

ANNALES DE L'I. H. P., SECTION B

PRZEMYSŁAW MATUŁA

ZDZISŁAW RYCHLIK

The invariance principle for nonstationary sequences of associated random variables

Annales de l'I. H. P., section B, tome 26, n° 3 (1990), p. 387-397

http://www.numdam.org/item?id=AIHPB_1990__26_3_387_0

© Gauthier-Villars, 1990, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section B » (<http://www.elsevier.com/locate/anihpb>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

The invariance principle for nonstationary sequences of associated random variables

by

Przemysław MATUŁA

and

Zdzisław RYCHLIK

Institute of Mathematics, Maria Curie-Skłodowska University
Plac M. Curie-Skłodowskiej 1, 20-031 Lublin, Poland

ABSTRACT. — In this paper we present the invariance principle for nonstationary sequences of associated random variables. The presented results extend the invariance principles given by T. Birkel (1988).

RÉSUMÉ. — Dans cet article nous présentons le principe d'invariance pour des suites non-stationnaires des variables aléatoires associées. Les résultats présentés étendent les principes d'invariances donnés par T. Birkel (1988).

Key words : sequences of associated random variables; invariance principles; nonstationary sequences of random variables; central limit theorem

Classification A.M.S : AMS 1980 Subject Classifications : Primary 60 B 10; Secondary 60 F 05

I. INTRODUCTION

Let $\{X_n, n \geq 1\}$ be a sequence of random variables, defined on some probability space (Ω, \mathcal{F}, P) , such that $EX_n = 0, EX_n^2 < \infty, n \geq 1$.

Let us put :

$$S_0 = 0, \quad S_n = \sum_{k=1}^n X_k, \quad s_n^2 = ES_n^2, \quad n \geq 1.$$

Let $\{k_n, n \geq 0\}$ be an increasing sequence of real number such that

$$(1.1) \quad 0 = k_0 < k_1 < k_2 < \dots$$

and

$$(1.2) \quad \lim_{n \rightarrow \infty} \max_{1 \leq i \leq n} (k_i - k_{i-1})/k_n = 0.$$

Let us define $m(t) = \max\{i : k_i \leq t\}, t \geq 0$,

and

$$(1.3) \quad W_n(t) = S_{m_n(t)}/s_n, \quad t \in [0, 1], \quad n \geq 1,$$

where $m_n(t) = m(n; t) = m(tk_n)$. Without loss of generality we may and do assume that $s_n^2 > 0, n \geq 1$.

Let $D[0, 1]$ be the space of functions defined on $[0, 1]$ that are right-continuous and have left-hand limits. We give the Skorohod J_1 -topology in $D[0, 1]$ (cf. [1, §14]). By W we will denote the Wiener measure on $D[0, 1]$ with the corresponding Wiener process $\{W(t), 0 \leq t \leq 1\}$.

Let us observe that for every $n \geq 1$ the function $\omega \rightarrow W_n(t, \omega)$, defined by (1.3), is a measurable map from (Ω, \mathcal{F}) into $(D[0, 1], \mathcal{B}(D))$, where $\mathcal{B}(D)$ is the Borel α -algebra induced by the Skorohod J_1 -topology.

In this paper we present some sufficient and necessary conditions for weak convergence, in the space $D[0, 1]$, of sequences of random elements $\{W_n, n \geq 1\}$ to the Wiener measure W . We investigate sequences $\{X_n, n \geq 1\}$ that satisfy a condition of positive dependence called association. No stationary is required. We obtain the invariance principle that is much more general than that given by Birkel [2]. The presented theorems generalize the results given in [2] and, at the same time, give similar results for associated processes which are known for φ -mixing sequences of random variables [8] and independent random variables [7, p. 221].

We remind that a finite collection $\{X_1, \dots, X_n\}$ of random variables is associated if for any two coordinatewise nondecreasing functions f_1, f_2

on \mathbb{R}^n such that $\hat{f}_i = f_i(X_1, \dots, X_n)$ has finite variance for $i = 1, 2$ there holds $\text{Cov}(\hat{f}_1, \hat{f}_2) \geq 0$. An infinite collection is associated if every finite subcollection is associated (cf. [3]).

Many recent papers have been concerned with invariance principles for associated processes (cf. [2] and the references given there). But the authors have only considered the following process :

$$(1.4) \quad W_n^*(t) = S_{[nt]}/S_n, \quad t \in [0, 1]$$

Let us observe that in the case $k_n = n, n \geq 1$ the process $\{W_n(t), t \in [0, 1]\}$ defined by (1.3) is the same as the process (1.4).

Newman and Wright [6] obtained an invariance principle for stationary sequences of associated random variables satisfying a summability criterion on their covariances. From their result one can get the following :

THEOREM A (Newman, Wright) : *Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence of associated random variables with $EX_1 = 0$ and $EX_1^2 < \infty$.*

$$\text{If } 0 < \sigma^2 = \text{Cov}(X_1, X_1) + 2 \sum_{n=2}^{\infty} \text{Cov}(X_1, X_n) < \infty,$$

$$\text{then } W_n^* \xrightarrow[n \rightarrow \infty]{\mathcal{D}} W.$$

An invariance principle for nonstationary associated processes has been studied by Birkel [2]. He proved the following theorems :

THEOREM B (Birkel) : *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with $EX_n = 0$ and $EX_n^2 < \infty, n \geq 1$. Assume :*

$$(1.5) \quad \lim_{n \rightarrow \infty} s_n^{-2} E(S_{nk} S_{nl}) = \min(k, l) \quad \text{for } k, l \in \mathbb{N},$$

$$(1.6) \quad \{s_n^{-2} (S_{n+m} - S_m)^2 : m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}\} \text{ is uniformly integrable.}$$

$$\text{Then } W_n^* \xrightarrow[n \rightarrow \infty]{\mathcal{D}} W.$$

THEOREM C (Birkel) : *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with $EX_n = 0$ and $EX_n^2 < \infty, n \geq 1$. Then the following assertions are equivalent :*

$$(1.7) \quad \text{condition (1.5) is fulfilled and } \{X_n, n \geq 1\} \text{ satisfies the central limit theorem,}$$

$$(1.8) \quad W_n^* \xrightarrow[n \rightarrow \infty]{\mathcal{D}} W.$$

Let us observe that by Remark 2.3 [4] (see also Remark 1 [2]) if

$$W_n^* \xrightarrow[n \rightarrow \infty]{\mathcal{D}} W, \text{ then}$$

$$(1.9) \quad s_n^2 = nh(n),$$

where $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is slowly varying.

Thus now the natural question arises, how to define and prove an invariance principle for sequences $\{X_n, n \geq 1\}$ which do not satisfy the condition (1.9). As the example given in Section 5 shows, we can prove the convergence of $\{W_n, n \geq 1\}$ defined by (1.3) to W for nonstationary sequences such that $\{s_n^2, n \geq 1\}$ is not of the form (1.9). Of course (1.9) is a consequence of (1.5), thus we have to replace Birkel's conditions (1.5) and (1.6) by more general ones, which would be more appropried for nonstationary sequences. Such conditions are presented in Section 2. In section 3 we present some lemmas which are needed in the proofs of the results. The proofs of our theorems are given in Section 4, while in Section 5 we give an example of associated process to which the results of this paper apply but Birkel's theorems do not.

II. RESULTS

We shall now state the main results of the paper.

THEOREM 1 : *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with $EX_n = 0$ and $EX_n^2 < \infty, n \geq 1$. Let $\{k_n, n \geq 1\}$ be a sequence of real numbers satysfying (1.1) and (1.2). Assume :*

$$(2.1) \quad \text{for } p, q \in \mathbb{N} \quad \lim_{n \rightarrow \infty} s_n^{-2} ES_{m_n(p)} S_{m_n(q)} = \min(p, q),$$

$$(2.2) \quad \{(s_{n+m}^2 - s_m^2)^{-1} (S_{n+m} - S_m)^2 : n \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}\}$$

is uniformly integrable.

Then $W_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} W$.

Remark 1 : Let us observe that condition (2.2) can be replaced by the following condition :

$$(2.3) \quad \{(s_{n+m} - s_m)^2 / E(S_{n+m} - S_m)^2 : n \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}\}$$

is uniformly integrable.

This fact follows from the inequality

$$E(S_{n+m} - S_m)^2 / (s_{n+m}^2 - s_m^2) \leq 1,$$

which is satisfied for nonnegatively correlated random variables.

Remark 2 : We also remark that condition (2.1), which by our opinion is much more appropriated for nonstationary sequences than that given in [2], is necessary for our invariance principle.

Namely if $W_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} W$, then $s_n^2 = k_n h(k_n)$, where $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is slowly varying (cf. [8], p. 617). Hence by (1.1) and (1.2), for every $i \in \mathbb{N}$, $\lim_{n \rightarrow \infty} s_n^2 s_{m_n(i)}^2 = i$. Thus it is enough to prove condition (v) of Lemma 2, but this can be similarly done as in [2, p. 60], we omit details

Remark 3 : Let us observe that conditions (2.2) or (2.3) are much more appropriated for nonstationary sequences than condition (2.2) of Birkel [2]. It is enough to consider a sequence of independent random variables. But, taking into account the proof of theorem 1 (cf. (4.4)-(4.8)), one can note, that, for example, if $k_n = n^\alpha$, $\alpha \in \mathbb{N}$, then condition (2.2) can be replaced by the following condition :

$$\{s_n^{-2}(S_{n+m} - S_m)^2 : m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}\} \text{ is uniformly integrable}$$

so the same one as given by Birkel [2].

THEOREM 2 : Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with $EX_n = 0$ and $EX_n^2 < \infty$, $n \geq 1$. Let $\{k_n, n \geq 0\}$ be a sequence of real numbers satisfying (1.1) and (1.2), then the following assertions are equivalent :

- (i) condition (2.1) holds and $s_n^{-1} S_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} N(0, 1)$;
- (ii) $W_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} W$.

Let us observe that from Theorem 2 we can easily get well known theorem of Prohorov [7, p. 221]. It is enough to put $k_n = s_n^2$, $n \geq 1$.

III. AUXILIARY LEMMAS

LEMMA 1 : Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with $EX_n = 0$ and $EX_n^2 < \infty$, $n \geq 1$. If $\{k_n, n \geq 0\}$ satisfies (1.1) and (1.2), then the following conditions are equivalent :

- (i) for every $i \in \mathbb{N}$, $\lim_{n \rightarrow \infty} s_n^{-2} s_{m_n(i)}^2 = i$;

(ii) for every $t > 0$, $\lim_{n \rightarrow \infty} s_n^{-2} s_{m_n(t)}^2 = t$.

Proof : it suffice to show (i) \Rightarrow (ii). In the proof of this implication we use the idea introduced by Birkel [2, Lemma 1]. First we consider the special case $t = q/p$, $p, q \in \mathbb{N}$.

It is easy to see that by (1.1) and (1.2) $k_{n+1}/k_n \xrightarrow[n \rightarrow \infty]{} 1$. Futhermore, for every $n, p, q \in \mathbb{N}$, we get

$$(m(m_n(q/p); p) \leq m_n(q).$$

On the other hand, since the random variables $\{X_n, n \geq 1\}$ are nonnegatively correlated, the sequence $\{s_n^2, n \geq 1\}$ is nondecreasing. Hence we obtain

$$(3.1) \quad \limsup_{n \rightarrow \infty} (s_n^{-2} s_{m_n(q/p)}^2 \leq \limsup_{n \rightarrow \infty} \{ (s_n^{-2} s_{m_n(q)}^2 / (s_{m_n(q/p)}^{-2} s_{m(m_n(q/p); p)}^2)) \} = q/p.$$

Now let us consider the following subsequences

$$\{m_r(p) + 1 : r \geq 1\}, \quad 1 \leq 1 \leq m_{r+1}(p) - m_r(p).$$

We have $k_{m(r;p)} \leq pk_r < k_{m(r;p)+1}$ and for every $1 \leq 1 \leq m_{r+1}(p) - m_r(p)$

$$(q/p)k_{m(r;p)+1} \geq (q/p)k_{m(r;p)+1} > (q/p)pk_r \geq k_{m(r;q)}.$$

Thus

$$(3.2) \quad m(r; q) \leq m(m_r(p) + 1; q/p), \quad \text{for } 1 \leq 1 \leq m_{r+1}(p) - m_r(p).$$

Let $i \in \mathbb{N}$, $i \geq 2$ be fixed, and let

$$n^* = n(r, q, i) = m_r(1/q(i - 1)) = m(k_r/q(i - 1)).$$

Of course $n^* \leq r$ and $n^* \rightarrow \infty$ as $r \rightarrow \infty$. Since $k_{r+1}/k_r \xrightarrow[r \rightarrow \infty]{} 1$, therefore there exists $j_0 \in \mathbb{N}$ such that for every $j \geq j_0$ $k_{j+1}/k_j < (i + q - 1)/(i - 1)$. Let $r_0 \in \mathbb{N}$, $r_0 \geq j_0$, be such an integer that for $r \geq r_0$ and $n^* \geq i_0$

$$(3.3) \quad k_{r+1}/k_r < (i + q - 1)/(i - 1) \quad \text{and} \quad k_{n^*+1}/k_{n^*} < (i + q - 1)/(i - 1).$$

Thus, by definition of n^* , we get

$$(3.4) \quad k_{n^*} \leq k_r/q(i - 1) < k_{n^*+1}.$$

Hence

$$(3.5) \quad r \geq m_{n^*}(q(i - 1)).$$

On the other hand, by (3.3) and (3.4), we obtain

$$k_{r+1} < k_r(i + q - 1)/(i - 1) < q(i + q - 1)k_{n^*+1} < k_{n^*}q(i + q - 1)^2 / (i - 1) \leq ([q(i + q - 1)^2/(i - 1)] + 1)k_{n^*},$$

so that

$$(3.6) \quad r + 1 \leq m_{n^*}([q(i + q - 1)^2/(i - 1)] + 1).$$

Hence, by (3.2), (3.5), (3.6) and (i), we get

$$(3.7) \quad \liminf_{r \rightarrow \infty} s_{m(r;p)+1}^{-2} s_{m(m_r(p)+1;q/p)}^2 \geq \liminf_{r \rightarrow \infty} s_{m(r+1;p)}^{-2} s_{m(r;q)}^2 \\ \geq \liminf_{r \rightarrow \infty} s_{m(m_{n^*};[q(i+q-1)^2/(i-1)]+1);p}^{-2} s_{m(m_{n^*};q(i-1));q}^2 \\ = q^2(i - 1)/p([q(i + q - 1)^2/(i - 1)] + 1) \xrightarrow{i \rightarrow \infty} q/p$$

Since $i \geq 2$ can be chosen arbitrarily large, (3.1) and (3.7) imply (ii) for $t = q/p$ $q, p \in \mathbb{N}$. For arbitrarily $t > 0$ (ii) follows from the rational case, as the sequence $\{s_n^2, n \geq 1\}$ is nondecreasing.

LEMMA 2 : Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with $EX_n = 0$ and $EX_n^2 < \infty, n \geq 1$. If $\{k_n, n \geq 0\}$ satisfies (1.1), (1.2) and for every $i \in \mathbb{N}, \lim_{n \rightarrow \infty} s_n^{-2} s_{m_n(i)}^2 = i$ then the following conditions are equivalent :

- (i) $\lim_{n \rightarrow \infty} s_n^{-2} E(S_{m_n(j)} - S_{m_n(i)})(S_{m_n(q)} - S_{m_n(p)}) = 0,$
for $i \leq j \leq p \leq q \in \mathbb{N} \cup \{0\}$;
- (ii) $\lim_{n \rightarrow \infty} s_n^{-2} E(S_{m_n(p)} S_{m_n(q)}) = \min(p, q),$
for $p, q \in \mathbb{N}$;
- (iii) $\lim_{n \rightarrow \infty} s_n^{-2} E(S_n S_{m_n(t)}) = t, \text{ for } t \in [0, 1]$;
- (iv) $\lim_{n \rightarrow \infty} s_n^{-2} E(S_{m_n(t)} - S_{m_n(s)})(S_{m_n(v)} - S_{m_n(u)}) = 0,$
for $0 \leq s \leq t \leq u \leq v$;
- (v) $\lim_{n \rightarrow \infty} s_n^{-2} E(S_{m_n(t)} - S_{m_n(s)})(S_{m_n(v)} - S_{m_n(u)}) = 0,$
for $0 \leq s \leq t \leq u \leq v \leq 1$.

Proof : At first we note that by Lemma 1, for every $t \geq 0,$

$$(3.8) \quad s_n^{-2} s_{m_n(t)}^2 \rightarrow t, \quad \text{as } n \rightarrow \infty.$$

Thus (ii) follows from (3.8) and (i). It is enough to put $i = 0, j = p, p \leq q.$

$$(ii) \Rightarrow (iii)$$

If $t = 0$, then (iii) holds. Let $0 < t \leq 1$ be given. Then $m_n(t) \leq n$ and, by (1.2), for sufficiently large n , $n \geq n_0 = n_0(t)$,

$$k_{m_n(t)} + 1/k_{m_n(t)} < 2, \quad \text{so that}$$

$$k_{m_n(t)} \leq tk_n < k_{m_n(t)+1} < 2k_{m_n(t)}.$$

Hence $k_n \leq (2/t)k_{m_n(t)} < ([2/t] + 1)k_{m_n(t)}$ and, in consequence $m_n(t) \leq n \leq m(m_n(t); [2/t] + 1)$. Since the random variables are nonnegatively correlated, for $n \geq n_0$, we have

$$1 \leq s_{m_n(t)}^{-2} ES_n S_{m_n(t)} \leq s_{m_n(t)}^{-2} ES_{m(m_n(t); [2/t]+1)} S_{m(m_n(t), 1)}$$

and the right hand side of the last inequality, by (ii) tends to 1 as $n \rightarrow \infty$. Thus, by (3.8), we get (iii).

The implication (iii) \Rightarrow (iv) can be proved similarly to (ii) \Rightarrow (iii). Now let us observe that implications (iv) \Rightarrow (v), (ii) \Rightarrow (i) and (v) \Rightarrow (iii) are trivial, thus the proof of Lemma 2 will be ended if we show that (iii) \Rightarrow (ii).

Let $p, q \in \mathbb{N}$ be given. If $p = q$ then (ii) holds. Assume that $p < q$. Since $k_{m(m_n(q); p/q)} \leq (p/q)k_{m_n(q)} \leq pk_n$, so that $m_n(p) \geq m(m_n(q); p/q)$. Choose $\varepsilon > 0$ such that $p(1 + \varepsilon)/q \leq 1$.

By (1.2) there exists $n_0 = n_0(\varepsilon)$ such that for every $n \geq n_0$ there holds

$$k_{m_n(q)+1}/k_{m_n(q)} < 1 + \varepsilon.$$

Hence for every $n \geq n_0$ we have

$$(q/p)k_{m_n(p)} \leq qk_n < k_{m_n(q)+1} < (1 + \varepsilon)k_{m_n(q)},$$

so that $m_n(p) \leq m(m_n(q); p(1 + \varepsilon)/q)$.

Since the random variables are nonnegatively correlated, we have for $n \geq n_0$

$$I_n(p, q) = s_n^{-2} ES_{m(m_n(q); p/q)} S_{m_n(q)} \leq s_n^{-2} ES_{m_n(p)} S_{m_n(q)}$$

$$\leq s_n^{-2} ES_{m(m_n(q); p(1+\varepsilon)/q)} S_{m_n(q)} = J_n(p, q, \varepsilon).$$

Futhermore, by (iii) and (3.8), $\lim_{n \rightarrow \infty} I_n(p, q) = p$ and $\lim_{n \rightarrow \infty} J_n(p, q, \varepsilon) = p(1 + \varepsilon)$. Since $\varepsilon > 0$ can be chosen arbitrary, we get (ii).

LEMMA 3 : Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with $EX_n = 0$ and $EX_n^2 < \infty, n \geq 1$. If $\{k_n, n \geq 0\}$ satisfies (1.1) and (1.2), then the following conditions are equivalent :

(3.9) for all $p, q \in \mathbb{N}, \lim_{n \rightarrow \infty} s_n^{-2} ES_{m_n(p)} S_{m_n(q)} = \min(p, q)$

(3.10) for all $p, q \in \mathbb{N}, p < q, \lim_{n \rightarrow \infty} s_{m_n(q-p)}^{-2} E(S_{m_n(q)} - S_{m_n(p)})^2 = 1$
and $\lim_{n \rightarrow \infty} s_n^{-2} s_{m_n(p)}^2 = p$.

The proof is simple, so we omit the details.

IV. PROOFS OF THEOREMS

Proof of theorem 1 : By Lemma 1 and Lemma 3, for every $t > 0$ we have

$$(4.1) \quad \lim_{n \rightarrow \infty} s_n^{-2} s_{m_n(t)}^2 = t$$

Hence, according to Lemma 2 for $0 \leq s \leq t \leq u \leq v$, we get

$$(4.2) \quad \lim_{n \rightarrow \infty} s_n^{-2} E(S_{m_n(t)} - S_{m_n(s)})(S_{m_n(v)} - S_{m_n(u)}) = 0.$$

At first we prove that the sequence $\{W_n, n \geq 1\}$ is tight. We will apply Theorem 15.5 of Billingsley [1]. Since $W_n(0) = 0, n \geq 1$, it is enough to show that for each positive ε and η , there exist a $\delta, 0 < \delta < 1$, and an integer n_0 , such that

$$(4.3) \quad P(w(W_n, \delta) \geq \varepsilon) \leq \eta \quad \text{for } n \geq n_0,$$

where $w(W_n, \delta) = \sup_{|s-t| < \delta} |W_n(s) - W_n(t)|$.

Let $\varepsilon > 0$ be given. Using the corollary to Theorem 8.3 [1, p.56], for every $r \in \mathbb{N}$, we obtain

$$(4.4) \quad P(w(W_n, 1/r) > \varepsilon) \leq \sum_{i=0}^{r-1} P\left(\max_{1 \in D_n(i, r)} |S_1 - S_{m_n(1/r)}| > \varepsilon s_n/3\right) \\ = \sum_{i=0}^{r-1} P_n(i, r, \varepsilon),$$

where $D_n(i, r) = \{1 : m_n(i/r) < 1 \leq m_n((i+1)/r)\}$.

Let $\alpha_1 = \alpha_2/2$, where $\alpha_2 = s_n \varepsilon/3 (E(S_{m_n((i+1)/r)} - S_{m_n(i/r)})^2)^{1/2}$. Since the random variables $\{X_n, n \geq 1\}$ are nonnegatively correlated and (4.1) holds, we have

$$s_n^{-2} E(S_{m_n((i+1)/r)} - S_{m_n(i/r)})^2 \leq s_n^{-2} (s_{m_n((i+1)/r)}^2 - s_{m_n(i/r)}^2) \leq 2/r,$$

for sufficiently large n , say $n \geq n_0(i, r)$. Thus, for $n \geq n_0(i, r)$ $\alpha_2 - \alpha_1 > 1$ iff $r > 72\varepsilon^{-2}$. Hence, by Corollary 5 [5], for $n \geq n_0(i, r)$ and $r > 72\varepsilon^{-2}$

$$(4.5) \quad P_n(i, r, \varepsilon) \leq \{1 - 36s_n^{-2}\varepsilon^{-2} E(S_{m_n((i+1)/r)} - S_{m_n(i/r)})^2\}^{-1} \\ \times P(|S_{m_n((i+1)/r)} - S_{m_n(i/r)}| \geq \varepsilon s_n/6) \\ \leq (1 - 72/r\varepsilon^2)^{-1} P(|S_{m_n((i+1)/r)} - S_{m_n(i/r)}| \geq \varepsilon s_n/6).$$

Now (4.4) and (4.5), for every $n \geq \max_{0 \leq i \leq r-1} n_0(i, r)$ and $r > 72\varepsilon^{-2}$, imply :

$$(4.6) \quad P(w(W_n, 1/r) > \varepsilon) \leq (1 - 72/r\varepsilon^2)^{-1} \sum_{i=0}^{r-1} P(|S_{m_n((i+1)/r)} - S_{m_n(i/r)}| \geq \varepsilon s_n/6).$$

Let $\varepsilon_0 < \varepsilon/6$, $r > 72\varepsilon^{-2}$ and $1 \leq i \leq r$ be fixed. Then, according to (4.1), there exists $n_0(i, r, \varepsilon_0)$ such that for $n \geq n_0(i, r, \varepsilon_0)$ we get

$$(4.7) \quad \begin{aligned} P(|S_{m_n((i+1)/r)} - S_{m_n(i/r)}| \geq \varepsilon s_n/6) \\ \leq P(|S_{m_n((i+1)/r)} - S_{m_n(i/r)}| \geq \varepsilon_0 r^{1/2} (s_{m_n((i+1)/r)}^2 - s_{m_n(i/r)}^2)^{1/2}). \end{aligned}$$

Thus (4.6) and (4.7) for $n \geq \max_{0 \leq i < r} n_0(i, r, \varepsilon_0)$ yield

$$(4.8) \quad \begin{aligned} P(w(W_n, 1/r) > \varepsilon) &\leq (1 - 72/r\varepsilon^2)^{-1} r(\varepsilon_0^2 r)^{-1} \\ &\times \sup_{n \in \mathbf{N}, m \in \mathbf{N} \cup \{0\}} E(S_{n+m} - S_m)^2 I(|S_{n+m} - S_m| \\ &\geq \varepsilon_0 r^{1/2} (s_{n+m}^2 - s_m^2))(s_{n+m}^2 - s_m^2)^{-1}, \end{aligned}$$

so that (4.8) and (2.2) imply (4.3).

Let X be a limit distribution of a subsequence $\{W_{n'}, n' \geq 1\}$ of $\{W_n, n \geq 1\}$. Then, by theorem 15.5 [1], $P(X \in C) = 1$. Thus by Theorem 19.1 [1] the proof of Theorem 1 will be ended if we show $EX(t) = 0$, $EX^2(t) = t$ and X has independent increments. But by (2.2) and (4.1), for every $t \in [0, 1]$, the sequence $\{W_n^2(t), n \geq 1\}$ is uniformly integrable and $W_{n'}(t) \xrightarrow[n' \rightarrow \infty]{\mathcal{D}} X(t)$, so that by Theorem 5.4 [1], $EX(t) = 0$, $EX^2(t) = t$. The proof that X has independent increments is the same as in [2], it is enough to use (4.2), so we omit the details.

Proof of theorem 2 : The proof is similar to the proof of Theorem 2 of Birkel [2] so will be omitted.

V. EXAMPLE

Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables such that $P(X_n = \pm n) = 1/(2n^{1/2})$, $P(X_n) = 0 = 1 - n^{-1/2}$, $n \geq 1$; of course $\{X_n, n \geq 1\}$ is an associated process (cf. [3]) and

$$s_n^2 = \sum_{k=1}^n k^{3/2} \approx 3n^{5/2}/5 \neq nh(n).$$

Futhermore the sequence $\{X_n, n \geq 1\}$ satisfies the Lindeberg condition. Thus if $W_n(t) = S_{m_n(t)}/s_n$, $t \in [0, 1]$, where $m_n(t) = \max\{0 \leq i \leq n : s_i^2 \leq ts_n^2\}$, $S_0 = 0$; $S_0^2 = 0$, then by Theorem 2 $W_n \xrightarrow[n \rightarrow \infty]{D} W$.

But for this sequence the condition (1.9) (and therefore (1.5)) is not fulfilled, so by Theorem C, the invariance principle in the form considered by Birkel fails.

Taking into account Theorems B, C and the example given above one can note that in general for a given sequence $\{X_n, n \geq 1\}$ of random variables the problem is, whether there exists a sequence $\{m_n(t), n \geq 1\}$ $t \in [0, 1]$ of positive integers such that if $W_n(t) = S_{m_n(t)}/s_n$, then $W_n \xrightarrow[n \rightarrow \infty]{D} W$. The example shows that in general the sequence $m_n(t) = [nt]$ is not an appropriate one.

ACKNOWLEDGEMENT

The authors thank the referee for useful remark and comments on the original manuscript.

REFERENCES

- [1] P. BILLINGSLEY, *Convergence of Probability Measures*, (Wiley, New York, 1968).
- [2] T. BIRKEL, The invariance principle for associated processes, *Stoch. Process. Appl.*, **27** (1988), 57-71.
- [3] J. ESARY, F. PROSCHAN and D. WALKUP, Association of random variables with applications, *Ann. Math. Statist.*, **38** (1967), 1466-1474.
- [4] N. HERRNDORF, The invariance principle for φ -mixing sequences, *Z. Warsch. Verw. Geb.* **63** (1983), 97-108.
- [5] C.M. NEWMAN and A.L. WRIGHT, Associated variables random variables and martingale inequalities, *Z. Warsch. Verw. Geb.*, **59** (1982), 361-371.
- [6] C.M. NEWMAN and A.L. WRIGHT, An invariance principle for certain dependent sequences, *Ann. Probab.*, **9** (1981), 671-675.
- [7] K.R. PARTASARATHY, *Probability Measures on Metric Spaces*, (Academic Press, New York, 1967).
- [8] Z. RYCHLIK and I. SZYSZKOWSKI, The invariance principle for φ -mixing sequences, *Teor. Verojatnost. Primenen.*, **32** (1987), 616-619.

(Manuscript received February 22, 1989;
in revised form : September 25, 1989)