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Wavelet coefficients of a Gaussian process and applications

by

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ABSTRACT. — The wavelet transform of Gaussian stationary continuous-time processes is studied. At each resolution, two Gaussian discrete stationary processes are obtained: the sequence of the approximations and the sequence of the details. Using the sequence of approximations of a continuous-time process, an approximation of the original one is reconstructed. We give the link between these sequences and the original process. Then we obtain the exact error between the approximation and the process, and the rate of convergence of this error. The weak convergence of the approximated process to the original process is proved. The asymptotic behaviour of the variance of the details and of the covariance between the details and the approximations are explained. This induced a choice of a best wavelet basis for data compression.

Key words : Data-compression; Gaussian stationary process; Wavelet.

RÉSUMÉ. — Nous étudions la transformée en ondelette d'un processus Gaussien stationnaire à temps continu. A chaque résolution, nous obtenons deux processus stationnaires Gaussiens à temps discret : la suite des approximations et la suite des détails. Nous reconstruisons, à l'aide de la suite des approximations d'un processus à temps continu, une approximation du processus original. Puis nous obtenons l'erreur entre l'approximation et le processus, et la vitesse de convergence de cette erreur. Nous montrons la convergence en loi du processus approché vers le processus original. Nous explicitons le comportement asymptotique de la variance des détails ainsi que la covariance entre les détails et les approximations.

Ceci induit un choix de base d'ondelette optimale en compression de données.

1. INTRODUCTION

Orthogonal expansions of random variables has been proposed for a long time in the case of the Brownian motion (Levy [8]) or for more general stochastic processes (Loève [9], Dacunha-Castelle and Duflo [5]). Among these orthogonal expansions, the family of orthonormal functions that uncorrelate a given random field is called the Karhunen-Loève expansion (Rosenfeld and Kak [13]). This transformation is of great interest in image compression. Unfortunately, since the Karhunen-Loève transformation requires the inversion of a covariance matrix, it involves a very large number of computations. It may also be useful to have to disposal of more suitable orthonormal bases. The Fourier base and the Haar base have been proposed. We aim here to study the decomposition of a continuous-time Gaussian stationary process $x(t)_{t \in \mathbf{R}}$ on specific orthonormal bases of $L^2(\mathbf{R})$: the compactly supported wavelet bases (Meyer [11], Mallat [10], Daubechies [6], [7]). Actually they provide a general method to study the behaviour of these processes at various resolutions.

The model is presented in Section 2. In Section 3, we study the links between the original process x and its wavelet transform. In Section 4, we propose some applications of the results of Section 3 to the image analysis.

Section 2.1 contains a general description of the multi-resolution analysis of $L^2(\mathbf{R})$ and $l^2(\mathbf{Z})$ obtained with wavelet bases. A multiresolution analysis of $L^2(\mathbf{R})$ is a sequence of subspaces $(V_j)_{j \in \mathbf{Z}}$ such that $V_j \subset V_{j+1}$, $\bigcup_{j \in \mathbf{Z}} V_j = L^2(\mathbf{R})$ and $\bigcap_{j \in \mathbf{Z}} V_j = \{0\}$. One studies a function f of $L^2(\mathbf{R})$ at a resolution 2^j by projecting this function on V_j . We denote by P_j this projection. The subspace V_j is called the approximation at the resolution 2^j . The orthogonal complement W_j of V_{j+1} is called the subspace of details.

The multi-resolution analyses of $L^2(\mathbf{R}^2)$ are introduced in Section 2.2. They are obtained as tensorial products of multi-resolution analysis of $L^2(\mathbf{R})$.

Section 2.3 contains the notations and assumptions concerning the covariance function and the spectral density of the stationary Gaussian process that will be analysed.

The sequence of the wavelet coefficients, approximations and details, of a stationary Gaussian process constitute two stationary Gaussian series at a given resolution. We study in Section 2.4 the links between these various series and the original process.

The sample path $x(t, \omega)$ of a stationary Gaussian process does not almost surely belong to $L^2(\mathbf{R})$. Therefore, since the wavelet is compactly supported, we can define the projection $P_j(x)$ on $[-T, T]$. In practical situations, we observe the restriction \tilde{x} of x to the time interval $[-T, T]$, for which the projection $P_j(\tilde{x})$ can be easily defined. We define in Section 3.1 two integrated square error on $[-T, T]$, ε_j and $\tilde{\varepsilon}_j$, between $x(t)$ and $P_j(x)(t)$, and between $\tilde{x}(t)$ and $P_j(\tilde{x})(t)$. We show in Theorem 1 that these two square errors converge, as $T \rightarrow \infty$, to a constant C_j which depends explicitly on the spectral density of the process x , the analysing wavelet and the resolution 2^j .

In Section 3.2 we study the rate of convergence of these integrated square errors. Theorem 2 shows that $(2T)^{1/2}(\varepsilon_j - C_j)$ and $(2T)^{1/2}(\tilde{\varepsilon}_j - C_j)$ converge, as $T \rightarrow \infty$, to a centered normal variable.

With an additional assumption on the wavelet, it is shown in Section 3.3 that the projection $P_j(x)$ converges in distribution with respect to the Skohorod topology to the original process as the resolution 2^j converges to $+\infty$.

This work was first motivated by the general problem of pictures compression using wavelet bases. The previous integrated square error are a possible measure of the error of compression. In Section 4, we apply the previous results to this problem. Following the investigation of Cohen, Froment and Istas [3], we prove in Section 4.1 that the asymptotic behaviour, as $j \rightarrow \infty$, of the variance of the detail is driven by the minimum of the regularity of the analysing wavelet and the rate of the decreasing of the spectral density of the process at infinity. This induces a choice of a best wavelet base in order to minimize the two square errors.

It can be useful in application in image analysis (Rosenfeld and Kak [13]) to uncorrelate the approximations and the details. We study then in Section 4.2 the asymptotic correlation between the approximations and the details. This leads to choose another wavelet base. However, we show that these two choices are consistent.

2. WAVELETS COEFFICIENTS OF GAUSSIAN PROCESSES

In order to analyse Gaussian processes at different resolutions, we introduce, according to Meyer [11] and Mallat [10], the wavelet transform and the orthogonal multiresolution representation of a one dimensional signal.

2.1. Orthogonal multiresolution representations

DEFINITION 1. — *A multiresolution representation is given by a sequence $(V_j)_{j \in \mathbf{Z}}$ of closed subspaces of $L^2(\mathbf{R})$ verifying the following properties:*

- (P 1) $V_j \subset V_{j+1}$.
- (P 2) $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$.
- (P 3) $\forall k \in \mathbf{Z}, f(t) \in V_0 \Leftrightarrow f(t-k) \in V_0$.
- (P 4) $\bigcup_{j \in \mathbf{Z}} V_j = L^2(\mathbf{R}) \quad \bigcap_{j \in \mathbf{Z}} V_j = \{0\}$.

(P 5) *There exists a function g in V_0 such that $(g(t-k))_{k \in \mathbf{Z}}$ is a Riesz base (Meyer [11], p. 22) of V_0 .*

One can show that there exists a function ϕ in V_0 such that $(\phi(t-k))_{k \in \mathbf{Z}}$ is an orthonormal base of V_0 . The function ϕ is called the scaling function of the multiresolution representation.

The orthogonal complement W_j of V_j in V_{j+1} is defined as:

$$V_{j+1} = W_j \oplus V_j. \tag{1}$$

There is a second function ψ , called a wavelet, such that $(\psi(t-k))_{k \in \mathbf{Z}}$ is an orthonormal base of W_0 . Now let us define the functions

$$\phi_k^j(t) = 2^{j/2} \phi(2^j t - k), \quad \psi_k^j(t) = 2^{j/2} \psi(2^j t - k). \tag{2}$$

The collection $(\phi_k^j)_{k \in \mathbf{Z}}$ is an orthonormal base of V_j . The collection $(\psi_k^j)_{k, j \in \mathbf{Z}}$ is an orthonormal base of $L^2(\mathbf{R})$.

$$L^2(\mathbf{R}) = \bigoplus_{j \in \mathbf{Z}} W_j.$$

Property (P 1) is a causality property: the projection of a function at resolution 2^{j+1} contains all the information necessary to build the projection of this function at a smaller resolution 2^j .

Property (P 2) means that the spaces of approximated functions at successive resolutions should be derived from each other by scaling each approximated function by the ratio of the resolution values, here equal to 2.

By property (P 3), no information is lost when f is translated.

By property (P 4), the approximated function converges to the original function, when the resolution increases to $+\infty$ and when the resolution decreases, the approximated function contains less and less information and converges to zero.

Let \hat{f} denote the Fourier transform of f , denote by $\langle f, g \rangle$ the inner product of $L^2(\mathbf{R})$ and denote by C^s the Holder spaces (Meyer [11], p. 175) of order s .

DEFINITION 2. — *A multiresolution analysis is said to be r -regular if ϕ is in C^r , and if its derivations verify:*

$$|\partial^\alpha \phi(t)| \leq C_m (1 + |t|)^{-m}, \quad \forall m \in \mathbf{N}, \quad \forall \alpha \in \mathbf{N} \text{ such that } \alpha \leq r.$$

where ∂^α is the derivative of order α and C_m is a constant.

The projection f_j of a function f of $L^2(\mathbf{R})$ on V_j is called the approximation of f at the resolution 2^j . This yields using (1) and (2):

$$f_j = P_j(f) = \sum_{k \in \mathbf{Z}} \langle f, \phi_k^j \rangle \phi_k^j. \tag{3}$$

According to Meyer [11], p. 182, the quality of an approximated function by a multiresolution analysis is directly linked with the regularity of the wavelet: for instance, if f belongs to some Hölder spaces C^s ($0 < s < r$):

$$|\langle f, \psi_k^j \rangle| = O(2^{-j(s+1/2)}) \quad k \in \mathbf{Z}.$$

Moreover, if the multiresolution analysis is r -regular, the wavelet ψ is also C^r and verifies:

$$\int t^k \psi(t) dt = \hat{\psi}^{(k)}(0) = 0 \quad \text{for } 0 \leq k \leq r. \tag{4}$$

Here for sake of simplicity we restrict our attention to wavelets which are compactly supported. Examples of wavelets having a compact support and an arbitrarily great regularity r have been constructed by Daubechies ([6], [7]).

We shall assume:

(H 1) Functions ϕ and ψ are associated with a r -regular multiresolution analysis of $L^2(\mathbf{R})$ and are compactly supported.

From now on, we shall use the notations:

$$\text{supp}(\psi) = [a, b], \quad \text{supp}(\phi) = [A, B].$$

Now, following Mallat [10], a multiresolution analysis of $l^2(\mathbf{Z})$ is associated to a multiresolution analysis of $L^2(\mathbf{R})$ by quadrature mirror filters.

Indeed, let h and g be the sequences of $l^2(\mathbf{Z})$ defined by:

$$h(n) = 1/2 \langle \phi_0^{-1}(u), \phi(u-n) \rangle, \quad n \in \mathbf{Z}, \tag{5}$$

$$g(n) = 1/2 \langle \phi_0^{-1}(u), \psi(u-n) \rangle, \quad n \in \mathbf{Z}. \tag{6}$$

The sequence $(h(n)_{n \in \mathbf{Z}})$ [respectively $(g(n)_{n \in \mathbf{Z}})$] is called the impulse response associated with ϕ (respectively ψ).

Therefore, the action of the impulse response h (or g) on any sequence $S(n)_{n \in \mathbf{Z}}$ of $l^2(\mathbf{Z})$ is defined by:

$$l^2(\mathbf{Z}) \rightarrow l^2(\mathbf{Z}) \quad S(n) \rightarrow \sum_{k \in \mathbf{Z}} S(k) h(k-2n) \tag{7}$$

This transformation is called by the pyramidal algorithm of Mallat. It is of common use in image analysis (Mallat [10]).

Using (5) and (6), the discrete Fourier transform of h and g is defined as:

$$H(\omega) = \sum_{n \in \mathbf{Z}} h(n) e^{-in\omega}, \quad G(\omega) = \sum_{n \in \mathbf{Z}} g(n) e^{-in\omega}.$$

Then, according to Mallat [10], the following properties, derived from the previous properties on ϕ and ψ , hold for H and G :

$$\begin{aligned} H(0) = 1 \quad \text{and} \quad |H(\omega)|^2 + |H(\omega + \pi)|^2 = 1, \\ H(\omega) \neq 0 \quad \text{if} \quad \omega \in [-\pi/2, \pi/2], \\ \hat{\phi}(\omega) = \prod_{k=1}^{\infty} H(2^{-k}\omega) \quad \text{and} \quad \hat{\psi}(2\omega) = G(\omega)\hat{\phi}(\omega), \end{aligned} \quad (8)$$

This relation 8 allows to reconstruct ϕ and ψ from H and G .

$$\left. \begin{aligned} |H(\omega)|^2 + |G(\omega)|^2 = 1, \\ H(\omega)\overline{G(\omega)} + H(\omega + \pi)\overline{G(\omega + \pi)} = 0. \end{aligned} \right\} \quad (9)$$

Moreover, if the multiresolution analysis of $L^2(\mathbf{R})$ is r -regular, so is the multiresolution analysis of $l^2(\mathbf{Z})$:

$$\frac{d^n G(0)}{d\omega^n} = 0 \quad \text{for} \quad 0 \leq n \leq r. \quad (10)$$

2.2. Extension to $L^2(\mathbf{R})$

We shall use in the applications multiresolution analysis of $L^2(\mathbf{R}^2)$ and $l^2(\mathbf{Z}^2)$.

In order to obtain a multiresolution analysis of $L^2(\mathbf{R}^2)$, a separable analysis of $L^2(\mathbf{R}^2)$ is generally used: let $(V_j^1)_{j \in \mathbf{Z}}$ be a multiresolution analysis of $L^2(\mathbf{R})$, and $(W_j^1)_{j \in \mathbf{Z}}$ the associated subspaces of details, as defined by (1). Then a separable multiresolution analysis of $L^2(\mathbf{R}^2)$ is obtained by setting:

$$V_j = V_j^1 \otimes V_j^1.$$

where \otimes denotes the tensorial product. We then have three subspaces of details which are defined by:

$$W_{1,j} = V_j^1 \otimes W_j^1, \quad W_{2,j} = W_j^1 \otimes V_j^1, \quad W_{3,j} = W_j^1 \otimes W_j^1.$$

Hence $V_{j+1} = V_j \oplus W_{1,j} \oplus W_{2,j} \oplus W_{3,j}$.

There is then a natural extension to multi-resolution of $l^2(\mathbf{Z}^2)$, taking as impulse responses $h(\cdot) \times h(\cdot)$ for the subspace of approximations and $h(\cdot) \times g(\cdot)$, $g(\cdot) \times h(\cdot)$, $g(\cdot) \times g(\cdot)$ for the three subspaces of details. This induces of course a two-dimensional algorithm of Mallat.

2.3. Notations and assumptions

In the sequel, we study the wavelet coefficients of a Gaussian process $(x(u))_{u \in \mathbf{R}}$. We shall assume:

(H2) The process $(x(u))_{u \in \mathbf{R}}$ is a one dimensional continuous stationary Gaussian real process with mean zero and covariance function:

$$\Gamma(t) = E[x(u+t)x(u)]. \tag{11}$$

(H3) The process $(x(u))_{u \in \mathbf{R}}$ has a spectral density f :

$$\Gamma(t) = \int_{\mathbf{R}} e^{-it\lambda} f(\lambda) d\lambda. \tag{12}$$

Let us point out that the sample paths of the process x introduced above are not in $L^2(\mathbf{R})$ (a. s.). Hence one cannot define the projection of x on V_j . However, since we consider here compactly supported wavelets, we can define for all l the following discrete-time processes X^l , Y^l and the continuous-time process Z^l :

$$X^l(n) = \langle x, \phi_n^l \rangle, \quad Y^l(n) = \langle x, \psi_n^l \rangle, \quad n \in \mathbf{Z}, \tag{13}$$

$$Z^l(t) = \sum_{n \in \mathbf{Z}} X^l(n) \phi_n^l(t), \quad t \in \mathbf{R}. \tag{14}$$

We shall call $(X^l(n))_{n \in \mathbf{Z}}$ the approximation of $(x(u))_{u \in \mathbf{R}}$ and $(Y^l(n))_{n \in \mathbf{Z}}$ the detail of $(x(u))_{u \in \mathbf{R}}$ at the resolution 2^l .

Since these three processes are obtained by linear combination of $(x(u))_{u \in \mathbf{R}}$, assumption (H1) implies that X , Y , and Z are real centered Gaussian processes.

Hence we define the covariance function and the spectral density of X^l and Y^l :

$$\Gamma^l(n) = E[X^l(n+p)X^l(p)], \tag{15}$$

$$\Lambda^l(n) = E[Y^l(n+p)Y^l(p)], \tag{16}$$

and if $T = [0, 2\pi]$,

$$\Gamma^l(n) = \frac{1}{2\pi} \int_T f^l(\lambda) e^{-in\lambda} d\lambda, \tag{17}$$

$$\Lambda^l(n) = \frac{1}{2\pi} \int_T g^l(\lambda) e^{-in\lambda} d\lambda. \tag{18}$$

We shall use in the sequel the following assumptions on the process x :

(H4) The covariances of Y^l are such that: $\sum_{l \geq 0} 2^l \Lambda^l(0) < \infty$

(H5) The covariances of Y^l are such that: $\forall l \in \mathbf{Z}, \lim_{p \rightarrow \infty} \Lambda^l(p) = 0$.

It follows from Rozanov [14], p. 163, that $(Y^l(n))_{n \in \mathbf{Z}}$ is an ergodic process, for each l . We shall give in Section 4. 1 conditions on the process x that imply assumptions (H 4) and (H 5).

Finally, let us define the following semi-norm, when it exists, for any real function a :

$$\|a\|_s^2 = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T a^2(t) dt. \tag{19}$$

2.4. Relations between the analysis at various resolutions and the original process

First, we shall study the relations between the covariance functions Γ^l and Λ^l and the spectral density f of x , defined in (12), (15), (16), (17) and (18).

PROPOSITION 1. — *Assuming (H 1)-(H 3), we have the following relations:*

$$\Gamma^l(n) = 2^{-l} \int_{\mathbf{R}} \exp(i 2^{-l} n \lambda) f(\lambda) |\hat{\phi}(2^{-l} \lambda)|^2 d\lambda, \tag{20}$$

$$\Lambda^l(n) = 2^{-l} \int_{\mathbf{R}} \exp(i 2^{-l} n \lambda) f(\lambda) |\hat{\psi}(2^{-l} \lambda)|^2 d\lambda, \tag{21}$$

$$f^l(\lambda) = 2\pi \sum_{k \in \mathbf{Z}} f(2^l(2\pi k - \lambda)) |\hat{\phi}(2\pi k - \lambda)|^2. \tag{22}$$

Proof of Proposition 1. — Let us first prove equation (20). From (H 1) and the Fubini theorem

$$\Gamma^l(n) = \int_{\mathbf{R}^2} \Gamma(u - u') \phi_0^l(u) \phi_n^l(u') du du',$$

Equation (20) is obtained using the Parseval identity and noting that:

$$\hat{\phi}_n^l(u) = 2^{-l/2} \exp(i 2^{-l} nu) \hat{\phi}(2^{-l} u).$$

We omit the proof of equation (21) which is similar.

Replacing $\Gamma^l(n)$ by its expression in (20) and applying the Poisson formula (Schwartz [15]) leads to equation (22).

Let us now consider the links between the analyses of the process x taken at various resolutions.

PROPOSITION 2. — *Consider two resolutions l and l' with $l' \geq l$. Then we have the following relations:*

$$\Gamma^l(n) = \int_{\mathbf{T}} \left| \prod_{k=1}^{l'-1} H(2^{k-1}) \right|^2 \exp(i 2^{l'-l} n \omega) f^{l'}(\omega) d\omega \tag{23}$$

$$\Lambda^l(n) = \int_{\mathbf{T}} \left| \prod_{k=1}^{l'-l-1} H(2^{k-1}) \right|^2 |G(2^{l'-l-1} \omega)|^2 \times \exp(i 2^{l'-l} n \omega) f^{l'}(\omega) d\omega \quad (24)$$

Proof of Proposition 2. – First note that the result of the impulse response h defined in (7) is almost a convolution product. Indeed, the sequence $(X_n^l)_{n \in \mathbf{Z}}$ is obtained by filtering recursively the sequence $(X_n^{l'})_{n \in \mathbf{Z}}$ ($l' - l$)-times with the impulse response h , as defined in (7).

The proof is then similar to the proof of (20): following the steps of the proof of (20), but for discrete functions, (23) and (24) is proved.

3. STUDY OF THE WAVELET TRANSFORM OF THE PROCESS x

In this section, we study the behaviour of the integrated square error between the restriction \tilde{x} of the signal x to the time-interval $[-T, T]$ and its projection $P_j(\tilde{x})$, measured with the semi-norm defined in (19).

3.1. Almost sure convergence of the integrated square error

Assume that the process x is observed on a time-interval $[-T, T]$ and define \tilde{x} by:

$$\begin{cases} \tilde{x}(t, \omega) = x(t, \omega) & \text{for } t \text{ in } [-T, T] \\ \tilde{x} = 0 & \text{elsewhere.} \end{cases}$$

For sake of simplicity, ω is omitted in the sequel. The sample path \tilde{x} is now in $L^2(\mathbf{R})$ and using (3), \tilde{x} is associated with its projection $P_j(\tilde{x})$ for all resolutions 2^j . Then it seems natural to define the quadratic error of compression by the quantity:

$$\tilde{\varepsilon}_j^2(T) = \frac{1}{2T} \int_{-T}^T (\tilde{x}(t) - P_j(\tilde{x})(t))^2 dt.$$

Considering the process x , another quantity may also be studied: the quadratic error between x and Z_j , defined in (14):

$$\varepsilon_j^2(T) = \frac{1}{2T} \int_{-T}^T (x(t) - Z_j(t))^2 dt.$$

Because of edge effects, $P_j(\tilde{x})$ and Z_j are different even on $[-T, T]$.

THEOREM 1. – *Under (H 1)-(H 5), we have:*

$$1. \quad \tilde{\varepsilon}_j^2(T) \rightarrow \sum_{l \geq j} 2^l \Lambda^l(0) \text{ as } T \rightarrow \infty \quad (\text{a. s.}),$$

$$2. \quad \varepsilon_j^2(T) \rightarrow \|x - Z_j\|_s^2 = \sum_{l \geq j} 2^l \Lambda^l(0) \quad \text{as } T \rightarrow \infty \quad (\text{a. s.}).$$

Note that an immediate consequence of Theorem (1) is that:
 $\lim_{j \rightarrow \infty} \|x - Z_j\|_s = 0.$

Proof of Theorem 1. — One has:

$$\tilde{\varepsilon}_j^2(T) = \sum_{l \geq j, l' \geq j, n, n' \in \mathbf{Z}^2} A_{n, n'}^{l, l'}$$

where

$$A_{n, n'}^{l, l'} = \left(\int_{\mathbf{R}^2} \tilde{x}(u) \tilde{x}(u') \psi_n^l(u) \psi_{n'}^{l'}(u') du du' \right) \left(\int_{-T}^T \psi_n^l(u) \psi_{n'}^{l'}(u') du du' \right).$$

For each l, l' such that $l \geq l' \geq j$, let us introduce the following subspace of \mathbf{Z}^2 or \mathbf{Z} in order to study the $A_{n, n'}^{l, l'}$:

$$E_l^1 = \{n \in \mathbf{Z}, \text{supp}(\psi_n^l) \subset [-T, T]\},$$

the set E_l^1 represents the main term of $\tilde{\varepsilon}_j^2(T)$.

$$\begin{aligned} E_{l, l'}^2 &= \{n, n' \in \mathbf{Z}^2, \text{supp}(\psi_n^l) \cap [-T, T] = \emptyset\} \\ &\cup \{n, n' \in \mathbf{Z}^2, \text{supp}(\psi_{n'}^{l'}) \cap \text{supp}(\psi_n^l) = \emptyset\} \\ &\cup \{n, n' \in \mathbf{Z}^2, \text{supp}(\psi_{n'}^{l'}) \subset \text{supp}(\psi_n^l) \text{ and } \text{supp}(\psi_{n'}^{l'}) \subset [-T, T]\} \\ &\cup \{n, n' \in \mathbf{Z}^2, \text{supp}(\psi_n^l) \subset [-T, T] \text{ and } \text{supp}(\psi_{n'}^{l'}) \subset [-T, T] \\ &\quad \text{and } l \neq l' \text{ or } n \neq n'\}. \end{aligned}$$

Clearly, using the orthonormality of functions ψ_n^l , the integral $\int_{-T}^T \psi_n^l(u) \psi_{n'}^{l'}(u') du du'$ will be equal to 0 if (n, n') are in $E_{l, l'}^2$.

If $l = l'$ let $E_{l, l'}^3$ be: $E_{l, l'}^3 = \mathbf{Z}^2 - E_l^1 \times E_l^1 - E_{l, l'}^2$.

Elsewhere let $E_{l, l'}^3$ be: $E_{l, l'}^3 = \mathbf{Z}^2 - E_{l, l'}^2$.

The set $E_{l, l'}^3$ contains all the elements n and n' of \mathbf{Z}^2 such that the support of ψ_n^l or $\psi_{n'}^{l'}$ intersect $[-T, T]$ and are not included in $[-T, T]$. They induce what may be called the edges effect. Clearly the number of elements of $E_{l, l'}^3$ is such that $\#E_{l, l'}^3 \leq 2([b-a]+1)^2$, where $[x]$ denote the integer part of x and $\#E$ the cardinal of the set E . Note that this cardinal is bounded by a quantity which is independent of l, l' and T . Let us now evaluate $A_{n, n'}^{l, l'}$. The continuous sample path $x(u)$ is bounded in probability. We overestimate $x(u)$ by $\sup\{|x(u)|, u \in \text{supp}(\psi_n^l)\}$. Since x is stationary, the distribution of this supremum is independent of the translation index n . Then we take $\sup_{\text{supp}(\psi_b)} |x(u)|$ as an upper bound. Therefore using the above mentioned results and the definition of $A_{n, n'}^{l, l'}$, one obtains:

$$\Pr[|A_{n, n'}^{l, l'}| > M] \leq (\Pr[\sup_{\text{supp}(\psi_b)} |x(u)| > M])^2 2^{-l/2} 2^{-l'/2} (b-a)^2 \|\Psi\|_\infty^4.$$

The summation $\sum_{E_{l,l'}^1}$, $A_{n,n'}^{l,l'}$ is therefore bounded in probability. Hence,

when $T \rightarrow \infty$, the edge effect, which are equal to $\frac{1}{2T} \sum_{l \geq j, l' \geq j, n, n' \in E_{l,l'}^1} A_{n,n'}^{l,l'}$

are asymptotically negligible.

It remains to study the asymptotic behaviour of the main term of $\varepsilon_j^2(T)$, which is equal to:

$$\frac{1}{2T} \sum_{l \geq j} \sum_{n \in E_l^1} \int_{\mathbf{R}^2} x(u) x(u') \psi_n^l(u) \psi_n^l(u') du du'.$$

Clearly, for any fixed $l \in \mathbf{Z}$:

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \# E_l^1 = 2^l$$

It follows from (H 5), that the process Y^l is ergodic for each l and an application of the law of large numbers yields:

$$\frac{1}{2T} \sum_{n \in E_l^1} \int_{\mathbf{R}^2} x(u) x(u') \psi_n^l(u) \psi_n^l(u') du du' \rightarrow 2^l \Lambda^l(0) \quad (\text{a. s.})$$

One obtains the first part of Theorem 1, concerning the asymptotic behaviour of $\varepsilon_j^2(T)$, using (H 4) and the dominated convergence theorem.

Noting that:

$$\begin{aligned} \tilde{\varepsilon}_j^2(T) &= \left(T + \frac{b-a}{2^j}\right) \frac{1}{2T} \varepsilon_j^2\left(T + \frac{b-a}{2^j}\right) \\ &\quad + \frac{1}{2T} \int_T^{T + ((b-a)/2^j)} (\tilde{x} - \sum_{n \in \mathbf{Z}} \langle \tilde{x}, \phi_n^j \rangle \phi_n^j)^2 \\ &\quad - \frac{1}{2T} \int_{-T - ((b-a)/2^j)}^{-T} (\tilde{x} - \sum_{n \in \mathbf{Z}} \langle \tilde{x}, \phi_n^j \rangle \phi_n^j)^2, \end{aligned}$$

and using the same evaluation as for the edges effect, one obtains that the difference between $\varepsilon_j^2(T)$ and $\tilde{\varepsilon}_j^2(T)$ is asymptotically negligible. Hence one obtains the second part of Theorem 1.

3.2. Rate of convergence of the integrated quadratic error

We shall now study the rate of convergence of the square error $\tilde{\varepsilon}_j^2(T)$ to $\sum_{l \geq j} 2^l \Lambda^l(0)$.

Here we shall assume that the covariance function of x verifies the following ergodic assumption:

$$(H6) \quad \int_{\mathbf{R}} |t \Gamma(t)| dt < \infty$$

Then we have the following lemma:

LEMMA 1. — Under (H6), the covariance function of X^l verifies:

$$\sum_{k \in \mathbf{Z}} |k \Gamma^l(k)| < \infty.$$

Proof of Lemma 1. — From Proposition 1:

$$\Gamma^l(n) = \int_{\mathbf{R}^2} \Gamma(u-u') \phi_0^l(u) \phi_n^l(u') du du'.$$

Since $\phi_{u_i}^l$ is compactly supported and continuous

$$\sum_{k \in \mathbf{Z}} |k \Gamma^l(k)| \leq 2^l (B-A) \|\phi\|_{\infty}^2 \int_{\mathbf{R}} |u \Gamma(u)| du < \infty$$

which ens the proof of Lemma 1.

Let us now consider the square error of subsampling defined by:

$$\tilde{\xi}_{j,J}^2(T) = \frac{1}{2T} \int_{-T}^T (P_j(\tilde{x}) - P_j(\tilde{x}))^2(t) dt \quad \text{for } j \leq J.$$

LEMMA 2. — Under (H6)

$\forall k, j \leq k \leq J (2T)^{1/2} (\tilde{\xi}_{k,k+1}^2(T) - 2^k \Lambda^k(0))_{j \leq k \leq J}$ converges in distribution to $(\xi_k)_{j \leq k \leq J}$ where $(\xi_k)_{j \leq k \leq J}$ is a Gaussian vector, with mean 0 and covariance matrix equal to:

$$\text{cov}(\xi_k, \xi_{k'}) = 4\pi 2^{-J} 2^k 2^{k'} \int_{\mathbf{T}} A_k^J(\alpha) A_{k'}^J(\alpha) (f^J(\alpha))^2 d\alpha$$

where $A_l^J(\omega) = \left| \prod_{k=1}^{l'-l-1} H(2^{k-1}\omega) G(2^{l'-l-1}\omega) \right|^2$

From relation (24) $\Lambda^J(0) = \int_{\mathbf{T}} A_l^J(\omega) f^J(\omega) d\omega.$

A central limit theorem can be applied to the spectrogram $\Lambda^l(0)$ (see for instance Azencot and Dacunha-Castelle [1], p. 108), and Lemma 2 is obtained with a rate of convergence equal to $(2T)^{1/2}$.

Note that $(2T)^{1/2} \left(\frac{1}{2T} \sum_{l \geq j, l' \geq j, n, n' \in E_{l,l'}^2} A_{n,n'}^{l,l'} \right) \rightarrow 0$ (a. s.). Hence there is no edge effects.

LEMMA 3. — *The serie $V_j = \sum_{k \geq j, k' \geq j} \text{cov}(\xi_k, \xi_{k'})$ is convergent.*

Proof of Lemma 3. — Relation (9) yields:

$$A_k^J(\omega) \leq 1 \cdot \int_{\mathbf{T}} A_k^J(\alpha) A_{k'}^J(\alpha) f^J(\alpha) f^J(\alpha) d\alpha$$

is then overestimated by

$$\int_{\mathbf{T}} f^J(\alpha) f^J(\alpha) d\alpha \leq \sup_{[0, 2\pi]} f^J(\alpha) \times \Gamma^J(0).$$

Using relation (22) leads to:

$$f^l(\lambda) = 2\pi \sum_{k \in \mathbf{Z}} f(2^l(2\pi k - \lambda)) |\hat{\phi}(2\pi k - \lambda)|^2.$$

Then, when $J \rightarrow \infty$, $\sup_{[0, 2\pi]} f^J(\alpha) \rightarrow 2\pi f(0)$.

Using relation (20), $\lim_{x \rightarrow 0} \hat{\phi}(x) = 1$ and the dominated convergence theorem yields: $\Gamma^J(0) \leq K 2^{-J}$ for a constant K which is independent of J .

To sum up, $\int_{\mathbf{T}} A_k^J(\alpha) A_{k'}^J(\alpha) f^J(\alpha) f^J(\alpha) d\alpha$ is of order $O(2^{-J})$.

We can now state the main result of this section: the rate of convergence of the integrated square error is of order $(2T)^{1/2}$.

THEOREM 2. — *Under assumptions (H 1)-(H 6), one has, as $T \rightarrow \infty$:*

$$(2T)^{1/2} (\tilde{\varepsilon}_j^2(T) - \sum_{l \geq j} 2^l \Lambda^l(0)) \xrightarrow{D} N(0, V_j)$$

Proof of Theorem 2. — By Lemma 2

$$(2T)^{1/2} ((\tilde{\varepsilon}_{j,1}^2(T) - \sum_{J \geq l \geq j} 2^k \Lambda^k(0)) \xrightarrow{D} N(0, \sum_{k \geq j, k' \geq j} \text{cov}(\xi_k, \xi_{k'}))$$

By Lemma 3:

$$\sum_{k \geq j, k' \geq j} \text{cov}(\xi_k, \xi_{k'}) \rightarrow V_j$$

Therefore, using Billingsley [2] p. 25, yields:

$$(2T)^{1/2} (\tilde{\varepsilon}_j^2(T) - \sum_{l \geq j} 2^l \Lambda^l(0)) \xrightarrow{D} N(0, V_j)$$

This completes the study of the integrated square error between the process and its wavelet transform. This study will be of great use in the next section, in which we explain the influence of the regularity of the wavelet on this error.

3.3. Convergence in distribution

Let us now study the convergence of Z_l to x in the space C equipped with the topology of the continuous function from \mathbf{R} to \mathbf{R} , as defined in Billingsley [2] or Pollard [12].

Let us introduce an additional assumption on the wavelet base:

(H7) The regularity of the wavelet verifies: $r \geq 1$.

Clearly, under (H7), the processes Z_l have sample paths which are continuous and differentiable.

THEOREM 3. — Under (H1), (H2) and (H7), Z_l converges in distribution to the process x .

Proof of Theorem 3. — Let us first study the convergence of the finite-dimensional distributions.

Since Z_l is a Gaussian process, we just need to prove that $\text{cov}(Z_l(t), Z_l(t'))$ converges for all (t, t') . One has:

$$\text{cov}(Z_l(t), Z_l(t')) = 2^l \sum_{n, n' \in \mathbf{Z}} \Gamma^l(n-n') \phi(2^l t - n) \phi(2^l t' - n')$$

This summation is done on the (n, n') such that:

$$t - t' - \frac{B-A}{2^l} \leq \frac{n-n'}{2^l} \leq t - t' + \frac{B-A}{2^l}$$

(Recall that $\text{supp}(\phi) = [A, B]$.) It is well known that: $\lim_{x \rightarrow 0} \hat{\phi}(x) = 1$.

Applying the dominated convergence to relation (20) yields for l_0 large enough:

$$l \geq l_0 \quad |2^{-l} \Gamma^l(n-n') - \Gamma(t-t')| \leq \varepsilon \text{ when } \varepsilon \text{ is independent of } n \text{ and } n'.$$

$$\text{According to Meyer [11], } \forall x \in \mathbf{R}, \sum_{n \in \mathbf{Z}} \phi(x-n) \equiv 1.$$

It follows that, for l large enough, $\text{cov}(Z_l(t), Z_l(t')) \sim \Gamma(t-t')$.

It remains now to prove the tightness of the sequence Z_l .

Recall that the modulus of continuity of x on a compact set K is defined as:

$$\Delta > 0 \quad \omega(x, \Delta, K) = \sup_{|t-t'| \leq \Delta, t, t' \in K} |x(t) - x(t')|,$$

and note that: $\lim_{l \rightarrow \infty} \omega\left(x, \frac{1}{2^l}, K\right) = 0$.

For all $(t, t') \in K$, introduce p and q such that: $p = [2^l t] + 1$ and $q = [2^l t']$.

One has:

$$|Z_l(t) - Z_l(t')| \leq \left| Z_l\left(\frac{p}{2^l}\right) - Z_l\left(\frac{q}{2^l}\right) \right| + \left| Z_l\left(\frac{p}{2^l}\right) - Z_l\left(\frac{q}{2^l}\right) \right| + \left| Z_l\left(\frac{q}{2^l}\right) - Z_l(t') \right|.$$

As $Z_l\left(\frac{p}{2^l}\right) - Z_l\left(\frac{q}{2^l}\right) = 2^{l/2} \sum_{n \in \mathbf{Z}} [X^l(n+p-q) - X^l(n)] \phi(p-n)$ we obtain the inequality:

$$\begin{aligned} & \left| Z_l\left(\frac{p}{2^l}\right) - Z_l\left(\frac{q}{2^l}\right) \right| \\ & \leq \|\phi\|_\infty \sum_{\substack{n, \phi(p-n) \neq 0, \phi(q-n) \neq 0}} \int \left| x\left(\frac{v+n}{2^l}\right) - x\left(\frac{v+n+q-p}{2^l}\right) \right| |\phi(v)| dv \\ & \leq \|\phi\|_\infty^2 2([\mathbf{B} - \mathbf{A}] + 1) \omega\left(x, \frac{1}{2^l}, \mathbf{K}\right). \end{aligned}$$

Then, for $l \geq l_0$

$$\left| Z_l\left(\frac{p}{2^l}\right) - Z_l\left(\frac{q}{2^l}\right) \right| \leq \|\phi\|_\infty^2 2([\mathbf{B} - \mathbf{A}] + 1) \omega\left(x, \frac{1}{2^{l_0}}, \mathbf{K}\right).$$

Using the Taylor expansion of $Z_l(t)$ up to the first order:

$$\left| Z_l(t) - Z_l\left(\frac{p}{2^l}\right) \right| \leq 2^{l/2} \left| \sum_{n \in \mathbf{Z}} X^l(n) \phi'(2^l c - n) \right|,$$

with $t \leq c \leq \frac{p}{2^l}$ and ϕ' being the derivative of ϕ .

Note that $\sum_{n \in \mathbf{Z}} \phi(x-n) \equiv 1$ implies $\sum_{n \in \mathbf{Z}} \phi'(x-n) \equiv 0$.

Let \mathbf{N}_c^1 be equal to $[2^l c - \mathbf{A}]$ and let \mathbf{N}_c^2 be equal to $[2^{l'} c - \mathbf{B}]$.

One has:
$$\left| Z_l(t) - Z_l\left(\frac{p}{2^l}\right) \right| \leq 2^{l/2} \sum_{n=\mathbf{N}_c^1}^{\mathbf{N}_c^2-1} |X^l(n) - X^l(\mathbf{N}_c^2)| |\phi'(2^l c - n)|.$$

For $l \geq l_0$:

$$\left| Z_l(t) - Z_l\left(\frac{p}{2^l}\right) \right| \leq \|\phi\|_\infty \|\phi'\|_\infty 2([\mathbf{B} - \mathbf{A}] + 1) \omega\left(x, \frac{1}{2^{l_0}}, \mathbf{K}\right).$$

This proves the tightness of the sequence Z_l on every compact set of \mathbf{R} , which together with the convergence of the finite-dimensional distributions, achieves the proof of Theorem 3.

4. APPLICATIONS

This work was first motivated by picture compression with wavelets: a picture can be compressed using the pyramidal algorithm of Mallat described in Section 2.1. We first study the compression error of a Gaussian stationary process. Since the multiresolution analysis of $L^2(\mathbf{R}^2)$ is constructed by a tensorial product of a multiresolution of $L^2(\mathbf{R})$, we shall study the one-dimensional case. This quadratic error of compression is described by Theorem 1 and Theorem 2 when the time-interval T increases to ∞ . We prove in this Section that this error depends on the asymptotic behaviour of the variance of the details of this process, when the resolution 2^j increases to ∞ . This result allows to choose the wavelet which minimizes the compression error of a Gaussian stationary process.

Another study will be the uncorrelation of the approximation and the details of a Gaussian stationary process. Sometimes, when a wavelet decomposition is applied to a process, it may be interesting to get the smallest correlation between the approximation of x and the detail (Rosenfeld and Kak [13], ch. 5). Indeed the more the details and the summary are uncorrelated, the more a separate analysis of these two quantities is justified: it is easier to work separately on the approximation and on the details. This is in fact an important goal in image analysis to obtain uncorrelated pictures (Rosenfeld and Kak [13], ch. 5)

Let us first recall what we mean by data-compression (*see* for instance Rosenfeld and Kak [13], ch. 5): let $x(t)$ be a continuous signal observed on $[-T, T]$. The signal x is usually compressed in the following way: first an orthonormal base ϕ_n of $L^2([-T, T])$ is chosen. Then the signal is sampled using: $a_n = \langle x, \phi_n \rangle$, $n = 1, \dots, N$. Finally the sequence $(a_n)_{n=1, \dots, N}$ is stored, and the original signal is approximated by: $\tilde{x}(t) = \sum_{n=1}^N a_n \phi_n(t)$. Hence the mean square error is: $\varepsilon_N^2 = E \left(\sum_{n=1}^N a_n^2 \right)$.

We are now able to study the properties of various wavelet orthonormal bases according to the regularity assumed on the original signal x .

4.1. Data compression with wavelets

A wavelet base does not generate $L^2(-T, T)$ but $L^2(\mathbf{R})$. Nevertheless, we can compress a signal observed on an interval $[-T, T]$ by storing the $\langle \tilde{x}, \phi_n^j \rangle$ at resolution 2^j . The integrated quadratic error of compression is characterised in Theorem 1 and the asymptotic behaviour of the square error of compression in Theorem 2. We obtain an exact expression of the limit of this integrated square error when the process is Gaussian and

stationary.

$$\tilde{\xi}_j^2(T) \rightarrow \sum_{l \geq j} 2^l \Lambda^l(0) \quad (\text{a. s.})$$

Therefore, the performance of the compression depends on the asymptotic properties of $\Lambda^l(0)$.

We shall need the following assumption:

(H8) The spectral density of x satisfies: $|f(\lambda)| \leq C|\lambda|^{-\alpha}$, $\alpha > 1$.

Assumption (H8) ensures that for all the integer d such that $2d + 1 < \alpha$, the covariance function of x possesses derivatives at 0 up to order $2d$; according to Cramer and Leadbetter [4] x is then d -times derivable in quadratic mean.

We can now state the following results on $\Lambda^l(0)$, which was directly obtained by Cohen, Froment and Istas [3]:

PROPOSITION 3. — *Let r be the regularity of the multiresolution analysis. Under (H 1)-(H 5) and (H 8)*

1. if $\alpha < 2r + 3$, $\Lambda^l(0) \leq K 2^{-\alpha l}$,
2. if $\alpha > 2r + 3$, $\Lambda^l(0) \sim 2^{-(2r+3)l} \left| \frac{\hat{\Psi}^{(r+1)}(0)}{(r+1)!} \right|^2 \Gamma^{2r+2}(0) (-1)^{r+1}$,
3. if $\alpha = 2r + 3$, $\Lambda^l(0) \leq K 2^{-(\alpha+\varepsilon)l}$, $\forall \varepsilon > 0$.

An immediate consequence of Proposition 3 is the following one: if the regularity of the analysing wavelet is lower than the derivability of the process (in quadratic mean), the rate of convergence is driven by r . Therefore, it is better to increase the regularity of the analysing wavelet in order to minimize the integrated square error of the compression. This leads to the choice of a wavelet base which has regularity $r = \inf \{s \in \mathbf{N}, 2s + 3 > \alpha\}$. From Daubechies [6], the size of the support of the wavelet increases with its regularity: there exists c and C such that $[-cr, cr] \subset \text{supp}(\psi) \subset [-Cr, Cr]$. That is the reason why it is better to choose in practice r such that $r = \inf \{s \in \mathbf{N}, 2s + 3 > \alpha\}$ in order to minimize the effects of the edges and the computations.

Proof of Proposition 3. — 1. We have:

$$\Lambda^l(0) = \int f(2^l \omega) |\hat{\Psi}(\omega)|^2 d\omega \leq C 2^{-\alpha l} \int \frac{|\hat{\Psi}(\omega)|^2}{|\omega|^2} d\omega$$

Noting that $|\hat{\Psi}(\omega)|^2$ is bounded on \mathbf{R} and that near 0 $\hat{\Psi}(\omega) = O(\omega^{r+1})$, one obtains that the integral $\int \frac{|\hat{\Psi}(\omega)|^2}{|\omega|^2} d\omega$ converges.

2. The integral $\Lambda^l(0)$ splits into two terms: $\Lambda^l(0) = \Lambda_1^l + \Lambda_2^l$ with

$$\Lambda_1^l = 2^{-l} \int_{|\omega| < \Lambda 2^l} f(\omega) |\hat{\Psi}(2^{-l} \omega)|^2 d\omega$$

and with

$$\Lambda_2^l = \int_{|\omega| > A} f(2^{-l}\omega) |\hat{\Psi}(\omega)|^2 d\omega.$$

For an arbitrary $\varepsilon > 0$, the constant A is chosen in order to replace $\hat{\Psi}(2^{-l}\omega)$ by its Taylor expansion at 0:

$$\Lambda_1^l = 2^{-(2r+3)l} \left(\int_{|\omega| < A2^l} f(\omega) \left| \frac{\hat{\Psi}^{(n+1)}(0)}{(n+1)!} \right|^2 |\omega|^{2n+2} d\omega + \varepsilon(A) \right).$$

Λ_2^l is bounded by $C 2^{-l\alpha} \int_{|\omega| > A} \frac{|\hat{\Psi}(\omega)|^2}{|\omega|^\alpha} d\omega$ and therefore is an $o(2^{-(2r+3)l})$.

3. Noting that for all $\varepsilon > 0$, we have: $f(\omega) \leq C |\omega|^{(-\alpha+\varepsilon)}$ and

$$\Lambda^l(0) \leq C 2^{-(\alpha+\varepsilon)l} \int \frac{|\hat{\Psi}(\omega)|^2}{|\omega|^{(\alpha-\varepsilon)}} d\omega$$

At point 0, the integral converges because $|\hat{\Psi}(\omega)|^2 = O(|\omega|^{\alpha-1})$.

Remark 1. — Clearly, Proposition 3 shows that (H 8) implies (H 4).

Remark 2. — Assumption (H 6) implies assumption (H 5).

Indeed, we know from (21) that:

$$\Lambda^l(n) = \int_{\mathbb{R}^2} \Gamma(u-u') \psi_0^l(u) \psi_n^l(u') du du'.$$

The effective interval of integration is equal to:

$$I_n = \text{supp}(\psi_0^l) \times \text{supp}(\psi_n^l) = \left[\frac{a}{2^l}, \frac{b}{2^l} \right] \times \left[\frac{a+n}{2^l}, \frac{b+n}{2^l} \right].$$

When $n \rightarrow \infty$, we have $\sup_{(u, u') \in I_n} |u - u'| \rightarrow \infty$. We know that ψ is bounded

so that: $\Lambda^l(n) \leq (\sup \psi)^2 \left(\frac{b-a}{2^l} \right)^2 \sup_{I_n} \Gamma(u-u')$.

4.2. Asymptotic uncorrelation

The correlation between the approximation $x - P_j(x)$ and the detail $P_{j+1}(x) - P_j(x)$ is given by:

$$\begin{aligned} C(n, p) &= E[\langle x, \phi_n^l \rangle \langle x, \psi_p^l \rangle] \\ &= 2^{-l} \int f(\lambda) \hat{\phi}(2^{-l}\lambda) \hat{\psi}(2^{-l}\lambda) \exp\left(i \frac{n-p}{2^l} \lambda\right) d\lambda. \end{aligned}$$

TABLE. — Value of the constant in the square error of compression.

Base	Number of zero-moment	$ \hat{\psi}^{r+1}(0) ^2$
haar	1	1
db04	2	0.094
db06	3	0.176
db08	4	0.616
db10	5	3.461
db12	6	28.55

Indeed, the numer $\frac{n-p}{2^l}$ is the distance between the two samples at resolution 2^l . We shall study the asymptotic behaviour of $C(n, p)$ by considering two points located at the same distance into the various resolutions. It is well known that $\lim_{x \rightarrow 0} \hat{\phi}(x) = 1$.

Therefore, as in Proposition 3, three cases are distinguished:

1. $\alpha < r + 3, \quad C(n, p) \leq K 2^{-\alpha l};$
2. $\alpha > r + 3, \quad C(n, p) \sim O(2^{-(r+3)l});$
3. $\alpha = r + 3, \quad \forall \varepsilon > 0, \quad C(n, p) \leq K' 2^{-(\alpha+\varepsilon)l}.$

Here our interest is to increases the regularity of the optimal wavelet with regard to the choice in data-compression of $r = \inf \{s \in \mathbb{N}, s + 3 > \alpha\}$. Let us remark that this choice is consistent with the previous choice: increasing the regularity of the wavelet does not improve the data-compression but does not make it worse.

We have seen how to choose the reguarity of a wavelet for data-compression or for asymptotic uncorrelation. Now it remains to choose the best wavelet for a given regularity, that is to say the wavelets for which $\hat{\psi}^{r+1}(0)$ is minimal.

Remark 3. — When the impulse response is known, we propose a method to compute this constant $\hat{\psi}^{r+1}(0)$. In fact, it follows from (8) and (10) that:

$$2^{r+1} \hat{\psi}^{r+1}(0) = G^{r+1}(0) = \sum_{n \in \mathbb{Z}} g(n) n^{r+1} \bar{r}^{r+1}$$

In Table, we denote by dbk the wavelet constructed by Daubechies [6] the impulse response of which has k non-zero coefficients.

The number of zero-moment is the number of moments of the wavelet which are equal to zero.

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