a rate so that, for some $0 < \varepsilon < 1$,

$$m(n) = o(n^{(\varepsilon+1)/2}/(\log n)^{1/2}),$$

then with probability one for all $x \geqq 0$

$$(2.2) \quad P\{m(n)(X_{n,n} - X^*_{m(n),m(n)})/X_{n,n} > x \mid \mathbb{F}_n\} \to \exp(-x) \quad \text{as } n \to \infty.$$

Here is a refined version of his result.

THEOREM 2.1. – *Assume that* $m(n) \to \infty$ *and* $m(n)/n \to 0$ *as* $n \to \infty$. *Then* (2.2) *holds in probability. In addition, if*

$$(2.3) \qquad (m(n) \log\log n)/n \to 0 \quad \text{as } n \to \infty,$$

then (2.2) *holds almost surely.*

Proof. – Let $U_1, U_2, \ldots$, be a sequence of independent Uniform (0,1) random variables and for each integer $n \geqq 1$, let $U_{1,n} \leqq \ldots \leqq U_{n,n}$ denote the order statistics of the first n of these random variables. Obviously, we can write $X_i = \theta U_i$, $i = 1, 2, \ldots$, and $X_{n,n} = \theta U_{n,n}$.

We shall first assume that (2.3) holds and show that for all $x > 0$ as $n \to \infty$

$$(2.4) \qquad P\{m(n)(X_{n,n} - X^*_{m(n),m(n)})/\theta > x \mid \mathbb{F}_n\} \to \exp(-x) \quad \text{a.s.}$$

It is known, *cf.* Kiefer (1972), that

$$(2.5) \qquad 1 - U_{n,n} = O((\log\log n)/n) \quad \text{a.s.}$$

Hence, almost surely as $n \to \infty$, for any $x > 0$,

$$\begin{aligned} X_{n,n} - \theta x/m(n) &= O((\log\log n)/n) + \theta - \theta x/m(n) \\ &= \theta - (\theta x/m(n))\{1 + O((m(n) \log\log n)/n)\}, \end{aligned}$$

which on account of (2.3) gives

$$(2.6) \qquad X_{n,n} - \theta x/m(n) = \theta - (\theta x/m(n))\{1 + o(1)\} \quad \text{a.s.}$$

From Theorem 2 on page 604 of Shorack and Wellner (1986) it can be readily inferred that whenever (2.3) holds, almost surely for all $0 < c_1 < c_2 < \infty$

$$\begin{aligned} (2.7) \quad \sup\{|\mathbb{F}_n(\theta - \theta t) - (1-t)| : 0 < c_1/m(n) \leqq t \leqq c_2/m(n)\} \\ = O(\{(\log\log n)/(nm(n))\}^{1/2}) = o(1/m(n)), \end{aligned}$$

which in combination with (2.6) yields

$$\mathbb{F}_n(X_{n,n}-\theta x/m(n))-(X_{n,n}/\theta - x/m(n))=o(1/m(n)) \quad \text{a.s.}$$

Thus

$$\mathbb{F}_n(X_{n,n}-\theta x/m(n))=1-(x/m(n))\{1+o(1)\} \quad \text{a.s.}$$

Hence for all $x>0$

$$P\{m(n)(X_{n,n}-X^*_{m(n),m(n)})/\theta > x \mid \mathbb{F}_n\}=(1-(x/m(n))\{1+o(1)\})^{m(n)} \quad \text{a.s.}$$

This immediatly yields (2.4). Now (2.2) follows from the fact that $X_{n,n}\to\theta$ a.s. and (2.4) in combination with Slutzky's Theorem.

We note that if "almost surely" is replaced by "in probability" and the "log log n" is removed everywhere in the above proof then the resulting argument also suffices for the in probability version of Theorem 2.1. Observe that the term in (2.5) is $O_P(1)$ by (2.1). The fact that the term in (2.7) is $o_P(1/m(n))$ follows from an application of Inequality 4, p. 873, in Shorack and Wellner (1986). With this remark, the proof of Theorem 2.1 is complete. □

Our treatment of the problem of bootstrapping extremes in the general case shall require the following special case of Theorem 2.1 and its proof. Let $U^*_1, \ldots, U^*_n$ be sampled from $\mathbb{G}_n$, the empirical distribution function of $U_1, \ldots, U_n$ (we define $\mathbb{G}_n(x)=n^{-1}\#\{i: U_i \leqq x, 1\leqq i \leqq n\}$ for $-\infty<x<\infty$). Assume that $m(n)\to\infty$ and $m(n)/n\to 0$ as $n\to\infty$. Then, for all $x\geqq 0$,

$$(2.8) \qquad P\{m(n)(1-U^*_{m(n),m(n)})\leqq x \mid \mathbb{G}_n\}\to P\{Y\leqq x\},$$

in probability as $n\to\infty$. In addition, if (2.3) holds, then (2.8) is true almost surely.

We now turn to bootstrapping general extremes. First, we record some well-known facts from extreme value theory. Let $X_1, X_2, \ldots,$ be independent random variables with common distribution function $F(x)=P(X_1\leqq x)$. Define the left-continuous inverse (or quantile) function of F to be

$$(2.9) \qquad Q(u)=\inf\{t: F(t)\geqq u\}, \qquad 0<u\leqq 1.$$

There exist sequences of norming and centering constants $a_n>0$ and b_n such that

$$(2.10) \qquad (X_{n,n}-b_n)/a_n \underset{d}{\to} W. \quad \text{as } n\to\infty$$

where W is a nondegenerate random variable, if and only if, for some $-\infty < c < \infty$.

$$(2.11)\quad \begin{cases} \dfrac{Q(1-xs)-Q(1-ys)}{Q(1-ws)-Q(1-zs)} \to \dfrac{x^{-c}-y^{-c}}{w^{-c}-z^{-c}} =: H_c(x, y, w, z) \quad \text{as } s \downarrow 0. \\ \dfrac{x^0-y^0}{w^0-z^0} := \dfrac{\log x - \log y}{\log w - \log z} \quad \text{for all } 0 < x, y, w, z < \infty \text{ with } w \neq z. \end{cases}$$

Moreover, whenever (2.10) and (2.11) hold, we can choose

$$(2.12)\quad a_n = Q\left(1-\frac{1}{n}\right) - Q\left(1-\frac{2}{n}\right) \quad \text{and} \quad b_n = Q\left(1-\frac{1}{n}\right),$$

and W becomes the random variable $H_c(Y, 1, 1, 2)$. We shall say that F is in the domain of attraction of $\Lambda(c)$, written $F \in \Lambda(c)$ if (2.11) holds for a given choice of c. All of these facts can be found in or easily derived from those in the monograph by de Haan (1971).

The following theorem shows that whenever $F \in \Lambda(c)$ for some c, then the distribution of $X_{n,n}$ can be bootstrapped.

THEOREM 22. – *Assume that* $F \in \Lambda(c)$ *for some* c. *Whenever* $m(n) \to \infty$ *and* $m(n)/n \to 0$ *as* $n \to \infty$, *then, in probability, for all* x,

$$(2.13)\quad P\{(X^*_{m(n), m(n)} - B_n)/A_n \leq x' \mid \mathbb{F}_n\} \to P\{H_c(Y, 1, 1, 2) \leq x\} \quad \text{as } n \to \infty,$$

for all x, *where*

$$A_n = X_{n - \lfloor n/m(n) \rfloor, n} - X_{n - 2\lfloor n/m(n) \rfloor, n} \quad \text{and} \quad B_n = X_{n - \lfloor n/m(n) \rfloor, n},$$

with $\lfloor x \rfloor$ *denoting the integer part of* x.

In addition, if (2.3) *holds then* (2.13) *is true almost surely.*

Proof. – Since $\{X_i, i \geq 1\} \underset{d}{=} \{Q(U_i), i \geq 1\}$, we can, from now on, write $X_i = Q(U_i)$, $i \geq 1$. It is also easy to see that, conditioned on $X_i = Q(U_i)$, $1 \leq i \leq n$.

$$(2.14)\quad \{X^*_1, \ldots, X^*_{m(n)}\} \underset{d}{=} \{Q(U^*_1), \ldots, Q(U^*_{m(n)})\}.$$

From (2.14), (2.8) and (2.11), it is trivial now to conclude that

$$(2.15)\quad (Q(U^*_{m(n), m(n)}) - b_{k(n)})/a_{k(n)} \underset{d}{\to} H_c(Y, 1, 1, 2) \quad \text{as } n \to \infty,$$

in probability conditioned on $\mathbb{F}_n$, where $k(n) = \lfloor n/m(n) \rfloor$,

$$a_{k(n)} = Q\left(1-\frac{1}{k(n)}\right) - Q\left(1-\frac{2}{k(n)}\right) \quad \text{and} \quad b_{k(n)} = Q\left(1-\frac{1}{k(n)}\right),$$

and furthermore that, in addition, if (2.3) holds, then (2.15) is true almost surely conditioned on $\mathbb{F}_n$.

To complete the proof of Theorem 2.2, we shall need the following fact. Let $j(n)$ be a sequence of positive integers satisfying $1 \leqq j(n) < n$ for $n \geqq 2$, $j(n) \to \infty$ and $j(n)/n \to 0$ as $n \to \infty$. Then

$$n(1 - U_{n-j(n), n})/j(n) \underset{P}{\to} 1 \quad \text{as } n \to \infty. \tag{2.16}$$

In addition, if $(\log \log n)/j(n) \to 0$, then (2.16) holds almost surely (refer to page 424 of Shorack and Wellner (1986) for results that imply this fact).

Writing

$$A_n = Q(U_{n-k(n), n}) - Q(U_{n-2k(n), n}) \quad \text{and} \quad B_n = Q(U_{n-k(n), n}),$$

we readily conclude from our fact and (2.11) that, in probability.

$$A_n/a_n \to 1 \quad \text{and} \quad (B_n - b_n)/a_n \to 0 \quad \text{as } n \to \infty, \tag{2.17}$$

and that, in addition, if (2.3) holds, then (2.17) is true almost surely.

The proof of Theorem 2.2 now follows from (2.15), (2.17) and Slutzky's Theorem. □

3. BOOTSTRAPPING INTERMEDIATE TRIMMED MEANS

Let $X_{1,n} \leqq \ldots \leqq X_{n,n}$ denote the order statistics from an i.i.d. sample of size n with common distribution function F_n. A *trimmed mean* is formed as follows. Choose integers $k(n)$ and $m(n)$ such that $0 \leqq k(n) < n - m(n) \leqq n$, for $n \geqq 1$. We call

$$T_n = \frac{1}{n} \sum_{i=k(n)+1} X_{i,n} \tag{3.1}$$

a trimmed mean. If we assume further that

$$\min\{k(n), m(n)\} \to \infty \quad \text{and} \quad \max\{k(n), m(n)\}/n \to 0 \text{ as } n \to \infty, \tag{3.2}$$

we call T_n an *intermediate trimmed mean.*

The asymptotic distribution of intermediate trimmed means when $F_n = F$ stays fixed in n was thoroughly investigated by S. Csörgő, Haeusler and Mason [CsHM] (1988 *a*) and Griffin and Pruitt (1989). In particular, they characterized when such means can be centered and normed so as to be asymptotically normally distributed.

In this section, we study the consistency of the bootstrapped intermediate trimmed mean. We shall assume throughout that the intermediate trimmed mean to be bootstrapped is asymptotically normal. Toward this end, we must first show that the CsHM (1988 *a*) necessary and sufficient condition for the asymptotic normality of intermediate trimmed means is sufficient for the asymptotic normality of a triangular array of intermediate trimmed means in which the common distribution can be different on each row.

Let Q, Q_n and $\mathbb{Q}_n$ denote the left-continuous inverse (or quantile) functions of F, F_n and $\mathbb{F}_n$, respectively, defined just as in (2.9).

Set

$$(3.3) \qquad \mu_n = \int_{[a(n),\, 1-b(n))} Q_n(u)\, du,$$

where here and elsewhere $a(n)=k(n)/n$ and $b(n)=m(n)/n$. Let K_n denote Q_n *Winsorized* outside of $[a(n), 1-b(n))$, that is

$$(3.4) \qquad K_n(u) = \begin{cases} Q_n(a(n)) & \text{if } u < a(n), \\ Q_n(u) & \text{if } a(n) \leqq u < 1-b(n), \\ Q_n(1-b(n)) & \text{if } u \geqq 1-b(n). \end{cases}$$

In the same way, define the Winsorized version $\mathbb{K}_n$ of $\mathbb{Q}_n$. The Winsorized mean and variance of F_n are

$$(3.5) \qquad m_n = \int_{(0,\,1)} K_n(u)\, du \quad \text{and} \quad \sigma_n^2 = \int_{(0,\,1)} K_n^2(u)\, du - m_n^2.$$

We Winsorized mean and variance of the empirical distribution function $\mathbb{F}_n$ are likewise

$$(3.6) \qquad \mathbb{M}_n = \int_{(0,\,1)} \mathbb{K}_n(u)\, du \quad \text{and} \quad \mathbb{S}_n^2 = \int_{(0,\,1)} \mathbb{K}_n^2(u)\, du - \mathbb{M}_n^2.$$

We now introduce the condition for asymptotic normality. For all c, as $n \to \infty$,

$$(3.7) \qquad \psi_{0,n}(c) = \frac{\sqrt{a(n)}}{\sigma_n} \left(Q_n\left(a(n) \left\{ 1 - \frac{c}{\sqrt{k(n)}} \right\} \right) - Q_n(a(n)) \right) \to 0,$$

and

$$(3.8) \qquad \psi_{1,n}(c) = \frac{\sqrt{b(n)}}{\sigma_n} \times \left(Q_n\left(1 - b(n) \left\{ 1 - \frac{c}{\sqrt{m(n)}} \right\} \right) - Q_n(1-b(n)) \right) \to 0.$$

This is from CsHM (1988 *a*), but now with Q subscripted. To this, we add the condition for the consistency of $\mathbb{S}_n^2$ as estimator of σ_n^2. As $n \to \infty$

$$(3.9) \qquad \chi_{0,n} = \frac{\sqrt{a(n)}}{\sigma_n} \{ Q_n(a(n)) - m_n \} = O(1).$$

and

$$\chi_{1,n} = \frac{\sqrt{b(n)}}{\sigma_n} \{ Q_n(1-b(n)) - m_n \} = O(1). \tag{3.10}$$

THEOREM 3.1. – *If* (3.2), (3.7) *and* (3.8) *hold, then, as* $n \to \infty$,

$$n^{1/2}(T_n - \mu_n)/\sigma_n \xrightarrow[d]{} N(0, 1). \tag{3.11}$$

In addition, if (3.9) *and* (3.10) *are satisfied, then*

$$\mathbb{S}_n/\sigma_n \to 1 \quad \textit{in probability.} \tag{3.12}$$

If, further,

$$\min\{k(n), m(n)\}/\log\log n \to \infty \quad \textit{as } n \to \infty, \tag{3.13}$$

and, for some $c > 2$, *as* $n \to \infty$,

$$\psi_{0,n}(\pm c(\log\log n)^{1/2}) \to 0, \tag{3.14}$$

and

$$\psi_{1,n}(\pm c(\log\log n)^{1/2}) \to 0, \tag{3.15}$$

then (3.12) *holds almost surely.*

Now, suppose that $X_1^*, \ldots, X_n^*$ are i.i.d. $\mathbb{F}_n$, with order statistics $X_{1,n}^* \leq \ldots X_{n,n}^*$, empirical distribution function $\mathbb{F}_n^*$ and empirical inverse (or quantile) function $\mathbb{Q}_n^*$ having Winsorized version $\mathbb{K}_n^*$. Let T_n^*, $\mathbb{M}_n^*$ and $\mathbb{S}_n^{*2}$ be the starred versions of the terms in (3.1) and (3.6).

THEOREM 3.2. – *Assume that* $F_n = F$, $n \geq 1$, *and that the conditions* (3.2) *and* (3.7) *through* (3.10) *hold. Then, as* $n \to \infty$, *we have both*

$$n^{1/2}(T_n^* - T_n)/\mathbb{S}_n^* \xrightarrow[d]{} N(0, 1), \tag{3.16}$$

and

$$\mathbb{S}_n^*/\mathbb{S}_n \xrightarrow[P]{} 1, \tag{3.17}$$

in probability, conditioned on $\mathbb{F}_n$.

Remark 3.1. – Assume that F is in the domain of attraction of a stable law of index $0 < \alpha \leq 2$, written $F \in D(\alpha)$, and choose $k(n) = m(n)$. It is readily verified that (3.7)-(3.10) hold if $k(n)$ satisfies (3.2) and, further, that (3.14) and (3.15) hold, if, in addition, one has (3.13) [*see* the argument on pages 11-12 of S. Csörgő, Horváth and Mason (1986)].

Remark 3.2. – If $F_n = F$ and $k(n) = m(n)$ for all $n \geq 1$, then (3.9) and (3.10) are always satisfied. This can be easily derived from Lemma 2.1 of CsHM (1988 *a*).

Proof of the theorems. — Let $U_1, \ldots, U_n$, as in Section 1, be Uniform (0, 1) with empirical distribution function $\mathbb{G}_n$. Introduce the uniform empirical process

$$\alpha_n(s)=n^{1/2}\{\mathbb{G}_n(s)-s\}, \qquad 0\leqq s\leqq 1.$$

There exists a triangular array of uniform empirical processes α_n, $n\geqq 1$, and a fixed Brownian bridge B such that, for all $0\leqq \upsilon<1/2$,

$$(3.18) \qquad \Delta_n(\upsilon)=\sup_{0\leqq s\leqq 1} n^{\upsilon}|\alpha_n(s)-B(s)|/(s(1-s))^{1/2-\upsilon}=O_P(1),$$

as shown in Mason and Van Zwet (1987) [*see* also Lemma 5 in Deheuvels and Mason (1990)].

We shall also have need of the following two quantile inequalities of Shorack. For any quantile function Q and with the above notation,

$$\text{(I)} \quad \int_{[a(n),\,1-b(n))} n^{-\upsilon}\{t(1-t)\}^{1/2-\upsilon}dQ(t)/\sigma_n\leqq \frac{3}{\sqrt{\upsilon}}(\min\{k(n), m(n)\})^{-\upsilon}$$

for all $0<\upsilon<1/2$,

and with $v_n(t)=-((Q(t)-m_n)^-)^2+((Q(t)-m_n)^+)^2$,

$$\text{(II)} \quad \int_{[a(n),\,1-b(n))} n^{-\upsilon}\{t(1-t)\}^{1-\upsilon}dv_n(t)/\sigma_n^2\leqq 2(\min\{k(n), m(n)\})^{-\upsilon}$$

for all $0<\upsilon<1$.

Inequality (I) is proved on page 388 of Shorack (1991 *a*) and Inequality (II) is Inequality 1.1 (4) of Shorack (1991 *b*).

Proof of Theorem 3.1. — Integrating by parts, using the convention $\int_{[a,b)}=-\int_{[b,a)}$ if $a>b$, we see that

$$(3.19) \quad n^{1/2}(T_n-\mu_n)/\sigma_n=-\int_{[a(n),\,1-b(n))}\alpha_n(t)\,dQ_n(t)/\sigma_n \\ -\int_{I(n)} n^{1/2}\{\mathbb{G}_n(t)-a(n)\}\,dQ_n(t)/\sigma_n \\ -\int_{J(n)} n^{1/2}\{\mathbb{G}_n(t)-(1-b(n))\}\,dQ_n(t)/\sigma_n=:Z_n+\gamma_{0,n}+\gamma_{1,n},$$

where here and elsewhere

$$(3.20) \quad I(n)=[U_{k(n),n}, a(n)) \quad \text{and} \quad J(n)=[1-b(n), U_{n-k(n),n}).$$

Notice that

$$(3.21) \qquad Z_{0,n}=-\int_{[a(n),\,1-b(n))}B(t)\,dQ_n(t)/\sigma_n \underset{d}{=} N(0, 1).$$

We see trivially that, for any $0<\upsilon<1/2$,

$$(3.22)\quad |Z_n - Z_{0,n}| \leqq \Delta_n(\upsilon) \int_{[a(n),\,1-b(n))} n^{-\upsilon}\{t(1-t)\}^{1/2-\upsilon}\, dQ_n(t)/\sigma_n,$$

which by (3.18), Inequality (I) and (3.2), is equal to

$$(3.23)\quad O_P(1)\times O((\min\{k(n),\, m(n)\})^{-\upsilon}) = o_P(1).$$

Therefore, on account of (3.19) through (3.23), to finish the proof of (3.11), it suffices to show that

$$(3.24)\quad \gamma_{i,n} = o_P(1), \qquad i=0,\,1.$$

First, consider the case $i=0$. Observe that

$$(3.25)\quad \gamma_{0,n} \leqq \left|\frac{\alpha_n(a(n))}{\sqrt{a(n)}}\right| \times |\psi_{0,n}(z(n))|,$$

where

$$z(n):=\frac{n\{a(n)-U_{k(n),\,n}\}}{\sqrt{k(n)}}.$$

It is well-known that whenever (3.2) holds,

$$(3.26)\quad z(n) \underset{d}{\rightarrow} N(0,\,1) \quad \text{as } n\rightarrow\infty,$$

[*cf.* Balkema and de Haan (1975)]. Moreover, an application of Chebyshev's inequality shows that

$$(3.27)\quad \frac{\alpha_n(a(n))}{\sqrt{a(n)}} = O_P(1).$$

Now (3.24) in the case $i=0$ follows readily from (3.25)-(3.27) in combination with the assumption (3.7). The case $i=1$ is treated in the same way. This completes the proof of (3.11).

We now turn to (3.12), the "consistency" of $\mathbb{S}_n$. We shall use the fact, given in Mason (1983) that for all $0<\upsilon<1/2$

$$(3.28)\quad D_n(\upsilon) = \sup_{0\leqq s\leqq 1}\left|\frac{n^{\upsilon}\{\mathbb{G}_n(s)-s\}}{(s(1-s))^{1-\upsilon}}\right| = O_P(1).$$

Setting $v_{0,n}=((Q-m_n)^-)^2$ and $v_{1,n}=((Q-m_n)^+)^2$, we see by the "same" integration by parts identity given in (3.19) that

$$(3.29)\quad \theta_n := \int_{[a(n),\,1-b(n))} ((\mathbb{Q}_n(t)-m_n)^2-(Q_n(t)-m_n)^2)\,\frac{dt}{\sigma_n^2}$$
$$= -\{Y_{0,n}+\delta_{1,n}(0)+\delta_{2,n}(0)\}+\{Y_{1,n}+\delta_{1,n}(1)+\delta_{2,n}(1)\},$$

where

$$Y_{i,n}=\int_{[a(n),\,1-b(n))}\{t-\mathbb{G}_n(t)\}\frac{dv_{i,n}(t)}{\sigma_n^2},\qquad i=0,\,1,$$

and, for, $i=0, 1$,

$$\delta_{1,n}(i)+\delta_{2,n}(i)=-\int_{I(n)}\{\mathbb{G}_n(t)-a(n)\}\frac{dv_{i,n}(t)}{\sigma_n^2}-\int_{J(n)}\{\mathbb{G}_n(t)-(1-b(n))\}\frac{dv_{i,n}(t)}{\sigma_n^2}.$$

The intervals $I(n)$ and $J(n)$ are defined as in (3.20); it is straightforward that with $v_n=-v_{0,n}+v_{1,n}$, for any $0<\upsilon<1/2$,

$$(3.30)\quad |Y_{0,n}|+|Y_{1,n}|\leqq D_n(\upsilon)\int_{[a(n),\,1-b(n))}n^{-\upsilon}(t(1-t))^{1-\upsilon}\frac{dv_n(t)}{\sigma_n^2},$$

which by (3.28) and Inequality (II) is equal to

$$(3.31)\qquad O_P(1)\times O((\min\{k(n),\,m(n)\}^{-\upsilon})=o_P(1).$$

Notice that

$$(3.32)\qquad |\delta_{1,n}(0)+\delta_{1,n}(1)|\leqq\left|\frac{\alpha_n(a(n))}{\sqrt{a(n)}}\right|\frac{\sqrt{k(n)}}{n}\int_{I(n)}\frac{dv_n(t)}{\sigma_n^2},$$

which by (3.26), (3.27), assumptions (3.7) and (3.9) and a little thought is less than or equal to

$$(3.33)\quad O_P(k(n)^{-1/2})\{|\psi_{0,n}(z(n))|(|\psi_{0,n}(z(n))|+2|\chi_{0,n}|)\}=:O_P(k(n)^{-1/2})\,\varepsilon_n=o_P(1),$$

which says that

$$(3.34)\qquad |\delta_{1,n}(0)+\delta_{1,n}(1)|=o_P(1).$$

In exactly the same way it can be shown that

$$(3.35)\qquad |\delta_{2,n}(0)+\delta_{2,n}(1)|=o_P(1).$$

Obviously now (3.30), (3.31), (3.34) and (3.35) imply

$$(3.36)\qquad \boldsymbol{\theta}_n=o_P(1).$$

To finish the proof of the in probability version of (3.12) we need to show that

$$(3.37)\qquad a(n)\{(\mathbb{Q}_n(a(n))-m_n)^2-(Q_n(a(n))-m_n)^2\}/\sigma_n^2\to 0,$$

$$(3.38)\quad b(n)\{(\mathbb{Q}_n(1-b(n))-m_n)^2-(Q_n(1-b(n))-m_n)^2\}/\sigma_n^2\to 0,$$

and

$$(\mathbb{M}_n - m_n)/\sigma_n \to 0 \tag{3.39}$$

in probability as $n \to \infty$.

That (3.37) holds follows in a straightforward manner from (3.26) and from the assumptions (3.7) and (3.9). The upper tail version (3.38) is proved the same way. The proof that (3.39) holds in probability is similar to the proof of (3.36). At the step corresponding to (3.30) one uses the fact that, for any $0 < \upsilon < 1/2$,

$$S_n(\upsilon) := \sup_{0 \leq s \leq 1} |\alpha_n(s)|/(s(1-s))^{1/2-\upsilon} = O_P(1), \tag{3.40}$$

cf. O'Reilly (1974), in combination with Inequality (I). This completes the proof of the in probability version of (3.12).

For the proof of the almost sure version of (3.12), we need the fact that, with $c_n = n^{-1} \log \log n$,

$$R_n := \sup_{c_n \leq s \leq 1 - c_n} |\alpha_n(s)|/(s(1-s))^{1/2} = O((\log \log n)^{1/2}) \quad \text{a.s.} \tag{3.41}$$

[*see* Theorem 2, p. 604, of Shorack and Wellner (1986)]. We will also need the fact, readily inferred from Theorem 2 of Einmahl and Mason (1988), that under (3.13)

$$\limsup_{n \to \infty} |z(n)|/(\log \log n)^{1/2} \leq 2 \quad \text{a.s.,} \tag{3.42}$$

with the same being true for the analogue of $z(n)$ defined in terms of $U_{n-m(n),n}$.

As in (3.30), we have

$$|Y_{0,n}| + |Y_{1,n}| \leq R_n \int_{[a(n),\, 1-b(n))} n^{-1/2} (t(1-t))^{1/2} \frac{dv_n(t)}{\sigma_n^2}, \tag{3.43}$$

which by (3.41), Inequality (II) and (3.13), equals

$$O((\log \log n)^{1/2}) \times O((\min\{k(n), m(n)\})^{-1/2}) = o(1) \quad \text{a.s.} \tag{3.44}$$

Fact (3.41), in combination with (3.42) and the assumption (3.15), implies that (3.32) equals

$$O(((\log \log n)/k(n))^{1/2})\, \varepsilon_n = o(1) \quad \text{a.s.,} \tag{3.45}$$

where ε_n is as in (3.33). This says that

$$|\delta_{1,n}(0) + \delta_{1,n}(1)| = o(1) \quad \text{a.s.} \tag{3.46}$$

Arguing in the same way, we get

$$|\delta_{2,n}(0) + \delta_{2,n}(1)| = o(1) \quad \text{a.s.} \tag{3.47}$$

Of course, (3.44)-(3.47) imply the almost sure version of (3.36).

The almost sure version of (3.37) and (3.38) is easy. The proof of the almost sure version of (3.39) requires, at the step (3.30), the fact that

$$(3.48) \qquad S_n(\upsilon) = O((\log\log n)^{1/2}) \quad \text{a.s.},$$

cf. James (1975), used in conjunction with Inequality (I). This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2. – Conditioned on $\mathbb{F}_n$, $X_1^*, \ldots, X_n^*$ are i.i.d. with inverse (or quantile) distribution $\mathbb{Q}_n$ and Winsorized variance $\mathbb{S}_n^2$. Therefore, by Theorem 3.1, to establish (3.7)-(3.10) in probability conditioned on $\mathbb{F}_n$, it suffices to show that, for each c, as $c \to \infty$,

$$(3.49) \qquad \psi_{i,n}^*(c) \underset{P}{\to} 0, \qquad i=0, 1,$$

and

$$(3.50) \qquad \chi_{i,n}^* = O_P(1), \qquad i=0, 1,$$

where $\psi_{i,n}^*$ and $\chi_{i,n}^*$, $i=0, 1$ are formed by replacing Q_n by $\mathbb{Q}_n$ (recall that we are assuming $F_n = F$) and σ_n by $\mathbb{S}_n$ in (3.7)-(3.10).

Now, for any c,

$$(3.51) \quad \mathbb{Q}_n\left(a(n)\left\{1-\frac{c}{\sqrt{k(n)}}\right\}\right) - \mathbb{Q}_n(a(n)) = Q\left(\mathbb{U}_n\left(a(n)\left\{1-\frac{c}{\sqrt{k(n)}}\right\}\right)\right) - Q(\mathbb{U}_n(a(n))),$$

where $\mathbb{U}_n$ is the left-continuous inverse of $\mathbb{G}_n$ as defined in (2.9).

From the fact (3.26), it is straightforward to infer that, for any d, one has, as $n \to \infty$,

$$(3.52) \quad \frac{n}{\sqrt{k(n)}}\left(a(n)\left\{1-\frac{d}{\sqrt{k(n)}}\right\} - \mathbb{U}_n\left(a(n)\left\{1-\frac{d}{\sqrt{k(n)}}\right\}\right)\right) \underset{d}{\to} N(0, 1).$$

Notice that the assumptions (3.7)-(3.10) imply that (3.12) holds. It is routine now to argue from (3.51), (3.52), (3.39), (3.12), (3.7) and (3.9) that (3.49) and (3.50) are satisfied in the case $i=0$. The case $i=1$ follows in exactly the same manner. This completes the proof of Theorem 3.2. □

4. BOOTSTRAPPING MEANS

We shall use the methodology developed in Sections 2 and 3 to investigate the consistency of the bootstrapped mean when the underlying distribution function F is in the domain of attraction of a stable law of index $0<\alpha\leq 2$, written $F \in D(\alpha)$. In doing so, we obtain an alternative formulation of the results of Athreya (1985) and Arcones and Giné (1989) on this problem. We then show that our version has an unsuspected connection to the intermediate trimmed means studied in the previous section.

First, we must derive a triangular array version of certain necessary and sufficient conditions for $F \in D(\alpha)$ given in S. Csörgő, Haeusler and Mason [CsHM] (1988 *b*) (refer to their Corollaires 1 and 3). As in Section 3, let $X_1, \ldots, X_n$ be i.i.d. F_n with inverse function Q_n, and order statistics $X_{1,n} \leqq \ldots \leqq X_{n,n}$. Consider the means

$$(4.1) \qquad M_n = \frac{1}{n} \sum_{i=1}^{n} X_i, \qquad n \geqq 1.$$

Introduce the centering constants, for $n \geqq 1$,

$$(4.2) \qquad b_n = \int_{[1/n,\, 1-1/n)} Q_n(u)\, du.$$

For $t > 0$, define the Winsorized variance

$$(4.3) \qquad \sigma_n^2(t) = \int_{(0,\,1)} K_{n,t}^2(u)\, du - \left\{ \int_{(0,\,1)} K_{n,t}(u)\, du \right\}^2,$$

where

$$(4.4) \qquad K_{n,t}(u) = \begin{cases} Q_n(t/n) & \text{if } u < t/n, \\ Q_n(u) & \text{if } t/n \leqq u < 1 - t/n, \\ Q_n(1 - t/n) & \text{if } u \geqq 1 - t/n. \end{cases}$$

Set for $s > 0$

$$(4.5) \qquad Z_{0,n}(s) = \frac{Q_n(s/n)}{a_n \sqrt{n}} \quad \text{and} \quad Z_{1,n}(s) = -\frac{Q_n(1 - s/n)}{a_n \sqrt{n}},$$

where $a_n = \sigma_n(1)$.

The following theorem provides sufficient conditions for a triangular array of means $\{M_n, n \geqq 1\}$, properly normalized. to converge in distribution to a stable law.

THEOREM 4.1. — I. *Whenever one has, for all* $s > 0$, *as* $n \to \infty$,

$$(4.6) \qquad Z_{i,n}(s) \to 0, \qquad \textit{for} \quad i = 0, 1,$$

then, as $n \to \infty$,

$$(4.7) \qquad n^{1/2} \{ M_n - b_n \} / a_n \underset{d}{\to} N(0, 1).$$

II. *Whenever, for some* $0 < \alpha < 2$ *and* $\delta_0 \geqq 0$, $\delta_1 \geqq 0$ *with at least one of the two strictly positive, one has, for all* $s > 0$, *as* $n \to \infty$,

$$(4.8) \qquad Z_{i,n}(s) \to -\delta_i s^{-1/\alpha} \qquad \textit{for} \quad i = 0, 1,$$

then, as $n \to \infty$,

$$n^{1/2}\{M_n - b_n\}/a_n \underset{d}{\to} V(\alpha, \delta_0, \delta_1), \tag{4.9}$$

where $V(\alpha, \delta_0, \delta_1)$ *is the stable random variable of index* $0<\alpha<2$ *given by*

$$V(\alpha, \delta_0, \delta_1) = \eta_1 - \eta_0, \tag{4.10}$$

where η_0 *and* η_1 *are independent random variables such that, for* $i=0, 1$,

$$\eta_i \underset{d}{=} \delta_i \left\{ \int_{(0, 1]} \alpha^{-1} \Pi(t)\, t^{-(1+1/\alpha)}\, dt + 1 + \int_{(1, \infty)} \alpha^{-1} (\Pi(t) - t)\, t^{-(1+1/\alpha)}\, dt \right\},$$

with $\{\Pi(t), t \geqq 0\}$ *being a standard, right-continuous, Poisson process with rate one.*

We now turn to the problem of bootstrapping means, in the case where the underlying distributions are in the domain of attraction of a stable law. Let $X_1^*, \ldots, X_{m(n)}^*$ be i.i.d. $\mathbb{F}_n$ with empirical inverse function $\mathbb{Q}_n$. Define the bootstrapped mean

$$M_{m(n)}^* = \frac{1}{m(n)} \sum_{i=1}^{m(n)} X_i^*, \tag{4.11}$$

and, as in (3.1), introduce the trimmed mean

$$T_n = \int_{[1/m(n),\, 1-1/m(n))} \mathbb{Q}_n(u)\, du = \frac{1}{b} \sum_{i=k(n)+1}^{n-k(n)} X_{i, n}, \tag{4.12}$$

where $k(n) = \lfloor n/m(n) \rfloor$ and, as in (3.6), define the empirical Winsorized variance by

$$\mathbb{S}_n^2 = \int_{(0, 1)} \mathbb{K}_n^2(u)\, du - \left\{ \int_{(0, 1)} \mathbb{K}_n(u)\, du \right\}^2, \tag{4.13}$$

where, as in (3.4),

$$\mathbb{K}_n(u) = \begin{cases} \mathbb{Q}_n(k(n)/n) & \text{if } u < k(n)/n, \\ \mathbb{Q}_n(u) & \text{if } k(n)/n \leqq u < 1 - k(n)/n, \\ \mathbb{Q}_n(1 - k(n)/n) & \text{if } u \geqq 1 - k(n)/n. \end{cases} \tag{4.14}$$

Let $W(2)$ denote a standard normal random variable, and, for $0<\alpha<2$, let $W(\alpha)$ be the stable random variable of the form given in (4.10).

THEOREM 4.2. — *Assume that* $F_n = F$ *for* $n \geqq 1$, *that* $F \in D(\alpha)$, $0<\alpha<2$, *and that* (3.2) *holds. Then, as* $n \to \infty$,

$$\sqrt{m(n)}\, \{M_{m(n)}^* - T_n\}/\mathbb{S}_n \underset{d}{\to} W(\alpha), \tag{4.15}$$

in probability conditioned on $\mathbb{F}_n$. *In addition, if* (2.3) *is satisfied, then* (4.15) *holds almost surely conditioned on* $\mathbb{F}_n$.

Remark 4.1. – An in probability version of (4.15) was first proved by Athreya (1985). The almost sure version of (4.15) due to Arcones and Giné (1989) requires a little more regularity on the sampling rate $m(n)$. Namely, they must impose the conditions that for some $c>0$, $m(n)/m(2n)>c$, and that $m(n)$ be nondecreasing. Both Athreya and Arcones and Giné use a non-random norming constant, and their centering constant, though random, requires knowledge of their norming constant. They also show that the rate (2.3) is necessary for the almost sure bootstrap to hold when $E(X^2)=\infty$. In Arcones and Giné (1992 *b*), centering and norming constants are also found that do not depend upon the distribution, as long that it is in the right class.

Remark 4.2. – From Athreya (1987) and Knight (1989), it can readily be inferred that the in probability version of (4.15) cannot hold if the bootstrap sampling rate $m(n)$ is of the order γn for some $\gamma>0$. A more precise statement of this is given in Theorem 3 of Giné and Zinn (1989). For a closely related work, *see* Hall (1990).

A Curious Distributional Comparison with Intermediate Trimmed Means. – Notice that, when $F\in D(\alpha)$, by Theorem 3.1 and Remark 3.1, under (3.2) and with μ_n defined as in (3.3), we get the curious result that, as $n\to\infty$,

$$\sqrt{n}\,\{T_n-\mu_n\}/\mathbb{S}_n \underset{d}{\to} N(0,\,1), \tag{4.16}$$

whereas

$$\sqrt{m(n)}\,\{M^*_{m(n)}-T_n\}/\mathbb{S}_n \underset{d}{\to} W(\alpha), \tag{4.17}$$

which is not $N(0, 1)$ if $0<\alpha<2$.

Proof of the theorems

Proof of Theorem 4.1. – The proof of Theorem 4.1 can be obtained by carefully going through the proofs of the closely related results in M. Csörgő, S. Csörgő, Horváth and Mason [CsCsHM] (1986), being sure to always verify that placing a subscript n on Q does not essentially change things. For the sake of completeness, however, we will sketch a proof here.

Integrating by parts as in (3.19), we get

$$(4.18)\quad \sqrt{n}\{M_n - b_n\}/a_n = \frac{1}{a_n\sqrt{n}}\{Q_n(U_{1,n}) + Q_n(U_{n,n})\}$$
$$- \int_{[1/n,\,1-1/n)} \alpha_n(t)\,\frac{dQ_n(t)}{a_n} - \int_{I(n)} \sqrt{n}\left\{\mathbb{G}_{n(t)} - \frac{1}{n}\right\}\frac{dQ_n(t)}{a_n}$$
$$- \int_{J(n)} \sqrt{n}\left\{\mathbb{G}_n(t) - \left(1 - \frac{1}{n}\right)\right\}\frac{dQ_n(t)}{a_n},$$

where here $I(n) = [U_{1,n}, 1/n)$ and $J(n) = [1 - 1/n, U_{n,n})$. Now, for any $T > 1$ and $n > T$, the right side of (4.18) is equal to

$$-\eta_{0,n}(T) + \eta_{1,n}(T) + Z_n(T),$$

where

$$\eta_{0,n}(T) := \int_{[0,\,1/n)} n\,\mathbb{G}_n(t)\,\frac{dQ_n(t)}{a_n\sqrt{n}} - \frac{Q_n(1/n)}{a_n\sqrt{n}} + \int_{[1/n,\,T/n)} \alpha_n(t)\,\frac{dQ_n(t)}{a_n},$$

$$\eta_{1,n}(T) := \int_{[1-1/n,\,1)} n(1 - \mathbb{G}_n(t))\,\frac{dQ_n(t)}{a_n\sqrt{n}}$$
$$- \frac{Q_n(1-1/n)}{a_n\sqrt{n}} + \int_{[1-T/n,\,1-1/n)} \alpha_n(t)\,\frac{dQ_n(t)}{a_n},$$

and

$$Z_n(T) := -\int_{[T/n,\,1-T/n)} \alpha_n(t)\,\frac{dQ_n(t)}{a_n}.$$

In Case II, we obtain by arguing just as on pages 96-99 of CsCsHM (1986) that for each $T > 1$, as $n \to \infty$,

$$(4.19)\quad \eta_{i,n}(T) \underset{d}{\to} \delta_i\left\{\int_{(0,\,1]} \alpha^{-1}\Pi(t)\,t^{-(1+1/\alpha)}\,dt\right.$$
$$\left. + 1 + \int_{[1,\,T)} \alpha^{-1}(\Pi(t) - t)\,t^{-(1+1/\alpha)}\,dt\right\} =: \eta_i(T) \qquad \text{for} \quad i = 0, 1.$$

In Case I, an argument very similar to that used to prove (4.19) shows that, for every $T > 1$, as $n \to \infty$,

$$(4.20)\qquad \eta_{i,n}(T) \underset{P}{\to} 0 \qquad \text{for} \quad i = 0, 1.$$

Next, in either case, just as in (3.20)-(3.23), we get

$$(4.21)\qquad Z_n(T) \underset{d}{=} O_P(1)\, T^{\upsilon} + (\sigma_n(T)/\sigma_n(1))\, Z,$$

where Z is N(0, 1). Now, assuming that (4.8) holds, it is straightforward to verify

$$(4.22)\qquad \lim_{T\to\infty} \{\limsup_{n\to\infty} (\sigma_n(T)/\sigma_n(1))\} = 0,$$

and, when (4.6) is satisfied,

$$(4.23)\qquad \lim_{n\to\infty} (\sigma_n(T)/\sigma_n(1)) = 1 \quad \text{for all } T > 1.$$

Finally, we point out that Lemma 3.5 of CsCsHM (1986) implies that, for each $i=0, 1$,

$$(4.24)\qquad \eta_i(T) \underset{P}{\to} \eta_i \quad \text{as } T \to \infty.$$

Putting all of these pieces together, it is routine to see that (4.6) implies (4.7) and (4.8) implies (4.9). This completes the sketch of Theorem 4.1 □

Proof of Theorem 4.2. – First, assume that (3.2) holds. It suffices to show for all $s > 0$ that, in probability conditioned on $\mathbb{F}_n$ as $n \to \infty$,

$$(4.25)\qquad Z^*_{i, m(n)}(s) \to \delta_i s^{-1/\alpha}, \qquad \text{for} \quad i=0, 1,$$

whenever $F \in D(\alpha)$, $0 < \alpha < 2$, with $\delta_i \geqq 0$, $i=0, 1$ at least one of the two being positive; and

$$(4.26)\qquad Z^*_{i, m(n)}(s) \to 0,$$

whenever $F \in D(2)$. Here, the $Z^*_{i, m(n)}$ are formed by replacing $a_n = \sigma_n(1)$ by $\mathbb{S}_n$, s/n by $s/m(n)$, $\sqrt{n}$ by $\sqrt{m(n)}$ and Q_n by the empirical quantile function $\mathbb{Q}_n$ in the definition (4.5) of the $Z_{i,n}$. Namely, we set

$$(4.27)\qquad Z_{0,n} = \frac{\mathbb{Q}_n(s/m(n))}{\mathbb{S}_n \sqrt{m(n)}} \qquad \text{and} \qquad Z_{1,n} = -\frac{\mathbb{Q}_n(1-s/m(n))}{\mathbb{S}_n \sqrt{m(n)}}.$$

If $F \in D(\alpha)$, it can be readily inferred from Corollaries 1 and 3 of CsHM (1988 *b*) that (4.6) holds in probability. Therefore, it follows by a simple argument much like the proof of Theorem 2.3, that (4.15) holds in probability given $\mathbb{F}_n$. If (2.3) is satisfied too, then, by Theorem 3.1, (3.12) holds almost surely. The almost sure version of (4.15) conditioned on $\mathbb{F}_n$ is then readily deduced using the almost sure version of (2.16). This completes the proof of Theorem 4.2 □

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