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Some remarks and examples concerning the transience and recurrence of random diffusions

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ABSTRACT. – Let $k(t)$ be an ergodic Markov chain on $E=\{1,2,\ldots,n\}$ and let

$$L_k=\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x;k)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d} b_i(x;k)\frac{\partial}{\partial x_i},$$

for $k=1,2,\ldots,n$.

Then for each realization $k(t)=k(t,\omega)$ of the Markov chain, $L_{k(t)}$ may be thought of as a time inhomogeneous diffusion generator. The process it generates, $X(t)=X(t;k(\cdot))$, will be called a random diffusion. We give examples such that each L_k generates a positive recurrent (transient) diffusion but such that the random diffusion is a. s. transient (positive recurrent). We also give some results and examples for transience and recurrence in the special case that $(X(t),k(t))$ is a reversible process. Finally, we consider the effects of speeding up or slowing down the jump rate of the Markov chain.

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1. INTRODUCTION

Let $k(t)$, $t \in \mathbb{R}_+$, be an ergodic Markov chain on the finite state space $E=\{1,2,\ldots,n\}$ with generator G and, for each $k \in E$, define on R^d the second order elliptic operator

$$L_k = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x;k) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x;k) \frac{\partial}{\partial x_i}.$$

Then for each realization $k(t)=k(t,\omega)$ of the Markov chain, $L_{k(t)}$ may be thought of as a time inhomogeneous diffusion generator (with coefficients $a_{ij}(x;k(t,\omega))$ and $b_i(x;k(t,\omega))$. We will always assume that $\{a_{ij}(x;k)\}$ is continuous and positive definite and that the $b_i(x;k)$, $i=1,2,\ldots,d$, are bounded on compacts and measurable; it then follows from [8] that $L_{k(t)}$ generates a unique time-inhomogeneous diffusion process $X(t)=X(t;k(\cdot))$. (We allow for the possibility of explosion; if the process explodes then $X(t;k(\cdot))$ is defined up to the explosion time.) We call such a process a diffusion in a random temporal environment or simply a random diffusion. Note that in fact $X(t)$ may be thought of as the first component of the Markov process $(X(t),k(t)) \in R^d \times E$ generated by $L+G$. $\Big($ $L+G$ acts on smooth functions $f(x,k)$ defined on $R^d \times E$ by $(L+G)(f)(x,k)=(L_k f)(x,k)+\sum_{j=1}^{n} G_{kj} f(x,j).\Big)$ From this, the compactness of E and the ergodicity of $k(t)$, it is easy to see that the random diffusion $X(t)$ is almost surely recurrent (transient) if $(X(t),k(t))$ is recurrent (transient). In [6], necessary and sufficient conditions were obtained for the transience or recurrence of a certain class of random diffusions. In certain cases, it turned out that each L_k generated a transient diffusion, but that the random diffusion was recurrent. This was because the directions of transience of the various L_k's cancelled one another out. The question was then raised whether the random diffusion must necessarily be recurrent if all the L_k's are recurrent generators.

In section 2, we construct two somewhat surprising examples on R^+, where the topology plays no role. In the first example, we construct two positive recurrent generators such that the random diffusion obtained by switching from one generator to the other at random times is transient. Then we construct two transient generators such that the random diffusion constructed from switching from one to the other at random times is positive recurrent.

These examples could easily be constructed on R by defining everything symmetrically on R^-. The reason we choose to consider R^+ is that, with regard to example two, it is easy to construct two transient generators whose drifts point in opposite directions and which get cancelled out in the random diffusion. The topology of R^+ does not allow for such cancellation.

It is usually difficult to establish transience or recurrence results for random diffusions and the above discussion shows that in general one cannot deduce transience or recurrence from the corresponding behavior of the L_k's. However, it turns out that if the Markov process $(X(t), k(t))$ is reversible, then the situation is more orderly. The class of random diffusions for which $(X(t), k(t))$ is reversible is described in the following proposition whose proof is given in section 3.

PROPOSITION 1. — *In addition to the ellipticity assumption made above, assume that* $a_{ij}(x;\, k) \in C^2(R^d)$ *and* $b_i(x;\, k) \in C^1(R^d)$ *for each* $k = 1, 2, \ldots, n$. *Then the Markov process* $(X(t), k(t))$ *is reversible if and only if* G *is reversible and the* L_k *are all reversible with a common (σ-finite) invariant density; that is, the* L_k *have the form*

$$L_k = \frac{1}{2} \nabla \cdot a(x; k) \nabla + a(x; k) \nabla Q(x) \cdot \nabla = \frac{1}{2} e^{-2Q(x)} \nabla \cdot a(x; k) e^{2Q(x)} \nabla.$$

The measure with respect to which the process is reversible is the product measure $e^{2Q(x)} dx \times \mu$ *on* $R^d \times E$, *where* μ *is the measure with respect to which* G *is reversible.*

In the reversible case, define $\bar{a}(x) = \sum_{k=1}^{n} a(x;\, k)\, \mu_k$, where $\mu = \{\mu_k\}_{k=1}^{n}$ and $a(x;\, k)$ are as in Proposition 1, and define

$$\bar{L} = \frac{1}{2} \nabla . \bar{a}(x) \nabla + \bar{a}(x) \nabla Q(x) . \nabla.$$

Thus $\bar{L}$ is the operator obtained by averaging the coefficients of L_k with respect to μ, the invariant measure of $k(t)$. We have the following theorem.

THEOREM 1. — *Assume that the Markov process* $(X(t), k(t))$ *is reversible.*

(i) *If at least one of the* L_k*'s is transient, then the random diffusion is a. s. transient.*

(ii) *All of the* L_k*'s are positive recurrent if and only if the random diffusion is a. s. positive recurrent.*

(iii) *If the averaged operator* $\bar{L}$ *is recurrent, then the random diffusion is a. s. recurrent.*

The diffusion generated by $L_k = \frac{1}{2} \nabla . a(x;\, k) \nabla + a(x;\, k) \nabla Q(x) \cdot \nabla$ is positive recurrent if and only if it does not explode and $\int_{R^d} e^{2Q(x)} dx < \infty$. Therefore, in the reversible case, if all the L_k's are recurrent, then they will either all be null-recurrent or all positive recurrent. Theorem 1 then leaves open two questions:

(1) If all the L_k's are null-recurrent, must the random diffusion necessarily be recurrent?

(2) Does the transience or recurrence of $\bar{L}$ determine that of the random diffusion? That is, if $\bar{L}$ is transient must the random diffusion necessarily be transient?

We will answer each of these questions in the negative with an example. The reversible case will be studied in section 3.

In section 4, we consider the random diffusion $X(t)$ corresponding to the Markov process $(X(t), k(t))$ generated by $L+\lambda G$, where λ is a positive parameter. Thus, the larger (smaller) λ is, the faster (slower) the switching rate is. We discuss the behavior of $\nu_\lambda(dx, k)$, the invariant probability density for $(X(t), k(t))$ (assuming that $(X(t), k(t))$ is positive recurrent) as $\lambda \to 0$ and $\lambda \to \infty$.

2. TWO EXAMPLES

Example 1. — L_1 *and* L_2 *are positive recurrent, but the random diffusion is transient.*

Before constructing the example, we provide a bit of intuition. We will construct two diffusions on R^+ with reflection at 0 with generators $L_k = \frac{1}{2}\frac{d^2}{dx^2} + b_k(x)\frac{d}{dx}$, $k=1, 2$. The idea is to construct b_1 such that it is periodic on R^+ with long low mesas interspersed with deep canyons such that the integral of b_1 over a period is negative, which makes L_1 positive recurrent. Then b_2 is chosen to be b_1 shifted by half a period. In effect, the chances are high that when the medium switches from 1 to 2 the process is close to the negative side of a canyon of b_1 and after switching feels the positive drift of b_2, and therefore moves to the next canyon of b_2 in the positive direction, and so on.

Let α denote a positive number to be chosen later. Choose $b_1^{(\alpha)}(x) \in C^\infty(R)$ satisfying

(i) $b_1^{(\alpha)}(x) = \alpha, x \leqq \frac{4}{3}$;

(ii) $\int_0^2 b_1^{(\alpha)}(s)\, ds < 0$;

(iii) $b_1^{(\alpha)}(x+2) = b_1^{(\alpha)}(x)$, for $x \geqq -\frac{1}{3}$.

Now define $b_2^{(\alpha)}(x) \equiv b_1^{(\alpha)}(x+1)$, $x \in \mathrm{R}$. We consider the diffusion generators $\mathrm{L}_k^{(\alpha)} = \frac{1}{2}\frac{d^2}{dx^2} + b_k^{(\alpha)}\frac{d}{dx}$, $k=1, 2$, on R^+ with reflection at 0. [This corresponds to the boundary condition $u'(0)=0$.] The density of the invariant measure corresponding to $\mathrm{L}_k^{(\alpha)}$ with the boundary condition $u'(0)=0$ satisfies the adjoint equation $\tilde{\mathrm{L}}_k^{(\alpha)} v(x)=0$ for $0 \leqq x < \infty$ and $(v' - bv)(0)=0$, and is given by

$$v(y) = \exp\left(\int_0^y 2\, b_k^{(\alpha)}(x)\, dx\right), \qquad 0 \leqq y < \infty.$$

Since $\int_0^\infty \exp\left(\int_0^y 2\, b_k^{(\alpha)}(x)\, dx\right) dy < \infty$ for $k=1, 2$, it follows that an invariant probability measure exists for $\mathrm{L}_k^{(\alpha)}$ and thus $\mathrm{L}_k^{(\alpha)}$ generates a positive recurrent diffusion.

We will show that for α sufficiently large, the Markov process $(\mathrm{X}(t), k(t))$ generated by $\mathrm{L}^{(\alpha)} + \mathrm{G}$ is transient, where $\mathrm{G} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\mathrm{L}^{(\alpha)} = \begin{pmatrix} \mathrm{L}_1^{(\alpha)} \\ \mathrm{L}_2^{(\alpha)} \end{pmatrix}$. As noted in the introduction, it then follows that the random diffusion $\mathrm{X}(t) = \mathrm{X}(t, k(\cdot))$ is almost surely transient.

To prove the transience of $(\mathrm{X}(t), k(t))$, it is enough to find an $r \in [0, \infty)$ and a function $\mathrm{V} \in \mathrm{C}^\infty([r, \infty) \times \{1, 2\})$ satisfying

$$(2.1) \quad \begin{cases} (1)\ \mathrm{V}(r, k) = 0, \quad k=1, 2; \\ (2)\ 0 < \mathrm{V}(x, k) \leqq 1, \quad \text{for } x > r \quad \text{and} \quad k=1, 2; \\ (3)\ (\mathrm{L}_k^{(\alpha)} + \mathrm{G})\, \mathrm{V}(x, k) \geqq 0 \quad \text{for } x \geqq r \quad \text{and} \quad k=1, 2. \end{cases}$$

To see this, let $\tau_y = \inf\{t \geqq 0 : (\mathrm{X}(t), k(t)) \in (-\infty, y] \times \{1, 2\}\}$, for $y \in [0, \infty)$. Then by Ito's formula, for $n > x > r$,

$$\begin{aligned} \mathrm{P}_{x,k}(\tau_r > \tau_n \wedge t) &\geqq \mathrm{E}_{x,k} \mathrm{V}(\mathrm{X}(\tau_r \wedge \tau_n \wedge t), k(\tau_r \wedge \tau_n \wedge t)) \\ &= \mathrm{V}(x, k) + \mathrm{E}_{x,k} \int_0^{\tau_r \wedge \tau_n \wedge t} (\mathrm{L}_{k(s)}^{\alpha} + \mathrm{G})\, \mathrm{V}(\mathrm{X}(s), k(s))\, ds \geqq \mathrm{V}(x, k). \end{aligned}$$

Since the b_k are bounded, clearly $\tau_n \nearrow \infty$ as $n \to \infty$. Thus we obtain

$$\mathrm{P}_{x,k}(\tau_r = \infty) \geqq \mathrm{V}(x, k) > 0.$$

It remains to find a V satisfying properties 1-3 above. To do this, we first choose W_1 and W_2 in $\mathrm{C}^\infty([0, 1])$ satisfying

(i) $\mathrm{W}_k(x) = x$, $0 \leqq x \leqq \frac{1}{6}$, $k=1, 2$;

(ii) $\mathrm{W}_k(x) = \frac{1}{2}x$, $\frac{5}{6} \leqq x \leqq 1$, $k=1, 2$;

(iii) $W_2(x) = \frac{1}{4}, \frac{1}{3} \leqq x \leqq \frac{2}{3}$;

(iv) $W_1'(x) > 0, 0 \leqq x \leqq 1$;

(v) $W_2'(x) > 0$ on $\left[0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right]$;

(vi) $W_1(x) > W_2(x), \frac{1}{3} \leqq x \leqq \frac{2}{3}$.

Now define

$$V(x, k) = \begin{cases} 2^{-[x]} W_k(x - [x]) + \left(1 - \left(\frac{1}{2}\right)^{[x]}\right), & \text{if } [x] \text{ is even} \\ 2^{-[x]} W_{3-k}(x - [x]) + \left(1 - \left(\frac{1}{2}\right)^{[x]}\right), & \text{if } [x] \text{ is odd.} \end{cases}$$

Clearly $V \in C^\infty(R^+)$ and conditions (1) and (2) on V are satisfied for $r = 0$. To verify condition (3), we consider two cases.

Case 1. — $[x]$ *is even and* $k = 1$ *or* $[x]$ *is odd and* $k = 2$. Then

$$V(x, k) = 2^{-[x]} W_1(x - [x]) + \left(1 - \left(\frac{1}{2}\right)^{[x]}\right)$$

and

$$\begin{aligned}(L_k^{(\alpha)} + G) V(x, k) &= L_k^{(\alpha)} V + V(x, 3-k) - V(x, k) \\ &= \alpha V'(x, k) + \frac{1}{2} V''(x, k) + V(x, 3-k) - V(x, k) \\ &= 2^{-[x]}\left(\alpha W_1'(x - [x]) + \frac{1}{2} W_1''(x - [x]) + W_2(x - [x]) - W_1(x - [x])\right).\end{aligned}$$

Thus, by condition (iv) above, $(L_k^{(\alpha)} + G) V(x, k) \geqq 0$ for all $x \geqq 0$ and $k = 1, 2$ provided α is sufficiently large.

Case 2. — $[x]$ *is odd and* $k = 1$ *or* $[x]$ *is even and* $k = 2$. Then

$$V(x, k) = 2^{-[x]} W_2(x - [x]) + \left(1 - \left(\frac{1}{2}\right)^{[x]}\right)$$

and

$$\begin{aligned}(L_k^{(\alpha)} + G) V(x, k) = 2^{-[x]}\Big[& b_k^{(\alpha)}(x) W_2'(x - [x]) \\ & + \frac{1}{2} W_2''(x - [x]) + W_1(x - [x]) - W_2(x - [x]) \Big].\end{aligned}$$

For $x-[x]\in\left[\frac{1}{3},\frac{2}{3}\right]$, W_2 is constant and hence

$$(L_k^{(\alpha)}+G)V(x,k)=2^{-[x]}(W_1(x-[x])-W_2(x-[x]))\geqq 0$$

by condition (vi) above.

For

$$x-[x]\in\left[0,\frac{1}{3}\right)\cup\left(\frac{2}{3},1\right),\ (L_k^{(\alpha)}+G)V(x,k)$$
$$=2^{-[x]}\left(\alpha W_2'(x-[x])+\frac{1}{2}W_2''(x-[x])+W_1(x-[x])-W_2(x-[x])\right).$$

From conditions (iii) and (vi) and the fact that W_2 is smooth, it follows that $\frac{1}{2}W_2''(y)+W_1(y)-W_2(y)>0$ for $y=\frac{1}{3}$ and $y=\frac{2}{3}$. Thus using condition (v), it follows that $(L_k^{(\alpha)}+G)V(x,k)\geqq 0$ for all $x\geqq 0$ and $k=1,2$ provided α is sufficiently large. This completes the proof.

Example 2. – L_1 *and* L_2 *are transient, but the random diffusion is positive recurrent.*

For this example we switch the direction of the drift in the previous example. Let $B_k^{(\alpha)}(x)=-b_k^{(\alpha)}(x)$, $k=1,2$, where $b_k^{(\alpha)}(x)$ is as in Example 1. We consider the diffusion generators $L_k^{(\alpha)}=\frac{1}{2}\frac{d^2}{dx^2}+B_k^{(\alpha)}\frac{d}{dx}$, $k=1,2$ on $[0,\infty)$ with reflection at 0. Since $\int_0^\infty \exp\left(-2\int_0^t B_k^{(\alpha)}(x)\,dx\right)dt<\infty$, $k=1,2$, it follows that the $L_k^{(\alpha)}$ are transient generators [2, Chap. 9]. We will show that for α sufficiently large, the Markov process $(X(t),k(t))$ generated by $L^{(\alpha)}+G$ is positive recurrent, where $G=\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}$ and $L^{(\alpha)}=\begin{pmatrix}L_1^{(\alpha)}\\ L_2^{(\alpha)}\end{pmatrix}$. To demonstrate positive recurrence it is enough to find a $V\in C^\infty(R^+\times\{1,2\})$ satisfying

(i) $$\inf_{x\in R^+} V(x,k)>-\infty,\qquad k=1,2$$

and

(ii) $$\overline{\lim_{x\to\infty}}\,(L_k^{(\alpha)}+G)V(x,k)<0,\qquad k=1,2.$$

To see that this is a sufficient condition for positive recurrence, let V satisfy (i) and (ii) above and, say,

$$(L_k^{(\alpha)}+G)V(x,k)\leqq -\delta<0\quad\text{for}\quad x>c\quad\text{and}\quad k=1,2.$$

Then by Ito's Formula, for $x>c$,

$$E_{x,k}V(X(t\wedge\tau_c),k(t\wedge\tau_c))\leqq V(x,k)-\delta E_{x,k}(\tau_c\wedge t),$$

and letting $t \to \infty$, one obtains $E_{x,k}\tau_c < \infty$ for $x > c$. By Khasminskii's construction [4], $E_{x,k}\tau_c < \infty$ is equivalent to positive recurrence.

Define $W_1, W_2 \in C^\infty([0, 1], R)$ with the following properties:

(i) $W_k(x) = x,\ 0 \leqq x \leqq \frac{1}{6},\ k = 1, 2;$

(ii) $W_k(x) = 2x,\ \frac{5}{6} \leqq x \leqq 1,\ k = 1, 2;$

(iii) $W_2(x) = 1,\ \frac{1}{3} \leqq x \leqq \frac{2}{3};$

(iv) $W_1'(x) > 0,\ 0 \leqq x \leqq 1;$

(v) $W_2'(x) > 0,\ x \in \left[0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right];$

(vi) $W_1(x) \leqq W_2(x),\ 0 \leqq x \leqq 1;$

(vii) $W_1(x) < W_2(x),\ \frac{1}{3} \leqq x \leqq \frac{2}{3}.$

Now define

$$V(x, k) = \begin{cases} W_k(x - [x]) \,.\, 2^{[x]} + 2^{[x]+1}, x \geqq 0 & \text{and} \quad [x] \text{ even} \\ W_{3-k}(x - [x]) \,.\, 2^{[x]} + 2^{[x]+1}, x \geqq 0 & \text{and} \quad [x] \text{ odd.} \end{cases}$$

A calculation similar to that in example one shows that V satisfies $\lim_{x \to \infty} (L_k^{(\alpha)} + G) V(x, k) = -\infty$, if α is sufficiently large.

3. THE REVERSIBLE CASE

For the processes we are considering, a necessary and sufficient condition for reversibility is that there exist a reference measure $\nu(dx, k) \neq 0$ such that

$$(3.1) \quad \sum_{k=1}^{n} \int_{R^d} f_k(x) [L_k g_k + (Gg)_k](x) \nu(dx, k) = \sum_{k=1}^{n} \int_{R^d} g_k(x) [L_k f_k + (Gf)_k](x) \nu(dx, k), \quad \text{for all } f_k, g_k \in C_c^\infty(R^d).$$

We begin with the

Proof of Proposition 1. — We must show that the equality (3.1) holds for all $f_k, g_k \in C_c^\infty(R^d)$ if and only if L is of the form specified in the statement of the proposition and G is reversible. One direction is clear: if μ is the invariant probability measure of G and $e^{2Q(x)}$ is the common invariant density of the L_k's, then (3.1) holds with $\nu(dx, k) = e^{2Q(x)} \mu_k \, dx$.

Now assume that (3.1) holds. First we will show that for all $k \in \mathrm{E}$, $\nu(dx, k) \not\equiv 0$. Let $\mathrm{E}_0 = \{k \in \mathrm{E} : \nu(dx, k) \equiv 0\}$ and assume that E_0 is not empty. Let $k_0 \in \mathrm{E}_0$ and set $f_k(x) \equiv 0$ for $k \neq k_0$ in (3.1). Then the lefthand side of (3.1) vanishes and the righthand side equals

$$\sum_{k \in \mathrm{E} - \mathrm{E}_0} \int_{\mathbf{R}^d} g_k(x) \, \mathrm{G}_{k, k_0} f_{k_0}(x) \, \nu(dx, k).$$

Thus, this must vanish for all f_{k_0}, $g_k \in \mathrm{C}_c^\infty(\mathbf{R}^d)$, all $k \in \mathrm{E} - \mathrm{E}_0$ and all $k_0 \in \mathrm{E}_0$. This implies that $\mathrm{G}_{k, k_0} = 0$, for $k \in \mathrm{E} - \mathrm{E}_0$ and $k_0 \in \mathrm{E}_0$ which contradicts the ergodicity of $k(t)$.

By fixing a $k_0 \in \mathrm{E}$ and taking $g_k = f_k \equiv 0$ for $k \neq k_0$, one sees that each L_k must be reversible, that is, L_k must be of the form

$$\mathrm{L}_k = \frac{1}{2} \nabla \cdot a(x; k) \nabla + a(x; k) \nabla \mathrm{Q}_k(x) \cdot \nabla,$$

in which case the density with respect to which L_k is reversible is $e^{2\mathrm{Q}_k}$. Therefore, ν is of the form $\nu(dx, k) = e^{2\mathrm{Q}_k(x)} \gamma_k \, dx$ where $\gamma_k > 0$ for all $k \in \mathrm{E}$. Without loss of generality, we may assume that $\mathrm{Q}_k(0) = 0$ for all $k \in \mathrm{E}$. For ν of this form, we have

$$\sum_{k=1}^{n} \int_{\mathbf{R}^d} f_k(x) \, \mathrm{L}_k g_k(x) \, \nu(dx, k) = \sum_{k=1}^{n} \int_{\mathbf{R}^d} g_k(x) \, \mathrm{L}_k f_k(x) \, \nu(dx, k),$$

for all $f_k, g_k \in \mathrm{C}_c^\infty(\mathbf{R}^d)$. Thus for (3.1) to hold, we need

$$(3.2) \quad \sum_{k=1}^{n} \int_{\mathbf{R}^d} f_k(x) \, (\mathrm{G} g)_k(x) \, \nu(dx, k) = \sum_{k=1}^{n} \int_{\mathbf{R}^d} g_k(x) \, (\mathrm{G} f)_k \, \nu(dx, k),$$

for all $f_k, g_k \in \mathrm{C}_c^\infty(\mathbf{R}^d)$. Fix k, $m \in \mathrm{E}$ and choose $f_j \equiv 0$ for $j \neq k$ and $g_j \equiv 0$ for $j \neq m$. Substituting this in (3.2) allows us to conclude that $\gamma_k \mathrm{G}_{km} e^{2\mathrm{Q}_k(x)} \equiv \gamma_m \mathrm{G}_{mk} e^{2\mathrm{Q}_m(x)}$. Letting $x = 0$ shows that G is reversible and that $\gamma = \{\gamma_k\}$ is a multiple of $\mu = \{\mu_k\}$. Furthermore, $\mathrm{Q}_k \equiv \mathrm{Q}_m$ whenever $\mathrm{G}_{km} > 0$. Since G is irreducible, it follows that $\mathrm{Q}_1 = \mathrm{Q}_2 = \ldots = \mathrm{Q}_n$. This completes the proof.

In the reversible case, transience or recurrence can be characterized by a simple variational principle. Let $\sum_m = \{|x| < m\}$ and for any function $v = \{v_k\}$ on E, let $\langle v \rangle = \sum_{k=1}^{n} v_k \mu_k$. Then $\langle v \mathrm{G} v \rangle \leqq 0$ with equality if and only if v is a multiple of the vector 1. In the reversible case, this follows from the Rayleigh-Ritz formula and the fact that, by ergodicity, zero is a simple eigenvalue. (In fact the result holds even in the nonreversible case.)

The quadratic form associated with L + G on any domain D, where

$$\mathrm{L} = \begin{matrix} \mathrm{L}_1 \\ \vdots \\ \mathrm{L}_n \end{matrix} \quad \text{and} \quad \mathrm{L}_k = \frac{1}{2}\nabla\cdot a(x;\,k)\nabla + a(x;\,k)\nabla \mathrm{Q}\cdot\nabla,$$

is

$$\mathrm{J_D}(u) = \int_{\mathrm{D}} (\langle \nabla u a \nabla u \rangle (x) - \langle u \mathrm{G} u \rangle (x))\, e^{2\mathrm{Q}(x)}\, dx,$$

where $u = (u_1, \ldots, u_n)$. Let $\Sigma_m = \{x \in \mathrm{R}^d : |x| < m\}$ and let $\mathrm{J}_m = \mathrm{J}_{\Sigma_m - \Sigma_1}$.

THEOREM (Ichihara [3]). – *Let*

$$\lambda_m = \inf_{\substack{u = (u_1, \ldots, u_n) \in \mathrm{W}^{1,2}(\Sigma_m - \Sigma_1,\, e^{2\mathrm{Q}}\, dx) \\ u = 1 \text{ on } \Sigma_1 \\ u = 0 \quad \text{on } \Sigma_m}} \mathrm{J}_m(u).$$

Then λ_m *is monotone decreasing and the reversible process* $(\mathrm{X}(t), k(t))$ *is recurrent if* $\lim_{m \to \infty} \lambda_m = 0$ *and transient if* $\lim_{m \to \infty} \lambda_m > 0$.

Remark. – Ichihara actually proved this theorem, for operators of the form $\frac{1}{2}\nabla\cdot a(x)\nabla$, but it is easy to extend it to the L + G where L is as above (*see* also [7]). We can now give the

Proof of Theorem 1. – Define

$$\mathrm{J}_m^k(v) = \int_{\Sigma_m - \Sigma_1} (\nabla v(x)\, a(x;\,k)\, \nabla v(x))\, e^{2\mathrm{Q}(x)}\, dx$$

for $k = 1, 2, \ldots, n$ and $v \in \mathrm{W}^{1,2}(\Sigma_m - \Sigma_1, e^{2\mathrm{Q}}\, dx)$ and define

$$\lambda_m^k = \inf_{\substack{v \in \mathrm{W}^{1,2}(\Sigma_m - \Sigma_1,\, e^{2\mathrm{Q}}\, dx) \\ v = 1 \text{ on } \Sigma_1 \\ v = 0 \text{ on } \Sigma_m}} \mathrm{J}_m^k(v),$$

Part (i). By Ichihara's theorem, L_k is recurrent or transient according to whether $\lim_{m \to \infty} \lambda_m^k = 0$ or $\lim_{m \to \infty} \lambda_m^k > 0$. Thus, by assumption, there exists a k_0 such that $\lim_{m \to \infty} \lambda_m^{k_0} > 0$. However, we have $\mathrm{J}_m(u) \geqq \mu_{k_0} \mathrm{J}_m^{k_0}(u_{k_0})$ for any $u = (u_1, \ldots, u_n)$, because $-\langle u \mathrm{G} u \rangle \geqq 0$.

We conclude that $\lambda_m \geqq \mu_{k_0} \lambda_m^{k_0}$ and thus $\lim_{m \to \infty} \lambda_m > 0$. By Ichihara's theorem again, it follows that $(\mathrm{X}(t), k(t))$ is transient and thus the random diffusion is almost surely transient.

Part (ii). It is easy to see that the random diffusion explodes if and only if at least one of the L_k's generates an explosive diffusion. Since explosion precludes recurrence, we may assume that the random diffusion and the diffusions generated by the L_k's are all nonexplosive. For reversible nonexplosive processes, positive recurrence is equivalent to the integrability of the measure with respect to which the semigroup and generator are reversible. (For diffusions, this follows from Chapter 31 in [9]. It can easily be extended to random diffusions.) Thus, under the assumption of no explosion, all of the L_k's generate positive recurrent diffusions if and only if $\int_{R^d} e^{2Q(x)} dx < \infty$ and the random diffusion is positive recurrent if and only if

$$\sum_{k=1}^{n} \mu_k \int_{R^d} e^{2Q(x)} dx = \int_{R^d} e^{2Q(x)} dx < \infty.$$

Part (iii). Let $\bar{J}_m(v) = \int_{\Sigma_m - \Sigma_1} (\nabla v \bar{a} \nabla v)(x) e^{2Q(x)} dx$ and let

$$\bar{\lambda}_m = \inf_{\substack{v \in W^{1,2}(\Sigma_m - \Sigma_1, e^{2Q} dx) \\ v = 1 \text{ on } \Sigma_1 \\ v = 0 \text{ on } \Sigma_m}} \bar{J}_m(v).$$

By Ichihara's Theorem, $\bar{L}$ is recurrent or transient according to whether $\lim_{m \to \infty} \bar{\lambda}_m = 0$ or $\lim_{m \to \infty} \bar{\lambda}_m > 0$. We have

$$\lambda_m \leqq \inf_{\substack{u = (u_1 u_2, \dots u_n) \in W^{1,2}(\Sigma_m - \Sigma_1, e^{2Q}(dx)) \\ u = 1 \text{ on } \Sigma_1, u = 0 \text{ on } \Sigma_m \\ u_1 = u_2 = \dots = u_n}} J_m(u) = \bar{\lambda}_m,$$

where the equality follows because $G u = 0$ if $u_1 = u_2 = \dots = u_n$.

Since by assumption, $\lim_{m \to \infty} \bar{\lambda}_m = 0$, we obtain $\lim_{m \to \infty} \lambda_m = 0$ and the random diffusion is almost surely recurrent.

We now turn to the two examples as mentioned in the introduction.

Example 3 (Reversible case). – L_1 *and* L_2 *are null recurrent, but the random diffusion is transient.*

Before constructing the example, we provide a bit of intuition. Choose $Q \equiv 0$ (that is, constant invariant density) and choose a_1 such that it has broad and very high mesas interspersed with very narrow canyons with a_1 almost zero at the bottom of the canyons. As x increases, these properties are to become more and more extreme. Now a_2 is chosen to have the same form but such that each canyon of a_2 is significantly closer to the next canyon of a_1 to the left than to the next canyon of a_1 to the right. Since the canyons are very deep, L_1 and L_2 are (null) recurrent. Starting at large x on a mesa, the distribution will almost immediately become almost uniform over the mesa and, as long as the medium does not switch, it will remain like that for a long time. Therefore the random diffusion is more likely to increase than to decrease between two successive switches. The following construction is based on this intuitive description although it does not follow it in every detail.

Let $=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and consider the generators

$$L_i = \frac{1}{2}\frac{d}{dx} a_i(x) \frac{d}{dx} = \frac{1}{2} a_i(x) \frac{d^2}{dx^2} + \frac{1}{2} a_i'(x) \frac{d}{dx}, \qquad 1 = 1, 2,$$

on the half line $[0, \infty)$ with reflection at 0. The recurrence of L_i, $i=1,2$, is equivalent to the condition

$$\int_0^\infty \frac{dx}{a_i(x)} = \infty, \qquad i=1,2 \quad [2, \text{Chap. } 9]. \tag{3.3}$$

As in (2.1), in order to show that the random diffusion is transient, it is enough to find functions $V_1, V_2 \in C^\infty([0, \infty))$ satisfying

$$\begin{cases} \frac{1}{2}(a_1 V_1')' + V_2 - V_1 = 0 \\ \frac{1}{2}(a_2 V_2')' + V_1 - V_2 = 0 \end{cases} \tag{3.4}$$

and

$$0 \leqq V_1, V_2 \leqq 1 \tag{3.5a}$$

$$V_1(0) = V_2(0) = 0 \tag{3.5b}$$

$$V_i' > 0, \qquad i=1,2. \tag{3.5c}$$

Solving (3.4) for a_i, $i=1,2$, gives

$$a_1(x) = \frac{2\left(\int_0^x (V_1 - V_2)(s)\, ds + c_1\right)}{V_1'(x)}$$

and

$$a_2(x) = \frac{2\left(\int_0^x (V_2 - V_1)(s)\, ds + c_2\right)}{V_2'(x)},$$

where c_1 and c_2 are arbitrary constants. We will set $c_1 = c_2 = 1$. Since $a_i > 0$, $i = 1, 2$, we obtain the condition

$$(3.5d)\quad \begin{cases} \dfrac{1+\displaystyle\int_0^x (V_1 - V_2)(s)\,ds}{V_1'(x)} > 0 \\ \text{and} \\ \dfrac{1+\displaystyle\int_0^x (V_2 - V_1)(s)\,ds}{V_2'(x)} > 0, \\ \text{for } x \geqq 0. \end{cases}$$

The condition (3.3) in terms of V_1 and V_2 now reads as

$$(3.5e)\quad \int_0^\infty \frac{V_1'(x)\,dx}{1+\displaystyle\int_0^x (V_1 - V_2)(s)\,ds} = \int_0^\infty \frac{V_2'(x)\,dx}{1+\displaystyle\int_0^x (V_2 - V_1)(s)\,ds} = \infty.$$

Thus to complete the example, we must find functions V_1 and V_2 satisfying (3.5*a*)-(3.5*e*).

We begin with

LEMMA 1. — *Let b_k, c_k, d_k, e_k, $k=0, 1$, be constants satisfying $b_0 < c_0 < d_0 < e_0$ and b_1, c_1, d_1, $e_1 > 0$. Define $A = \dfrac{c_0 - b_0}{4}$ and $B = \dfrac{e_0 - d_0}{4}$. Then there exist $F, G \in C^\infty([0, 1])$ such that*

(i) $F^{(k)}(0) = b_k$, $G^{(k)}(0) = c_k$, $F^{(k)}(1) = e_k$, $G^{(k)}(1) = d_k$, $k = 0, 1$.

(ii) $F^{(k)}(0) = G^{(k)}(0) = F^{(k)}(1) = G^{(k)}(1) = 0$, $k = 2, 3, \ldots$

(iii) $F^{(1)}(x)$; $G^{(1)}(x) > 0$, $x \in [0, 1]$

(iv) $G(x) \geqq F(x)$, $0 \leqq x \leqq \dfrac{1}{4}$

(v) $F(x) \geqq G(x)$, $\dfrac{3}{4} \leqq x \leqq 1$

(vi) $F(x) = G(x) = (d_0 - c_0)\left(x - \dfrac{1}{2}\right) + \dfrac{c_0 + d_0}{2}$, $\dfrac{1}{4} \leqq x \leqq \dfrac{3}{4}$

(vii) $\displaystyle\int_0^{1/4} (G(x) - F(x))\,dx = A$

(viii) $\displaystyle\int_{3/4}^1 (F(x) - G(x))\,dx = B$

Proof. — Obvious.

We now construct V_1 and V_2 as follows. For $i=1, 2, \ldots$, let

$$f_i = 1 - \frac{1}{2^i}, \qquad g_i = 1 - \frac{1}{2^i} + \frac{1}{2^{i+2}}, \qquad l_i = \frac{1}{2^{i+3}}, \qquad A_i = \frac{l_i}{4}, \qquad B_i = \frac{l_{i+1}}{4},$$

$$t_1 = 1, \qquad t_{i+1} = t_i + 1 + \left(2 - \frac{1}{2^i} - \frac{1}{2^{i+1}} - B_i - A_{i+1}\right)\frac{1}{l_{i+1}}, \qquad m_1 = 1$$

and

$$m_{i+1} = \frac{f_{i+1} - g_i}{t_{i+1} - t_i - 1}.$$

Clearly, $t_{i+1} - t_i - 1 > 0$ and $m_i > 0$ for all $i = 1, 2, \ldots$

Define V_1 and V_2 on $[0, t_1]$ such that

$$V_1,\ V_2 \in C^\infty([0, t_1]),$$
$$V_1(0) = V_2(0) = 0, \qquad V_2(x) \geqq V_1(x), \qquad V_1'(x) > 0, \qquad V_2'(x) > 0,$$
$$V_1'(t_1) = V_2'(t_1) = m_1 = 1, \qquad V_1^{(k)}(t_1) = V_2^{(k)}(t_1) = 0,$$

and

$$\int_0^{t_1} (V_2(s) - V_1(s))\, ds = \frac{1}{2} - \frac{1}{64}.$$

For $t_i + 1 \leqq x \leqq t_{i+1}$ define

$$V_1(x) = m_{i+1}(x - t_i - 1) + g_i, \qquad V_2(x) = V_1(x) + l_{i+1}, \quad \text{if } i \text{ is even}$$
$$V_2(x) = m_{i+1}(x - t_i - 1) + g_i, \qquad V_1(x) = V_2(x) + l_{i+1}, \quad \text{if } i \text{ is odd}.$$

For $x \in [t_i, t_i + 1]$ define

$$V_1(x) = G(x - t_i), \qquad V_2(x) = F(x - t_i), \quad \text{if } i \text{ is even}$$
$$V_2(x) = G(x - t_i), \qquad V_1(x) = F(x - t_i), \quad \text{if } i \text{ is odd},$$

where F and G are as in Lemma 1 with $b_0 = f_i$, $c_0 = f_i + l_i$, $d_0 = g_i$, $e_0 = g_i + l_{i+1}$, $b_1 = c_1 = m_i$, $d_1 = e_1 = m_{i+1}$. Note that the assumptions of the lemma are satisfied and that A, B in the lemma coincide with A_i, B_i. It is obvious that $V_1, V_2 \in C^\infty([0, \infty))$ and satisfy 3.5 (*a*), (*b*) and (*c*).

We now consider 3.5(*d*). By observing where $(V_1 - V_2)(x)$ changes sign, it suffices to show 3.5(*d*) for $x = t_i + \frac{1}{2}$, $i = 1, 2, \ldots$

For i even,

$$\int_{t_i + 1/2}^{t_{i+1} + 1/2} (V_2 - V_1)(x)\, dx = B_i + (t_{i+1} - t_i - 1)\, l_{i+1} + A_{i+1} = 2 - \frac{1}{2^i} - \frac{1}{2^{i+1}}.$$

For i odd,

$$\int_{t_i + 1/2}^{t_{i+1} + 1/2} (V_1 - V_2)(x)\, dx = 2 - \frac{1}{2^i} - \frac{1}{2^{i+1}}.$$

Furthermore,

$$\int_0^{t_1+1/2} (V_2-V_1)(x)\,dx = \frac{1}{2} - \frac{1}{64} + A_1 = \frac{1}{2}.$$

Hence, for i even,

$$\int_0^{t_{i+1}+1/2} (V_2-V_1)(x)\,dx = 1 - \frac{1}{2^i}$$

and for i odd,

$$\int_0^{t_{i+1}+1/2} (V_1-V_2)(x)\,dx = 1 - \frac{1}{2^i}.$$

This gives 3.5 (d).

The inequality

$$\int_0^\infty \frac{V_1'(x)\,dx}{1+\int_0^x (V_1-V_2)(s)\,ds} \geqq \sum_{i \text{ odd}} \int_{t_i+1/4}^{t_i+3/4} \frac{V_1'(x)\,dx}{1+\int_0^x (V_1-V_2)(s)\,ds}$$
$$= \sum_{i \text{ odd}} \frac{1}{2} \frac{g_i-f_i-l_i}{-1+(1/2^{i-1})+1} = \sum_{i \text{ odd}} \frac{1}{2} 2^{i-1} \frac{1}{2^{i+3}} = \infty,$$

and an analogous one for

$$\int_0^\infty \frac{V_2'(x)\,dx}{\int_0^x (V_2-V_1)(s)\,ds+1} \quad \text{give 3.5 } (e).$$

Example 4 (Reversible case). – *The averaged operator* L *is transient but the random diffusion is recurrent.*

Let $L_k = \frac{1}{2}\frac{d}{dx} a_k \frac{d}{dx} + a_k Q' \frac{d}{dx}$ on R for $k=1, 2$ and let $G = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$.

Then the averaged operator $\bar{L}$ is given by $\bar{L} = \frac{1}{2}\frac{d}{dx}\bar{a}\frac{d}{dx} + \bar{a}Q'\frac{d}{dx}$, where $\bar{a}(x) = \frac{1}{2}(a_1+a_2)(x)$. Recall that $L_k(\bar{L})$ is recurrent if and only if

$$\int_0^\infty \frac{e^{-2Q(x)}}{a_k(x)}\,dx = \int_{-\infty}^0 \frac{e^{-2Q(x)}}{a_k(x)}\,dx = \infty$$
$$\left(\int_0^\infty \frac{e^{-2Q(x)}}{\bar{a}(x)}\,dx = \int_{-\infty}^0 \frac{e^{-2Q(x)}}{\bar{a}(x)}\,dx = \infty\right) \quad [2].$$

Choose Q satisfying $\int_{-\infty}^{\infty} e^{2Q(x)}\,dx<\infty$ and $\{g_n\}_{n=-\infty}^{\infty}$ be a sequence of functions defined on $[0, 1)$ satisfying

(i) $g_n \in C^1([0, 1])$
(ii) $g_n(0)=g_n(1)=1$
(iii) $g'_0(0)=g'_n(1)=0$
(iv) $g_n(x)>0$ for $x\in[0, 1]$
(v) $\sum_{n=-\infty}^{\infty}\int_n^{n+1} g_n(x \bmod 1)\, e^{-2Q(x)}\,dx<\infty$

Define

$$a_1(x)=\begin{cases} 1, & \text{if } 2n\leqq x\leqq 2n+1,\ n=0,\ \pm 1,\ \pm 2,\ldots \\ \dfrac{1}{g_n(x \bmod 1)}, & \text{if } 2n-1\leqq x\leqq 2n,\ n=0,\ \pm 1,\ \pm 2,\ldots \end{cases}$$

and define $a_2(x)=a_1(x+1)$. Then $a_k\in C^1(R)$ for $k=1,2$ and it follows from the properties of $\{g_n\}$ and of Q and from an application of the Schwartz inequality on the function $1\equiv e^{Q}e^{-Q}$ that L_k is recurrent for $k=1,2$ and that $\bar{L}$ is transient. (In fact, it turns out that $\bar{L}$ is explosive.) Furthermore, the invariant density of L + G is $e^{2Q(x)}\mu_k$ which is integrable. Thus as long as $(X(t), k(t))$ is nonexplosive, then in fact it is positive recurrent and consequently the random diffusion is almost surely positive recurrent. To prove that $(X(t), k(t))$ is nonexplosive, note that since L_1 and L_2 are recurrent, $X(t)$ can not explode when $k(t)=1$ nor when $k(t)=2$. We have thus given an example where each L_k is recurrent, $\bar{L}$ is transient and the random diffusion is recurrent.

4. THE DEPENDENCE OF $L+\lambda G$ ON λ

The parameter λ can have an effect on transience or recurrence, but not in the reversible case. Indeed, from the variational principle of the previous section, it is easy to see that $L+\lambda G$ generates either a recurrent random diffusion for all $\lambda>0$ or a transient one for all $\lambda>0$. To see that this is not true in general, consider [6, Corollary 2 in the case $\delta=0$] with G replaced by λG. In this section we will discuss the following question for the general not necessarily reversible case. Assuming that $(X(t), k(t))$ is positive recurrent for all large λ or all small λ, how does the invariant probability measure behave as

$$\lambda\to\infty \qquad \text{or} \qquad \lambda\to 0?$$

The case $\lambda\to\infty$ is an example of the "averaging principle". It follows from Khasminskii [5] (*see* also [1], Chap. 4, Sect. 3) that the random diffusion $X(t)$ converges weakly to the diffusion generated by the averaged operator

$$\bar{L}=\frac{1}{2}\sum_{i,j=1}^{d}\langle a_{ij}\rangle(x)\frac{\partial^2}{\partial x_i\,\partial x_j}+\sum_{i=1}^{d}\langle b_i\rangle(x)\frac{\partial}{\partial x_i},$$

where $\langle\ \rangle$ is as defined in section 3. Assuming that $\bar{L}$ is positive recurrent and that $L+\lambda G$ is positive recurrent for large λ, it also follows from Khasminskii that the invariant measures ν_λ corresponding to $L+\lambda G$ converge weakly to the product measure $\bar{\nu}\times\mu$ on $R^d\times E$, where $\bar{\nu}$ is the invariant probability measure for $\bar{L}$.

For $\lambda\to 0$, we will prove the following theorem.

THEOREM 2. – (i) *The positive recurrence of all the* L_k*'s is a necessary but not a sufficient condition for the tightness of* $\{\nu_\lambda\}$ *as* $\lambda\to 0$.

(ii) *If* $\{\nu_\lambda\}$ *is tight as* $\lambda\to 0$, *then* $w-\lim_{\lambda\to 0}\nu_\lambda(dx,k)=\mu_k\,\nu_k(dx)$, $k=1,2,\ldots,n$ *where* ν_k *is the invariant probability measure for* L_k.

Proof. – Assume that $\{\nu_\lambda\}$ is tight as $\lambda\to 0$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence such that $\lim_{n\to\infty}\lambda_n=0$ and $\lim_{n\to\infty}\nu_{\lambda_n}=\nu$. Note that the marginal on E of ν_λ is μ for all $\lambda>0$. It is easy to see that the measure P_{λ_n} on $C([0,\infty),R^d\times E)$, induced by the process generated by $L+\lambda_n G$ with initial distribution ν_{λ_n}, converges weakly as $n\to\infty$ to the measure $P\equiv\sum_{k=1}^{n}\mu_k\,\delta_k(k(.))\,P_k$, where δ_k is the delta measure on $C([0,\infty),E)$ at $k(t)\equiv k$, $0\leqq t<\infty$ and P_k is the measure on $C([0,\infty),R^d)$ induced by the process generated by L_k with initial distribution $\nu(dx,k)$. It also follows that P is a stationary measure with marginal ν. But given the form of P, it is clear that this can occur only if each L_k is positive recurrent and if $\nu(dx,k)=\mu_k\,\nu_k(dx)$, where ν_k is the invariant probability measure for L_k.

It remains to show that the positive recurrence of all of the L_k's is not a sufficient condition for the tightness of $\{\nu_\lambda\}$ as $\lambda\to 0$. Return to the first example in section 2. Replace the generator $L+G$ by $L+\lambda G$. For each fixed $0<\lambda<1$, one can find an appropriate $\alpha=\alpha(\lambda)$ for which the random diffusion is transient despite the fact that L_1 and L_2 are positive recurrent. If we could pick one α to work simultaneously for all $0<\lambda<1$, we would be done. Running through the proof reveals that a problem occurs only in case 2 with $x-[x]\in\left[0,\frac{1}{3}\right]\cup\left(\frac{2}{3},1\right)$, where we obtain

$$(4.1)\quad (L^{(\alpha)}+\lambda G)\,V(x,k)=2^{-[x]}\left(\alpha W_2'(x-[x])+\frac{1}{2}W_2''(x-[x])\right.$$
$$\left.+\lambda\,(W_1(x-[x])-W_2(x-[x]))\right).$$

Since $W'(z)$ and $W''(z)$ vanish on $\frac{1}{3} \leqq z \leqq \frac{2}{3}$, and since from the construction $W_2''(z)$ can not be nonnegative in a lefthand neighborhood of $z=\frac{1}{3}$, it follows that no matter how large α is, $\alpha W_2'(z)+\frac{1}{2} W_2''(z)$ will be negative for $z<\frac{1}{3}$ and sufficiently close to $\frac{1}{3}$. Thus as $\lambda \to 0$, in order that (4.1) remain nonnegative, we must pick α larger and larger. Thus we amend example one as follows. Let $(\alpha_j)_{j=0}^{\infty}$ be a sequence of positive numbers such that $\lim_{j \to \infty} \alpha_j=\infty$. For each j, let $b_k^{(\alpha_j)}$, $k=1, 2$, be defined as in example one and chosen so that $\sup_{j=1,2,\ldots} \int_0^2 b_1^{(\alpha_j)}(x)\,dx<0$. For $k=1, 2$ define $b_k(x)=b_k^{(\alpha_j)}(x)$, if $[x]=j$, $j=0, 1, \ldots$, and define $L_k=\frac{1}{2}\frac{d^2}{dx^2}+b_k \frac{d}{dx}$. Now for each $\lambda>0$, the V of example one will satisfy (2.1) with $L+\lambda G$ instead of $L^{(\alpha)}+\lambda G$ and r sufficiently large depending on λ. To see this note that in (4.1), if $[x]=j$, then α_j instead of α will appear in (4.1) and thus for any fixed $\lambda>0$ (4.1) will remain nonnegative for sufficiently large $|x|$.

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