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ABSTRACT. — We study McKean-Vlasov diffusions with jumps, given by nonlinear martingale problems with integro-differential operators, which may not always be represented as strong solutions to stochastic differential equations. We show existence and uniqueness by a contraction method on probability measures, using a coupling obtained by an adequate sample-path representation.

Key words : Nonlinear martingale problems, couplings, fixed-point methods, interacting particle systems, propagation of chaos.

RÉSUMÉ. — Nous étudions des diffusions avec sauts de type McKean-Vlasov, données par des problèmes de martingales non-linéaires avec opérateurs intégro-différentiels, qui ne peuvent pas toujours être représentées en tant que solution forte d'équation différentielle stochastique. Nous montrons l'existence et l'unicité par une technique de contraction sur les mesures de probabilité, utilisant un couplage obtenu grâce à une représentation trajectorielle adéquate.

Mots clés : Problèmes de martingales non linéaires, couplages, méthodes de points fixes, systèmes de particules en interaction, propagation de chaos.

Classification A.M.S. : Primary: 60 K 35. Secondary: 60 B 10, 60 J 75, 47 H 10.

INTRODUCTION

Nonlinear diffusions with jumps abound in applications as propagation of chaos limits for interacting particle systems. Even some apparently continuous processes are modelled using jumps corresponding to changes of state. For instance in Chromatography one wishes to analyse or separate a mixture of L different species. An inert fluid pushes the mixture through a long column partly full of an adsorbant medium. The particles of each species have different mobilities and different affinities with the adsorbant medium, and thus take different lengths of time to cross the column. There is competition between molecules both for access to the adsorbant medium and for space to diffuse. Experimentally there is a nonlinear effect: each molecule does not see individual molecules but reacts to their distributions. A probabilistic model for a chromatographic column is thus given by a nonlinear system of L interacting diffusions with jumps.

There are few results on existence and uniqueness for such nonlinear processes. It is natural to use fixed-point methods. A major difficulty comes from the mixture of integral and differential terms in the generator. For integral generators, regularity of the operator may lead to a contraction result on the martingale problem itself, as in Shiga-Tanaka [8] or Sznitman [9]. Elsewise one usually considers a stochastic differential equation on which to obtain a pathwise contraction result, as in Sznitman [10] and Tanaka [11]. Given a specific problem, usually modelled by a martingale problem, it is not always possible to obtain an equation with Lipschitz coefficients. In Graham [4], this was done (in a special L^1 setting) in the case of discrete set of jumps; this solved the Chromatography problem above. General representation theorems as in El-Karoui-Lepeltier [1] are of no avail, since they give no clue as to the regularity of coefficients.

Coupling is central in obtaining fixed-point results, especially when there is not a nice stochastic differential equation; for instance in Graham [3] and Graham-Métivier [5], time-change is used to get the coupling. When there is such an equation, coupling is automatic using the same driving terms, but pathwise computations and use of the Yamada-Watanabe theorem obscure this fact.

We first give a result when the differential part of the operator is linear, using a coupling specified by a martingale problem. Regularity assumptions are minimal, and generalize in this direction the results for pure-jump processes in Gerardi-Romiti [2] (obtained by a stochastic differential equation representation). It also generalizes the result in Shiga-Tanaka [8], and is adequate for Boltzmann-like models, where the jumps occur in the speeds and apart from that there is free flow.

The main result considers general nonlinear integro-differential operators; the only restrictions are bounded jump rate and natural Lipschitz assumptions. We tailor a sample-path representation for the coupling which ensures the paths of the two marginals stay close. We then devise an original recursive L^1 contraction scheme able to handle the accumulation of deviations due to the jumps. This result enables to consider finer models for Chromatography, where for example sorption effects on the adsorbant medium lead to a continuous set of jumps in the adsorbed phase. It also enables to consider particles with a continuum of states, for instance when the number of species L goes to infinity, or when we finely model the adsorbing medium using a continuum of adsorbed states corresponding to thermodynamical considerations.

1. THE NONLINEAR PROBLEM AND ITS APPLICATIONS

Let $\Omega = D(\mathbb{R}+, \mathbb{R}^d)$ be the set of right continuous mappings with left-hand limits, X the canonical process, $(\mathscr{F}_t)_{t \geqq 0}$ its filtration, $\Omega^T = D([0, T], \mathbb{R}^d)$. We use the Skorohod topology, whose Borel σ-field coincides with the product σ-field; see Pollard [7]. $\langle \ , \ \rangle$ denotes duality brackets, $| \ |$ the Euclidian norm, and δ the Dirac measure. For a Borel space E, $\Pi(E)$ is the topological space of probability measures on E under weak convergence, and $M_b^+(E)$ is the set of positive bounded measures.

For $1 \leqq i, j \leqq d$, σ_{ij} and b_i are real measurable functions on $\mathbb{R}^d \times \Pi(\mathbb{R}^d)$, and a is the matrix $\sigma\sigma^*$. Let μ be measurable on $\mathbb{R}^d \times \Pi(\mathbb{R}^d)$ with values in $M_b^+(\mathbb{R}^d - \{0\})$. Let φ be in $C_b^2(\mathbb{R}^d)$, x in $\mathbb{R}^d$, p in $\Pi(\mathbb{R}^d)$. We define the diffusion operator $\mathscr{L}$ by

$$\mathscr{L}\varphi(x, \mathrm{p}) = \frac{1}{2} \sum_{i, j=1}^{d} a_{ij}(x, \mathrm{p}) \partial_{ij}^2 \varphi(x) + \sum_{i=1}^{d} b_i(x, \mathrm{p}) \partial_i \varphi(x), \qquad (1.1)$$

the pure jump operator $\mathscr{J}$ by

$$\mathscr{J}\varphi(x, \mathrm{p}) = \int_{\mathbb{R}^d - \{0\}} (\varphi(x+h) - \varphi(x)) \mu(x, \mathrm{p}, dh), \qquad (1.2)$$

and the diffusion-with-jumps operator $\mathscr{A} = \mathscr{L} + \mathscr{J}$. The total mass of μ, $\lambda(x, \mathrm{p}) = \mu(x, \mathrm{p}, \mathbb{R}^d - \{0\})$, is the intensity or rate of jumps, and when λ does not vanish, the probability measure $\pi = \mu/\lambda$ is the law of the amplitudes of jumps.

Definition. – Let $\mathrm{p} \in \Pi(\mathbb{R}^d)$. We say that $\mathrm{P} \in \Pi(\Omega)$ solves the nonlinear, or McKean-Vlasov, martingale problem starting at p if it solves the following inhomogenous martingale problem: $\mathrm{P}_0 = \mathrm{p}$, and for any φ in $C_b^2(\mathbb{R}^d)$

$$M_t^\varphi = \varphi(X_t) - \varphi(X_0) - \int_0^t \mathscr{A}\varphi(X_s, P_s)\, ds \qquad (1.3)$$

is a local martingale under P, where $(P_t)_{t \geqq 0}$ is the own set of time-marginals of P. Such a solution is called a McKean measure.

Remark. – This nonlinear problem can be seen as the Statistical Mechanics limit for a particle system. As soon as we prove uniqueness for the nonlinear problem, techniques developed in Sznitman ([9], [10]) apply to get propagation of chaos results in the mean-field, or Vlasov, limit. See Graham [4] for a result that is valid in the present context.

Let us give a probabilistic model for a chromatographic column. A particle position $x=(y, z)$ consists in its spatial position y in $\mathbb{R}^n$ and "state" z, 0 if adsorbed and 1 if desorbed. The system of particles evolves in $(\mathbb{R}^n \times \{0, 1\})^L \subset \mathbb{R}^{(n+1)L} = \mathbb{R}^d$ and jumps between 2^L possible states. Assume that the j-th particle, $1 \leqq j \leqq L$, is adsorbed at rate α^j and desorbed at rate β^j, and follows a nonlinear diffusion in $\mathbb{R}^n$ with operator $\mathscr{L}_1^j$ when desorbed and $\mathscr{L}_0^j$ when adsorbed. We then set $\mathscr{L}^j(x, p) = z\, \mathscr{L}_1^j(y, p) + (1-z)\, \mathscr{L}_0^j(y, p)$, acting on the y component, and $\mu^j(x, p) = z\, \alpha^j(y, p)\, \delta_{(0, -1)} + (1-z)\, \beta^j(y, p)\, \delta_{(0, 1)}$, where -1 and 1 correspond to the z component. Set $\mathscr{A}^j = \mathscr{L}^j + \mathscr{J}^j$. The particles follow a nonlinear system of martingale problems with jumps: $P \in \Pi(D(\mathbb{R}+, \mathbb{R}^{(n+1)L}))$ is such that for $1 \leqq j \leqq L$, φ in $C_b^2(\mathbb{R}^{n+1})$,

$$M_t^{\varphi\, j} = \varphi(X_t^j) - \varphi(X_0^j) - \int_0^t \mathscr{A}^j \varphi(X_s^j, P_s)\, ds \tag{1.4}$$

are orthogonal martingales, where P_s is the law of $(X_s^1, \ldots, X_s^L)$. This model englobes McKean's caricature of the Boltzmann equation by a two-speed model of Maxwellian molecules in McKean [6] and Shiga-Tanaka [8], with additional streaming and diffusing, and other discrete-speed Boltzmann equations.

Let E be a Polish space, with topology induced by a metric d. We may define on $M_b^+(E)$ the metric of total variation

$$V(\mu, \nu) = \sup\{\langle \varphi, \mu \rangle - \langle \varphi, \nu \rangle;\ \|\varphi\|_\infty \leqq 1\}. \tag{1.5}$$

To define the Kantorovitch-Rubinstein metric on $\Pi(E)$, we use the Lipschitz semi-norm $\|\varphi\|_L = \inf\{K \geqq 0;\ \forall x, y \in E,\ |\varphi(x) - \varphi(y)| \leqq K d(x, y)\}$ and set

$$\rho(P, Q) = \sup\{\langle \varphi, P \rangle - \langle \varphi, Q \rangle;\ \|\varphi\|_L \leqq 1\} \tag{1.6}$$

which is equal to the Vasserstein metric:

$$\rho(P, Q) = \inf\left\{\int_{E \times E} d(x, y)\, R(dx, dy);\ \text{R has marginals P and Q}\right\}. \tag{1.7}$$

If (E, d) is complete then so is $(\Pi(E), \rho)$, and ρ induces weak topology plus convergence of the first moment; see Zolotarev [12].

ρ shall denote the Vasserstein metric for $E=\mathbb{R}^d$ and $d(x, y)=|x-y|$. D^T is the Vasserstein metric for a complete metric inducing the Skorohod topology on Ω^T. We also consider the uniform metric $\|X-Y\|_\infty^T=\sup_{0\leqq t\leqq T}|X_t-Y_t|$ and the corresponding Vasserstein metric Δ^T on $\Pi(\Omega^T)$. We have $D^T\leqq K\Delta^T$. See Pollard [7] for details.

Adding a constant to φ does not change $\langle \varphi, \mathrm{P}\rangle-\langle \varphi, \mathrm{Q}\rangle$ for probability measures P and Q, and we may restrict the supremum in (1.6) to functions vanishing at the origin. This definition extends nicely to jump measures since they are defined on $\mathbb{R}^d-\{0\}$, and we set on $M_b^+(\mathbb{R}^d-\{0\})$

$$\rho(\mu, \nu)=\sup\{\langle \varphi, \mu\rangle-\langle \varphi, \nu\rangle;\ \|\varphi\|_L\leqq 1,\ \varphi(0)=0\}. \qquad (1.8)$$

2. THE RESULTS ON THE NONLINEAR PROBLEM

We now give two results, the first using the metric V and the second the metric ρ. We set

$$m(x, \mathrm{p})=\int_{\mathbb{R}^d-\{0\}} h\,\mu(x, \mathrm{p}, dh),\quad \nu^{ij}(x, \mathrm{p})=\int_{\mathbb{R}^d-\{0\}} h^i h^j \mu(x, \mathrm{p}, dh).$$

THEOREM 2.1. — *Assume that $\mathscr{L}(x, \mathrm{p})$ does not depend on* p. *Assume that for any* Q *in* $\Pi(\Omega^T)$, $\lambda(.,\mathrm{Q}_t)$ *is locally bounded and* $\sigma(.,\mathrm{Q}_t)$ *and* $b(.,\mathrm{Q}_t)$ *are locally Lipschitz on* $\mathbb{R}^d$, *uniformly in t, and that either* $\lambda(.,\mathrm{Q}_t)$, $\sigma(.,\mathrm{Q}_t)$, *and* $b(.,\mathrm{Q}_t)$ *are bounded on* $\mathbb{R}^d$ *uniformly in t, or* P_0 *has a second moment and the affine growth assumption holds:*

$$|b(x, \mathrm{Q}_t)|^2+\mathrm{tr}(a(x, \mathrm{Q}_t))+|m(x, \mathrm{Q}_t)|^2+\mathrm{tr}(v(x, \mathrm{Q}_t))\leqq K_{\mathrm{Q}}(1+|x|^2).$$

Assume furthermore that $V(\mu(x, \mathrm{p}), \mu(x, \mathrm{q}))\leqq KV(\mathrm{p}, \mathrm{q})$ *uniformly in x.*

Then there is a unique McKean measure starting at P_0.

Proof. — We denote by V^T both the metric of total variation on $\Pi(\Omega^T)$ and the semi-metric on $\Pi(\Omega)$ obtained by projection. Let Q^1 and Q^2 be two laws on Ω, P^1 and P^2 be the solutions to the linear inhomogenous martingale problems in which the nonlinearity in (1.3) is replaced by the time-marginals $(\mathrm{Q}_t^1)_{t\geqq 0}$ and $(\mathrm{Q}_t^2)_{t\geqq 0}$. These solutions are uniquely defined by classical results, the growth and moment assumptions being needed to prevent explosion and to enable L^2 stochastic calculus. We shall define a coupling of P^1 and P^2: this is a probability measure $\bar{\mathrm{P}}$ on $\bar{\Omega}=\Omega\otimes\Omega$ having first marginal P^1 and second marginal P^2. $\bar{\Omega}$ is identified with $\mathrm{D}(\mathbb{R}+, \mathbb{R}^d\times\mathbb{R}^d)$, with canonical process

$$\bar{X}=(X^1, X^2)=(X^{1,1}, \ldots, X^{1,d}, X^{2,1}, \ldots, X^{2,d}).$$

$\bar{P}$ will be defined as the solution to a linear inhomogeneous martingale problem with operator $\bar{\mathscr{A}}=\bar{\mathscr{L}}+\bar{\mathscr{J}}$. We define

$$\bar{\mathscr{L}}_t\varphi(\bar{x})=\bar{\mathscr{L}}\varphi(\bar{x})=\bar{\mathscr{L}}^1\varphi(\bar{x})+\bar{\mathscr{L}}^2\varphi(\bar{x})+\bar{\mathscr{L}}^{12}\varphi(\bar{x}) \tag{2.1}$$

with for $k=1, 2$, setting $\bar{x}=(x^1, x^2)=(x^{1,1}, \ldots, x^{1,d}, x^{2,1}, \ldots, x^{2,d})$,

$$\left.\begin{aligned}\bar{\mathscr{L}}^k\varphi(\bar{x})&=\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x^k)\,\partial^2_{(k,i)(k,j)}\varphi(\bar{x})+\sum_{i=1}^{d} b_i(x^k)\,\partial_{(k,i)}\varphi(\bar{x})\\ \bar{\mathscr{L}}^{12}\varphi(\bar{x})&=\frac{1}{2}\sum_{i,j=1}^{d}((\sigma(x^1)\sigma(x^2)^*)_{ij}\,\partial^2_{(1,i)(2,j)}\\ &\qquad\qquad+(\sigma(x^2)\sigma(x^1)^*)_{ij}\,\partial^2_{(2,i)(1,j)})\,\varphi(\bar{x}).\end{aligned}\right\} \tag{2.2}$$

By the Hahn-Jordan decomposition theorem, there are orthogonal positive measure kernels μ^{12}, ν^1, ν^2 such that $\mu(x^k, Q_t^k)=\mu_t^{12}(x^1, x^2)+\nu_t^k(x^1, x^2)$. We then define

$$\begin{aligned}\bar{\mathscr{J}}\varphi(\bar{x})=\int_{\mathbb{R}^d-\{0\}}&(\varphi(x^1+h, x^2+h)-\varphi(x^1, x^2))\,\mu_t^{12}(x^1, x^2, dh)\\ &+(\varphi(x^1+h, x^2)-\varphi(x^1, x^2))\,\nu_t^1(x^1, x^2, dh)\\ &\quad+(\varphi(x^1, x^2+h)-\varphi(x^1, x^2))\,\nu_t^2(x^1, x^2, dh).\end{aligned} \tag{2.3}$$

For $\bar{P}_0$ we take the law concentrated on the diagonal with marginals P_0.

It is easy to see this martingale problem has an unique solution which is indeed a coupling. A simple examination of the martingale problem satisfied by X^1-X^2 plus uniqueness shows that X^1 is equal to X^2 up to the first time τ that they do not jump together, and the corresponding jump measure is $\nu^1+\nu^2$. If $\|\varphi\|_\infty\leq 1$,

$$\langle\varphi, P^1\rangle-\langle\varphi, P^2\rangle=\bar{E}(\varphi(X^1)-\varphi(X^2))\leq 2\bar{P}(\tau\leq T) \tag{2.4}$$

and since the total mass of $\nu_t^1(x, x)+\nu_t^2(x, x)$ is $V(\mu(x, Q_t^1), \mu(x, Q_t^2))$, we have

$$\begin{aligned}V^T(P^1, P^2)\leq 2\bar{P}(\tau\leq T)\leq 2\left(1-\exp\left(\int_0^T KV(Q_t^1, Q_t^2))\,dt\right)\right)\\ \leq 2K\int_0^T V^t(Q^1, Q^2)\,dt\end{aligned} \tag{2.5}$$

and a standard contraction argument finishes the proof. □

THEOREM 2.2. — *Assume* σ, b, *and* μ *uniformly Lipschitz and* λ *uniformly bounded:* $\lambda(x, p)\leq c$ *and* $|\sigma(x, p)-\sigma(y, q)|+|b(x, p)-b(y, q)|+\rho(\mu(x, p), \mu(y, q))\leq K(|x-y|+\rho(p, q))$. *Assume that either* σ *and* b *are bounded, or* P_0 *has a second moment and* $|m(x, p)|^2+\mathrm{tr}(v(x, p))\leq K(1+|x|^2)$ *uniformly in* p.

Then there is a unique McKean measure starting at P_0.

Proof. – For laws Q^1 and Q^2 on Ω we shall give a pathwise representation X^1 and X^2 of the solutions P^1 and P^2 of the linear martingale problems corresponding to $(Q_t^1)_{t\geq 0}$ and $(Q_t^2)_{t\geq 0}$. These solutions exist and are unique by classical results, the growth and moment assumptions preventing explosion and ensuring L^2 stochastic integrands. We shall strive to construct X^1 and X^2 as close as possible.

Since the rate of jumps is bounded by c, we shall set the times of jumps in advance using a Poisson process of rate c. Between jumps the paths follow Ito stochastic differential equations. The Poisson process and Brownian motion will be chosen the same for X^1 and X^2. Since the law of jumps varies from point to point, we shall chose the joint law along the way in the best possible fashion.

We take a Brownian motion B, a random variable X_0 with law P_0, and a Poisson process N of rate c with jump-times $(T_n)_{n\geq 1}$, all independent. We define a probability kernel

$$\bar{\pi}(x, p) = \frac{1}{c}\mu(x, p) + \left(1 - \frac{\lambda(x, p)}{c}\right)\delta_0. \tag{2.6}$$

Thus all the space-time dependency has been put in the law of the jumps, and we replace a varying rate by a larger constant rate by allowing jumps of amplitude 0:

$$\mathscr{J}\varphi(x, p) = c\int_{\mathbb{R}^d} (\varphi(x+h) - \varphi(x))\,\bar{\pi}(x, p, dh). \tag{2.7}$$

By (1.8),

$$\rho(\bar{\pi}(x, p), \bar{\pi}(y, q)) = (1/c)\,\rho(\mu(x, p), \mu(y, q)) \leq (K/c)(|x-y| + \rho(p, q)).$$

We assume $c \geq 1$ for simplicity of notations.

Take $T_0 = 0$, $X_0^1 = X_0^2 = X_0$, and assume the processes are defined up to T_{n-1}. For $k=1, 2$, define X^k between T_{n-1} and T_n- as the solution of

$$dX_t^k = \sigma(X_t^k, Q_t^k)\,dB_t + b(X_t^k, Q_t^k)\,dt \tag{2.8}$$

starting at $X_{T_{n-1}}^k$. Then choose $(X_{T_n}^1 - X_{T_n-}^1,\ X_{T_n}^2 - X_{T_n-}^2)$ conditionally on the past, independently of the rest, according to the law R_n having marginals $\bar{\pi}(X_{T_n-}^1, Q_{T_n}^1)$ and $\bar{\pi}(X_{T_n-}^2, Q_{T_n}^2)$ such that

$$\rho(\bar{\pi}(X_{T_n-}^1, Q_{T_n}^1), \bar{\pi}(X_{T_n-}^2, Q_{T_n}^2)) = \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|\,R_n(dx, dy). \tag{2.9}$$

This choice is logical considering (1.7), and R_n exists according to a simple minimisation argument for continuous functions on compact sets. We thus have constructed processes X^1 and X^2 having laws P^1 and P^2.

Now take $T \in \mathbb{R}+$. $\Delta^T(P^1, P^2) \leq E(\|X^1 - X^2\|_\infty^T)$, and we wish to get a contraction estimate relating this to $\Delta^T(Q^1, Q^2)$. We denote $T_n \wedge T$ again by T_n, and $\|X^1-X^2\|_\infty^{T_n}$ by S_n. We wish to find constants C_n such that $E(S_n) \leqq C_n \Delta^T(Q^1, Q^2)$.

$$
\begin{aligned}
S_n = \sup_{0 \leqq t \leqq T_n} |X_t^1 - X_t^2| &= |X_{T_n}^1 - X_{T_n}^2| \vee \sup_{0 \leqq t < T_n} |X_t^1 - X_t^2| \\
&\leqq |(X_{T_n}^1 - X_{T_n-}^1) - (X_{T_n}^2 - X_{T_n-}^2)| + |X_{T_n-}^1 - X_{T_n-}^2|) \vee \sup_{0 \leqq t < T_n} |X_t^1 - X_t^2| \\
&\leqq |(X_{T_n}^1 - X_{T_n-}^1) - (X_{T_n}^2 - X_{T_n-}^2)| + \sup_{0 \leqq t < T_n} |X_t^1 - X_t^2|, \quad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
E(|(X_{T_n}^1 - X_{T_n-}^1) - (X_{T_n}^2 - X_{T_n-}^2)|) &= E\left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| R_n(dx, dy)\right) \\
&= E(\rho(\bar{\pi}(X_{T_n-}^1, Q_{T_n}^1), \bar{\pi}(X_{T_n-}^2, Q_{T_n}^2))) \\
&\leqq K E(|X_{T_n-}^1 - X_{T_n-}^2| + \Delta^T(Q^1, Q^2)) \\
&\leqq K(E(\sup_{0 \leqq t < T_n} |X_t^1 - X_t^2|) + \Delta^T(Q^1, Q^2)), \quad (2.11)
\end{aligned}
$$

and thus we get

$$
E(S_n) \leqq (1+K) E\left(\sup_{0 \leqq t < T_n} |X_t^1 - X_t^2|\right) + K\Delta^T(Q^1, Q^2). \quad (2.12)
$$

$$
\begin{aligned}
\sup_{0 \leqq t < T_n} |X_t^1 - X_t^2| &\leqq S_{n-1} \vee \sup_{T_{n-1} < t < T_n} |X_t^1 - X_t^2| \\
&\leqq S_{n-1} \vee (|X_{T_{n-1}}^1 - X_{T_{n-1}}^2| \\
&\quad + \sup_{T_{n-1} < t < T_n} |(X_t^1 - X_{T_{n-1}}^1) - (X_t^2 - X_{T_{n-1}}^2)|) \\
&\leqq S_{n-1} + \sup_{T_{n-1} < t < T_n} |(X_t^1 - X_{T_{n-1}}^1) - (X_t^2 - X_{T_{n-1}}^2)|, \quad (2.13)
\end{aligned}
$$

and since $(X_t^1 - X_{T_{n-1}}^1) - (X_t^2 - X_{T_{n-1}}^2)$ is equal to

$$
\int_{T_{n-1}}^t (\sigma(X_s^1, Q_s^1) - \sigma(X_s^2, Q_s^2)) dB_s + \int_{T_{n-1}}^t (b(X_s^1, Q_s^1) - b(X_s^2, Q_s^2)) ds
$$

using the Burkholder-Davis-Gundy inequalities we have

$$
\begin{aligned}
&E\left(\sup_{T_{n-1} < t < T_n} |(X_t^1 - X_{T_{n-1}}^1) - (X_t^2 - X_{T_{n-1}}^2)|\right) \\
&\qquad \leqq E\left(C\left(\int_{T_{n-1}}^{T_n} (\sigma(X_s^1, Q_s^1) - \sigma(X_s^2, Q_s^2))^2 ds\right)^{1/2}\right. \\
&\qquad\qquad \left. + \int_{T_{n-1}}^{T_n} |b(X_s^1, Q_s^1) - b(X_s^2, Q_s^2)| ds\right). \quad (2.14)
\end{aligned}
$$

Using the Lipschitz properties, for some constant K this is bounded by

$$
K(T+\sqrt{T})\left(E\left(\sup_{0 \leqq t < T_n} |X_t^1 - X_t^2|\right) + \Delta^T(Q^1, Q^2)\right). \quad (2.15)
$$

We take T small enough for $K(T+\sqrt{T})<1$. Using (2.13) and (2.15),

$$\mathrm{E}\Big(\sup_{0\leqq t<T_n}|X_t^1-X_t^2|\Big) \leqq \frac{\mathrm{E}(S_{n-1})}{1-K(T+\sqrt{T})}+\frac{K(T+\sqrt{T})}{1-K(T+\sqrt{T})}\Delta^T(\mathrm{Q}^1,\mathrm{Q}^2), \quad (2.16)$$

which together with (2.12) gives

$$\mathrm{E}(S_n)\leqq\frac{1+K}{1-K(T+\sqrt{T})}\mathrm{E}(S_{n-1})+\frac{K(1+T+\sqrt{T})}{1-K(T+\sqrt{T})}\Delta^T(\mathrm{Q}^1,\mathrm{Q}^2). \quad (2.17)$$

Set $a=\dfrac{1+K}{1-K(T+\sqrt{T})}$ and $b=\dfrac{K(1+T+\sqrt{T})}{1-K(T+\sqrt{T})}$. Notice that $a>1$ and $b/(a-1)=1$. By a simple affine recursion argument,

$$\mathrm{E}(S_n)\leqq a^n(\mathrm{E}(S_0)+\Delta^T(\mathrm{Q}^1,\mathrm{Q}^2))-\Delta^T(\mathrm{Q}^1,\mathrm{Q}^2) = (a^n-1)\,\Delta^T(\mathrm{Q}^1,\mathrm{Q}^2). \quad (2.18)$$

Now

$$\sup_{0\leqq t\leqq T}|X_t^1-X_t^2|\leqq\sup_{0\leqq t\leqq T_{N_T}}|X_t^1-X_t^2|\vee\sup_{T_{N_T}<t\leqq T}|X_t^1-X_t^2|. \quad (2.19)$$

and after reasoning as for (2.13), (2.14), (2.15), conditional expectations give

$$\mathrm{E}\Big(\sup_{0\leqq t\leqq T}|X_t^1-X_t^2|\Big) \leqq\mathrm{E}(S_{N_T})+K(T+\sqrt{T})(E(\sup_{0\leqq t<T}|X_t^1-X_t^2|)+\Delta^T(\mathrm{Q}^1,\mathrm{Q}^2)) \quad (2.20)$$

and using (2.18)

$$\begin{aligned}\mathrm{E}(\sup_{0\leqq t\leqq T}|X_t^1-X_t^2|)&\leqq\frac{\mathrm{E}(a^{N_T})-1+K(T+\sqrt{T})}{1-K(T+\sqrt{T})}\Delta^T(\mathrm{Q}^1,\mathrm{Q}^2)\\&\leqq\frac{e^{(a-1)cT}-1+K(T+\sqrt{T})}{1-K(T+\sqrt{T})}\Delta^T(\mathrm{Q}^1,\mathrm{Q}^2). \quad (2.21)\end{aligned}$$

We see that for a sufficiently small T, depending only on K and c, we have

$$\Delta^T(\mathrm{P}^1,\mathrm{P}^2)\leqq\mathrm{E}(\sup_{0\leqq t\leqq T}|X_t^1-X_t^2|)\leqq\frac{1}{2}\Delta^T(\mathrm{Q}^2,\mathrm{Q}^2). \quad (2.22)$$

This certainly proves uniqueness of the McKean measure on $[0, T]$. For existence, we must be careful: the Borel σ-field generated by Δ^T does not coincide with the product σ-field, and the probability laws we deal with are only defined on the latter. Nevertheless, the classical iteration method for a recursive sequence defined by a contracting mapping, applied in this context, shows that the corresponding sequence of laws is Cauchy for Δ^T and thus for D^T. Since this metric is defined through a complete Skorohod metric, it is itself complete, and the sequence converges in $\Pi(\Omega^T)$. The solutions to the martingale problems we consider are quasi-continuous, thus T, $2T$, etc., are a.s. continuity points and we may extend existence and uniqueness to $\mathbb{R}+$. □

We now use this theorem on the model for a chromatographic column defined in Section 1. Let σ_1^j, b_1^j, σ_0^j, b_0^j be the coefficients in $\mathscr{L}_1^j$ and $\mathscr{L}_0^j$.

THEOREM 2.3. – *If* σ_1^j, b_1^j, σ_0^j, b_0^j, α^j, β^j *are Lipschitz bounded for* ρ, *then for any initial condition there is a unique solution to the nonlinear system* (1.4).

Proof. – We use Theorem 2.2. The Lipschitz assumptions are only stated on the y coordinates, but since $z\in\{0, 1\}$, we can deduce a Lipschitz assumption on (y, z). For instance, if α^j is K-Lipschitz and bounded by A, then the jump rate for the jump $(0, -1)$ is $z^j\alpha^j(y, p)$, and $|z^j\alpha^j(y, p)-r^j\alpha^j(v, q)|\leqq \mathrm{A}|z^j-r^j|+\mathrm{K}(|y-v|+\rho(p, q))$. □

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