ANNALES DE L'I. H. P., SECTION B

M. MISIUREWICZ
E. VISINESCU

Kneading sequences of skew tent maps

Annales de l'I. H. P., section B, tome 27, nº 1 (1991), p. 125-140
<http://www.numdam.org/item?id=AIHPB_1991__27_1_125_0>

Ann. Inst. Henri Poincaré,

Vol. 27, n° 1, 1991, p. 125-140. *Probabilités et Statistiques*

Kneading sequences of skew tent maps

by

M. MISIUREWICZ

Institute of Mathematics,
Warsaw University, PKiN IX p.,
00-901 Warszawa, Poland

and

E. VISINESCU

Département de Mathématiques,
Université de Dijon,
21004 Dijon, France

ABSTRACT. — We investigate kneading sequences for expansive unimodal maps of the interval with constant slopes. We prove in particular monotonicity of the kneading sequence and thus of the topological entropy.

RÉSUMÉ. — Nous étudions les itinéraires des applications unimodales expansives à pentes constantes. Nous montrons en particulier la croissance de l'itinéraire (et donc de l'entropie topologique) en fonction des pentes.

1. INTRODUCTION

We investigate unimodal maps with slopes constant to the left and to the right of the turning point. Let these slopes be λ and $-\mu\,(\lambda,\ \mu>0)$.

The choice of λ and μ determines such map up to an affine conjugacy (and up to a choice of a smaller or larger interval). We choose the representation in which the map has the maximum at 0 and the image of 0 is 1. Then it is given by the formula

$$F_{\lambda,\mu}(x)=\begin{cases} 1+\lambda x & \text{if} \quad x\leqq 0, \\ 1-\mu x & \text{if} \quad x\geqq 0. \end{cases}$$

We shall call these maps the *skew tent maps* (the *tent maps* are $F_{\mu,\mu}$).

We shall investigate the dependence of the kneading sequence and topological entropy of $F_{\lambda,\mu}$ on the parameters. We choose them from the region

$$D=\left\{(\lambda,\mu):\lambda\geqq 1,\ \mu>1,\ \frac{1}{\lambda}+\frac{1}{\mu}\geqq 1\right\}.$$

The main result of the paper is that both kneading sequence and topological entropy are strictly increasing functions of λ and μ. The set of (λ, μ) where the kneading invariant attains any given value $\underline{M}$ is a graph of a decreasing function $\lambda=\beta_{\underline{M}}(\mu)$. The kneading invariant for our maps is determined by the topological entropy.

The discussion why we choose the region D, as well as the precise statement of the results, we postpone until the next section. Here we would like to discuss the relation of our results to the results of Helmberg [H].

The question of the monotonicity of the entropy for various classes of maps of the interval is very natural and appeared as soon as people started to think about the entropy of these maps. For the tent maps $F_{\mu,\mu}$ the answer is simple — the entropy is $\log\mu$ (*see* [MS]) and therefore it is an increasing function of μ. The answer for the skew tent maps was stated by Helmberg [H]. He also described the curves like our $\beta_{\underline{M}}$ (he worked with different parametrization and only with entropy, not with the kneading sequences). However, his proof of monotonicity of the entropy with respect to λ (Korollar 3.2 of [H]) contains a serious error. Namely, in the proof of Korollar 3.1 (from which Korollar 3.2 follows), on page 246, lines 9-10, he apparently uses already this monotonicity. Otherwise statement that there exists $\bar{s}_5<\bar{s}_4$ such that $P(s_3, \bar{s}_5)\in C(2^i h)$, is unjustified. In view of this, the proof of the monotonicity of the entropy with respect to λ, is *de facto* absent in [H]. We should remark also that the Helmberg's proof of the monotonicity of the entropy with respect to μ (*see* p. 224-235 of [H]) is much more complicated than ours (Lemma 3.3).

After this paper was written, we learned that a result equivalent to our Theorem C has been obtained independently by W. F. Darsow and M. J. Frank. They are using completely different methods than ours.

This paper was written during the visits of the first author to the University of Dijon and Sonderforschungsbereich 170 at Göttingen in 1987. He acknowledges both institutions for hospitality and support.

2. STATEMENT OF RESULTS

We start by discussing why we choose (λ, μ) from D. Since we want to have a unimodal map of an interval into itself, we have to make the assumption that $\frac{1}{\lambda}+\frac{1}{\mu}\geqq 1$ (*see* Lemma 3.1). If $\mu<1$ then there exists an attracting fixed point which attracts everything (except perhaps one point) and the kneading sequence is R^{∞}. If $\mu=1$ then there exists a whole interval of periodic points of period 2 (except one which has period 1) and the kneading sequence is RC. We want to avoid such situations, which are quite different from what happens for other parameters. Therefore we shall assume $\mu>1$. The only assumption made for technical reasons is $\lambda\geqq 1$. However, some regions where $\lambda<1$ can be studied also by the methods of this paper (*see* Remark 5.4).

We shall use the kneading theory. We shall keep more or less to the notations of [CE]. A reader not familiar with the basic notions and facts of the kneading theory can find them in [CE].

We shall denote by $\mathscr{M}$ the class of sequences $\underline{M}$ which occur as kneading sequences of $F_{\mu,\mu}$ for $1<\mu\leqq 2$. They can be characterized by the following properties:

(i) $\underline{M}$ is a maximal admissible sequence,

(ii) $\underline{M}>R^{*\infty}$,

(iii) if $\underline{M}=\underline{A}*\underline{B}$ with $\underline{A}\neq 0$, $\underline{B}\neq C$ then $\underline{A}=R^{*m}$ for some m.

We should call them probably *primary sequences*, but the use of this notion in [CE] is very unclear (are the primary sequences of [CE] only finite?, are they maximal?).

It is known that for each $h\in(0, \log 2]$ there is a unique $\underline{M}\in\mathscr{M}$ with the entropy equal to h (*i.e.* the topological entropy of a map with the kneading sequence $\underline{M}$ is equal to h). Since one of the results of this paper (Theorem B) is that all kneading sequences of the maps under consideration belong to $\mathscr{M}$, in the other results (Theorems A and C) one can replace kneading sequences by topological entropy.

We shall write $(\lambda', \mu') > (\lambda, \mu)$ if $\lambda' \geqq \lambda$, $\mu' \geqq \mu$ and at least one of these inequalities is sharp. The kneading sequence of a map F will be denoted by $K(F)$ and its topological entropy by $h(F)$. However, to simplify notations we shall write $K(\lambda, \mu)$ for $K(F_{\lambda, \mu})$ and $h(\lambda, \mu)$ for $h(F_{\lambda, \mu})$.

Now we state the main results of the paper.

In fact, Theorem C contains Theorems A and B, but it is useful to have the explicit statements of Theorems A and B separated from the whole description given in Theorem C.

THEOREM A. – *If* (λ, μ), $(\lambda', \mu') \in D$ *and* $(\lambda', \mu') > (\lambda, \mu)$ *then* $K(\lambda', \mu') > K(\lambda, \mu)$.

THEOREM B. – *If* $(\lambda, \mu) \in D$ *then* $K(\lambda, \mu) \in \mathcal{M}$.

COROLLARY. – *If* (λ, μ), $(\lambda', \mu') \in D$ *and* $(\lambda', \mu') > (\lambda, \mu)$ *then* $h(\lambda', \mu') > h(\lambda, \mu)$.

THEOREM C. – *For each* $\underline{M} \in \mathcal{M}$ *there exists a number* $\gamma(\underline{M})$ *and a continuous decreasing function* $\beta_{\underline{M}} : (1, \gamma(\underline{M})] \to [1, \infty)$ [*with one exception* $\underline{M} = RL^{\infty}$ *when* $\gamma(\underline{M}) = \infty$ *and the domain of* $\beta_{\underline{M}}$ *is* $(1, \infty)$] *such that for* $(\lambda, \mu) \in D$ *we have* $K(\lambda, \mu) = \underline{M}$ *if and only if* $\lambda = \beta_{\underline{M}}(\mu)$. *The function* γ *is increasing. The graphs of the functions* $\beta_{\underline{M}}$ *fill up the whole set* D. *Moreover, we have:*

$$\lim_{\underline{M} \to R^{*\infty}} \gamma(\underline{M}) = 1,$$

$$\lim_{\underline{M} \to RL^{\infty}} \gamma(\underline{M}) = \infty,$$

$$\lim_{\mu \searrow 1} \beta_{\underline{M}}(\mu) = \infty \quad \text{if} \quad \underline{M} \geqq RLR^{\infty},$$

$$\lim_{\mu \searrow 1} \beta_{\underline{M}}(\mu) = \gamma(\underline{J}) \quad \text{if} \quad \underline{M} < RLR^{\infty}$$

and $\underline{J}$ is given by

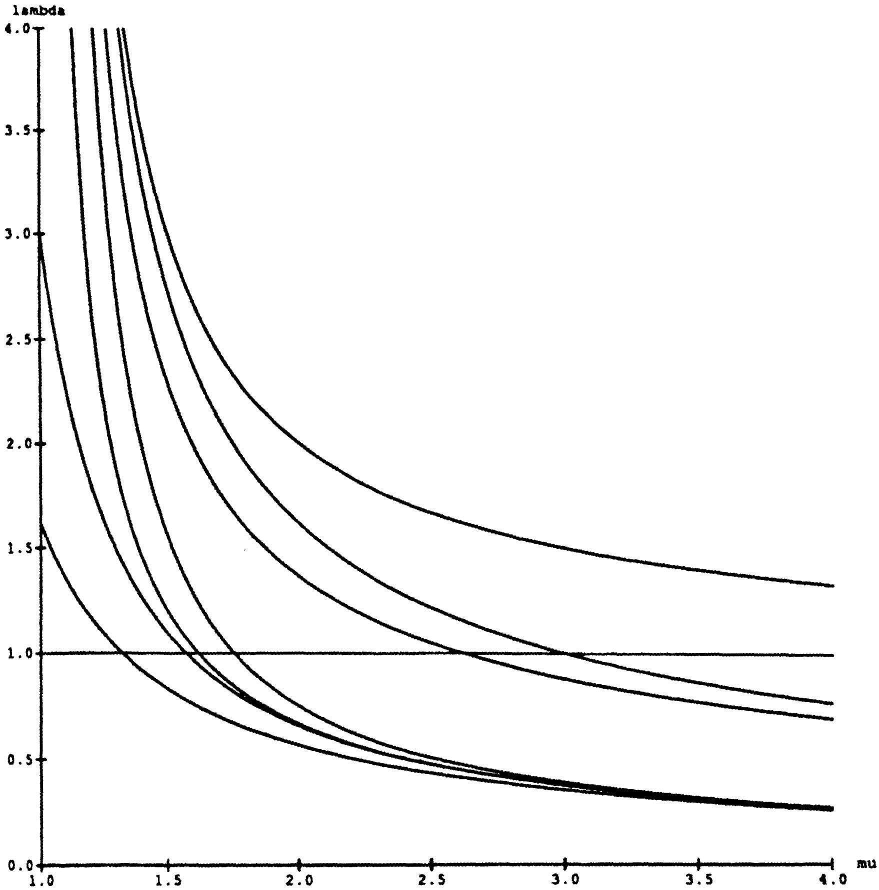
$$\underline{M} = R * \underline{J},$$

$$\beta_{\underline{M}}(\gamma(\underline{M})) = 1 \quad \text{if} \quad \underline{M} \neq RL^{\infty},$$

$$\lim_{\mu \to \infty} \beta_{\underline{M}}(\mu) = +\infty \quad \text{if} \quad \underline{M} = RL^{\infty}.$$

Remark. – In Theorem C, if we replace sequences $\underline{M} \in \mathcal{M}$ by numbers $s \in (0, \log 2]$ and $K(\lambda, \mu)$ by $h(\lambda, \mu)$ then the theorem remains true.

Their paper is organized as follows. In Sections 3 and 4 we investigate the skew tent maps with the kneading sequences larger than RLR^{∞}. In Section 5 we extend the previous results to all kneading sequences and prove Theorem A. In Section 6 we prove Theorems B and C.



The curves on which $K(\lambda, \mu) = M$ for the following values of M: RL^∞, RL^2C, RL^2RC, $RLR^\infty = R * RL^\infty$, $RLR^5C = \bar{R} * RL^2C$ and $RLR^2(RL)^\infty = R \mathbin{\bar{*}} R * RL^\infty$. Above the line $\lambda = 1$ they are the graphs of the functions $\beta_{\underline{M}}$.

3. ESTIMATES OF PARTIAL DERIVATIVES

Since in this section we will mainly work with one map $F_{\lambda, \mu}$ (although we will compute some derivatives with respect to λ and μ), we will write simply F for $F_{\lambda, \mu}$.

By an invariant interval we will understand an interval I such that $F(I) \subset I$.

LEMMA 3.1. — *Assume that* λ, $\mu>0$. *There exists an interval containing* 0 *in its interior and invariant for* F *if and only if*

$$\frac{1}{\lambda}+\frac{1}{\mu}\geqq 1. \tag{3.1}$$

Proof. — Let c, $d>0$. The interval $[-c, d]$ is invariant for F if and only if $F(-c)\geqq -c$, $F(0)\leqq d$ and $F(d)\geqq -c$. These inequalities are equivalent to

$$\lambda\leqq\frac{1+c}{c},\qquad 1\leqq d\qquad\text{and}\qquad \mu\leqq\frac{1+c}{d} \tag{3.2}$$

respectively.

If (3.2) is satisfied then

$$\frac{1}{\lambda}+\frac{1}{\mu}\geqq\frac{c+d}{1+c}\geqq 1,$$

so (3.1) is also satisfied.

If (3.1) is satisfied then in the case $\mu>1$ (3.2) holds with $c=\mu-1$, $d=1$ and in the case $\mu\leqq 1$ (3.2) holds with $d=1$ and any c sufficiently small. ■

In the above lemma we required that the invariant interval contains 0 in its interior because we are interested here only in unimodal maps (not homeomorphisms).

From now on we shall consider only $F=F_{\lambda,\ \mu}$ with $(\lambda, \mu)\in D$. Denote the kneading sequence of F by $K(F)=A_0A_1\ldots$ For $b\geqq 0$ we set $x_n=F^n(1)$. Then we have $A_n=L$, C or R when $x_n<0$, $x_n=0$ or $x_n>0$ respectively (of course unless $x_i=0$ for some $i<n$, in which case A_n is not defined).

LEMMA 3.2. — *For* $(\lambda, \mu)\in D$ *we have* $K(F)>RLR^{\infty}$ *if and only if* $\lambda>\dfrac{\mu}{\mu^2-1}$.

Proof. — The map F has a fixed point $z>0$. We have $1-\mu z=z$, *i.e.* $z=\dfrac{1}{\mu+1}$. Then $K(F)>RLR^{\infty}$ if and only if $x_2<z$ (notice that $x_0=1>0$ and $x_1=1-\mu<0$). Since $x_2=1+\lambda(1-\mu)$, the inequality $x_2<z$ is equivalent to $\lambda>\dfrac{\mu}{\mu^2-1}$. ■

We consider first only these λ, μ for which $K(F)>RLR^{\infty}$. We define

$$D_0=\left\{(\lambda, \mu)\in D:\lambda>\frac{\mu}{\mu^2-1}\right\}.$$

We shall estimate the partial derivatives of x_n with respect to λ and μ. To simplify the notation, we set

$$a_n=\frac{\partial x_n}{\partial \lambda}, \qquad b_n=\frac{\partial x_n}{\partial \mu}.$$

They do exist if $x_i \neq 0$ for all $i<n$. We have the recursive formulas:

$$a_0=\mathrm{b}_0=0,$$
$$a_{n+1}=\begin{cases} x_n+\lambda a_n & \text{if } x_n<0,\\ -\mu a_n & \text{if } x_n>0,\end{cases}$$
$$b_{n+1}=\begin{cases} \lambda b_n & \text{if } x_n<0,\\ -(x_n+\mu b_n) & \text{if } x_n>0.\end{cases}$$

It is important to keep track of the number of R's in the kneading sequence. For this we define $\theta(n)$ and ε_n as follows. If $x_i=0$ for some $i<n$ then we do not define $\theta(n)$ and we set $\varepsilon_n=0$. Otherwise, we define $\theta(n)$ as the number of R's among $\mathrm{A}_0, \ldots, \mathrm{A}_{n-1}$ and we set $\varepsilon_n=(-1)^{\theta(n)}$.

Now we estimate a_n and b_n. It is easier to do this for b_n.

LEMMA 3.3. – *Assume that* $(\lambda, \mu) \in \mathrm{D}_0$. *Set* $c=\lambda-\frac{\mu}{\mu^2-1}$. *Then* $c>0$ *and we have:*

(i) $b_1=-1$ *and* $\varepsilon_1=-1$,

(ii) *for* $n \geqq 2$:

if $\varepsilon_n=-1$ *then* $b_n \leqq -\left(\frac{\mu}{\mu^2-1}+c\,\mu^{\theta(n)-1}\right)$,

if $\varepsilon_n=+1$ *then* $b_n \geqq \frac{1}{\mu^2-1}+c\,\mu^{\theta(n)-1}$.

Proof. – The inequality $c>0$ follows from the assumption that $(\lambda, \mu) \in \mathrm{D}_0$. The statement (i) follows from the equalities $x_1=1-\mu$ and $\mathrm{A}_0=\mathrm{R}$.

We prove (ii) by induction. For $n=2$ we have $\theta(2)=1$ and $\varepsilon_2=-1$ because $\mathrm{A}_0=\mathrm{R}$ and $\mathrm{A}_1=\mathrm{L}$. We have also $x_2=1+\lambda(1-\mu)$ and hence $b_2=-\lambda=-\left(\frac{\mu}{\mu^2-1}+c\right)$. Therefore (ii) holds for $n=2$.

Assume that (ii) holds for some n and prove it for $n+1$ replacing n. Assume that $\varepsilon_n \neq 0$. If $x_n<0$ then $\theta(n+1)=\theta(n)$, $\varepsilon_{n+1}=\varepsilon_n$, $b_{n+1}=\lambda b_n$, and since $\lambda \geqq 1$, we are done. If $x_n>0$, we distinguish two cases.

Case 1. – $\varepsilon_n = +1$. Then $\varepsilon_{n+1} = -1$, $\theta(n+1) = \theta(n)+1$,

$$b_{n+1} = -x_n - \mu b_n \leqq -\mu b_n = -\frac{\mu}{\mu^2-1} - c\mu^{\theta(n)},$$

and we are done.

Case 2. – $\varepsilon_n = -1$. Then $\varepsilon_{n+1} = +1$, $\theta(n+1) = \theta(n)+1$, and we get since $x_n \leqq 1$:

$$b_{n+1} = -x_n - \mu b_n \geqq -1 - \mu b_n = \frac{1}{\mu^2-1} + c\mu^{\theta(n)},$$

and we are also done.

If $x_n = 0$ or $\varepsilon_n = 0$ then $\varepsilon_{n+1} = 0$ and there is nothing to prove. ■

The estimates for a_n are more complicated.

LEMMA 3.4. – *Assume that* $(\lambda, \mu) \in \mathrm{D}_0$ *and* $\mathrm{K(F)} = \mathrm{RL}^m\mathrm{R}\ldots$ *for some* $m \geqq 1$. *Set* $\alpha = \dfrac{\mu}{\mu^2\lambda - 1} a_{m+2}$ *and* $d = a_{m+2} - \alpha$. *Then* α, $d > 0$ *and we have*

(i) $a_0 = a_1 = 0 > a_2 > \ldots > a_{m+1} > -a_{m+2}$ *and* $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_{m+1} = -1$,

(ii) *for* $n \geqq m+2$:

if $\varepsilon_n = -1$ *then* $a_n \leqq -\left(\dfrac{\alpha}{\mu} + d\mu^{\theta(n)-2}\right) < 0$,

if $\varepsilon_n = +1$ *then* $a_n \geqq \lambda^k\alpha + a_{k+1} + d\mu^{\theta(n)-2} > 0$, *where* $k \in \{0, \ldots, m\}$ *is such that* $\mathrm{A}_{n-k-1} = \mathrm{R}$ *and* $\mathrm{A}_i = \mathrm{L}$ *for* $n-k \leqq i < n$.

Proof. – Since $\mathrm{K(F)} = \mathrm{RL}^m\mathrm{R}\ldots$, by the maximality of the kneading sequence there cannot be more than m consecutive L's in K (F). Therefore the definition of k in (ii) is correct.

We start by proving (i). We have $x_0 = 1$, $x_1 = 1-\mu$, so $a_0 = a_1 = 0$. Then for $i = 1, \ldots, m$ we have $x_i < 0$, and we get by induction $a_{i+1} = x_i + \lambda a_i < a_i < 0$ (induction is necessary to know that $a_i < 0$, so that we can use the inequality $\lambda a_i \leqq a_i$). We have $a_{m+2} = -\mu a_{m+1} > -a_{m+1}$. Since $\mathrm{A}_0 = \mathrm{R}$ and $\mathrm{A}_1 = \ldots = \mathrm{A}_m = \mathrm{L}$, then $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_{m+1} = -1$. Hence, (i) is proved.

Now the inequality $\alpha > 0$ follows from (i) and the assumptions $\lambda \geqq 1$, $\mu > 1$. By the definition of D_0, we have $\mu^2\lambda - \lambda > \mu$. Since $\lambda \geqq 1$, we get $\mu^2\lambda - 1 > \mu$, *i.e.* $1 - \dfrac{\mu}{\mu^2\lambda - 1} > 0$. Hence, $d > 0$.

We prove (ii) by induction. For $n = m+2$ we have $\theta(n) = 2$ and $\varepsilon_n = +1$. We have also $k = 0$ and $a_n = a_{m+2} = \alpha + d > 0$. Therefore (ii) holds for $n = m+2$.

Assume that (ii) holds for some n and prove it for $n+1$ replacing n. Assume that $\varepsilon_n \neq 0$. We distinguish four cases:

Case 1. – $\varepsilon_n = -1$, $x_n < 0$. Then $\varepsilon_{n+1} = -1$, $\theta(n+1) = \theta(n)$, and

$$a_{n+1} = x_n + \lambda a_n \leqq a_n \leqq -\left(\frac{\alpha}{\mu} + d\mu^{\theta(n)-2}\right).$$

Case 2. – $\varepsilon_n = -1$, $x_n > 0$. Then $\varepsilon_{n+1} = +1$, $\theta(n+1) = \theta(n)+1$, $k=0$ and

$$a_{n+1} = -\mu a_n \geqq \alpha + d\mu^{\theta(n)-1}.$$

Case 3. – $\varepsilon_n = +1$, $x_n < 0$. Then $\varepsilon_{n+1} = +1$, $\theta(n+1) = \theta(n)$, and if we replace n by $n+1$ then we have to replace also k by $k+1$. We have

$$a_{n+1} = x_n + \lambda a_n \geqq \lambda^{k+1} \alpha + d\mu^{\theta(n)-2} \lambda + (x_n + \lambda a_{k+1}).$$

We have $x_{n-k-1} \leqq 1 = x_0$ and then we get consecutively

$$x_{n-k} \geqq x_1, \ \ldots, \ x_n \geqq x_{k+1}.$$

Therefore

$$x_n + \lambda a_{k+1} \geqq x_{k+1} + \lambda a_{k+1} = a_{k+2}$$

(remeber that we have $1 \leqq k+1 \leqq m$, so $x_{k+1} < 0$). Hence, taking into account that $\lambda \geqq 1$, we obtain

$$a_{n+1} \geqq \lambda^{k+1} \alpha + a_{k+2} + d\mu^{\theta(n)-2}.$$

Case 4. – $\varepsilon_n = +1$, $x_n > 0$. Then $\varepsilon_{n+1} = -1$, $\theta(n+1) = \theta(n)+1$, and

$$a_{n+1} = -\mu a_n \leqq -\mu(\lambda^k \alpha + a_{k+1} + d\mu^{\theta(n)-2}).$$

By (i), we have $a_{k+1} \geqq a_{m+1} = -\frac{1}{\mu} a_{m+2}$. Therefore, if $k \geqq 1$ then

$$a_{n+1} \leqq -\mu\lambda\alpha + a_{m+2} - d\mu^{\theta(n)-1}.$$

Since

$$-\mu\lambda\alpha + a_{m+2} = \left(\frac{-\mu^2 \lambda}{\mu^2 \lambda - 1} + 1\right) a_{m+2} = \frac{-1}{\mu^2 \lambda - 1} a_{m+2} = -\frac{\alpha}{\mu},$$

we get

$$a_{n+1} \leqq -\left(\frac{\alpha}{\mu} + d\mu^{\theta(n)-1}\right).$$

If $k=0$, then

$$a_{n+1} = -\mu(\alpha + d\mu^{\theta(n)-2}) \leqq -\left(\frac{\alpha}{\mu} + d\mu^{\theta(n)-1}\right).$$

In all 4 cases we have obtained the required estimates of a_{n+1}. The inequality

$$-\left(\frac{\alpha}{\mu}+d\mu^{\theta(n+1)-2}\right)<0$$

(Cases 1 and 4) holds because α, d, $\mu>0$, and so does the inequality

$$\alpha+d\mu^{\theta(n+1)-2}>0$$

(Case 2). The inequality

$$\lambda^{k+1}\alpha+a_{k+2}+d\mu^{\theta(n+1)-2}>0$$

(Case 3) holds because

$$\lambda^{k+1}\alpha+a_{k+2}\geqq\lambda\alpha+a_{m+1}=\left(\frac{\lambda\mu}{\mu^2\lambda-1}-\frac{1}{\mu}\right)a_{m+2}=\frac{a_{m+2}}{\mu(\mu^2\lambda-1)}>0$$

and d, $\mu>0$.

If $x_n=0$ or $\varepsilon_n=0$ then $\varepsilon_{n+1}=0$ and there is nothing to prove. This completes the proof of (ii). ■

Remark 3.5. – If $K(F)=RL^mC$ then clearly (i) of Lemma 3.4 also holds, except the inequality $a_{m+1}>-a_{m+2}$ (since a_{m+2} is not well defined).

PROPOSITION 3.6. – *Assume that* $(\lambda,\mu)\in D_0$. *Then for* $n\geqq 2$ *we have either* $\varepsilon_n=0$ *or* $a_n\varepsilon_n>0$ *and* $b_n\varepsilon_n>0$. *Moreover, if* $\varepsilon_n\neq o$ *for all* $n\geqq 2$ *then*

$$\lim_{n\to\infty}|a_n|=\lim_{n\to\infty}|b_n|=\infty.$$

Proof. – This follows immediately from Lemmas 3.3 and 3.4 and Remark 3.5 (notice that in Lemma 3.4 (ii) k is bounded by m), except one case not considered in Lemma 3.4 and Remark 3.5, namely $K(F)=RL^\infty$.

If $K(F)=RL^\infty$ then $\varepsilon_n=-1$ for all $n\geqq 1$ and $a_1=0$, $a_{n+1}=x_n+\lambda a_n$ for $n\geqq 1$. Notice that in this case all $x_n(n\geqq 1)$ are negative and $\lambda>1$. Then $a_2=x_1$ and $a_n\leqq\lambda^{n-2}x_1$ for all $n\geqq 2$. Therefore $a_n<0$ for all $n\geqq 2$ and $\lim_{n\to\infty}|a_n|=\infty$. ■

4. MONOTONICITY OF THE KNEADING SEQUENCES FOR $(\lambda,\mu)\in D_0$

Assume that (λ,μ), $(\lambda',\mu')\in D_0$ and $(\lambda,\mu)<(\lambda',\mu')$. Set

$$F_t=F_{\lambda(1-t)+\lambda' t,\ \mu(1-t)+\mu' t}\quad\text{for}\quad t\in[0,1]. \tag{4.1}$$

Since the functions $s \mapsto \dfrac{s}{s^2-1}$ and $s \mapsto \dfrac{1}{1-1/s}$ are decreasing for $s>1$, we have $F_t \in D_0$ for all $t \in [0, 1]$. We have $F_0 = F_{\lambda,\,\mu}$ and $F_1 = F_{\lambda',\,\mu'}$.

Denote $x_n(t) = F_t^n(1)$ and let $K(F_t) = A_0(t) A_1(t) \ldots$ We have

$$\frac{dx_n(t)}{dt} = (\lambda' - \lambda) a_n + (\mu' - \mu) b_n.$$

With the above assumptions and notations we prove two lemmas.

LEMMA 4.1. — *Assume that* $0 \leqq v < w \leqq 1$ *and* $K(F_v) \neq K(F_w)$. *Then* $K(F_v) < K(F_w)$.

Proof. — Take the largest n such that $A_0, \ldots, A_{n-1}$ are defined and constant on the whole $[v, w]$. Since $K(F_v) \neq K(F_w)$, such n exists. If $A_{n-1}(t) = C$ on $[v, w]$ then $K(F)$ is constant on $[v, w]$, a contradiction. Therefore A_n is defined on the whole $[v, w]$, and it is not constant there. Since $A_0(t) = R$ and $A_1(t) = L$ for all t, we have $n \geqq 2$.

By proposition 3.6, $\dfrac{dx_n(t)}{dt}$ is positive if the number of R's among $A_0(t), \ldots, A_{n-1}(t)$ is even and negative if it is odd. In the first case we have $A_n(v) < A_n(w)$ and in the second case $A_n(v) > A_n(w)$. By the definition of the ordering of kneading sequences, we get in both cases $K(F_v) < K(F_w)$. ■

LEMMA 4.2. — *The function* $K(F)$ *cannot be constant on* $[0, 1]$.

Proof. — Suppose that it is constant. Assume first that $k(F_t)$ is infinite. By Proposition 3.6, for each $n \geqq 2$ the function $\dfrac{dx_n}{dt}$ has a constant sign. All $x_n(t)$ are contained in the interval $[1-\mu', 1]$. Therefore $\displaystyle\int_0^1 \left|\frac{dx_n(t)}{dt}\right| dt \leqq \mu'$ for all $n \geqq 2$. However, by Proposition 3.6, for each t we have $\displaystyle\lim_{n \to \infty} \left|\frac{dx_n(t)}{dt}\right| = \infty$. In view of Fatou's Lemma, we get a contradiction.

If $K(F_t)$ is finite then there is $n \geqq 2$ such that $x_n(t) = 0$ but $x_i(t) \neq 0$ for all $i < n$. By Proposition 3.5, $\dfrac{dx_n(t)}{dt} \neq 0$, and therefore that cannot happen for all $t \in [0, 1]$. ■

Now we can easily prove the monotonicity of the kneading sequences for $(\lambda, \mu) \in D_0$.

PROPOSITION 4.3. — *Assume that* $(\lambda, \mu), (\lambda', \mu') \in D_0$ *and* $(\lambda, \mu) < (\lambda', \mu')$. *Then* $K(\lambda, \mu) < K(\lambda', \mu')$.

Proof. — Let F_t be given by (4.1). Then by Lemma 4.1, $K(F_0) \leqq K(F_t) \leqq K(F_1)$ for all $t \in [0, 1]$. If $K(F_0) = K(F_1)$ then $K(F_t)$ is constant, which is impossible by Lemma 4.2. Therefore $K(F_1) > K(F_0)$. ■

5. RENORMALIZATION

We shall use the well known renormalization method for the unimodal maps with the kneading sequence smaller or equal to RLR^∞. It is known that for such map G one has $K(G) = R * K(H)$, where H equals to G^2 restricted to the interval $[G^2(0), G^4(0)]$. The map H is also unimodal, but with a minimum instead of the maximum at 0, and when writing the kneading sequence for H, we have to replace R by L and vice versa. Then we can rescale H linearily by taking

$$\tilde{H}(x) = \frac{1}{H(0)} \cdot H(x \cdot H(0)).$$

Since $H(0) < 0$, the map $\tilde{H}$ has again a maximum at 0. Clearly, $K(\tilde{H}) = K(H)$. If we start with $G = F_{\lambda, \mu}$ then we end up with $\tilde{H} = F_{\mu^2, \lambda\mu}$.

Denote $\varphi(\lambda, \mu) = (\mu^2, \lambda\mu)$. Then the construction described above gives the following result.

LEMMA 5.1. — *If* $(\lambda, \mu) \in D \setminus D_0$ *then* $K(\lambda, \mu) = R * K(\varphi(\lambda, \mu))$. ■

To be able to use Lemma 5.1, we have to know that $\varphi(\lambda, \mu) \in D$.

LEMMA 5.2. — *If* $(\lambda, \mu) \in D \setminus D_0$ *then* $\varphi(\lambda, \mu) \in D$.

Proof. — If $\lambda \geqq 1$, $\mu > 1$, then we have $\mu^2 \geqq 1$, $\lambda\mu > 1$. If $(\lambda, \mu) \notin D_0$ then $\lambda \leqq \dfrac{\mu}{\mu^2 - 1}$ and then

$$\frac{1}{\mu^2} + \frac{1}{\lambda\mu} \geqq \frac{1}{\mu^2} + \frac{\mu^2 - 1}{\mu^2} = 1. \quad ■$$

If F is unimodal, its topological entropy satisfies the inequality $h(F) \leqq \log 2$. We know that $K(F) > RLR^\infty$ is equivalent to $h(F) > \frac{1}{2} \log 2$. We know also that the renormalization procedure doubles the entropy of the map: $h(\tilde{H})=h(H)=2h(G)$. Therefore if $(\lambda, \mu)\in D$ then

$$(5.1)\qquad h(\lambda, \mu)\leqq \log 2,$$

$$(5.2)\qquad h(\lambda, \mu)>\frac{1}{2}\log 2 \quad \text{if and only if} \quad (\lambda, \mu)\in D_0,$$

$$(5.3)\qquad \text{if } (\lambda, \mu)\notin D_0 \quad \text{then } h(\varphi(\lambda, \mu))=2h(\lambda, \mu).$$

LEMMA 5.3. – *If* $(\lambda, \mu)\in D$ *then* $h(\lambda, \mu)>0$.

Proof. – If $(\lambda, \mu)\in D_0$ then it follows from (5.2) that $h(\lambda, \mu)>0$. Assume that $(\lambda, \mu)\in D\setminus D_0$. Then the map $F_{\varphi(\lambda, \mu)}=F_{\mu^2, \lambda_\mu}$ has both slopes with absolute values at least μ. Therefore the variation of $F^n_{\varphi(\lambda, \mu)}$ on the interval $[1-\lambda\mu, 1]$ is at least $\lambda\mu.\mu^n$ for all $n\geqq 1$, and by [MS] we have $h(\varphi(\lambda, \mu))\geqq \log\mu$. Then by (5.3), $h(\lambda, \mu)\geqq\frac{1}{2}\log\mu$. ■

Proof of Theorem A. – Assume that (λ, μ), $(\lambda', \mu')\in D$ and $(\lambda', \mu')>(\lambda, \mu)$ but $K(\lambda', \mu')\leqq K(\lambda, \mu)$. By Lemma 5.3 and (5.1), we have

$$\frac{1}{2^{m+1}}\log 2<h(\lambda, \mu)\leqq\frac{1}{2^m}\log 2$$

for some integer $m\geqq 0$. In view of (5.2), (5.3) and Lemma 5.2, we can apply Lemma 5.1 m times and we get

$$(5.4)\qquad K(\lambda, \mu)=R^{*m} * K(\varphi^m(\lambda, \mu)) \qquad \text{and} \qquad \varphi^m(\lambda, \mu)\in D_0.$$

Since we assumed that $K(\lambda', \mu')\leqq K(\lambda, \mu)$, we have

$$h(\lambda', \mu')\leqq h(\lambda, \mu)\leqq\frac{1}{2^m}\log 2,$$

and therefore

$$(5.5)\qquad K(\lambda', \mu')=R^{*m} * K(\varphi^m(\lambda', \mu')) \qquad \text{and} \qquad \varphi^m(\lambda', \mu')\in D.$$

From the definition of φ it follows that it preserves the "$<$" relation. Therefore $\varphi^m(\lambda, \mu)<\varphi^m(\lambda', \mu')$. Since the function $s\mapsto\frac{s}{s^2-1}$ is decreasing for $s>1$, we get from the definition of D_0 that $\varphi^m(\lambda', \mu')\in D_0$. By Proposition 4.3 we obtain $K(\varphi^m(\lambda', \mu'))>K(\varphi^m(\lambda, \mu))$. In view of (5.4) and (5.5) it follows that $K(\lambda', \mu')>K(\lambda, \mu)$. ■

Remark 5.4. – If $\mu>1$ but $\lambda<1$ then if $K(\lambda, \mu)\leqq RLR^\infty$ then we can also use the renormalization procedure to get $F_{\varphi(\lambda, \mu)}=F_{\mu^2, \lambda_\mu}$. If then $\lambda\mu>1$, we can apply Theorem A to it. If $\lambda\mu\leqq 1$ then we get for $F^2_{\lambda, \mu}$ a periodic orbit of period 2, attracting or indifferent. For example, if we study the one-parameter family of maps $F_{\lambda,2}$ (*see* [BGG]) then our methods work for $\frac{1}{2}<\lambda\leqq 2$ and we get strict monotonicity of the kneading invariant (and therefore topological entropy) there. For $\lambda\leqq\frac{1}{2}$ we get a very simple behaviour and topological entropy zero.

6. PROOFS OF THEOREMS B AND C

Proof of Theorem B. — Assume first that $(\lambda, \mu)\in D_0$. Let $F=F_{\lambda,\mu}$ and suppose that $K(F)\notin \mathcal{M}$. Then $K(F)=\underline{A} * \underline{B}$ for some sequences $\underline{A}$ and $\underline{B}$ and the length p of $\underline{A}$ is finite and larger than 0. In this case there exist intervals $I_0, \ldots, I_p$ with disjoint interiors and such that $F(I_i)\subset I_{i+1}$ for $i=0, \ldots, p-1$; $F(I_p)\subset I_0$ and $0\in \operatorname{int}(I_0)$. The map $F^{p+1}|_{I_0}$ is linearily conjugated to the restriction of some $F_{\kappa,\nu}$ to some invariant interval. If k of the intervals $I_1, \ldots, I_p$ are to the right of 0 then:

$\kappa=\lambda^{p+1-k}\mu^k$ and $\nu=\lambda^{p-k}\mu^{k+1}$ if k is even;

$\kappa=\lambda^{p-k}\mu^{k+1}$ and $\nu=\lambda^{p+1-k}\mu^k$ if k is odd.

Since $F(0)=1>0$, the interval I_1 is to the right of 0 and consequently $k\geqq 1$. Therefore κ, $\nu>1$. Since $F_{\kappa,\nu}$ has an invariant interval, by Lemma 1.1 we have $\frac{1}{\kappa}+\frac{1}{\nu}\geqq 1$. Hence

$$\frac{1}{\lambda^{p+1-k}\mu^k}+\frac{1}{\lambda^{p-k}\mu^{k+1}}\geqq 1.$$

Since $p\geqq 1$ and $1\leqq k\leqq p$, we have $\lambda^{p+1-k}\mu^k\geqq\lambda\mu$ and $\lambda^{p-k}\mu^{k+1}\geqq\mu^2$. Therefore

$$\frac{1}{\lambda\mu}+\frac{1}{\mu^2}\geqq 1,$$

which is equivalent to

$$\lambda\leqq\frac{\mu}{\mu^2-1}.$$

This contradicts the assumption that $(\lambda, \mu)\in D_0$. Hence, if $(\lambda, \mu)\in D_0$ then $K(\lambda, \mu)\in\mathcal{M}$.

If $(\lambda, \mu)\in D$ then we have (5.4) for some $m\geqq 0$ and since $K(\varphi^m(\varphi^m(\lambda, \mu))\in\mathcal{M}$, then $K(\lambda, \mu)\in\mathcal{M}$. ■

We shall use a theorem of [M] saying that if a one-parameter family G_t of continuous unimodal maps depends continuously on t and $h(G_t)>0$ for all t then if $K(G_{t_0})<\underline{K}<K(G_{t_1})$ and $\underline{K}\in\mathcal{M}$ then there exists t between t_0 and t_1 with $K(G_t)=\underline{K}$. We shall refer to this result as the intermediate value theorem. We can use it for our maps in view of Theorem B. Clearly, the dependence of $F_{\lambda,\mu}$ on λ and μ is continuous (even if we rescale $F_{\lambda,\mu}$ to get the same invariant interval for all $(\lambda, \mu)\in D$).

LEMMA 6.1. – *For each* $\underline{M}\in\mathcal{M}$ *except* RL^∞ *there exists a unique* $\gamma(\underline{M})>1$ *such that* $K(1, \gamma(\underline{M}))=\underline{M}$.

Proof. – If $\lambda=1$ then $F_{\lambda,\mu}(x)=x+1$ for $x<0$. Therefore if $\mu>n+1$ then $F_{1,\mu}(1)<-n$ and $K(1, \mu)=RL^n\ldots$ Therefore, $\lim_{\mu\to\infty} K(1, \mu)=RL^\infty$. On the other hand, by Theorem A, $K(1, \mu)<K(\mu, \mu)$ and since $\lim_{\mu\searrow 1} K(\mu, \mu)=R^{*\infty}$, we have also $\lim_{\mu\searrow 1} K(1, \mu)=R^{*\infty}$. Hence, by the intermediate value theorem, if $\underline{M}\neq RL^\infty$ and $\underline{M}\in\mathcal{M}$ then there exists $\gamma(\underline{M})>1$ with $K(1, \gamma(\underline{M}))=\underline{M}$. The uniqueness of $\gamma(\underline{M})$ follows from Theorem A. ■

Proof of Theorem C. – Let $\underline{M}\in\mathcal{M}$. The function γ is given by Lemma 6.1. If $1<\mu\leqq\gamma(\underline{M})$ then we have by Theorem A,

$$K(1, \mu)\leqq K(1, \gamma(\underline{M}))=\underline{M},$$

and

$$K\left(\frac{\mu}{\mu-1}, \mu\right)=RL^\infty$$

(because $F_{\mu/(\mu-1),\mu}(1-\mu)=1-\mu$). Therefore, by the intermediate value theorem, there exists $\beta_{\underline{M}}(\mu)$ with $K(\beta_{\underline{M}}(\mu), \mu)=\underline{M}$. Its uniqueness follows from Theorem A. If $\mu>\gamma(\underline{M})$ then there is no λ with $K(\lambda, \mu)=\underline{M}$ by Theorem A (we have $(\lambda, \mu)>(1, \gamma(\underline{M}))$). Hence, $K(\lambda, \mu)=\underline{M}$ if and only if $\lambda=\beta_{\underline{M}}(\mu)$. Moreover, $\beta_{\underline{M}}$ is a function from $(1, \gamma(\underline{M})]$ into $(1, \infty)$. In the exceptional case $K=F_{\lambda,\mu}$, the function $\beta_{\underline{M}}$ has the domain $(1, \infty)$ and is given by the formula $\beta_{\underline{M}}(\mu)=\dfrac{\mu}{\mu-1}$.

The functions $\beta_{\underline{M}}$ are decreasing by Theorem A. Their graphs fill up the set D by Theorem B.

Since $\beta_{\underline{M}}$ is decreasing, to prove that it is continuous it is enough to show that if $\mu'<\mu''$ are in the domain of $\beta_{\underline{M}}$ and $\beta_{\underline{M}}(\mu\pi£0)>\lambda>\beta_{\underline{M}}(\mu'')$ then there exists $\mu\in(\mu', \mu'')$ such that $\beta_{\underline{M}}(\mu)=\lambda$. For μ', μ'' and λ as above we have by Theorem A, $K(\lambda, \mu')<\underline{M}<K(\lambda, \mu'')$, and the existence of μ with the required properties follows from the intermediate value theorem. Hence, $\beta_{\underline{M}}$ is continuous.

The function γ is increasing by Theorem A. Since its inverse $\mu \mapsto K(1, \mu)$ is defined on $(1, \infty)$, we get

$$\lim_{\underline{M} \to R*^{\infty}} \gamma(\underline{M}) = 1 \quad \text{and} \quad \lim_{\underline{M} \to RL^{\infty}} \gamma(\underline{M}) = \infty.$$

If $\underline{M} = RLR^{\infty}$ then $\beta_{\underline{M}}(\mu) = \dfrac{\mu}{\mu^2 - 1}$ (*see* Lemma 3.2) and we get

$$\lim_{\mu \searrow 1} \beta_{\underline{M}}(\mu) = \infty.$$

Then by Theorem A, this limit is infinite also for all $\underline{M} > RLR^{\infty}$. If $\underline{M} < RLR^{\infty}$ then we perform the renormalization construction of Section 5 and we get $\underline{M} = R * \underline{J}$ for $\underline{J} = K(\mu^2, \mu . \beta_{\underline{M}}(\mu))$. We have $\mu^2 = \beta_{\underline{J}}(\mu . \beta_{\underline{M}}(\mu))$, and hence

$$\lim_{\mu \searrow 1} \beta_{\underline{J}}(\mu . \beta_{\underline{M}}(\mu)) = 1.$$

Since $\beta_{\underline{J}}$ is continuous and decreasing and $\beta_{\underline{J}}(\gamma(\underline{J})) = 1$, we get

$$\lim_{\mu \searrow 1} \beta_{\underline{M}}(\mu) = \lim_{\mu \searrow 1} (\mu . \beta_{\underline{M}}(\mu)) = \gamma(\underline{J}).$$

Clearly, if $\underline{M} \neq RL^{\infty}$ then $\beta_{\underline{M}}(\gamma(\underline{M})) = 1$. If $\underline{M} = RL^{\infty}$ then $\beta_{\underline{M}}(\mu) = \dfrac{\mu}{\mu - 1}$ and therefore

$$\lim_{\mu \to \infty} \beta_{\underline{M}}(\mu) = \infty. \quad \blacksquare$$

REFERENCES

[BGG] C. Brodiscou, Ch. Gillot and G. Gillot, Variations d'entropie d'une famille de transformations de l'intervalle unité, unimodales et linéaires par morceaux, *C. R. Acad. Sci. Paris*, T. **300**, Series I, 1985, pp. 585-588.

[CE] P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems*, Birkhäuser, Boston, 1980.

[H] G. Helmberg, Die Entropie einer linear gebrochenen Funktion, *Monatshefte Math.*, Vol. **94**, 1982, pp. 213-248.

[M] M. Misiurewicz, Jumps of Entropy in One Dimension, *Fund. Math.*, Vol. **132**, 1989, pp. 215-226.

[MS] M. Misiurewicz and W. Szlenk, Entropy of Piecewise Monotone Mappins, *Studia Math.*, Vol. **67**, 1980, pp. 45-63.

(*Manuscript received March 1st*, 1990.)