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Hydrodynamical limit for asymmetric attractive particle systems on $\mathbf{Z}^d$

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ABSTRACT. — We prove conservation of local equilibrium for asymmetric attractive particle systems on $\mathbf{Z}^d$ starting from some initial configurations. These initial profiles have density β in some cone H and density α in the complement. We assume that the flow is a vector in H or in $-$H. The d-dimensionnal results allow us to prove hydrodynamical behavior for systems on $\mathbf{Z}$ starting from some non monotone initial profiles.

Key words : Local equilibrium, hydrodynamical equation, attractive systems, entropy solution.

RÉSUMÉ. — Nous démontrons la conservation de l'équilibre local pour des systèmes de particules asymétriques et attractifs sur $\mathbf{Z}^d$ pour certaines configurations initiales. Ces profils initiaux ont densité β dans un cone H et densité α dans le complémentaire. Nous supposons que la dérive moyenne est un vecteur de $H \cup (-H)$. Ces résultats d-dimensionnels nous permettent de démontrer le comportement hydrodynamique de systèmes sur $\mathbf{Z}$ dont le profil initial n'est pas monotone.

Classification A.M.S. : 60 K 35.

INTRODUCTION

The zero range process is one of the most studied interacting particle systems. It can be informally described as follows. Consider undistinguishables particles moving on $\mathbf{Z}^d$. Let $g:\mathbf{N}\to\mathbf{R}$ be a non negative function with $g(0)=0$ and $\mathrm{P}(x, y)$ transition probabilities on $\mathbf{Z}^d$. Suppose there are k particles on a site x of $\mathbf{Z}^d$. These particles wait a mean $1/g(k)$ exponential time at the end of which one of them jump to y with probability $\mathrm{P}(x, y)$.

In this paper we are interested in conservation of local equilibrium. To describe the results, consider a system of particles moving on $\mathbf{Z}^d$. Suppose that this Markov process in $\mathbf{N}^{\mathbf{Z}^d}$ has an infinite family of extremal invariant measures ν_p caracterized by a parameter p in some open subset P of $\mathbf{R}^n$.

In the sequel, for $x\in\mathbf{Z}^d$, we denote by τ_x the translation by x in $\mathbf{N}^{\mathbf{Z}^d}$ and extend them to the functions and to the measures in the natural way. Hence, for $\eta\in\mathbf{N}^{\mathbf{Z}^d}$, $\tau_x\eta(z)=\eta(x+z)$, $\tau_x f(\eta)=f(\tau_x\eta)$ and $\int f\,d(\tau_x\mu)=\int\tau_x f\,d\mu$.

Let μ_ε be a sequence of probability measures on $\mathbf{N}^{\mathbf{Z}^d}$. We shall say that the sequence (μ_ε) satisfies the local equilibrium property if there is a regular function $p:\mathbf{R}^d\to\mathrm{P}$ such that:

$$\lim_{\varepsilon\to 0}\tau_{[x\varepsilon^{-1}]}\mu_\varepsilon=\nu_{p(x)}\quad\text{for every continuity point } x \text{ of } p,$$

where $[r]$ denotes the interger part of r and the limit, as all measure limits in this paper, is taken in the weak* sense.

Let S_t be the semigroup of the Markov process. We shall say that there is conservation of local equilibrium if there exists a time renormalisation $\mathrm{T}(\varepsilon)$ and a regular function $p:\mathbf{R}_+\times\mathbf{R}^d\to\mathrm{P}$ such that,

$$\lim_{\varepsilon\to 0}\tau_{[x\varepsilon^{-1}]}\mathrm{S}_{\mathrm{T}(\varepsilon)t}\mu_\varepsilon=\nu_{p(t,x)}\quad\text{for every continuity point } (t, x) \text{ of } p.$$

We expect $p(t, x)$ to be the solution of some P.D.E. with initial condition given by $p(x)$. This partial differential equation is called the hydrodynamical equation of the process. The time renormalisation $\mathrm{T}(\varepsilon)$ is usually ε^{-1} in the asymmetric case and ε^{-2} in the symmetric case.

In [DIPP] one can find precise statements about hydrodynamical behavior of many particles systems and a list of references.

In asymmetric particle processes under the Euler rescaling $\mathrm{T}(\varepsilon)=\varepsilon^{-1}$, the hydrodynamical equations obtained are quasi-linear hyperbolic equations of first order:

$$\frac{\partial\rho}{\partial t}+\sum_{i=1}^{d}\gamma_i\frac{\partial}{\partial v_i}\varphi(\rho(t, v))=0, \tag{0.0}$$

where φ is a concave function.

Theses P.D.E. do not admit, in general, smooth solutions. Therefore we have to consider weak solutions, in which case we loose unicity. Kružkov proposed in [K] a criterion to pick up among the weak solutions the one with physical meaning, called the entropy solution. The function $\rho(t, x)$ which describes the hydrodynamical behavior of attractive asymmetric particle processes is exactly the entropy solution of (0.0).

In this paper we prove conservation of local equilibrium for attractive particle systems on $\mathbf{Z}^d$ for some initial configurations.

This problem has been considered before by several autors. Rost in 1981 [R] gave the first contribution to the field considering the totally asymmetric simple exclusion process. His results were later improved in [AK], [BF1], [AV], [BF2], [L], [BFSV] and [Re]. There is a problem in the proof of Proposition 1 in [BF1] which is being corrected by the authors. This paper is the sequel of [L], where we considered attractive particle systems on $\mathbf{Z}^d$.

1. RESULTS AND NOTATION

Let (η_t) be the zero range process. This is the Markov process on $\mathbf{N}^{\mathbf{Z}^2}=\mathbf{X}$ whose generator acts on cylindrical functions as

$$\mathrm{L}f(\eta)=\sum_{x,\, y\in\mathbf{Z}^2} g(\eta(x))\,\mathrm{P}(x, y)\,[f(\eta^{x,\,y})-f(\eta)]$$

where

$$\eta^{x,\,y}(z)=\begin{cases}\eta(z) & \text{if } z\neq x, y\\ \eta(x)-1 & \text{if } z=x\\ \eta(y)+1 & \text{if } z=y.\end{cases} \tag{1.1}$$

All over this paper we will make the following assumptions on g and P:

(i) The function g is nondecreasing and bounded, $0=g(0)<g(1)$.

(ii) $\mathrm{P}(x, y)=\mathrm{P}(0, y-x)=p(y-x)$ and there is $\mathrm{A}\in\mathbf{N}$ such that:

$$p(x)=0 \qquad \text{if } |x|\geqq \mathrm{A}.$$

The existence of this Markov process, under more general assumptions than ours, is proved in [A]. Before proceeding, we introduce some notation which will be used troughout this paper.

(*a*) $\mathscr{M}$ denotes the set of probability measures on $\mathbf{N}^{\mathbf{Z}^2}$.

(*b*) S_t denotes the semigroup of the Markov process and $\mathscr{I}$ the set of probability measures invariant under (S_t).

(*c*) $\{\tau_y, y\in\mathbf{Z}^2\}$ denote the shifts on X: $\tau_y\eta(z)=\eta(y+z)$ for every y, z in $\mathbf{Z}^2$, η in X. We extend the shift to the functions and to the measures in the natural way: $(\tau_y f)(\eta)=f(\tau_y \eta)$ and $\int f\, d(\tau_y \mu)=\int (\tau_y f)\, d\mu$. $\mathscr{S}$ denotes the probability measures invariant under the group $\{\tau_y,\ y \in \mathbf{Z}^2\}$. Remark that the assumptions on the process imply that τ_y and S_t commute.

(*d*) For r in $\mathbf{R}$, $[r]$ will denote the integer part of r.

(*e*) $\gamma_j=\sum_{(z_1, z_2) \in \mathbf{Z}^2} z_j p(z)$, for $j=1, 2$.

(*f*) For a subset A of $\mathbf{R}^d$, A^0 is the interior of A.

(*g*) H is a closed cone with non empty interior.

We introduce in X the partial order defined by $\eta \leqq \xi$ if $\eta(x) \leqq \xi(x)$ for every x in $\mathbf{Z}^2$. Denote by M the class of continuous functions on X which are monotone in the sense that $f(\eta) \leqq f(\xi)$ whenever $\eta \leqq \xi$. We extend the partial order to $\mathscr{M}$ in the natural way: $\mu \leqq \nu$ if $\int f\, d\mu \leqq \int f\, d\nu$ for every $f \in M$. A Feller process, with semigroup S_t is said to be attractive if S_t preserves the order in $\mathscr{M}$, *i.e.*, if for every $t>0$, $\mu \leqq \nu \Rightarrow S_t \mu \leqq S_t \nu$. It is proved in [A] that the monotonicity of g implies the attractiveness of the zero range process. This property is the crucial point in the proof of conservation of local equilibrium as we shall see.

With the same proof as the one of Corollary II.2.8 of [Li], if $\mu \leqq \nu$, in order to prove that $\mu=\nu$, we only have to show that

$$\mu[\eta(x)]=\nu[\eta(x)] \quad \text{for every } x \in \mathbf{Z}^2. \tag{1.2}$$

It is proved in [A] that the set of extremal measures in $\mathscr{I} \cap \mathscr{S}$ is the weakly continuous family of translation invariant product measures $\{\nu_\rho,\ 0 \leqq \rho < \infty\}$, such that:

$$\nu_\rho[\eta,\ \eta(x)=k]=\begin{cases} \dfrac{[\varphi(\rho)]^k}{g(1)\dots g(k)}\,\dfrac{1}{\chi(\rho)} & \text{if } k \geqq 1, \\ \dfrac{1}{\chi(\rho)} & \text{if } k=0, \end{cases}$$

where $\chi(\rho)$ is a normalizing factor, $\nu_\rho[\eta(0, 0)]=\rho$ and

$$\varphi(\rho)=\nu_\rho[g(\eta(0))]. \tag{1.3}$$

Therefore, every measure in $\mathscr{I} \cap \mathscr{S}$ can be written as

$$\int_0^\infty \nu_\rho \lambda(\rho), \tag{1.4}$$

for some probability measure λ on $\mathbf{R}_+$.

To state the theorems, we define the product measures $\mu_{\alpha,\beta}$ on $\mathbf{N}^{\mathbf{Z}^2}$:

$$\mu_{\alpha,\beta}\{\eta;\ \eta(k,j)=n\}=\begin{cases} \nu_\beta\{\eta;\ \eta(k,j)=n\} & \text{if } (k,j) \in H, \\ \nu_\alpha\{\eta;\ \eta(k,j)=n\} & \text{otherwise.} \end{cases}$$

In section 2, we will prove the following

THEOREM 1. — *Suppose that:*

(*a*) φ *given by* (1.3) *is concave*, $\gamma(-\gamma)$ *is in* H *and* $\alpha \leqq \beta (\beta \leqq \alpha)$ *or suppose that*

(*b*) φ *given by* (1.3) *is strictly concave*, $\gamma(-\gamma)$ *is in* H *and* $\beta \leqq \alpha (\alpha \leqq \beta)$.

Then,

$$\lim_{\varepsilon \to 0} \mu_{\alpha, \beta} S_{t\varepsilon^{-1}} \tau_{([v_1 \varepsilon^{-1}], [v_2 \varepsilon^{-1}])} = \nu_{\rho(t, (v_1, v_2))},$$

for every continuity point $(t, (v_1, v_2))$ of ρ, the entropy solution of the 2-dimensionnal P.D.E. (0.0) with initial condition

$$\rho(0, (v_1, v_2)) = \begin{cases} \beta & \text{if } (v_1, v_2) \in \mathrm{H}, \\ \alpha & \text{otherwise.} \end{cases}$$

The important assumption of the theorem is that the shocks all diffuses ($\gamma \in \mathrm{H}$, $\beta < \alpha$) or they all propagates ($\gamma \in \mathrm{H}$, $\alpha < \beta$). Our techniques at their present stage do not apply to the case where one front diffuses and the other propagates.

Consider the 1-dimensionnal nearest neighbor zero range process. This is the Markov process whose generator acts on cylindrical functions as

$$\mathrm{L}f(\eta) = \sum_{k, j \in \mathbf{Z}} g(\eta(k)) p(j-k) [f(\eta^{k, j}) - f(\eta)],$$

where $\eta^{k, j}$ is given by (1.1) and $p(1) = 1 - p(-1) = p > 1/2$.

Define $m^{\varepsilon}_{\alpha, \beta}$ as the product measure on $\mathbf{N}^{\mathbf{Z}}$:

$$m^{\varepsilon}_{\alpha, \beta}\{\eta;\ \eta(k) = n\} = \begin{cases} \nu_\beta\{\eta;\ \eta(k) = n\} & \text{if } -a\varepsilon^{-1} \leqq k \leqq 0, \\ \nu_\alpha\{\eta;\ \eta(k) = n\} & \text{otherwise.} \end{cases}$$

In section 3, we prove

THEOREM 2. — *Suppose that* φ *given by* (1.3) *is strictly concave and that* $\alpha < \beta$. *Then,*

$$\lim_{\varepsilon \to 0} m^{\varepsilon}_{\alpha, \beta} S_{t\varepsilon^{-1}} \tau_{[v\varepsilon^{-1}]} = \nu_{\rho(t, v)},$$

for every continuity point (t, v) *of* ρ, *the entropy solution of the* 1-*dimensionnal P.D.E.* (0.0) *with initial condition*

$$\rho(0, v) = \begin{cases} \beta & \textit{if } -a \leqq v \leqq 0, \\ \alpha & \textit{otherwise.} \end{cases}$$

Remark 1.1. — With a change of variables $\mathrm{T} = t\varepsilon^{-1}$, we see that in order to prove Theorem 1 for the case where $\gamma \in \mathrm{H}$, $\alpha \leqq \beta$ and φ is concave, it is enough to prove that

$$\lim_{t\to\infty} \mu_{\alpha,\beta} S_t \tau_{([v_1 t],[v_2 t])} = \begin{cases} \nu_\beta & \text{if } (v_1, v_2) \in v_c + H^\circ \\ \nu_\alpha & \text{if } (v_1, v_2) \in v_c + H^c, \end{cases}$$

where $v_c = (v_{1,c}, v_{2,c}) = [\varphi(\beta) - \varphi(\alpha)]/[\beta - \alpha](\gamma_1, \gamma_2)$. In the other cases ($\gamma \notin H, \ldots$), analogous remarks can be stated.

Remark 1.2. – Remarks 5.1 and 5.2 in [AV] remain valid in our context. We can construct similar couplings to the one presented in section 3 for the nearest neighbor "misanthrope" process.

Theorem 1 can be proved in higher dimensions with the same arguments.

2. PROOF OF THEOREM 1

In this section we prove Theorem 1, proving the assertion made in Remark 1.1. We consider the case where φ given by (1.3) is concave, γ is in H and $\alpha \leqq \beta$. The other situations are handled in the same way.

The method presented here was introduced in [AV] to prove conservation of local equilibrium for attractive particle systems on $\mathbf{Z}$. It was extended in [L] to attractive processes on $\mathbf{Z}^d$.

The proof of Theorem 1 follows from a sequence of lemmas. Some of them appeared before in [AV] or in [L]. Since we need slight modifications in the statements of these lemmas and in the sake of clearness, we present the staments and omit partially or totally the proofs when they appear in [AV] or in [L].

Remark 2.1. – In the proof of Theorem 1, we need only to consider the case where $\gamma \in Q_1 = \{(x, y) \in \mathbf{R}^2;\ x \geqq 0,\ y \geqq 0\} \subset H^0$. Indeed, if $\gamma \in H^0$, with a change of variables which sends $\mathbf{Z}^2$ onto $\mathbf{Z}^2$, we can transform the process and the initial condition in order that they satisfy the above assumption (*cf.* [L], Remark 7.2, for a similar problem). On the other hand, if $\gamma \in H - H^0$, approximating H by cones H_N such that $H \subset H_N^0$, we see that we can restrict our study to the case where $\gamma \in H^0$ (*cf.* Lemma 3.2 of this paper).

LEMMA 2.1. – *Under the hypotheses of Theorem* 1 *and under the assumptions presented in Remark* 2.1,

$$\lim_{t\to\infty} \mu_{\alpha,\beta} S_t \tau_{([v_1 t],[v_2 t])} = \nu_\alpha \qquad \textit{if} \quad (v_1, v_2) \notin v_c + H.$$

The proof is the same as the one of Proposition 5.1 of [L].

With the next lemma, we begin the proof that $\lim_{t\to\infty} \mu_{\alpha,\beta} S_t \tau_{([v_1 t],[v_2 t])} = \nu_\beta$ if $(v_1, v_2) \in v_c + H^0$. We should observe that we do not use in Lemmas 2.2 and 2.4 the assumption that $\gamma \in H$.

LEMMA 2.2. — *Let* μ *be a probability on* $\mathbf{N}^{\mathbf{Z}^2}$ *and* H *be a closed cone which interior contains* Q_1. *Suppose that there are* $0 \leqq \theta_1 \leqq \theta_2 < \infty$ *such that;*

(i) $\nu_{\theta_1} \leqq \mu \leqq \nu_{\theta_2}$

and

(ii) $\mu \leqq \mu \tau_{(k, j)}$ *for every* $(k, j) \in H$.

Then, or every sequence $T_N \uparrow \infty$ *there exists a subsequence* T_{N_k} *and two countables and denses subsets* D_1, D_2 *of* **R**, *such that, for every* $(v_1, v_2) \in D = D_1 \times D_2$,

$$\lim_k \frac{2}{T_{N_k}^2} \int_0^{T_{N_k}} t\, \mu\, S_t\, \tau_{([v_1 t], [v_2 t])}\, dt = \mu_{(v_1, v_2)}$$

for some $\mu_{(v_1, v_2)} \in \mathscr{I} \cap \mathscr{S}$.

Proof. — Almost all the proof is omitted since it is similar to the one of Lemma 3.1 of [AV]. Consider a countable and dense subset A of $\mathbf{R}^2$. As in [AV], we obtain a subsequence T_{N_k} for which for every $(v_1, v_2) \in A$,

$$\lim_k \frac{2}{T_{N_k}^2} \int_0^{T_{N_k}} t\, \mu\, S_t\, \tau_{([v_1 t], [v_2 t])}\, dt = \mu_{(v_1, v_2)} \in \mathscr{M}.$$

For $(v_1, v_2) \in A$, let $F(v_1, v_2) = \mu_{(v_1, v_2)}[\eta(0, 0)]$. We extend F to $\mathbf{R}^2$ in the natural way:

$$F(v_1, v_2) = \inf_{(w_1, w_2) \in \{(v_1, v_2) + H\} \cap A} F(w_1, w_2).$$

F defined in this way is non decreasing in each coordinate and bounded. Let $D_2 \subset \mathbf{R}$ be a dense and countable subset of **R**. For each $v \in D_2$, define F_v as the real function such that $F_v(x) = F(x, v)$ for every $x \in \mathbf{R}$. For each $v \in D_2$, the set C_v of discontinuity points of F_v is countable. Therefore, $\bigcup_{v \in D_2} C_v$ is countable. Let D_1 be a dense and countable subset of **R**, $D_1 \cap \bigcup_{v \in D_2} C_v = \varnothing$. It is not difficult to see that F is continuous at every point (v_1, v_2) of $D = D_1 \times D_2$. The rest of the proof follows just as the one of Lemma 3.1 of [AV]. □

We fix some notation which will be needed later. For (v_1, v_2) in $D_1 \times D_2$, denote

$$F(v_1, v_2) = \mu_{(v_1, v_2)}[\eta(0, 0)] \tag{2.1}$$

and

$$G(v_1, v_2) = \mu_{(v_1, v_2)}[g(\eta(0, 0))]. \tag{2.2}$$

We extend those functions to $\mathbf{R}^2$ as we did in Lemma 2.2.

Remark 2.2. — In Lemma 2.2, the hypothesis that $Q_1 \subset H^0$ was necessary to insure that the dense and countable subset D of $\mathbf{R}^2$ can be taken as the product of two dense and countable subsets of $\mathbf{R}$. Under the weaker assumption that $Q_1 \subset H$, a similar proof shows that there is a countable and dense subset D of $\mathbf{R}^2$ with the properties listed in Lemma 2.2.

By (1.4), since $\mu_{(v_1, v_2)} \in \mathcal{I} \cap \mathcal{S}$ for $(v_1, v_2) \in D_1 \times D_2$, there exists a probability measure $\lambda_{(v_1, v_2)}$ on $[\theta_1, \theta_2]$ such that,

$$\mu_{(v_1, v_2)} = \int_{\theta_1}^{\theta_2} \nu_\rho \lambda_{(v_1, v_2)}(d\rho).$$

LEMMA 2.3. – *With the notation of Lemma* 2.2 *and with* $\mu = \mu_{\alpha, \beta}$, *there exist* $\bar{v}_1$ *and* $\bar{v}_2$, *such that,*

$$\mu_{(v_1, v_2)} = \nu_\beta \begin{cases} \textit{if} \quad v_1 > \bar{v}_1 \textit{ and } v_2 > v_{2, c} \\ \textit{or} \\ \textit{if} \quad v_2 > \bar{v}_2 \textit{ and } v_1 > v_{1, c}. \end{cases}$$

The proof is omitted since it is similar to the one of Lemma 2.2 of [L].

Remark 2.3. – If $\gamma \notin H \cup (-H)$ and $\gamma_2 < 0 < \gamma_1$, with the notation of Lemma 2.2 and with $\mu = \mu_{\alpha, \beta}$, the same proof gives that there exist $\bar{v}_1$ and $\bar{v}_2$, such that,

$$\mu_{(v_1, v_2)} = \nu_\beta \begin{cases} \text{if} \quad v_1 > \bar{v}_1 \text{ and } v_2 > \gamma_2 \varphi'(\beta) \\ \text{or} \\ \text{if} \quad v_2 > \bar{v}_2 \text{ and } v_1 > v_{1, c}. \end{cases}$$

Now, we arrive at the main point in the proof of Theorem 1. Lemma 2.4 is also the most important modification needed in the arguments of [L] to prove Theorem 1. This lemma computes the density in a 2-dimensionnal macroscopic box.

LEMMA 2.4. – *Let* μ *be a probability on* $\mathbf{N}^{\mathbf{Z}^2}$ *and suppose that there are two dense subsets* D_1 *and* D_2 *of* $\mathbf{R}$ *and a sequence* $T_N \uparrow \infty$ *such that:*

(i) $\nu_{\theta_1} \leqq \mu \leqq \nu_{\theta_2}$;

(ii) $\mu \leqq \mu\tau_{(1, 0)}$ *and* $\mu \leqq \mu\tau_{(0, 1)}$

and

(iii) $\lim_N \dfrac{2}{T_N^2} \displaystyle\int_0^{T_N} t\, \mu S_t \tau_{([w_1 t], [w_2 t])}\, dt = \mu_{(w_1, w_2)}$ for every

$$(w_1, w_2) \in D_1 \times D_2,$$

where

$$\mu_{(w_1, w_2)} = \int_{\theta_1}^{\theta_2} \nu_\rho \lambda_{(w_1, w_2)}(d\rho)$$

Then, for every (u_1, u_2), (v_1, v_2) *in* $D_1 \times D_2$ *with* $u_i < v_i$, $i=1,2$,

$$\lim_N \frac{1}{T_N^2} \sum_{k=[u_1 T_N]+1}^{[v_1 T_N]} \sum_{j=[u_2 T_N]+1}^{[v_2 T_N]} \mu S_{T_N}[\eta(k,j)]$$
$$= \frac{1}{2}\int_{u_2}^{v_2} \{[v_1 F(v_1, r) - \gamma_1 G(v_1, r)] - [u_1 F(u_1, r) - \gamma_1 G(u_1, r)]\}\, dr$$
$$+ \frac{1}{2}\int_{u_1}^{v_1} \{[v_2 F(r, v_2) - \gamma_2 G(r, v_2)] - [u_2 F(r, u_2) - \gamma_2 G(r, u_2)]\}\, dr, \quad (2.3)$$

where F *and* G *are given by* (2.1) *and* (2.2).

The proof of this lemma, which relies on a long computation on the generator, is postponned to the appendix. The reader should remark that no assumption is made on γ.

LEMMA 2.5. – *With the notation of Lemma* 2.2 *and with* $\mu = \mu_{\alpha, \beta}$, *we have:*

$$\mu_{(v_1, v_2)} = \nu_\beta \qquad \text{if} \quad (v_1, v_2) \in (D_1 \times D_2) \cap (v_c + H^0).$$

Proof. – First, take (u_1, u_2) in $D_1 \times D_2$ such that $u_i > v_{i,c}$, for $i=1,2$. Consider (v_1, v_2) in $D_1 \times D_2$ such that $v_i > \bar{v}_i$ and $v_i > u_i$ for $i=1,2$. By attractiveness, we know that

$$\limsup_N \frac{1}{T_N^2} \sum_{k=[u_1 T_N]+1}^{[v_1 T_N]} \sum_{j=[u_2 T_N]+1}^{[v_2 T_N]} \mu S_{T_N}[\eta(k,j)] \leqq \beta (v_1 - u_1)(v_2 - u_2).$$

Therefore, Lemma 2.4 applied to the sequence T_{N_k} obtained in Lemma 2.2 states that

$$\frac{1}{2}\int_{u_2}^{v_2} \{[v_1 F(v_1, r) - \gamma_1 G(v_1, r)] - [u_1 F(u_1, r) - \gamma_1 G(u_1, r)]\}\, dr$$
$$+ \frac{1}{2}\int_{u_1}^{v_1} \{[v_2 F(r, v_2) - \gamma_2 G(r, v_2)] - [u_2 F(r, u_2) - \gamma_2 G(r, u_2)]\}\, dr$$

is bounded above by $\beta (v_1 - u_1)(v_2 - u_2)$.

Applying Lemma 2.3, we obtain that

$$u_1 \int_{u_2}^{v_2} [\beta - F(u_1, r)]\, dr + u_2 \int_{u_1}^{v_1} [\beta - F(r, u_2)]\, dr$$
$$\leqq \gamma_1 \int_{u_2}^{v_2} [\varphi(\beta) - G(u_1, r)]\, dr + \gamma_2 \int_{u_1}^{v_1} [\varphi(\beta) - G(r, u_2)]\, dr. \quad (2.4)$$

Now, we consider partitions $u_2 = \sigma_0 < \sigma_1 \ldots < \sigma_M = v_2$, such that, $\sigma_i \in D_2$ for $0 \leqq i \leqq M$. Thus, the first integral in the right hand side of the last expression is equal to

$$\gamma_1 \lim_{\max(\sigma_{i+1}-\sigma_i)\to 0} \sum_{i=0}^{M-1} \left\{ \int_\alpha^\beta [\varphi(\beta)-\varphi(\rho)]\,\lambda_{(u_1,\sigma_i)}(d\rho) \right\} (\sigma_{i+1}-\sigma_i).$$

Since φ is a concave function and $\gamma_1>0$, this expression is bounded above by

$$\gamma_1 \frac{\varphi(\beta)-\varphi(\alpha)}{\beta-\alpha} \lim_{\max(\sigma_{i+1}-\sigma_i)\to 0} \sum_{i=0}^{M-1} \left\{ \int_\alpha^\beta [\beta-\rho]\,\lambda_{(u_1,\sigma_i)}(d\rho) \right\} (\sigma_{i+1}-\sigma_i),$$

and this term is equal to

$$\gamma_1 \frac{\varphi(\beta)-\varphi(\alpha)}{\beta-\alpha} \int_{u_2}^{v_2} [\beta-F(u_1, r)]\,dr.$$

Handling in the same way the expression $\int_{u_1}^{v_1} [\varphi(\beta)-G(r, u_2)]\,dr$, from (2.4), we obtain that

$$u_1 \int_{u_2}^{v_2} [\beta-F(u_1, r)]\,dr + u_2 \int_{u_1}^{v_1} [\beta-F(r, u_2)]\,dr$$
$$\leqq \gamma_1 \frac{\varphi(\beta)-\varphi(\alpha)}{\beta-\alpha} \int_{u_2}^{v_2} [\beta-F(u_1, r)]\,dr + \gamma_2 \frac{\varphi(\beta)-\varphi(\alpha)}{\beta-\alpha} \int_{u_1}^{v_1} [\beta-F(r, u_2)]\,dr.$$

Since F is increasing in each coordinate and bounded above by β, remembering that $v_{i,c}=\gamma_i[\varphi(\beta)-\varphi(\alpha)]/[\beta-\alpha]$, we obtain from the definition of u_i, $i=1,2$, that $F(u_1,\sigma)=\beta$ if $\sigma>u_2$ and that $F(\sigma, u_2)=\beta$ if $\sigma>u_1$. This shows that $\mu_{(w_1,w_2)}[\eta(0,0)]=\beta$ for every $(w_1, w_2)\in D_1\times D_2$ such that, $w_i>v_{i,c}$ for $i=1,2$. Since by attractiveness $\mu_{(w_1,w_2)}\leqq\nu_\beta$, from (1.2) we obtain that $\mu_{(w_1,w_2)}=\nu_\beta$ if $w_i>v_{i,c}$ for $i=1,2$, because F is continuous at $D_1\times D_2$.

Finally, since $\mu_{\alpha,\beta}\leqq\mu_{\alpha,\beta}\tau_{(k,j)}$ for every $(k,j)\in H$, by attractiveness we obtain that $\mu_{(v_1,v_2)}=\nu_\beta$ if $(v_1, v_2)\in(D_1\times D_2)\cap(v_c+H^0)$. □

Remark 2.4. – If $\gamma\notin H\cup(-H)$ and $\gamma_2<0<\gamma_1$, with the notation of Lemma 2.2 and with $\mu=\mu_{\alpha,\beta}$, Remark 2.3 and the same proof gives that

$$\mu_{(v_1,v_2)}=\nu_\beta \quad \text{if} \quad (v_1, v_2)\in(v_{1,c}, \gamma_2\varphi'(\beta))+H^0.$$

Remark 2.5. – Assume that for every sequence T_N and every subsequence T_{N_k} the measures $\mu_{(v_1,v_2)}$ obtained in Lemma 2.2 are equal to some ν_θ in an open set U of $\mathbf{R}^2$. The proof below of Theorem 1 shows that in this case $\lim_t \mu_{\alpha,\beta} S_t \tau_{([v_1 t],[v_2 t])}=\nu_\theta$ for every (v_1, v_2) in the open set U. This method will be used in the next section to prove conservation of local equilibrium without further comments.

Proof of Theorem 1. – Choose (v_1, v_2) such that $(v_1, v_2) \in v_c + H^0$. By attractiveness and from the inequality $\mu_{\alpha, \beta} \leqq \nu_\beta$, we know that the set

$$\{ \mu_{\alpha, \beta} S_t \tau_{([v_1 t], [v_2 t])}, t \geqq 0 \}$$

is relatively compact. Let $\tilde{\mu}$ be a cluster point of this set and T_N a subsequence associated to this cluster point. From this sequence we obtain by Lemma 2.2 a subsequence T_{N_k} and subsets D_1 and D_2 of $\mathbf{R}$ for which, by Lemma 2.5,

$$\lim_k \frac{2}{T_{N_k}^2} \int_0^{T_{N_k}} t \, \mu_{\alpha, \beta} S_t \tau_{([w_1 t], [w_2 t])} \, dt = \nu_\beta$$

for every (w_1, w_2) in $D_1 \times D_2$ such that (w_1, w_2) is in $v_c + H^0$.

On the other hand,

$$\lim_k \frac{2}{T_{N_k}^2} \int_0^{T_{N_k}} t \, \mu_{\alpha, \beta} S_t \tau_{([v_1 t], [v_2 t])} \, dt = \tilde{\mu}.$$

Since we can find (w_1, w_2) in $D_1 \times D_2$ such that (w_1, w_2) is in $v_c + H^0$ and (v_1, v_2) is in $w + H^0$, we get that $\nu_\beta = \mu_{(w_1, w_2)} \leqq \tilde{\mu}$. □

Remark 2.6. – If $\gamma \notin H \cup (-H)$ and $\gamma_2 < 0 < \gamma_1$, the same proof together with the result stated in Remark 2.5 show that

$$\lim_{t \to \infty} \mu_{\alpha, \beta} S_t \tau_{([v_1 t], [v_2 t])} = \nu_\beta$$

if $(v_1, v_2) \in (v_{1, c}, \gamma_2 \varphi'(\beta)) + H^0$.

3. PROOF OF THEOREM 2

In this section, with the results obtained in the previous section for 2-dimensionnal attractive processes, we prove Theorem 2. Throughout this section, the product measure $\mu_{\alpha, \beta}$ is

$$\mu_{\alpha, \beta} \{ \eta; \eta(k, j) = n \} = \begin{cases} \nu_\beta \{ \eta; \eta(k, j) = n \} & \text{if } \min\{k, j\} \geqq 0, \\ \nu_\alpha \{ \eta; \eta(k, j) = n \} & \text{otherwise,} \end{cases}$$

and the zero range process is the one with jump rates equal to $p(1, -1) = p = 1 - p(-1, 1) > 1/2$. In this case, $\gamma_1 = -\gamma_2 = \gamma = (2p - 1)$. The reader should remark that the one-dimensionnality and the nearest neighbour assumption is only needed in the proof of Lemmas 3.3 and 3.5.

Theorem 2 is a corollary of the next theorem which proof is divided in 5 lemmas. Before stating the theorem, we introduce some notation. Define

$\sigma : [-\gamma\varphi'(\alpha), -\gamma\varphi'(\beta)] \to \mathbf{R}$ as the continuous and increasing function

$$\sigma(v) = [\varphi']^{-1}\left(\frac{-v}{\gamma}\right).$$

Throughout this section, for a fixed $(v_1, v_2) \in \mathbf{R} \times [-\gamma\varphi'(\alpha), -\gamma\varphi'(\beta)]$, $\sigma = \sigma(v_2)$. We omit v_2 when no confusion arises.

Consider $\mathrm{J} : [-\gamma\varphi'(\alpha), -\gamma\varphi'(\beta)] \to \mathbf{R}$ the continuous function:

$$\mathrm{J}(v_2) = \gamma \frac{\varphi(\sigma) - \varphi(\alpha) - (\sigma - \beta)\varphi'(\sigma)}{\beta - \alpha}.$$

THEOREM 3. – *With the notation just introduced,*

$$\lim_{t \to \infty} \mu_{\alpha,\beta} S_t \tau_{([v_1 t], [v_2 t])}$$

$$= \nu_\alpha \quad \text{if} \begin{cases} v_1 < v_{1,c} \quad \text{and} \quad v_2 \geqq -\gamma\varphi'(\beta) \quad \text{or} \\ v_1 < \mathrm{J}(v_2) \quad \text{and} \quad -\gamma\varphi'(\alpha) \leqq v_2 \leqq -\gamma\varphi'(\beta) \quad \text{or} \\ v_2 \leqq -\gamma\varphi'(\alpha). \end{cases}$$

$$= \nu_\beta \quad \text{if} \quad v_{1,c} < v_1 \quad \text{and} \quad v_2 \geqq -\gamma\varphi'(\beta).$$

$$= \nu_{[\varphi']^{-1}(-v_2/\gamma)} \quad \text{if} \quad \mathrm{J}(v_2) < v_1 \quad \text{and} \quad -\gamma\varphi'(\alpha) \leqq v_2 \leqq -\gamma\varphi'(\beta).$$

The proof of this theorem is divided in 5 Lemmas. The proof of the first lemma is omitted since it is similar to the one of Proposition 5.1 in [L].

LEMMA 3.1:

$$\lim_{t \to \infty} \mu_{\alpha,\beta} S_t \tau_{([v_1 t], [v_2 t])} = \nu_\alpha \quad \text{if} \begin{cases} v_1 < v_{1,c} \quad \text{or} \\ v_1 + v_2 < 0 \quad \text{or} \\ v_2 < -\gamma\varphi'(\alpha). \end{cases}$$

LEMMA 3.2:

$$\lim_{t \to \infty} \mu_{\alpha,\beta} S_t \tau_{([v_1 t], [v_2 t])} = \nu_\beta \quad \text{if} \quad v_{1,c} < v_1 \quad \text{and} \quad v_2 > -\gamma\varphi'(\beta).$$

Proof. – For $\mathrm{N} \in \mathbf{N}$, let H_N be the cone

$$\mathrm{H}_\mathrm{N} = \{(x, y) \in \mathbf{R}^2;\ y \geqq \max(-\mathrm{N}x, -x/\mathrm{N})\}$$

and μ_N be the product measure

$$\mu_\mathrm{N}\{\eta;\ \eta(k, j) = n\} = \begin{cases} \nu_\beta\{\eta;\ \eta(k, j) = n\} & \text{if} \quad (k, j) \in \mathrm{H}_\mathrm{N} \\ \nu_\alpha\{\eta;\ \eta(k, j) = n\} & \text{if} \quad (k, j) \notin \mathrm{H}_\mathrm{N}. \end{cases}$$

From Remark 2.6, for N sufficiently large,

$$\lim_{t \to \infty} \mu_N S_t \tau_{([v_1 t], [v_2 t])} = \nu_\beta \tag{3.1}$$

if

$$(v_1, v_2) \in \gamma \left(\frac{\varphi(\beta) - \varphi(\alpha)}{\beta - \alpha}, \ -\varphi'(\beta) \right) + H_N^0.$$

Fix $(v_1, v_2) \in \mathbf{R}^2$ such that $v_1 > \gamma [\varphi(\beta) - \varphi(\alpha)]/[\beta - \alpha]$, $v_2 > -\gamma \varphi'(\beta)$. From (1.2) and by attractiveness, to prove that $\lim_{t \to \infty} \mu_{\alpha, \beta} S_t \tau_{([v_1 t], [v_2 t])} = \nu_\beta$, it is enough to prove that $\lim_{t \to \infty} \mu_{\alpha, \beta} S_t \tau_{([v_1 t], [v_2 t])} [\eta(k_1, k_2)] = \beta$ for every $(k_1, k_2) \in \mathbf{Z}^2$.

Fix $(k_1, k_2) \in \mathbf{Z}^2$. Let $u_i < w_i < v_i$, $i = 1, 2$ and $u_1 > \gamma [\varphi(\beta) - \varphi(\alpha)]/[\beta - \alpha]$, $u_2 > -\gamma \varphi'(\beta)$. For t sufficiently large,

$$\mu_{\alpha, \beta} S_t \tau_{([v_1 t], [v_2 t])} [\eta(k_1, k_2)] \geq \frac{1}{([w_2 t] - [u_2 t])([w_1 t] - [u_1 t])} \sum_{j_1 = [u_1 t] + 1}^{[w_1 t]} \sum_{j_2 = [u_2 t] + 1}^{[w_2 t]} \mu_{\alpha, \beta} S_t [\eta(j_1, j_2)].$$

We couple the measure $\mu_{\alpha, \beta}$ and μ_N in the following way. First, we place η-particles on $\mathbf{Z}^2$ distributed according to $\mu_{\alpha, \beta}$. Then we add ξ-particles so that $\eta + \xi$ is distributed according to μ_N. The particles jump in order that η and $\eta + \xi$ evolve as zero-range processes with generator introduced in the begining of section 2. Let $\tilde{\mu}_N$ be the coupling measure.

From (3.1), we obtain that for every integer N,

$$\lim_{t \to \infty} \frac{1}{([w_2 t] - [u_2 t])([w_1 t] - [u_1 t])} \times \sum_{j_1 = [u_1 t] + 1}^{[w_1 t]} \sum_{j_2 = [u_2 t] + 1}^{[w_2 t]} \tilde{\mu}_N S_t [\eta(j_1, j_2) + \xi(j_1, j_2)] = \beta.$$

Therefore, to prove this Lemma, it is enough to show that

$$\lim_{N \to \infty} \lim_{t \to \infty} \frac{1}{([w_2 t] - [u_2 t])([w_1 t] - [u_1 t])} \times \sum_{j_1 = [u_1 t] + 1}^{[w_1 t]} \sum_{j_2 = [u_2 t] + 1}^{[w_2 t]} \tilde{\mu}_N S_t [\xi(j_1, j_2)] = 0.$$

This is done with similar arguments to the ones used in the proof of Lemma 2.2 in [L]. □

In lemmas 3.3 and 3.5, we proceed as described in Remark 2.5. We consider a sequence $T_N \uparrow \infty$. Applying Lemma 2.2, we obtain a subset D of $\mathbf{R}^2$, a subsequence T_{N_k} and measures $\mu_{(v_1, v_2)} \in \mathcal{I} \cap \mathcal{S}$ for $(v_1, v_2) \in D$. We will show that these measures $\mu_{(v_1, v_2)}$ are equal to some measure ν_θ in open sets of $\mathbf{R}^2$, what is enough, as observed in Remark 2.5, to prove Theorem 3.

Lemma 3.3 consists on obtaining the true α-region:

LEMMA 3.3:

$$\lim_{t \to \infty} \mu_{\alpha, \beta} S_t \tau_{([v_1 t], [v_2 t])} = \nu_\alpha$$

if

$$(v_1, v_2) \in \Gamma = \{ (w_1, w_2);\ w_1 < J(w_2),\ -\gamma \varphi'(\alpha) < w_2 < -\gamma \varphi'(\beta) \}.$$

Proof. – Since Γ is an open set, by Remark 2.5, it is enough to show that $\mu_{(v_1, v_2)} = \nu_\alpha$ for every $(v_1, v_2) \in D \cap \Gamma$.

To prove this assertion, we only have to show that $F(v_1, v_2) = \mu_{(v_1, v_2)}[\eta(0, 0)] = \alpha$ for every $(v_1, v_2) \in D \cap \Gamma$. Indeed, suppose that this last statement is true. Fix such a (v_1, v_2). In view of (1.2), since $\nu_\alpha \leqq \mu_{(v_1, v_2)}$, to prove that $\nu_\alpha = \mu_{(v_1, v_2)}$, it is enough to show that $\mu_{(v_1, v_2)}[\eta(k, j)] = \alpha$ for every $(k, j) \in \mathbf{Z}^2$. Since Γ is open and D dense in $\mathbf{R}^2$, there is $(w_1, w_2), (u_1, u_2) \in D \cap \Gamma$; $u_i < v_i < w_i$, $i = 1, 2$. By attractiveness,

$$\alpha = F(u_1, u_2) \leqq \mu_{(v_1, v_2)}[\eta(k, j)] \leqq F(w_1, w_2) = \alpha.$$

To prove that $F(v_1, v_2) = \alpha$ for every $(v_1, v_2) \in D \cap \Gamma$, fix $(v_1, v_2) \in D \cap \Gamma$ such that $v_1 + v_2 > 0$.

Holding in mind that $\sigma = \sigma(v_2)$, define H_v as the cone

$$H_v = \left\{ a(1, 0) + b\left(\frac{J(v_2) + v_1}{2} - \gamma \frac{\varphi(\sigma) - \varphi(\alpha)}{\sigma - \alpha},\ \gamma\left[\frac{\varphi(\sigma) - \varphi(\alpha)}{\sigma - \alpha} - \varphi'(\sigma) \right] \right),\ a, b \geqq 0 \right\}.$$

Consider the product measure μ_σ,

$$\mu_\sigma\{\eta;\ \eta(k, j) = n\} = \begin{cases} \nu_\sigma\{\eta;\ \eta(k, j) = n\} & \text{if } (k, j) \in H_v \\ \nu_\alpha\{\eta;\ \eta(k, j) = n\} & \text{otherwise.} \end{cases}$$

As in Lemma 3.1, with a proof similar to the one of Proposition 5.1 in [L], it is easy to show that

$$\lim_{t \to \infty} \mu_\sigma S_t \tau_{([w_1 t], [w_2 t])} = \nu_\alpha$$

if

$$w_2 \geqq -\gamma \frac{\varphi(\sigma) - \varphi(\alpha)}{\sigma - \alpha}$$

and

$$(w_1, w_2) \notin \frac{\varphi(\sigma) - \varphi(\alpha)}{\sigma - \alpha} (\gamma, -\gamma) + H_v.$$

Since $v_1 + v_2 > 0$, let c_1 and c_2 be constants such that $0 < c_1 < v_1 + v_2 < c_2 - 1$. We couple the measures $\mu_{\alpha,\beta}$ and μ_σ in the following way. We place first on $\mathbf{Z}^2$ η-particles distributed according to ν_α. Then we add particles, called χ_1-particles, in such a way that the particles $\eta + \chi_1$ are distributed according to $\mu_{\alpha, \sigma}$. Now, we add χ_2 and χ_3-particles for $\eta + \chi_1 + \chi_2$ being distributed according to $\mu_{\alpha, \beta}$ and $\eta + \chi_1 + \chi_3$ being distributed according to μ_σ. Let $\zeta = \chi_1 + \chi_2$ and $\xi = \chi_1 + \chi_3$ and denote by $\bar{\mu}$ this coupling measure. With the notation just introduced we have:

CLAIM. – *There exist positive constants* K_1 *and* K_2 *which depend only on* c_1 *and* c_2, *such that, for every* $t > 0$

$$\sum_{j \leqq k} \bar{\mu}[\zeta_t(j, K-j)] \leqq \sum_{j \leqq k} \bar{\mu}[\xi_t(j, K-j)] + K_1 e^{-K_2 t},$$

for every $k \in \mathbf{Z}$ *and for every* $[c_1 t] \leqq K \leqq [c_2 t]$.

The proof of this claim is deferred to the end of the proof of this Lemma. We proceed with the proof of the Lemma.

Let $0 < \Delta < 1/2$, such that $\left(v_1 + \frac{3\Delta}{2}, v_2 + \frac{\Delta}{2}\right)$ and $\left(v_1 + \frac{\Delta}{2}, v_2 - \frac{\Delta}{2}\right)$ are in the set

$$\left\{(w_1, w_2) \in \mathbf{R}^2;\ w_2 \geqq -\gamma \frac{\varphi(\sigma) - \varphi(\alpha)}{\sigma - \alpha}\right\} - \left[\frac{\varphi(\sigma) - \varphi(\alpha)}{\sigma - \alpha}(\gamma, -\gamma) + H_v\right].$$

Then,

$$F(v_1, v_2) = \lim_k \frac{2}{T_{N_k}^2} \int_0^{T_{N_k}} t\, \mu_{\alpha,\beta} S_t \tau_{([v_1 t], [v_2 t])} [\eta(0, 0)]\, dt$$

$$= \alpha + \lim_k \frac{2}{T_{N_k}^2} \int_0^{T_{N_k}} t\, \bar{\mu}\, \tau_{([v_1 t], [v_2 t])} [\zeta_t(0, 0)]\, dt. \quad (3.2)$$

By attractiveness, for a fixed $t>1$,

$$\bar{\mu}\tau_{([v_1 t], [v_2 t])}[\zeta_t(0, 0)] \leqq \frac{1}{[\Delta t]^2} \sum_{k=0}^{[\Delta t]-1} \sum_{j=0}^{[\Delta t]-1} \bar{\mu}[\zeta_t([v_1 t]+k, [v_2 t]+j)]$$

$$\leqq \frac{1}{[\Delta t]^2} \sum_{K=[v_1 t]+[v_2 t]}^{[v_1 t]+[v_2 t]+2[\Delta t]-2} \sum_{j \leqq (1/2)\{K-[v_2 t]+[v_1 t]+[\Delta t]\}} \bar{\mu}[\zeta_t(j, K-j)]. \quad (3.3)$$

Since $0<c_1<v_1+v_2<v_1+v_2+2\Delta<c_2$, from the claim we have that the last expression is bounded above by

$$\frac{1}{[\Delta t]^2} \sum_{K=[v_1 t]+[v_2 t]}^{[v_1 t]+[v_2 t]+2[\Delta t]-2} \sum_{j \leqq (1/2)\{K-[v_2 t]+[v_1 t]+[\Delta t]\}} \bar{\mu}[\xi_t(j, K-j)] + \frac{2 K_1}{[\Delta t]} e^{-K_2 t}.$$

Since for every (w_1, w_2) such that $v_1+v_2 \leqq w_1+w_2 \leqq v_1+v_2+2\Delta$ and $w_1 \leqq w_2 - v_2 + v_1 + \Delta$,

$$(w_1, w_2) \in \left\{ (u_1, u_2) \in \mathbf{R}^2;\ u_2 \geqq -\gamma \frac{\varphi(\sigma)-\varphi(\alpha)}{\sigma-\alpha} \right\} - \left[\frac{\varphi(\sigma)-\varphi(\alpha)}{\sigma-\alpha}(\gamma, -\gamma) + H_v \right].$$

Therefore, for every such a point,

$$\lim_{t\to\infty} \mu_\sigma S_t \tau([w_1 t], [w_2 t]) = \nu_\alpha.$$

Hence, it is easy to show that

$$\lim_{t\to\infty} \frac{1}{[\Delta t]^2} \times \sum_{K=[v_1 t]+[v_2 t]}^{[v_1 t]+[v_2 t]+2[\Delta t]-2} \sum_{j \leqq (1/2)\{K-[v_2 t]+[v_1 t]+[\Delta t]\}} \bar{\mu}[\xi_t(j, K-j)] = 0. \quad (3.4)$$

From (3.2), (3.3) and (3.4), we obtain that $\mu_{(v_1, v_2)}[\eta(0, 0)] = \alpha$. Therefore, to conclude the proof of this lemma, we only have to prove the Claim.

Proof of the Claim. – Fix $t>0$, $0<[c_1 t] \leqq K \leqq [c_2 t]$. Define A as the set

$$A^c = \{ \sum_{k\in\mathbf{Z}} \chi_2(k, K-k) > \sum_{k\in\mathbf{Z}} \chi_3(k, K-k) \}.$$

It is easy to see that there are positive constants $\tilde{K}_1$ and $\tilde{K}_2$ which depend only on c_1 and c_2 such that,

$$\bar{\mu}[A^c] \leqq \tilde{K}_1 e^{-\tilde{K}_2 t}.$$

We have that

$$A = \bigcap_{k\in\mathbf{Z}} \{ \sum_{j\leqq k} \zeta(j, K-j) \leqq \sum_{j\leqq k} \xi(j, K-j) \}.$$

Therefore, to prove the claim, we only have to construct a dynamic for which $\eta+\zeta$ and $\eta+\xi$ evolve as a zero range processes and for which the set A is absorbing. Indeed, if we construct such a coupling, we have that

$$\begin{aligned}\sum_{j\leq k}\bar{\mu}[\zeta_t(j, K-j)] &= \bar{\mu}[\sum_{j\leq k}\zeta_t(j, K-j)\,1_A]+\bar{\mu}[\sum_{j\leq k}\zeta_t(j, K-j)\,1_{A^c}]\\ &\leq \bar{\mu}[\sum_{j\leq k}\xi_t(j, K-j)]+[c_2 t]\tilde{K}_3 e^{-\tilde{K}_2 t}\\ &\leq \sum_{j\leq k}\bar{\mu}[\xi_t(j, K-j)]+K_1 e^{-K_2 t}.\end{aligned}$$

Now, we construct the coupled process. To construct this coupled process, we use the one-dimensionnality of the zero range and the nearest neighbour assumption. Since the particules evolve only on a line, we describe the coupling in dimension 1.

Since there exist an integer L such that

$$\sum_{k\notin[-L, L]}\zeta_0(k)=\sum_{k\notin[-L, L]}\xi_0(k)=0,$$

we label the ζ and the ξ-particles from the left to the right. In order to have the same number of labeled ξ and ζ-particles, we place, if necessary, ξ or ζ-particles in $+\infty$. These particles do not move. A configuration (ξ, ζ) is in the set A if and only if for every $j\in\mathbf{N}$, the jth labeled ξ particle is not at the right of the jth labeled ζ particle. We construct a coupling which preserves this order. Let $(\xi, \zeta)\in A$.

The η-particles evolve as first class particles, while the others as second class particles. Fix a site i in $\mathbf{Z}$. Let k_0 be the number of first class particles on i. To keep notation simple, let $h(j)=g(k_0+j)-g(k_0)$, for $j\geq 0$. Let a and b (c and d) be the least and the greatest label of the ξ-particles (ζ-particles) on i. Thus, $c\leq a$ and $d\leq b$. There are four possible cases. Suppose that $a=c$ and $b=d$. In this case, the particles a and c (b and d) jump together to $i-1$ ($i+1$) at rate $(1-p)h(b-a+1)$ ($ph(b-a+1)$). If $c<a$ and $d<b$, then the particules a, b, c and d jump, independently one from the others, to $i-1$, $i+1$, $i-1$ and $i+1$ at rate $(1-p)h(b-a+1)$, $ph(b-a+1)$, $(1-p)h(d-c+1)$ and $ph(d-c+1)$, respectively. If $c=a$ and $d<b$, then the particules a and c jump together to $i-1$ at a rate $(1-p)h(d-c+1)$; the particules a, b and d jump, independently one from the others, to $i-1$, $i+1$ and $i+1$ at rates equals respectively to $(1-p)[h(b-a+1)-h(d-c+1)]$, $ph(b-a+1)$ and $ph(d-c+1)$. The last case is similar to the third one and is omitted. It is simple to see that this coupling has the required properties. □

The proof of the next lemma is omitted since it is similar to the second part of the proof of Theorem 2 in [L]. It is a corollary of Lemmas 3.1 and 3.2.

LEMMA 3.4. – *For* (v_1, v_2) *such that* $-\gamma\varphi'(\alpha)<v_2<-\gamma\varphi'(\beta)$ *and* $v_1>\gamma\dfrac{\varphi(\sigma)-\varphi(\alpha)}{\sigma-\alpha}$,

$$\lim_{t\to\infty} \mu_{\alpha,\beta} S_t \tau_{([v_1 t],[v_2 t])} = \nu_\sigma.$$

It remains to study the region $-\gamma\varphi'(\alpha)<v_2<-\gamma\varphi'(\beta)$, $J(\sigma)<v_1<\gamma\dfrac{\varphi(\sigma)-\varphi(\alpha)}{\sigma-\alpha}$:

LEMMA 3.5. – *For* (v_1, v_2) *such that* $-\gamma\varphi'(\alpha)<v_2<-\gamma\varphi'(\beta)$ *and* $J(\sigma)<v_1<\gamma\dfrac{\varphi(\sigma)-\varphi(\alpha)}{\sigma-\alpha}$,

$$\lim_{t\to\infty} \mu_{\alpha,\beta} S_t \tau_{([v_1 t],[v_2 t])} = \nu_{\sigma(v_2)}.$$

Proof. – To prove this lemma, we will argue that if it was not true, we would have lost mass, which is impossible since the system is conservative.

Fix $(v_1, v_2)\in D$ such that $J(\sigma(v_2))<v_1<\gamma[\varphi'(\sigma)-\varphi'(\alpha)]/[\sigma-\alpha]$. Define $\mu^*_{\alpha,\beta}$ as the product measure

$$\mu^*_{\alpha,\beta}\{\eta;\ \eta(k,j)=n\}=\begin{cases}\nu_\beta\{\eta;\eta(k,j)=n\} & \text{if } j\geqq 0\\ \nu_\alpha\{\eta;\eta(k,j)=n\} & \text{otherwise.}\end{cases}$$

We know from [L] that $\lim_{t\to\infty}\mu^*_{\alpha,\beta}S_t\tau_{([w_1 t],[w_2 t])}=\nu_{\sigma(w_2)}$ for every (w_1, w_2) such that $-\varphi'(\alpha)\leqq w_2\leqq-\varphi'(\beta)$. Since $\mu_{\alpha,\beta}\leqq\mu^*_{\alpha,\beta}$, by attractiveness, $\mu_{(v_1,v_2)}\leqq\nu_{\sigma(v_2)}$. Thus, from (1.2) and with similar arguments to those used in the begining of the proof of Lemma 3.3, to prove that $\mu_{(v_1,v_2)}=\nu_{\sigma(v_2)}$, we only have to show that $F(v_1, v_2)=\sigma$.

Let

$$\Pi=\{(v_1, v_2);\ -\gamma\varphi'(\alpha)\leqq v_2\leqq-\gamma\varphi'(\beta),\ J(\sigma)<v_1<[\varphi(\sigma)-\varphi(\alpha)]/[\sigma-\alpha]\},$$

c_2 such that $\Pi\subset E=\{(x, y);\ x+y\leqq c_2\}$ and

$$\tilde F(v_1, v_2)=\begin{cases}\sigma(v_2) & \text{if } (v_1, v_2)\in\Pi\\ F(v_1, v_2) & \text{otherwise.}\end{cases}$$

We claim that $\int_E [F(r, s)-\alpha]\,dr\,ds=[\beta-\alpha]\,c_2^2/2$. Since $F\leqq\tilde F$ and $\int_E[\tilde F(r, s)-\alpha]\,dr\,ds=[\beta-\alpha]\,c_2^2/2$, we have one inequality. The other is obtained in the following way. Fix $\varepsilon>0$. Let $\{r_j, j\in\mathbf{Z}\}$, $\{s_i, i\in\mathbf{Z}\}$ be two partitions of $\mathbf{R}$ such that $r_j < r_{j+1}$, $s_i < s_{i+1}$ for every i, j and

$$\int_E [F(r, s) - \alpha]\, dr\, ds \geqq \sum_{(i, j);\, [r_j, r_{j+1}] \times [s_i, s_{i+1}] \cap E \neq \varnothing} [r_{j+1} - r_j][s_{i+1} - s_i][F(u_{i,j}, v_{i,j}) - \alpha] - \varepsilon,$$

where $(u_{i,j}, v_{i,j}) \in D$ and $u_{i,j} > r_{j+1}$, $v_{i,j} > s_{i+1}$ for every i,j. From the definition of the function F and by attractiveness, the right hand side of the last expression without ε is bounded below by

$$\lim_k \frac{2}{T_{N_k}^2} \int_1^{T_{N_k}} dt \sum_{(i,j)} \frac{t(r_{j+1} - r_j)(s_{i+1} - s_i)}{[(r_{j+1} - r_j)t][(s_{i+1} - s_i)t]} \times \sum_{m=[s_i t]+1}^{[s_{i+1} t]} \sum_{l=[r_j t]+1}^{[r_{j+1} t]} \{\mu_{\alpha,\beta} S_t[\eta(l, m)] - \alpha\}.$$

A simple computation leads that this term is bounded below by

$$\lim_k \frac{2}{T_{N_k}^2} \int_1^{T_{N_k}} dt \left(\frac{1}{t} - \frac{K(\varepsilon)}{t^2}\right) \sum_{(k, j) \in E_t} \{\mu_{\alpha,\beta} S_t[\eta(k, j)] - \alpha\},$$

where $E_t = \{(k, j) \in \mathbf{Z}^2;\ 0 \leqq k + j \leqq [c_2 t]\}$. But,

$$\sum_{(k,j) \in E_t} \{\mu_{\alpha,\beta} S_t[\eta(k, j)] - \alpha\} = \sum_{(k,j) \in E_t} \{\mu_{\alpha,\beta}[\eta(k, j)] - \alpha\} = \frac{1}{2}(\beta - \alpha)[c_2 t]^2 + O(t).$$

Therefore,

$$\int_E [F(r, s) - \alpha]\, dr\, ds \geqq \frac{1}{2}(\beta - \alpha) c_2^2.$$

Since F and $\tilde{F}$ increase in each variable, $F = \tilde{F}$ a.e. Since $D \cap \Pi$ are continuity points of boths F and $\tilde{F}$, $F(v_1, v_2) = \sigma(v_2)$ for every $(v_1, v_2) \in D \cap \Pi$. □

4. APPENDIX

Proof of Lemma 3. — Since the proof of this lemma is similar to the one of Lemma 3.1 of [L], we will omit several details. Fix (u_2, u_2) and (v_1, v_2) in $D_1 \times D_2$. We first rewrite (2.3) as

$$\lim_N \frac{1}{T_N^2} \Big\{ (v_1 T_N - [v_1 T_N])(v_2 T_N - [v_2 T_N]) \times \mu S_{T_N}[\eta([v_1 T_N], [v_2 T_N])] + (v_1 T_N - [v_1 T_N])$$

$$\times \sum_{j=[u_2 T_N]+1}^{[v_2 T_N]} \mu S_{T_N}[\eta([v_1 T_N], j)] + (v_1 T_N - [v_1 T_N])$$
$$\times ([u_2 T_N] + 1 - u_2 T_N) \mu S_{T_N}[\eta([v_1 T_N], [u_2 T_N]+1)]$$
$$+ (v_2 T_N - [v_2 T_N]) \sum_{k=[u_1 T_N]+1}^{[v_1 T_N]} \mu S_{T_N}[\eta(k, [v_2 T_N])]$$
$$+ \sum_{k=[u_1 T_N]+1}^{[v_1 T_N]} \sum_{j=[u_2 T_N]+1}^{[v_2 T_N]} \mu S_{T_N}[\eta(k, j)] + ([u_2 T_N] + 1 - u_2 T_N)$$
$$\times \sum_{k=[u_1 T_N]+1}^{[v_1 T_N]} \mu S_{T_N}[\eta(k, ([u_2 T_N]+1))] + ([u_1 T_N] + 1 - u_1 T_N)$$
$$\times (v_2 T_N + 1 - [v_2 T_N]) \mu S_{T_N}[\eta([u_1 T_N]+1, [v_2 T_N])]$$
$$+ ([u_1 T_N] + 1 - u_1 T_N) \sum_{j=[u_2 T_N]+1}^{[v_2 T_N]} \mu S_{T_N}[\eta([u_1 T_N]+1, j)]$$
$$+ ([u_1 T_N] + 1 - u_1 T_N)([u_2 T_N] + 1 - u_2 T_N)$$
$$\left. \mu S_{T_N}[\eta([u_1 T_N]+1, [u_2 T_N]+1)] \right\} = \lim_N \frac{1}{T_N^2} H(T_N).$$

It is easy to see that the function H is differentiable for every t such that $u_1 t, u_2 t, v_1 t$ and $v_2 t$ are not in **Z**. Moreover, $\lim_N \frac{1}{T_N^2} H(T_N) = \lim_N \frac{1}{T_N^2} \int_0^{T_N} H'(t)\, dt$.

We compute the integral $\lim_N \frac{1}{T_N^2} \int_0^{T_N} H'(t)\, dt$. 21 terms of 5 different kinds appear. In what follows, we show how to deal with each kind of term. The first kind is of the form:

$$\frac{1}{T_N^2} \int_0^{T_N} (v_1 t - [v_1 t])(v_2 t - [v_2 t]) \mu S_t L[\eta([v_1 t], [v_2 t])]\, dt.$$

It is easy to show that this term converges to zero by hypothesis (i) and by attractiveness.

The second kind of term is of the following form:

$$\frac{1}{T_N^2} \int_0^{T_N} v_1 (v_2 t - [v_2 t]) \mu S_t [\eta([v_1 t], [v_2 t])]\, dt.$$

By the same reasons, this kind of term converges to zero.

The third kind of term is of the following form:

$$\frac{1}{T_N^2}\int_0^{T_N} (v_1 t-[v_1 t])\,\mu S_t L\left[\sum_{j=[u_2 t]+1}^{[v_2 t]} \eta([v_1 t], j)\right] dt. \tag{4.1}$$

Let $\varepsilon>0$ be fixed and consider a partition $u_2=\sigma_0<\sigma_1<\ldots<\sigma_M=v_2$ such that $\sigma_i\in D_2$ for $0\leqq i\leqq M$ and $\max_i(\sigma_{i+1}-\sigma_i)\leqq\varepsilon$. After simple computations using that the process is of finite range and the attractiveness, we obtain that this expression is bounded above by

$$\frac{1}{T_N^2}\int_0^{T_N}\sum_{i=1}^{M}([\sigma_i t]-[\sigma_{i-1} t])\,\{\mu S_t[g(\eta([v_1 t]+A, [\sigma_i t]+A))]$$
$$-\mu S_t[g(\eta([v_1 t], [\sigma_{i-1} t]-A))]\}\,dt.$$

By hypothesis (ii) and since the function $G(v_1, .)$ is bounded and increasing, this expression converges to zero when $\varepsilon\to 0$. In the same way, we can bound below the expression (4.1) by a term which converges to zero.

The fourth kind of term is of the following type:

$$\frac{1}{T_N^2}\int_0^{T_N} v_1\,\mu S_t\left[\sum_{j=[u_2 t]+1}^{[v_2 t]} \eta([v_1 t], j)\right] dt. \tag{4.2}$$

Fix $\varepsilon>0$. Consider the partition σ_i, with the same properties as before. If $v_1>0$, the expression (4.2) is bounded above by

$$\frac{1}{T_N^2}\int_0^{T_N} v_1\sum_{i=1}^{M}([\sigma_i t]-[\sigma_{i-1} t])\,\mu S_t[\eta([v_1 t], [\sigma_i t])].$$

By hypothesis (iii), by attractiveness and since the function $F(v_1, .)$ is bounded and increasing, when ε goes to zero, this last expression converges to

$$\frac{v_1}{2}\int_{u_2}^{v_1} F(v_1, \sigma)\,d\sigma.$$

Finally, the last term is

$$\frac{1}{T_N^2}\int_0^{T_N}\sum_{k=[u_1 t]+1}^{[v_1 t]}\ \sum_{j=[u_2 t]+1}^{[v_2 t]} \mu S_t L[\eta(k, j)]\,dt. \tag{4.3}$$

This expression is similar to the one which appears in the proof of lemma 5.1 of [L]. We therefore omit some details. After some computations, we obtain that (4.3) is bounded above by a sum of expressions of the form

$$\frac{1}{T_N^2}\int_0^{T_N}\Big\{E[(-Y)^+]\sum_{k=[u_1 t]+A+1}^{[v_1 t]-A}\mu S_t[g(\eta(k,[v_2 t]+A))] - E[Y^+]\sum_{k=[u_1 t]+A+1}^{[v_1 t]-A}\mu S_t[g(\eta(k,[v_2 t]-A))]\Big\}dt + O\Big(\frac{1}{T_N}\Big),$$

where Y is an integer random variable with $P[Y=k]=\sum_j p(j,k)$. Now, we proceed just as we did in the last two cases and obtain that the expression (4.3) converges to

$$-\Big\{\frac{\gamma_1}{2}\int_{u_2}^{v_2}[G(v_1 r)-G(u_1 r)]\,dr+\frac{\gamma_2}{2}\int_{u_1}^{v_1}[G(r,v_2)-G(r,u_2)]\,dr\Big\}.$$

This concludes the proof. □

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