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Edge processes of one dimensional stochastic growth models

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ABSTRACT. — The following results are proven for one dimensional Stochastic Growth Models :

1) The infinite system converges exponentially fast to the empty configuration if and only if the edge speed α is strictly negative (which is equivalent to $\alpha = -\infty$).

2) The corresponding system seen from the edge has at least one invariant probability measure if and only if $\alpha \geq 0$.

3) For the supercritical basic Contact Process the unique invariant

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probability for the system seen from the edge can be coupled to the upper invariant measure of the infinite system in such a way that they only differ at finitely many sites (to the left of the origin). At the critical point the asymptotic density of any invariant probability measure for the system seen from the edge is zero.

RÉSUMÉ. — On démontre les résultats suivants, pour les processus stochastiques de croissance unidimensionnels :

1) Le système infini converge exponentiellement vers le vide si et seulement si la vitesse du bord α est strictement négative (ceci est équivalent à $\alpha = -\infty$).

2) Le système correspondant vu du bord a au moins une probabilité invariante si et seulement si $\alpha \geq 0$.

3) Pour le processus de contact surcritique, l'unique mesure invariante du processus vu du bord peut être couplée avec la mesure invariante supérieure du processus infini, de telle façon qu'elles diffèrent en un nombre fini de points (à gauche de l'origine). Au point critique la densité asymptotique de toute probabilité invariante pour le système vu du bord est nulle.

I. INTRODUCTION AND STATEMENT OF RESULTS

A one dimensional Stochastic Growth Model is a Markov process whose state at time t , ξ_t is in $\{0, 1\}^Z$. If $\xi_t(x) = 1$ this means that we have a particle on $x \in Z$, if $\xi_t(x) = 0$ this means that the site x is empty. The process ξ_t evolves according to the following rules : particles die at rate 1, i.e., if $\xi_t(x) = 1$ then

$$(1) \quad \lim_{s \rightarrow 0} \frac{1}{s} P(\xi_{t+s}(x) = 0 \mid \xi_t) = 1$$

and particles are born at rate $\lambda b_x(\xi)$, i.e., if $\xi_t(x) = 0$ then

$$(2) \quad \lim_{s \rightarrow 0} \frac{1}{s} P(\xi_{t+s}(x) = 1 \mid \xi_t) = \lambda b_x(\xi_t)$$

where $\lambda > 0$ is a real parameter.

Throughout this paper we will assume that there is no spontaneous generation of particles, that the process is attractive and invariant under

translations and reflection, and that the interaction between particles has a finite range. This mean that the birth rates will satisfy the conditions (3)-(7), stated below.

$$(3) \quad b_x(\eta_0) = 0,$$

where η_0 is the configuration with no particles,

$$(4) \quad b_x(\eta) = b_0(\tau_x \eta),$$

where $(\tau_x \eta)(y) = \eta(y + x)$

$$(5) \quad b_0(\xi^-) = b_0(\xi),$$

where $\xi^-(x) = \xi(-x)$ for all x ,

$$(6) \quad b_0(\eta) \leq b_0(\xi) \quad \text{if} \quad \eta \leq \xi,$$

where we say that $\eta \leq \xi$ if $\eta(x) \leq \xi(x)$ for all x ,

$$(7) \quad \text{for some positive integer } R, b_0(\eta) = b_0(\xi) \text{ if } \eta(x) = \xi(x) \forall x \in [-R, R].$$

When $R = 1$ we say that we have a Nearest-Neighbor model. The Basic Contact process is a special Nearest-Neighbor model where :

$$b_x(\xi) = \xi(x - 1) + \xi(x + 1).$$

These processes can be constructed by Harris' graphical methods as was indicates for instance in Durrett (1985). For this purpose, for each $x \in Z$ let $(S_n^x)_{n \geq 1}$ and $(T_n^x)_{n > 1}$ be the arrival times of two independent Poisson processes having rates 1 and $b = \lambda b_0(Z)$ respectively (Z is the configuration with one particle on every site of Z), and let $(U_n^x)_{n \geq 1}$ be a sequence of independent random variables each one with an uniform distribution on $[0,1]$. At times S_n^x , $n \geq 1$ we kill a particle at x if one is present there. At times T_n^x , $n \geq 1$ there will be a birth at x if it is not already occupied and if the configuration satisfies $b_x(\xi_{T_n^x-}) \geq \frac{b}{\lambda} U_n^x$, where $\xi_{T_n^x-}$ is the limit of ξ_t as t increases to T_n^x .

Using properties of the Poisson process one can show that the construction above for any initial configuration gives rise to a Markov process which evolves according to the rules given before.

We will denote the Stochastic Growth model by $\xi_{t,\lambda}^\eta$, with initial configuration η and birth rates : $\lambda b_x(\xi)$.

We also define $*$, the configuration with only one particle which is on site 0 and

$$\tau^*(\lambda) = \inf\{t > 0 : \sum_{x \in Z} \xi_{t,\lambda}^*(x) = 0\}.$$

When no confusion is possible we will omit λ in all our notation. We define now two well known critical parameters, first the critical parameters for finite systems :

$$\lambda_f = \inf\{\lambda > 0 : P(\tau^*(\lambda) = \infty) > 0\}.$$

It follows from the attractiveness assumption that $\xi_{t,\lambda}^Z$ converges in law, as $t \rightarrow \infty$, to a probability μ_λ . Let δ_0 be the Dirac probability which concentrates on the empty configuration. The critical parameter for infinite systems is :

$$\lambda_i = \inf\{\lambda > 0 : \mu(\lambda) \neq \delta_0\}.$$

We can now state our results. Let 1 be the configuration with a particle on every site on the left of the origin and no particle on the right of the origin. Let r_t^1 be the position of the right edge of the process with initial configuration 1. From an appropriate version of the Subadditive Ergodic Theorem (Liggett (1985) Chap. VI Th. 2.6 and Th. 2.19) it is easy to see that there exists a real constant $\alpha(\lambda)$ in $[-\infty; +\infty[$ such that :

$$\lim_{t \rightarrow \infty} \frac{r_t^1}{t} = \lim_{t \rightarrow \infty} E\left(\frac{r_t^1}{t}\right) = \alpha(\lambda) = \inf_{t > 0} E\left(\frac{r_t^1}{t}\right) \quad a.s.$$

From the last expression above and monotonicity, $\alpha(\cdot)$ is a right-continuous function.

THEOREM 1 : (a) *If $\alpha(\lambda) < 0$, then there exist strictly positive real numbers C and γ such that*

$$P(\xi_{t,\lambda}^Z(0) = 1) \leq Ce^{-\gamma t}.$$

(b) *If $\alpha(\lambda) > -\infty$, then*

$$\lim_{t \rightarrow \infty} tP(\xi_{t,\lambda}^Z(0) = 1) = +\infty.$$

Both parts of this theorem were already known for the compact process (See Th. 3.4 and Th 3.10 in Chap. VI of Liggett (1985)), but their proofs relied on properties as self duality and additivity which are not satisfied

by all the processes we consider here. As an immediate consequence of this theorem we obtain :

COROLLARY 1 : *If $\alpha(\lambda) < 0$, then $\alpha(\lambda) = -\infty$.*

Because of Theorem 1, it seems natural to define the following critical values of λ :

$$\lambda_\alpha = \inf\{\lambda > 0 : \alpha(\lambda) \geq 0\},$$

$$\lambda_{f,e} = \sup\{\lambda > 0 : \text{there exist } C, \gamma \in]0, +\infty[\text{ such that } P(\tau^*(\lambda) > t) \leq Ce^{-\gamma t}\}$$

and

$$\lambda_{i,e} = \sup\{\lambda > 0 : \text{there exist } C, \gamma \in]0, +\infty[\text{ such that } P(\xi_{t,\lambda}^Z(0) = 1) \leq Ce^{-\gamma t}\}.$$

Our next corollary concerns these critical values.

COROLLARY 2 : (a) $\lambda_\alpha = \lambda_{i,e} \leq \lambda_{f,e}$,

(b) $\lim_{t \rightarrow \infty} tP(\xi_{t,\lambda_{i,e}}^Z(0) = 1) = \infty$.

Note that the equality in part (a) follows immediately from Theorem 1, and that part (b) is an easy consequence of part (a), the right continuity of α and part (b) of Theorem 1. The inequality in part (a) will be proved in Section 2, after we proof of Theorem 1. Finally, observe that part (b) implies that if $P(\xi_{t,\lambda_{i,e}}^Z(0) = 1)$ behaves as $t^{-\kappa}$ when $t \rightarrow \infty$, then we must have $\kappa < 1$.

We consider now the process $\xi_{t,\lambda}$ seen from the edge with an initial configuration with no particles on the right of the origin. This mean that when the rightmost particle dies or when a particle is born on the right of the rightmost particle then we make a translation to have at all times the rightmost particle at the origin. Let us call the process seen from the edge : $\tilde{\xi}_{t,\lambda}$. We have :

THEOREM 2 : *The process $\tilde{\xi}_{t,\lambda}$ has an invariant probability measure $\tilde{\mu}(\lambda)$ if and only if $\alpha(\lambda) > -\infty$.*

For the contact process in discrete time, this result is not new ; Durrett (1984) showed that the invariants measure exists if $\alpha(\lambda) > -\infty$ and Schonmann (1987) proved the converse. The proof we give in Section 3 simplifies Durrett's for the "if" part, and uses the ideas in Schonmann (1987), but is not a straightforward generalization of his proof, for the "only if" part.

We are now going to state some results which we proved only for the Contact Process. We will say that $\mu_1 \leq \mu_2$ where μ_1 and μ_2 are two probability measures, if for any continuous function f we have :

$$\int f(\eta)d\mu_1(\eta) \leq \int f(\eta)d\mu_2(\eta)$$

Let us recall that μ_λ is the upper invariant probability for the system not seen from the edge, and let $\tilde{\mu}_\lambda$ be an invariant for the process seen

form the edge. In the following two theorems we will compare $\tilde{\mu}_\lambda$ with the restriction of μ_λ to the non-positive integers. For this purpose, to any configuration η we associate $\hat{\eta}$, defined by : $\hat{\eta}(x) = 0$ if $x > 0$ and $\hat{\eta}(x) = \eta(x)$ if $x \leq 0$. Then, given a measure μ , we define $\hat{\mu}$ by

$$\hat{\mu}_\lambda(A) = \mu_\lambda(\eta : \hat{\eta} \in A)$$

for every cylinder set A . Recall that for the contact process $\lambda_\alpha = \lambda_i = \lambda_f$ and denote by λ_c their common value.

Theorems 3 and 4 below will be proved in Section 4. As shown there, the first of these theorems is an easy consequence of a result of Gray (1986), and the second one follows from the first and from the lemma appearing in Section 3.

THEOREM 3 : *Let $\lambda \geq \lambda_c$, then $\tilde{\mu}_\lambda \geq \hat{\mu}_\lambda$.*

THEOREM 4 : *Let $\lambda \geq \lambda_c$, then there exist constants $C, \gamma, a \in]0, \infty[$ such that any invariant measure $\tilde{\mu}_\lambda$ satisfies :*

$$0 \leq \tilde{\mu}_\lambda(\eta : \eta(-x) = 1) - \hat{\mu}_\lambda(\eta : \eta(-x) = 1) \leq Ce^{-\gamma x} + P(x/a < \tau^* < \infty) \quad \forall x > 0.$$

Galves and Presutti (1987) proved that if $\lambda > \lambda_c$ then there is a unique invariant measure for the contact process seen from the edge. Since in this case we have the following estimates due to Durrett and Griffeath (see Liggett (1985), Chap. VI, Th. 3.23) :

$$P(t < \tau^* < \infty) \leq Ce^{-\gamma t}$$

we obtain

COROLLARY 3 : *Let $\lambda > \lambda_c$, then there exist constants $C, \gamma \in]0, \infty[$ such that the invariant measure $\tilde{\mu}_\lambda$ for the process seen from the edge satisfies :*

$$0 \leq \tilde{\mu}_\lambda(\eta : \eta(-x) = 1) - \hat{\mu}_\lambda(\eta : \eta(-x) = 1) \leq Ce^{-\gamma x} \quad \forall x > 0.$$

From Theorem 3, Corollary 3 and the Borel-Cantelli Lemma we conclude that for $\lambda > \lambda_c$, there exists a probability measure ν on $\{0, 1\}^{\mathbb{Z}^-} \times \{0, 1\}^{\mathbb{Z}^-}$ such that :

- (a) The first marginal of ν is $\tilde{\mu}_\lambda$,
- (b) The second marginal of ν is $\hat{\mu}_\lambda$,
- (c) $\nu((\eta, \xi) : \eta \geq \xi) = 1$,
- (d) $\nu((\eta, \xi) : \sum |\eta(x) - \xi(x)| < \infty) = 1$.

From Theorem 4 we can also conclude that

$$(8) \quad \lim_{x \rightarrow \infty} \tilde{\mu}_{\lambda_c}(\eta : \eta(-x) = 1) = \hat{\mu}_{\lambda_c}(\eta : \eta(0) = 1) = 0,$$

where the last equality has been recently proved by Bezuidenhout and Grimmett (1989). This last result also implies that there is no measure ν satisfying a) – d) above at the critical point.

Since for $\lambda > \lambda_c$ μ_λ is ergodic and satisfies a Central Limit Theorem, (see Liggett (1985) (Chap. III, Prop. 2.16) and Griffeath (1981)), the comment following Corollary 3 implies that $\tilde{\mu}_\lambda$ satisfies a strong law of large numbers and a Central Limit Theorem, i.e., that for $\lambda > \lambda_c$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{x=0}^{n-1} \eta(-x) = \rho_\lambda \quad \tilde{\mu}_\lambda \quad a.s.$$

where $\rho_\lambda = \mu_\lambda(\eta : \eta(0) = 1)$, and

$$\lim_{n \rightarrow \infty} \tilde{\mu}_\lambda(\eta : \frac{1}{\sqrt{n}} \sum_{x=0}^{n-1} (\eta(-x) - \rho_\lambda) \leq a) = F(a)$$

where ρ_λ is as above and F is the distribution function of a normal random variable with mean 0 and positive variance (which depends on λ).

II. PROOFS OF THEOREM 1 AND COROLLARY 2

We will prove part a) of Theorem 1 using a restart argument from Durrett (1984); Sect. 12. Let us define an auxiliary process ξ_t^r with one particle on every site at the initial time 0 and which evolves like the Stochastic Growth Model we are considering until time

$$\tau_1 = \inf\{t > 0 : g_t \leq -2R \quad \text{and} \quad d_t \geq 2R\},$$

where

$$g_t = \sup\{x \leq 0 : \xi_t^r(x) = 1\},$$

$$d_t = \inf\{x \geq 0 : \xi_t^r(x) = 1\}.$$

At time τ_1 we put a particle on every site on the left of $-2R$ and on the right of $2R$. We leave all the sites between $-2R$ and $2R$ empty. Recall that R is the range of the interaction. After τ_1 we let again the process ξ_t^r evolve like the Stochastic Growth Model until time

$$\sigma_1 \doteq \inf\{t > \tau_1 : g_t \geq -R \quad \text{or} \quad d_t \leq R\}$$

Note that τ_1 is finite with probability 1 but σ_1 is infinite with strictly positive probability. At time σ_1 (if it is finite) we put a particle on every site of Z and so ξ_t^r begins again from the initial configuration. We define by induction two sequences of random variables τ_k and σ_k , for $k \geq 2$

$$\begin{aligned} \tau_k &= \inf\{t > \sigma_{k-1} : g_t \leq -2R \text{ and } d_t \geq 2R\}, \\ \sigma_k &= \inf\{t > \tau_k : g_t \geq -R \text{ or } d_t \leq R\}. \end{aligned}$$

At time τ_k we put a particle on every site on the left of $-2R$ and on the right of $2R$, and we let sites between $-2R$ and $2R$ empty. At time σ_k we put a particle on every site of Z and so the system “restarts”. Between these random times the system ξ_t^r evolves like the Stochastic Growth Model. Let us define the random variable :

$$K = \sup\{k \geq 1 : \sigma_k < \infty\}.$$

We fix λ such that $\alpha(\lambda) < 0$ and will omit it in the rest of the proof. Let ξ_t^Z be the Stochastic Growth Model with initial configuration Z . By the attractiveness assumption (6), we have :

$$(2.1) \quad P(\xi_t^Z(0) = 1) \leq P(\xi_t^r(0) = 1) \leq P(\tau_{K+1} > t)$$

since after time τ_{K+1} we will not have any particle between the sites $-R$ and R . Let us define $X_k = \sigma_k - \tau_k$ for $1 \leq k \leq K$, $Y_1 = \tau_1$ and $Y_k = \tau_k - \sigma_{k-1}$ for $2 \leq k \leq K + 1$. We can write :

$$(2.2) \quad T_{K+1} = \sum_{i=1}^K X_i + \sum_{i=1}^{K+1} Y_i.$$

From (2.1) and (2.2) we have that :

$$(2.3) \quad P(\xi_t^Z(0) = 1) \leq P\left(\sum_{i=1}^K X_i \geq t/2\right) + P\left(\sum_{i=1}^{K+1} Y_i \geq t/2\right).$$

But conditioned on $\{K = k\}$ we have that $(X_i)_{i \leq k}$ are i.i.d. random variables with the same distribution as X_1 conditioned on $\{X_1 < \infty\}$. Let $(Z_i)_{i \geq 1}$ be an infinite sequence of i.i.d. random variables with this same distribution. For every $\epsilon > 0$ we have :

$$(2.4) \quad \begin{aligned} P\left(\sum_{i=1}^K X_i \geq t/2\right) &= \sum_{k=1}^{\infty} P\left(\sum_{i=1}^K X_i \geq t/2 \mid K = k\right) P(K = k) \\ &\leq P(K \geq [\epsilon t]) + P\left(\sum_{i=1}^{[\epsilon t]} Z_i \geq t/2\right) \end{aligned}$$

where $[a]$ is the largest integer less than or equal to a . Our next task is to show that Z_1 has an exponentially decaying tail, i.e.,

$$(2.5) \quad P(Z_1 > s) \leq C e^{-\gamma s}.$$

Here and in the sequel C and γ are strictly positive constants whose values may change from line to line. Since the range of the interaction is R , between times τ_1 and σ_1 , the system to the left of the origin does not interact with the system to its right. So by symmetry,

$$(2.6) \quad P(s < X_1 < \infty) \leq 2P(r_u^1 > 0 \text{ for some } u > s).$$

And since $\alpha < 0$, we have (see the proof of Th. 1 Sect.4.b of Durrett (1988))

$$(2.7) \quad P(r_u^1 > 0 \text{ for some } u > s) \leq C e^{-\gamma s}.$$

Now (2.5) follows from (2.6), (2.7) and the fact that $P(X_1 < \infty) > 0$.

Let m be the mean value of Z_1 . Due to (2.5), m is finite and the Cramer-Chernoff Theorem implies that if ϵ is chosen so that $1/2\epsilon > m$ then

$$(2.8) \quad P\left(\sum_{i=1}^{[\epsilon t]} Z_i \geq t/2\right) \leq C e^{-\gamma t}.$$

Since K has a geometric distribution, for every $\epsilon > 0$

$$(2.9) \quad P(K > [\epsilon t]) \leq C e^{-\gamma t}.$$

From (2.4), (2.8) and (2.9) we have that

$$P\left(\sum_{i=1}^K X_i \geq t/2\right) \leq C e^{-\gamma t}.$$

Comparing ξ_i^r with a system where every site on the left of $-R$ and every site on the right of R has a particle which cannot die (so we have a finite Markov Process), it is easy to see that :

$$P(Y_i > t) \leq C e^{-\gamma t}.$$

This shows that we can treat the second term on the r.h.s of (2.3) exactly as the first term and this concludes the proofs of part a).

Let us now prove part b). For every $\epsilon > 0$,

$$(2.10) \quad P(r_t^1 \in [t(\alpha(\lambda) - \epsilon); t(\alpha(\lambda) + \epsilon)]) \leq P(\exists x \in [t(\alpha(\lambda) - \epsilon); t(\alpha(\lambda) + \epsilon)] : \xi_t^1(x) = 1).$$

Using attractiveness and translation invariance of the model, we have that the r.h.s of (2.10) is less than :

$$(2\epsilon t + 1)P(\xi_t^z(0) = 1).$$

And so

$$\liminf_{t \rightarrow \infty} tP(\xi_t^z(0) = 1) \geq \frac{1}{2\epsilon} \liminf_{t \rightarrow \infty} P(r_t^1 \in [t(\alpha(\lambda) - \epsilon); t(\alpha(\lambda) + \epsilon)]) = \frac{1}{2\epsilon}.$$

Since ϵ can be taken arbitrarily small this conclude the proof of Theorem 1.

We will now prove the inequality in part a) of Corollary 2. Let $\lambda < \lambda_{i,e}$ and with this fixed λ we have for every $c > 0$,

$$(2.11) \quad P(\tau^* > t) \leq P(\exists x : |x| < ct \text{ and } \xi_t^*(x) = 1) + P(\exists x : |x| \geq ct \text{ and } \xi_t^*(x) = 1).$$

By attractiveness and translation invariance the first term in (6.1) is smaller than :

$$2ctP(\xi_t^z(0) = 1)$$

For the second term we use the fact that each edge of ξ_t^* increases at most (when there are no deaths) as R times a Poisson process of parameter λB , where $B = Rb_0(\mathbf{Z})$. So the right hand side of (2.11) is smaller than

$$2ctP(\xi_t^z(0) = 1) + 2P(RP(\lambda Bt) \geq ct)$$

where $\mathcal{P}(m)$ is a random variable with a Poisson distribution of parameter m . Since these two terms go to 0 exponentially fast as t goes to infinity, provided c is chosen large enough, there are two strictly positive constants C and γ such that :

$$P(\tau^* > t) \leq Ce^{-\gamma t}.$$

and so $\lambda \leq \lambda_{f,e}$. This completes the proof of Corollary 2.

III. PROOF OF THEOREM 2

First suppose that $\alpha(\lambda) > -\infty$, then set

$$X = \{\eta : \eta(0) = 1, \eta(x) = 0 \text{ for all } x > 0\}$$

$$\tilde{X} = \{\eta \in X : \sum_{x \leq 0} \eta(x) = \infty\}.$$

For $\eta \in \tilde{X}$ the process seen from the edge $\tilde{\xi}_t^\eta$ is well defined by

$$\tilde{\xi}_t^\eta(x) = \xi_t^\eta(x + \sup\{y \in Z : \tilde{\xi}_t^\eta(y) = 1\}).$$

Clearly $\tilde{\xi}_t^\eta \in \tilde{X}$ for every $t \geq 0$. Let νS_t be the distribution of $\tilde{\xi}_t^\eta$. Recall that 1 is the configuration with a particle on every site on the left of the origin and no particle on the right of the origin, δ_1 is the probability measure that concentrates on this configuration. And set $\tilde{\nu}_t = \delta_1 S_t$.

Since X is a compact space, there is a sequence t_n of real numbers going to infinity such that the sequence of probability measures :

$$\tilde{\mu}_n = \frac{1}{t_n} \int_0^{t_n} \tilde{\nu}_s ds$$

converges weakly to some probability $\tilde{\mu}$ on X when n goes to infinity. We will show now that $\tilde{\mu}$ concentrates on \tilde{X} . Let us define :

$$A_{i,j} = \{\eta : \sum_{x=-i}^0 \eta(x) < j\}.$$

Set

$$B = Rb_0(Z).$$

A simple comparison shows that :

$$(3.1) \quad E(\tau_{t+1}^1 - \tau_t^1)^+ \leq E(RP(\lambda B)) = \lambda RB$$

where $\mathcal{P}(m)$ has a Poisson distribution with parameter m . The important inequality is the following one :

$$(3.2) \quad E(\tau_{t+1}^1 - \tau_t^1)^- \geq ip(j)\tilde{\nu}_i(A_{i,j}) \text{ where } p(j) > 0.$$

This comes from the fact that if the process is at time t on $A_{i,j}$ and if all the particles which are between 0 and $-i$ at time t die before time $t + 1$ and if there are no births on the right of $-i$ during the interval $[t, t + 1]$

then the rightmost particle goes backward by at least i . The event we just described has a probability which is bounded below by a strictly positive constant which depends only on j . It is this constant that we have denoted by $p(j)$. From (3.1) and (3.2) we obtain

$$(3.3) \quad \int_0^t E(r_{s+1}^1 - r_s^1) ds \leq RB\lambda t - ip(j) \int_0^t \tilde{v}_s(A_{i,j}) ds.$$

The term on the left of (3.3) is also equal to :

$$\int_t^{t+1} Er_s^1 - \int_0^1 Er_s^1 ds.$$

But

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_t^{t+1} Er_s^1 ds &= \alpha(\lambda) > -\infty \\ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^1 Er_s^1 ds &= 0. \end{aligned}$$

We substitute t_n for t and let n go to infinity in (3.3), so that :

$$\tilde{\mu}(A_{i,j}) \leq \frac{RB\lambda - \alpha(\lambda)}{ip(j)}.$$

We let i go to infinity to conclude that :

$$\tilde{\mu} \left(\bigcap_{i=1}^{\infty} A_{i,j} \right) \quad \forall j, \quad \text{therefore}$$

$$\tilde{\mu} (\text{configurations with finitely many particles}) = 0.$$

Now we know that $\tilde{\mu}_n$ and $\tilde{\mu}$ concentrate on \tilde{X} the subset of X of infinite configurations. But \tilde{X} is a dense Borel subset of X and so many uniformly continuous function on \tilde{X} may be extended to a uniformly continuous function on all X and since $\tilde{\mu}_n$ converges to $\tilde{\mu}$ on X ,

$$\lim_{n \rightarrow \infty} \int f d\tilde{\mu}_n = \int f d\tilde{\mu}$$

for any f uniformly continuous on \tilde{X} . This implies that $\tilde{\mu}_n$ converges to $\tilde{\mu}$ on \tilde{X} . (See Theorem 3.3.1 in Ethier and Kurtz (1986)). For any function f on \tilde{X} which depends only on finitely many sites and any $t > 0$ it is not difficult to see that the function : $\eta \rightarrow E(f(\xi_t^\eta))$ is continuous on \tilde{X} . So

the process seen from the edge has the Feller property on \tilde{X} . To show now that $\tilde{\mu}$ is invariant it is enough to use a standart argument, since from the remarks above

$$\tilde{\mu}S_t = \lim_{n \rightarrow \infty} \tilde{\mu}_n S_t = \lim_{n \rightarrow \infty} \frac{1}{t_n} \int_t^{t+t_n} \tilde{\nu}_s ds = \tilde{\mu}.$$

The key of the proof of the converse is the following lemma which gives a relation between the process seen from the edge and the process not seen from the edge. This lemma will also be very useful for the proof of Theorem 4. Recall that \tilde{X} is the set of infinite configurations of X .

LEMMA 1 : Given $\lambda > 0$, there exist constants C, γ and a in $]0, \infty[$ such that :

$$\sup_{\eta \in \tilde{X}} P(\tilde{\xi}_t^\eta([-at]) = 1) \leq P(\xi_t^Z(0) = 1) + Ce^{-\gamma t},$$

where $[\]$ denotes integer part of.

Proof of the Lemma : We define a process ζ_t which is roughly going to play the role of the dual of ξ_t . To do so we start constructing for each $t > 0$ and each $x \in Z$ a process ${}_t\zeta_s^x$, for which the time parameter s runs in the interval $[0, t]$, in the following way : for each realization of $(\xi_u)_{0 \leq u \leq t}$ we have a corresponding realization of Poisson marks (deaths and births) as we indicated in construction of the process. The time for ${}_t\zeta_s$ is though of a running backwards : $s = t - u$. Now we define ${}_t\zeta_s^x$ as follows : ${}_t\zeta_0^x = x$ and for all $s \leq t$, ${}_t\zeta_t^x$ is an interval which increases with s in the following way : if at time s there is a birth mark at a site y of ${}_t\zeta_{s-}^x$ then

$${}_t\zeta_s^x = {}_t\zeta_{s-}^x \cup [y - R; y + R]$$

For the purpose of this construction it is understood that there is a birth mark at a site y at time s if and only if $T_n^y = t - s$ for some $n \geq 1$. Now we define ζ_t^x as ${}_t\zeta_t^x$. It is easy to see that ξ_t^x has the following property :

$$(3.4) \quad \text{If } \eta_1(y) = \eta_2(y) \text{ for all } y \in \zeta_t^x \text{ then } \xi_t^{\eta_1}(x) = \xi_t^{\eta_2}(x).$$

In particular form (3) we have

$$(3.5) \quad \{\xi_t^\eta(y) = 1\} \subset \{\exists x \in \zeta_t^y \text{ such that } \eta(x) = 1\}.$$

Recall that :

$$B = Rb_0(Z)$$

and let $l({}_t\zeta_s^x)$ and $r({}_t\zeta_s^x)$ be the right and left edge respectively of ${}_t\zeta_s^x$. Since $l({}_t\zeta_s^x)$ only decrease if there is a birth mark at a distance at most equal to R to the right of $l({}_t\zeta_s^x)$ we have for every $c > 0$:

$$(3.6) \quad P(l(\zeta_t^x) < x - c) \leq P(R\mathcal{P}(\lambda Bt) > c)$$

where $\mathcal{P}(m)$ is a variable having a Poisson distribution with parameter m . By symmetry a similar inequality is also true for the right edge. We have for any initial configuration η in \tilde{X} and any number a :

$$(3.7) \quad P(\tilde{\xi}_t^\eta(-at) = 1) = \sum_{x \in \mathbb{Z}} P(r_t^\eta = x, \xi_t^\eta(x - at) = 1).$$

(In this proof when a real number appears as a site, we are referring to its integer part) Let a and b be positive constants such that $a > 2b$ and define the two events :

$$B_1(x, t) = \{l(\zeta_t^y) \geq x - bt, \text{ for all } y \geq x\}$$

$$\text{and } B_2(x, t) = \{r(\zeta_t^{x-at}) \leq x - at + bt\}$$

and define $B(x, t) = B_1(x, t) \cap B_2(x, t)$. Each term of the sum in (3.7) is less than :

$$(3.8) \quad P(r_t^\eta = x; B_1(x, t); \xi_t^\eta(x - at) = 1; B_2(x, t)) + P(r_t^\eta = x; B^c(x, t)).$$

By attractiveness the first term in (3.8) is less than :

$$(3.9) \quad P(r_t^\eta = x; B_1(x, t); \xi_t^Z(x - at) = 1; B_2(x, t)).$$

From (3.4) we see that the occurrence of the event $\{r_t^\eta = a; B_1(x, t)\}$ depends only on the Poisson marks contained in $\{(z, s) : z \in \mathbb{Z}, z \geq x - bt, 0 \leq s \leq t\}$ and that the occurrence of the event $\{\xi_t^Z(x - at) = 1; B_2(x, t)\}$ depends only on the Poisson marks contained in $\{(z, s) : z \in \mathbb{Z}, z \leq x - at + bt, 0 \leq s \leq t\}$. Since these two regions of $\mathbb{Z} \times \mathbb{R}_+$ are disjoint, the events $\{r_t^\eta = x; B_1(x, t)\}$ and $\{\xi_t^Z(x - at) = 1; B_2(x, t)\}$ are independent. Hence, by translation invariance, (3.9) is less than

$$(3.10) \quad P(\xi_t^Z(0) = 1)P(r_t^\eta = x)$$

We now consider the second term of (3.8). We estimate first $P(B^c(x, t))$,

$$P(B^c(x, t)) \leq \sum_{y \geq x} P(l(\zeta_t^y) < x - bt) + P(r(\zeta_t^{x-at}) > x - at + bt).$$

But $-l(\zeta_t^y)$ increases less than R times a Poisson process of parameter λB , therefore

$$P(l(\zeta_t^y) \leq x - bt) \leq P(R\mathcal{P}(\lambda Bt) \geq y - x + bt).$$

For $c \geq \lambda$ we have :

$$\begin{aligned}
 P(\mathcal{P}(\lambda) \geq c) &= \lambda^c e^{-\lambda} \sum_{n \geq c} \frac{\lambda^{n-c}}{n!} \leq e^{-\lambda} \lambda^c \sum_{n=0}^{\infty} \frac{c^{n-c}}{n!} \\
 &= \left(\frac{\lambda}{c}\right)^c e^{c-\lambda} = e^{-(c \ln \frac{c}{\lambda} - c + \lambda)}.
 \end{aligned}$$

Therefore for c such that $\ln \frac{c}{\lambda} \geq 2$ we obtain :

$$(3.11) \quad P(\mathcal{P}(\lambda) \geq c) \leq e^{-c}.$$

Hence, for $b \geq \lambda B R \epsilon^2$

$$P(l(\zeta_t^y) \leq x - bt) \leq e^{-\frac{bt+y-x}{R}}$$

and since we can give an upper bound for $P(r(\zeta_t^{x-at}) \geq x - at + bt)$ exactly in the same way, we have :

$$(3.12) \quad P(B^c(x, t)) \leq C e^{-\frac{bt}{R}}$$

where C here depends only on R . Let $z_1 = 0$ and $z_{i+1} = \sup \{x < z_i : \eta(x) = 1\}$. We define $A_\eta(t)$ as :

$$A_\eta(t) = \{z_i, 1 \leq i \leq [te^t]\}, \quad \text{where } [te^t] = \inf \{k \in \mathbb{Z} : k \geq te^t\}.$$

Observe that if $\{r_t^\eta < z_{[te^t]}\}$ occurs, then this means that all the $[te^t]$ rightmost particles died before time t and since the death rate is 1, we have that :

$$(3.13) \quad P(r_t^\eta < z_{[te^t]}) \leq (1 - e^{-t})^{[te^t]} \leq e^{-t}.$$

Let c be a constant and

$$C_\eta(t) = \{z \in Z : |z - x| < ct \text{ for some } x \in A_\eta(t)\} \quad .$$

We now consider

$$\begin{aligned}
 (3.14) \quad \sum_{x \in Z} P(r_t^\eta = x; B^c(x, t)) &= \sum_{x \in C_\eta(t)} P(r_t^\eta = x; B^c(x, t)) \\
 &\quad + \sum_{x \notin C_\eta(t)} P(r_t^\eta = x; B^c(x, t)).
 \end{aligned}$$

The first term on the r.h.s. of (3.14) is less than :

$$|C_\eta(t)|P(B^c(x, t))$$

and using (3.12) this is less than :

$$(3.15) \quad (2ct + 1)[te^t]Ce^{-\frac{bt}{R}}.$$

The second term on the r.h.s. of (3.14) is less than :

$$(3.16) \quad P(r_t^\eta \notin C_\eta(t); r_t^\eta \geq z_{[te^t]}) + P(r_t^\eta < z_{[te^t]})$$

For the second term in (3.16) we will use (3.13), the first term in (3.16) is less than :

$$\sum_{x \notin C_\eta(t); x \geq z_{[te^t]}} P(\xi_t^\eta(x) = 1)$$

but using (3.5) this is less than :

$$\sum_{x \notin C_\eta(t); x \geq z_{[te^t]}} P(\exists y \in \zeta_t^x, \eta(y) = 1)$$

but ζ_t^x must contain a $z_i, 1 \leq i \leq [te^t]$. So the preceding term is less than

$$(3.17) \quad \sum_{1 \leq i \leq [te^t]} \sum_{x \notin C_\eta(t)} P(z_i \in \zeta_t^x) \leq \sum_{1 \leq i \leq [te^t]} 2 \sum_{y \geq 0} P\left(\mathcal{P}(\lambda Bt) \geq \frac{ct + y}{R}\right).$$

The last inequality comes from the fact that $-l(\zeta_t^x)$ and $r(\zeta_t^x)$ increase less than R times a Poisson process of parameter λB . For $c \geq \lambda BR e^2$, we use (3.11) and the r.h.s. of (3.17) is less than :

$$(3.18) \quad [te^t]Ce^{-\frac{ct}{R}}$$

From (3.10), (3.15), (3.13) and (3.18) we have that :

$$P(\tilde{\xi}(-at) = 1) \leq P(\xi_t^Z(0) = 1) + (2ct + 1)[te^t]Ce^{-\frac{bt}{R}} + e^{-t} + [te^t]Ce^{-\frac{ct}{R}}.$$

To complete the proof of Lemma 1 just pick $b = c \geq \max(e^2 \lambda BR, 2R)$ and $a > 2b$.

Now to complete the proof of Theorem 2, suppose that $\alpha(\lambda) = -\infty$ and that $\tilde{\mu}$ is an invariant probability for the process seen from the edge. From Lemma 1 we have :

$$\tilde{\mu}(\eta : \eta(-at) = 1) \leq P(\xi_t^Z)(0) = 1) + Ce^{-\gamma t}$$

but from Theorem 1, we have

$$P(\xi_t^Z(0) = 1) \leq Ce^{-\gamma t}.$$

Now, by Borel-Cantelli Lemma, $\tilde{\mu}$ concentrates on finite configurations, which is absurd.

IV. PROOF OF THEOREM 3 AND THEOREM 4

We start proving Theorem 3. Since the critical contact process is ergodic (see Bezuidenhout and Grimmett (1989)), we only need to consider the case in which $\lambda > \lambda_c$. Let $\tilde{\xi}_t$ and ν_t be as in Section 3 and let μ_t be the distribution of ξ_t^Z , where ξ_t represents the contact process. Then, define

$$\tilde{\xi}_{t,n}^1(x) = \tilde{\xi}_{t,n}^1(x - n) \quad n = 1, 2, \dots$$

and denote by $\tilde{\nu}_{t,n}$ the distribution of $\tilde{\xi}_{t,n}^1$, $n = 1, 2, \dots$. It now follows from Theorem 4 in Gray (1986), that $\hat{\nu}_{t,n} \leq \tilde{\nu}_t$. It is easy to see that

$$\lim_{n \rightarrow \infty} \tilde{\nu}_{t,n} = \mu_t,$$

hence

$$\hat{\mu}_t \leq \tilde{\nu}_t \quad \forall t \geq 0$$

and

$$(4.1) \quad \frac{1}{T} \int_0^T \hat{\mu}_t dt \leq \frac{1}{T} \int_0^T \tilde{\nu}_t dt \quad \forall T > 0.$$

Since we are considering the case in which $\lambda > \lambda_c$, it follows from the uniqueness of the invariant measure for the process seen from the edge and from the proof of its existence given in Section 3, that for some subsequence $T_n \uparrow \infty$ we have

$$\lim_n \frac{1}{T_n} \int_0^{T_n} \tilde{\nu}_t dt = \tilde{\mu}_\lambda.$$

Hence, the theorem follows from (4.1) and the fact that $\lim_t \mu_t = \mu_\lambda$.

Theorem 4 is an almost immediate consequence of Theorem 3 and Lemma 1. From these results we have :

$$(4.2) \quad \begin{aligned} 0 &\leq \tilde{\mu}_\lambda(\eta : \eta(-x) = 1) - \mu_\lambda(\eta : \eta(-x) = 1) \\ &\leq Ce^{-\gamma x} + P(\xi_{x/a}^Z(0) = 1) - \mu_\lambda(\eta : \eta(-x) = 1). \end{aligned}$$

From the fact that the Contact Process is self-dual we have :

$$P(\xi_{x/a}^Z(0) = 1) - \mu_\lambda(\eta : \eta(-x) = 1) = P(x/a < \tau^* < \infty).$$

The result follows immediately from this equality and (4.2).

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